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W-Algebras and Superalgebras from Constrained WZW Models: A Group Theoretical Classification

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Abstract. We present a classification of W algebras and superalgebras arising in Abelian as well as non Abelian Toda theories. Each model, obtained from a constrained WZW action, is related with an $Sl(2)$ subalgebra (resp. $OSp(1|2)$ superalgebra) of a simple Lie algebra (resp. superalgebra) $\mathcal{G}$. However, the determination of an $U(1)_Y$ factor, commuting with $Sl(2)$ (resp. $OSp(1|2)$), appears, when it exists, particularly useful to characterize the corresponding W algebra. The (super) conformal spin contents of each W (super) algebra is performed. The class of all the superconformal algebras (i.e. with conformal spins $s \leqq 2$) is easily obtained as a byproduct of our general results.

Contents

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1. Introduction

Lots of efforts have been done recently to detect and understand the infinite dimensional symmetries which underly two dimensional field theories. A particular role is played by Toda theories, since each of them possesses a W symmetry [1, 2]. More recently, it has been shown that in fact Toda models can be seen as constrained WZW models [3]. One can say that such a property reinforces the fundamental role of WZW models in the realm of conformal field theories. It also provides a natural framework to compute explicitly the W algebras which then appear.

In order to reduce a WZW model to a Toda one, some of the conserved current components have to be set to constants or zero. It can be realized that, from a given simple Lie algebra (or superalgebra) $\mathscr{G}$, different choices of constraints can be proposed, each of them giving rise to a different Toda model, to which will be associated a W (super)algebra. Actually, to each such a Toda model corresponds a (integral or half-integral) grading [4] of $\mathscr{G}$ specified by a Cartan element $H \in \mathscr{G}$. In other words, $\mathscr{G}$, which is chosen maximally non-compact, admits a vector space decomposition:

$$\mathscr{G} = \bigoplus_{h \in \frac{1}{2}\mathbf{Z}} \mathscr{G}_h \quad \text{with} \quad [H, X_h] = hX_h \quad \text{for any} \quad X_h \in \mathscr{G}_h \,. \tag{1.1}$$

As an example, the usual or Abelian Toda model associated to $\mathscr{G}$ is obtained by taking H as the Cartan generator of the principal $Sl(2)$ in the algebra (or superprincipal $OSp(1|2)$ in the superalgebra) $\mathscr{G}$.

For each such a grading H can be defined either a $Sl(2)$ [4–7] or an $Sl(2) \oplus U(1)$ [8] (resp. $OSp(1|2)$ [9, 10] or $OSp(1|2) \oplus U(1)$) sub(super)algebra of $\mathscr{G}$ generated by $\{M_0, M_\pm\} \oplus \{Y\}$ (resp. $\{M_0, M_\pm, F_\pm\} \oplus \{Y\}$) and such that $H = M_0 + Y$. More precisely, even when the $U(1)$ part is not zero, the $Sl(2)$ (resp. $OSp(1|2)$) subalgebra is sufficient to characterize the W algebra: one can then say that the different Toda models in $\mathscr{G}$ are classified by the different $Sl(2)$ (resp. $OSp(1|2)$) subalgebras of G. However, interesting information on the structure of the corresponding W algebra can be obtained when the Y generator exists. As will be shown below, a conserved hypercharge can be associated to it, which may greatly simplify the Poisson Bracket (PB) computation of the different primary fields constituting the W algebra. The usefulness of the conserved hypercharge Y is illustrated to calculate the PB of the algebra of spins 2, $\frac{3}{2}$, $\frac{3}{2}$, 1 first considered in [11, 12].

Once given the $Sl(2)$ (resp. $Osp(1|2)$) subalgebra of $\mathscr{G}$, the conformal spin content of the corresponding W algebra can easily be deduced, owing to the existence of the so-called highest weight Drinfeld–Sokolov gauge [13], from the decomposition of the $\mathscr{G}$-adjoint representation w.r.t. $Sl(2)$ (resp. $OSp(1|2)$). Since, as mentioned above, the existence of a $U(1)$ factor in $\mathscr{G}$ commuting with $Sl(2)$ (resp. $OSp(1|2)$) can help for the computation of the PB between W generators, it is the determination of $Sl(2) \oplus U(1)$ (resp. $OSp(1|2) \oplus U(1)$) subalgebras in $\mathscr{G}$ that we plan to perform, as well as the reduction of the $\mathscr{G}$-adjoint representation w.r.t. each $Sl(2) \oplus U(1)$ (resp. $OSp(1|2) \oplus U(1)$) algebras.

Let us distinguish for a while the Lie algebra case (or bosonic case), from the Lie superalgebra one. Much is known, owing to Dynkin, concerning the first point. Indeed, the determination of the semi-simple subalgebras of a simple Lie algebra has been considered by this author [14], and made explicit for algebras of rank up to 6 by Lorente and Gruber [15]. We have added the determination of $Sl(2) \oplus U(1)$ algebras and provided, by means of general formulae, the reduction of the adjoint representation of a classical algebra $\mathscr{G}$ w.r.t. each of its $Sl(2) \oplus U(1)$ subalgebras. In particular, in each case, the construction of the defining vector from which can immediately be deduced the gradation has been performed. Such a detailed study of the bosonic case was necessary to complete the W algebra part, and also to settle down some material for the super W case.

As already mentioned, in the supersymmetric case, when $\mathscr{G}$ is a simple Lie superalgebra, the $Sl(2)$ algebra is replaced by its supersymmetric "extension" $OSp(1|2)$ [9, 10]. It is therefore the classification of $OSp(1|2) \oplus U(1)$ subsuperalgebras in $\mathscr{G}$ which is now of interest. Contrarily to the bosonic case, not very much is known about the classification of $OSp(1|2)$ subalgebras in a simple Lie superalgebra. Note that a first attempt in that direction can be found in [9], and also that [10] deals only with Abelian super Toda models, in other words with the super principal $OSp(1|2)$ in a simple superalgebra. Hereafter, we explicitly achieve this classification in a way which, we believe, is clear and allows a direct use. As in the algebra case, general formulae for the decomposition of the fundamental and adjoint representations of a simple Lie superalgebra with respect to $OSp(1|2) \oplus U(1)$ subsuperalgebras are given, and the (super) conformal spin content of the super W algebras determined. In order to illustrate these results, and mainly to allow a comparison with the extended superconformal algebras [16], tables are constructed for superalgebras of rank up to 4.

2. *W* Algebras and (Half-)Integral Gradings

2.1. W Algebras in Toda Theories. It has been elegantly shown that, starting from a WZW model, the action of which is $S(g)$ and the fields $g(x)$ (resp. superfields $g(x, \theta)$) belong to the group (resp. supergroup) G, and imposing some of the components of the conserved (super) currents to be constant or zero leads to a Toda model.

Let us, at this point, briefly fix some notations.

As far as G is a group, the WZW conserved currents read:

$$J_+ = g^{-1}\partial_+ g \quad J_- = (\partial_- g)g^{-1} \tag{2.1}$$

with

$$\partial_- J_+ = \partial_+ J_- = 0 \,. \tag{2.2}$$

When considering a supersymmetric WZW model [10], a supergroup element will locally be defined as:

$$g(x, \theta) = \exp(\varphi^i B_i + \psi^j F_j) \,, \tag{2.3}$$

where the φ^i (resp. ψ^j) are bosonic (resp. fermionic) superfields, and the B_i (resp. F_j) commuting (resp. anticommuting) generators in the considered finite dimensional superalgebra $\mathscr{G}$. Then the corresponding supercurrents are:

$$J_+ = \hat{g}^{-1} D_+ g, \quad J_- = (D_- g) g^{-1} , \tag{2.4}$$

where $\hat{g}$ differs from g by the change of sign on its fermionic generator part, the bosonic ones staying unchanged. We note that the fermionic character of $D_\pm = \theta_\pm \partial_{x_\pm} + \partial_{\theta_\pm}$ implies the supercurrents to develop as:

$$J = \Psi^i B_i + \Phi^j F_j \tag{2.5}$$

the Ψ^i being fermionic and the Φ^j bosonic superfields.

The choice of the J components which are constrained to be constant with respect to those which are put to zero naturally defines a grading (see 1.1) on the (super)algebra $\mathscr{G}$. The simplest and most known example is the Abelian Toda model relative to $\mathscr{G}$. In this case the J components associated to the opposite of the simple roots have constant values while those relative to the other negative roots are put to zero. The grading is ruled by the generator H, sum of the Cartan generators in the Cartan Weyl basis. The $\mathscr{G}$ subalgebra $\mathscr{G}_0$ is exactly the Cartan subalgebra of $\mathscr{G}$ in this basis, the simple root generators $E_{+\alpha}$ form the $\mathscr{G}$ subspace $\mathscr{G}_{+1}$, and their partners $E_{-\alpha}$ the subspace $\mathscr{G}_{-1}$; finally $\mathscr{G}_+$ is constructed from the positive roots and $\mathscr{G}_-$ from the negative ones.

As could be expected, imposing a set of constraints reduces the huge symmetry provided by the Kac–Moody current algebra to a subset of quantities, polynomials in the current components and their derivatives, which will constitute a W-algebra. For example, the original conformal symmetry of the WZW model itself is broken when constraints corresponding to the grading H are imposed, and in order to construct the Virasoro symmetry for this Toda model a H dependent correction term has to be added to the former one.

More precisely, the stress energy tensor reads [3]:

$$T_H = \frac{1}{2} \mathrm{Tr}\, J^2 - \mathrm{Tr}\, H \partial J \tag{2.6}$$

when $\mathscr{G}$ is an algebra, and [10]:

$$T_H = \mathrm{Str}\left(\frac{1}{3} J \hat{J} J + \frac{1}{2} J D J\right) - \mathrm{Str}(H \cdot D^2 J) \tag{2.7}$$

when $\mathscr{G}$ is a superalgebra.

The determination of the other generators of the W algebra can be achieved as follows.

If $\mathscr{G}$ is an algebra, one selects in $\mathscr{G}_{-1}$ a (constant) element M_- such that [3]

$$\mathrm{Ker}(\mathrm{ad}\, M_-) \cap \mathscr{G}_+ = \{0\} . \tag{2.8}$$

Then one expresses J as:

$$J = M_- + J_{>-1} , \tag{2.9}$$

where the variable dependent part $J_{>-1}$ belongs to $\bigoplus_h \mathscr{G}_h$ with $h > -1$.

If $\mathcal{G}$ is a superalgebra, then one picks up in $\mathcal{G}_{-1/2}$ a fermionic (constant) element F_- with $\{F_-, F_-\} = M_- \neq 0$ such that:

$$\mathrm{Ker}(\mathrm{ad}\, F_-) \cap \mathcal{G}_+ = \{0\}\,, \tag{2.10}$$

and one expresses J as:

$$J = F_- + J_{>-\frac{1}{2}}\,. \tag{2.11}$$

Finally one has just to use the gauge transformations:

$$J \to g J g^{-1} + (\partial g) g^{-1}\,, \tag{2.12}$$

where g belongs to the local Lie groups generated by $\mathcal{G}_+$, or:

$$J \to \hat{g} J g^{-1} + (D_- g) g^{-1} \tag{2.13}$$

in the supersymmetric case, to transform J into:

$$J' = \mu_- + \sum_h W_{h+1}(J) X_h \quad \text{with} \quad \mu_- = M_- \text{ (resp. } F_-)\,, \tag{2.14}$$

where the $W_{h+1}(J)$ are gauge invariant polynomials generating the W algebra associated to the Toda theory, and $X_h \in \mathcal{G}$.

Note that the condition (2.8) expresses the non-degeneracy for $h > 0$, of the operator:

$$\mathrm{ad} M_- : \mathcal{G}_h \to \mathcal{G}_{h-1}\,. \tag{2.15}$$

Then Drinfeld–Sokolov (D.S.) gauges can be used to determine a complete set of gauge invariant quantities $W_{h+1}(J)$. In the highest weight D.S. gauge, each $W_{h+1}(J)$ is "carried" by the highest weight X_h of a given $Sl(2)$ subalgebra built from M_-.

The PB among W generators will be calculated from the PB:

$$\{J^a(x), J^b(y')\}_{PB} = i f^{ab}_c \delta(x - x') J^c(x') + k\eta^{ab} \partial_x \delta(x - x')\,, \tag{2.16}$$

when $\mathcal{G}$ is a Lie algebra and:

$$\begin{aligned}\{J^a(X), J^b(X')\}_{PB} = {} & i(-1)^{[a](1+[b])} f^{ab}_c\, \delta(X - X') J^c(X') \\ & + k\eta^{ab} D_x \delta(X - X')\,,\end{aligned} \tag{2.17}$$

when $\mathcal{G}$ is a superalgebra. f^{ab}_c are the structure constants, η^{ab} the scalar product and k the central extension parameter of the Kac Moody (super)algebra; by $[a]$ is expressed the $\mathbb{Z}_2$ grading of the generator T^a: $[a] = 0$ (resp. 1) if T^a is a commuting (resp. anticommuting) generator (see [10] for more details).

Using (2.6) (or 2.7) one understands that $W_{h+1}(J)$ has a (super) conformal weight $1 + h$ under T_H.

2.2. Properties of (Half) Integral Gradations. We have presented in [8] three propositions establishing a correspondence between (integral and half integral) gradings of a simple Lie algebra $\mathcal{G}$ which specify Toda theories, and $Sl(2) \oplus U(1)$ subalgebras of $\mathcal{G}$. The generalisation to the superalgebra case is straightforward, replacing the $Sl(2)$ part by its "supersymmetric extension" $OSp(1|2)$. Therefore, we limit ourselves to enounce hereafter these properties.

Let H be a grading operator of a (super)algebra $\mathcal{G}$. Then:

Proposition 1.

i) *$\mathscr{G}$ being an algebra, any element $M_- \in \mathscr{G}_-$ can be embedded in one of its $Sl(2)$ subalgebra.*

ii) *$\mathscr{G}$ being a superalgebra, any fermionic element $F_- \in \mathscr{G}_-$* with $\{F_-, F_-\} = M_- \neq 0$ *can be embedded in one of its $OSp(1|2)$ subalgebra.*

Proposition 2. *Let $M_- \in \mathscr{G}_{-1}$* (resp. *$F_- \in \mathscr{G}_{-1/2}$*). *Then, it is always possible to write H as:*

$$H = M_0 + Y \tag{2.18}$$

with M_0 being the Cartan part of an $Sl(2)$ algebra constructed from M_- (resp. *an $OSp(1|2)$ superalgebra built on F_-*), *and the generator Y commuting, when non-zero, with this three* (resp. *five*) *dimensional subalgebra.*

Moreover, the $Sl(2)$ part constructed from M_- (resp. $OSp(1|2)$ superalgebra built on F_-) is unique up to a conjugation by group elements generated from the subalgebra $\mathscr{G}_0 = \mathrm{Ker}(\mathrm{ad}\, M_-) \cap \mathscr{G}_0$.

Proposition 3.

i) *Let M_-, M_0, M_+ and Y generate an $Sl(2) \oplus U(1)$ subalgebra of $\mathscr{G}$ with $M_- \in \mathscr{G}_{-1}$ and $M_0 + Y = H$. Decompose $\mathscr{G}$, considered as a vector space, into irreducible representations $\mathscr{D}_{j_i}(y_i)$ of this algebra, where y_i denotes the eigenvalue of Y on the $Sl(2)$ representation $\mathscr{D}_{j_i}$. Then*

$$\mathrm{Ker}(\mathrm{ad}\, M_-) \cap \mathscr{G}_+ = \{0\} \quad \text{iff } |y_i| \leqq j_i \quad \textit{for any } \mathscr{D}_{j_i}(y_i) \textit{ in } \mathscr{G}\,. \tag{2.19}$$

ii) *Let M_-, F_-, M_0, F_+, M_+ and Y generate an $OSp(1|2) \oplus U(1)$ subsuperalgebra of $\mathscr{G}$ with $F_- \in \mathscr{G}_{-1/2}$ and $M_0 + Y = H$. Decompose $\mathscr{G}$, considered as a vector space, into irreducible representations $\mathscr{R}_{j_i}(y_i)$ of this algebra, where y_i denotes the eigenvalue of Y on the $OSp(1|2)$ representation $\mathscr{R}_{j_i} = \mathscr{D}_{j_i} \oplus \mathscr{D}_{j_i - 1/2}$. Then*

$$\mathrm{Ker}(\mathrm{ad}\, F_-) \cap \mathscr{G}_+ = \{0\} \quad \text{iff } |y_i| \leqq j_i \quad \textit{for any } \mathscr{R}_{j_i}(y_i) \textit{ in } \mathscr{G}\,. \tag{2.20}$$

In the following, we will call the condition (2.19) (resp. 2.20) *a non-degeneracy condition for* $\mathrm{ad}\, M_-$ (resp. *for* $\mathrm{ad}\, F_-$). *Of course, as the grades satisfy $h_i = j_i + y_i$, one must impose $h_i \in \frac{1}{2}\mathbb{Z}$ in the $\mathscr{G}$ adjoint representation to have (half)integral grading.*

These three propositions have to be completed by:

Proposition 4. *The gradations $H = M_0 + Y$ and M_0 lead to the same W algebra.*

This last proposition has been proven in [7]. From the point of view of the decomposition under $Sl(2) \oplus U(1)$, note that (2.19) ensures that the highest weight of the $Sl(2)$ subalgebra are in the $\mathscr{G}_{\geqq 0}$ part of $\mathscr{G}$ for both $H = M_0$ and $H = M_0 + Y$ gradations. This is in agreement with the "halving" used in [7].

We end this section by a property which characterizes the position of Y in $\mathscr{G}$.

Proposition 5. *Let $\mathscr{C}$ be the commutant of the chosen subalgebra $Sl(2)$* (resp. *$OSp(1|2)$*) *in $\mathscr{G}$. Then Y, when it exists, belongs to the commutant of the semi-simple part $\mathscr{C}_1$* of $\mathscr{C}$ in $\mathscr{G}$.

Before proving this proposition, let us first remark that, once $Sl(2)$ (resp. $OSp(1|2)$) is given, a necessary condition for Y to exist is the existence in the $\mathscr{G}$-decomposition w.r.t. $Sl(2)$ (resp. $OSp(1|2)$) of a $\mathscr{D}_0$ (resp. $\mathscr{R}_0 = \mathscr{D}_0$) part.

Now, let us remark that Y belongs obviously to the commutant $\mathscr{C}$ of the subalgebra $Sl(2)$ (resp. $OSp(1|2)$) under consideration, but cannot be any element of $\mathscr{C}$. Note that a subalgebra of a simple Lie algebra $\mathscr{G}$ is reductive, that is $\mathscr{C}$ decomposes as:

$$\mathscr{C} = \mathscr{C}_1 \oplus U(1) \oplus \cdots \oplus U(1)\,, \tag{2.21}$$

where $\mathscr{C}_1$ is a semi-simple Lie (super)algebra. The non-degeneracy condition implies that any element of $\mathscr{C}$ reads as $\mathscr{D}_0(0)$, that is Y commutes with any element in $\mathscr{C}$. It follows that Y must belong to the $U(1) \oplus \cdots \oplus U(1)$ part commuting with $\mathscr{C}_1$ in $\mathscr{C}$.

2.3. Primary Fields of W Algebras. The spin of the W generators corresponding to a given gradation H are obtained from the highest weights of the $Sl(2) \oplus U(1)$ (resp. $OSp(1|2) \oplus U(1)$) decomposition of the $\mathscr{G}$-adjoint representation (DS gauge). Now, we have to know whether the W generators are (super) primary fields under T_H.

(Super) primary fields satisfy the following Poisson bracket:

$$\{T_H(x),\, W_{h+1}(x')\}_{\mathrm{PB}} = (h+1)\, W_{h+1}(x')\partial_x\delta(x-x') + \partial W_{h+1}(x')\delta(x-x')\,, \tag{2.22}$$

$$\begin{aligned}\{T_H(X),\, W_{h+1/2}(X')\}_{\mathrm{PB}} &= \left(h+\frac{1}{2}\right)\partial_x\delta(X-X')\,W_{h+1}(X') \\ &\quad + \delta(X-X')\partial W_{h+1}(X') - \frac{1}{2}D_X\delta(X-X')\,W_{h+1/2}(X')\,,\end{aligned} \tag{2.23}$$

where we have used for the supersymmetric case the conventions

$$X = (x, \theta) \quad \text{and} \quad \delta(X - X') = (\theta - \theta')\,\delta(x - x')\,. \tag{2.24}$$

Note that (2.22) corresponds to PB between fields and (2.23) between superfields. We will say, in the former case, that W_{h+1} has *spin* $h+1$, whereas, in the latter case, $W_{h+1/2}$ carries a *superspin*[1] $h+\frac{1}{2}$. In fact, it is clear from the expression of T_H that the only generators W_{h+1}(resp. $W_{h+1/2}$) which are not primary are those which satisfy $\langle H, X_h\rangle \neq 0$, where $\langle\,,\rangle$ is the $\mathscr{G}$ non-degenerated scalar product and X_h is the generator of $\mathscr{G}$ carrying W_{h+1} (resp. $W_{h+1/2}$) in (2.14). This implies that X_h is a Cartan generator, so that $h = 0$ and $W_{h+1} \equiv W_1$ (resp. $W_{h+1/2} \equiv W_{1/2}$) forms a singlet representation of $Sl(2)$ (resp. $OSp(1|2)$). Actually, by linear combinations, one can always eliminate these non-primary generators, but one. Since for $H = M_0$ all the W generators are primary (except T_{M_0} of course), we can think of the non-primary generator as carried by Y itself [7]. This is ensured by the equality

$$\langle H, Y\rangle = \langle M_0 + Y, Y\rangle = \langle Y, Y\rangle \neq 0 \quad \text{iff } Y \neq 0\,. \tag{2.25}$$

We will call this (super) generator W_1^Y (resp. $W_{1/2}^Y$). Note that because of its spin 1 (resp. superspin $\frac{1}{2}$), the PB of T_H with W_1^Y ($W_{1/2}^Y$) differs from the PB of T_H with a (super) primary field only by a central extension term, corresponding to a second order derivative (resp. fermionic derivative D) of a (super)delta distribution.

[1] Note that the two components of the superfield $W_{h+1/2}$ are of conformal spins $h+\frac{1}{2}$ and $h+1$

Thus, all the W generators are primary with respect to T_H, except T_H itself and, when $Y \neq 0$, a spin 1 generator W_1^Y (resp. *a superspin $\frac{1}{2}$ generator $W_{1/2}^Y$*) *carried by Y. In that case, W_1^Y ($W_{1/2}^Y$) differs from a primary field* (resp. *superfield*) *by a central extension term.*

2.4. Classification of Constrained WZW Models. The above properties suggest a way to determine all the different (super)Toda models associated with (half-)integral gradings of a simple Lie (super)algebra $\mathscr{G}$, and their corresponding (super) W algebras, namely:

i) Classify all the $Sl(2)$ (resp. $OSp(1|2)$) sub(super) algebras of $\mathscr{G}$.

ii) Add to each of these simple sub(super) algebras a commuting $U(1)$ factor such that in the decomposition of the $\mathscr{G}$ adjoint representation into $Sl(2) \oplus U(1)$ representations $\mathscr{D}_{j_i}(y_i)$ (resp. $OSp(1|2) \oplus U(1)$ representations $\mathscr{R}_{j_i}(y_i)$), the following conditions hold:

$$|y_i| \leqq j_i \quad i = 1, \ldots, n \,, \tag{2.26}$$

$$j_i + y_i \in \mathbb{Z} \quad \text{(integral grading)} \quad j_i + y_i \in \frac{1}{2}\mathbb{Z} \quad \text{(half-integral grading)} \,. \tag{2.27}$$

Note that the y_i values are naturally restricted when calculating the $Sl(2) \oplus U(1)$ (resp. $OSp(1|2) \oplus U(1)$) decomposition of the adjoint representation of $\mathscr{G}$ coming from the product of fundamental representations already decomposed into $Sl(2) \oplus U(1)$ (resp. $OSp(1|2) \oplus U(1)$) representations: this will be made explicit in the following.

iii) Then to each such an $Sl(2) \oplus U(1)$ (resp. $OSp(1|2) \oplus U(1)$) sub(super)algebra of $\mathscr{G}$ satisfying (2.26) and (2.27) there will correspond a classical (i.e. PB) W algebra generated by the n elements $W_{h_1+1}, \ldots, W_{h_n+1}$ (resp. $W_{h_1+1/2}, \ldots, W_{h_n+1/2}$) of conformal (super)spin under the (super)Virasoro algebra defined in (2.6, 2.7) $h_1 + 1, \ldots, h_n + 1$ (resp. $h_1 + \frac{1}{2}, \ldots, h_n + \frac{1}{2}$) with h_i given by

$$h_i = y_i + j_i \tag{2.28}$$

as a consequence of a Drinfeld–Sokolov highest weight gauge [3, 13].

iv) Reconstruct the grading H from the $Sl(2) \oplus U(1)$ (resp. $OSp(1|2) \oplus U(1)$) decomposition. Varying Y for a fixed $Sl(2)$ ($OSp(1|2)$ super)algebra will give all the isomorphic gradations.

v) Deduce informations of the PB from the $Sl(2) \oplus U(1)$ (resp. $OSp(1|2) \oplus U(1)$) reduction.

These five steps will be made explicit in the following. In Part I, we will focus on the algebras case, while in Part II the previous results will be used to state the superalgebras case.

Part I. *W* Algebras Built on Lie Algebras

3. The Different *Sl*(2) Subalgebras in a Simple Lie Algebra $\mathscr{G}$

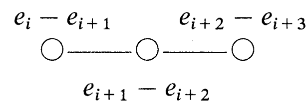
The classification of $Sl(2)$ subalgebras of a simple Lie algebra $\mathscr{G}$ has been achieved by Dynkin [14]. His techniques can be summarized as follows:

Any $Sl(2)$ subalgebra in $\mathcal{G}$ can be seen, up to a few exceptions occurring in D_n and $E_{6,7,8}$ algebras[2], *as the principal $Sl(2)$ algebra of a regular $\mathcal{G}$ subalgebra.*

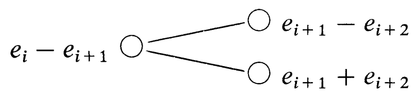
In the D_n case, one has to add $\left[\frac{n-2}{2}\right]$ $Sl(2)$ subalgebras, each of them being a principal subalgebra of the singular ones:

$$B_i \oplus B_j \quad \text{with } i+j=n-1 \quad \text{and } i \neq j\,. \tag{3.1}$$

For $\mathcal{G} = B_n$ and D_n, $n > 3$, the diagram ○—○—○ must be considered twice, one been related to an "algebra A_3," and the other one to "D_3." Indeed, the $\mathcal{G}$ subdiagram

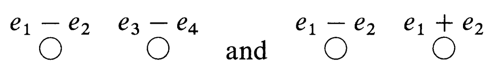


defines a system of simple roots for "A_3," while the subdiagram



provides a system of simple roots of "D_3." In order to convince the reader, we remark that the fundamental representation of D_n reduces with respect to A_3 as $\underline{2n} = \underline{4} + \bar{\underline{4}} + (2n-8)\underline{1}$, and with respect to D_3 as $\underline{2n} = \underline{6} + (2n-6)\underline{1}$.

Again B_n and D_n admit two different types of $2A_1$ subalgebras associated to the diagrams



The fundamental of D_n reduces with respect to the first algebras as $\underline{2n} = (\underline{2} + \bar{\underline{2}}, \underline{0}) + (\underline{0},\ \underline{2} + \bar{\underline{2}}) + (2n-8)\ (\underline{0}, \underline{0})$ and with respect to the second as $\underline{2n} = (\underline{2}, \underline{2}) + (2n-4)(\underline{0}, \underline{0})$. We can note that as well as in case 1), it is the bifurcation appearing in the (extended) DD of (B_n) D_n which is responsible for these doublings, the first reduction being associated with "$2A_1$," and the second with "D_2."

4. *Sl*(2) Decompositions of Simple Lie Algebras

Given any $Sl(2)$ subalgebra of a Lie algebra $\mathcal{G}$ in the A, B, C, D series, we need to know the decomposition of the adjoint representation of $\mathcal{G}$ with respect to this three dimensional subalgebra. For such a purpose, we will first compute the $Sl(2)$ decomposition of the $(1, 0, 0, \ldots, 0)$ fundamental representation of $\mathcal{G}$. We will deduce the $Sl(2)$ decomposition of the $\mathcal{G}$ adjoint representation by computing the product of the fundamental representation by its contragredient one: for the A_n series, the adjoint representation is given by this product, once throwing away a trivial representation, while in the B_n and D_n (resp. C_n) cases, one has to select the antisymmetric (resp. symmetric) part.

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[2] We will not discuss the $E_{6,7,8}$ cases: see [15]

4.1. The $\mathscr{G}$ Fundamental Representation with Respect to a $Sl(2)$ Subalgebra.

4.1.1. $Sl(n)$ case. Any $Sl(2)$ subalgebra is the principal subalgebra of a (sum of) $Sl(p)$ subalgebra(s) in $Sl(n)$. For each $Sl(p)$ subalgebra will correspond a $\mathscr{D}_{(p-1)/2}$ representation of $Sl(2)$ in the $\underline{n}$ of $Sl(n)$, which will be completed with singlets.

For instance, if we look at the $Sl(2)$ principal subalgebra of $Sl(p) \oplus Sl(q)$ in $Sl(n)$, we will have

$$\underline{n} = \mathscr{D}_{(p-1)/2} \oplus \mathscr{D}_{(q-1)/2} \oplus (n-p-q)\mathscr{D}_0 \,. \tag{4.1}$$

4.1.2. $Sp(2n)$ case. An $Sl(2)$ subalgebra is the principal subalgebra of a (sum of) $Sp(2p)$ subalgebra(s), $Sl(2)^1$ subalgebra(s), or $Sl(q)^2$ subalgebra(s). The superscript refers to the Dynkin index of the $Sl(m)$ subalgebra considered: it is 1 when the $Sl(2)$ subalgebra is constructed on a long root, and 2 in the other cases. The $Sp(2p)$ subalgebra contributes to the fundamental representation via a $\mathscr{D}_{p-(1/2)}$ $Sl(2)$ representation, while the $Sl(q)^2$ (resp. $Sl(2)^1$) yields the $\mathscr{D}_{(q-1)/2} + \bar{\mathscr{D}}_{(q-1)/2}$ (resp. $\mathscr{D}_{1/2}$) representations. The $\underline{2n}$ representation is then completed by singlets. For example, for the decomposition of $Sp(2n)$ under the principal $Sl(2)$ of $Sp(2p) \oplus Sl(q)^2 \oplus r\,Sl(2)^1$, we have:

$$\underline{2n} = \mathscr{D}_{p-(1/2)} \oplus (\mathscr{D}_{(q-1)/2} \oplus \bar{\mathscr{D}}_{(q-1)/2}) \oplus r\mathscr{D}_{1/2} \oplus (2n-2p-2q-2r)\mathscr{D}_0 \,. \tag{4.2}$$

4.1.3. $SO(n)$ case. When $Sl(2)$ is principal subalgebra of either an $SO(2p+1)$ or an $SO(2p+2)$ one, the $\underline{n}$ fundamental of $SO(n)$ contains a $\mathscr{D}_p$ representation. In the case of an $Sl(q)$, $q \neq 2$, subalgebra, then it is the sum $\mathscr{D}_{(q-1)/2} \oplus \bar{\mathscr{D}}_{(q-1)/2}$ which shows up. For $q=2$, one must distinguish the case $Sl(2)^1$ (long root) which leads to $\mathscr{D}_{1/2} \oplus \bar{\mathscr{D}}_{1/2}$ from the case $Sl(2)^2$ (short root) leading to $\mathscr{D}_1$.

Finally, we have mentioned in Sect. 3 the existence of two $Sl(2) \oplus Sl(2)$ and two $Sl(4) \equiv SO(6)$ algebras. The corresponding decompositions are:

$$Sl(2) \oplus Sl(2) \begin{cases} \underline{n} = 2(\mathscr{D}_{1/2} \oplus \bar{\mathscr{D}}_{1/2}) \oplus (n-8)\mathscr{D}_0 \\ \underline{n} = \mathscr{D}_1 \oplus (n-3)\mathscr{D}_0 \end{cases}, \tag{4.3}$$

$$Sl(4) \begin{cases} \underline{n} = \mathscr{D}_{3/2} \oplus \bar{\mathscr{D}}_{3/2} \oplus (n-8)\mathscr{D}_0 \\ \underline{n} = \mathscr{D}_2 \oplus (n-5)\mathscr{D}_0 \end{cases}. \tag{4.4}$$

We recall that for each $SO(2n)$ subalgebras, there exist $Sl(2)$ algebras related to the singular embeddings $SO(2k+1) \oplus SO(2n-2k-1)$, $0 < 2k < n$.

4.2. The $\mathscr{G}$ Adjoint Representation with Respect to $Sl(2)$ Subalgebras. To achieve the $Sl(2)$ reduction of the adjoint representation for any simple Lie algebra $\mathscr{G}$ from the knowledge of the fundamental representation, the following formulae are especially convenient:

$$(\mathscr{D}_n \times \mathscr{D}_n)_{\mathrm{A}} = \mathscr{D}_{2n-1} \oplus \mathscr{D}_{2n-3} \oplus \cdots \oplus \mathscr{D}_1 \quad n \in \mathbb{Z}\,, \tag{4.5}$$

$$(\mathscr{D}_{n-(1/2)} \times \mathscr{D}_{n-(1/2)})_{\mathrm{A}} = \mathscr{D}_{2n-2} \oplus \mathscr{D}_{2n-4} \oplus \cdots \oplus \mathscr{D}_0 \quad n \in \mathbb{Z}\,, \tag{4.6}$$

$$(\mathscr{D}_n \times \mathscr{D}_n)_{\mathrm{S}} = \mathscr{D}_{2n} \oplus \mathscr{D}_{2n-2} \oplus \cdots \oplus \mathscr{D}_0 \quad n \in \mathbb{Z}\,, \tag{4.7}$$

$$(\mathscr{D}_{n-(1/2)} \times \mathscr{D}_{n-(1/2)})_{\mathrm{S}} = \mathscr{D}_{2n-1} \oplus \mathscr{D}_{2n-3} \oplus \cdots \oplus \mathscr{D}_1 \quad n \in \mathbb{Z}\,, \tag{4.8}$$

the subscript (A) S standing for (Anti-)Symmetric part of the product. We have also, for $m, p \in \mathbb{Z}$ and $j, k \in \frac{1}{2}\mathbb{Z}$:

$$\{(m\mathscr{D}_j)\times(m\mathscr{D}_j)\}_{\mathrm{A}} = \frac{m(m+1)}{2}(\mathscr{D}_j\times\mathscr{D}_j)_{\mathrm{A}} \oplus \frac{m(m-1)}{2}(\mathscr{D}_j\times\mathscr{D}_j)_{\mathrm{S}}$$

$$= m(\mathscr{D}_j\times\mathscr{D}_j)_{\mathrm{A}} \oplus \frac{m(m-1)}{2}(\mathscr{D}_j\times\mathscr{D}_j)\,, \tag{4.9}$$

$$\{(m\mathscr{D}_j)\times(m\mathscr{D}_j)\}_{\mathrm{S}} = \frac{m(m+1)}{2}(\mathscr{D}_j\times\mathscr{D}_j)_{\mathrm{S}} \oplus \frac{m(m-1)}{2}(\mathscr{D}_j\times\mathscr{D}_j)_{\mathrm{A}}$$

$$= m(\mathscr{D}_j\times\mathscr{D}_j)_{\mathrm{S}} \oplus \frac{m(m-1)}{2}(\mathscr{D}_j\times\mathscr{D}_j)\,, \tag{4.10}$$

$$\{(m\mathscr{D}_j)\times(p\mathscr{D}_k)\oplus(p\mathscr{D}_k)\times(m\mathscr{D}_j)\}_{\mathrm{A}} = \{(m\mathscr{D}_j)\times(p\mathscr{D}_k)\oplus(p\mathscr{D}_k)\times(m\mathscr{D}_j)\}_{\mathrm{S}}$$

$$= mp(\mathscr{D}_j\times\mathscr{D}_k)\,, \tag{4.11}$$

where $m\mathscr{D}_j$ stands for the direct sum of m representations $\mathscr{D}_j$.

5. $Sl(2)\oplus U(1)_Y$ Decompositions of Simple Lie Algebras

5.1. $Sl(n)$ Algebras. We start by considering the case $\mathscr{G} = Sl(n)$, which has already been studied in some detail in [8]. Let us recall that, for such an algebra, all the $Sl(2)$ representations of equal dimension $\mathscr{D}_j$ have the same $U(1)_Y$ eigenvalue y_j in the $\underline{n}$ fundamental representation, so that a general decomposition reads

$$\underline{n} = \bigoplus_j n_j\mathscr{D}_j(y_j) \quad \text{with } j\text{'s all different}\,, \tag{5.1}$$

where n_j is the multiplicity of $\mathscr{D}_j$. One will have to impose to the product $\underline{n}\times\bar{n} - \mathscr{D}_0(0)$, the non-degeneracy condition $|y| \leq j$ for any representation $\mathscr{D}_j(y)$ in the $\mathscr{G}$ adjoint representation. Note that the condition $y \in \frac{1}{2}\mathbb{Z}$, which ensures a (half-)integral gradation, has to be imposed only in the adjoint representation, and *not* in the fundamental.

As an example, consider the $Sl(2)$ which is principal with respect to A_n in A_{n+2}. Then

$$\underline{n+3} = \mathscr{D}_{n/2}(y)\oplus 2\mathscr{D}_0\left(-\frac{n+1}{2}y\right), \tag{5.2}$$

$$\overline{n+3} = \mathscr{D}_{n/2}(-y)\oplus 2\mathscr{D}_0\left(\frac{n+1}{2}y\right), \tag{5.3}$$

where we have, of course, imposed the traceless condition for Y. It follows:

$$\underline{n+3}\times\overline{n+3} - \mathscr{D}_0(0) = \left(\bigoplus_{j=1}^{n}\mathscr{D}_j(0)\right)\oplus 2\mathscr{D}_{n/2}\left(\frac{n+3}{2}y\right)$$

$$\oplus\, 2\mathscr{D}_{n/2}\left(-\frac{n+3}{2}y\right)\oplus 4D_0(0) \tag{5.4}$$

with the condition $\left|\frac{n+3}{2}y\right| \leq \frac{n}{2}$ and $y' = \frac{n+3}{2}y \in \frac{1}{2}\mathbb{Z}$.

5.2. $SO(n)$ Algebras. Now, let us turn to the $\mathscr{G} = B_n$ or D_n case. These algebras have a real fundamental representation, so that if $\mathscr{D}_j(y)$, $y \neq 0$, appears in the decomposition, then $\mathscr{D}_j(-y)$ must also be present with the same multiplicity. To get the adjoint representation, we have to improve the formulae (4.5–4.11) by specifying the $U(1)$ dependence. Using the reality of the adjoint representation, one is led to

$$\begin{aligned}&\{[n\mathscr{D}_j(y) \oplus n\mathscr{D}_j(-y)] \times [n\mathscr{D}_j(y) \oplus n\mathscr{D}_j(-y)]\}_{\mathrm{A}}\\&\quad = n^2(\mathscr{D}_j \times \mathscr{D}_j)\,(0) \oplus ((n\mathscr{D}_j) \times (n\mathscr{D}_j))_{\mathrm{A}}\,(2y)\\&\quad\quad \oplus ((n\mathscr{D}_j) \times (n\mathscr{D}_j))_{\mathrm{A}}\,(-2y) \quad \text{for } j \in \frac{1}{2}\mathbb{Z} \text{ and } n \in \mathbb{Z}\end{aligned} \tag{5.5}$$

where $((n\mathscr{D}_j) \times (n\mathscr{D}_j))_{\mathrm{A}}$ is computed via (4.9). This formula shows that from a term $n\mathscr{D}_j(y)$ in the fundamental, we will always get a term $\mathscr{D}_0(2y)$ in the adjoint, except if $n = 1$ and j is integer. Moreover, when $n = 1$ and j is integer but non-zero, there will always exist a $\mathscr{D}_1(\pm 2y)$ term in the adjoint representation. The non-degeneracy condition $|y| \leqq j$ for $\mathscr{D}_j(y)$ will then lead to set $y = 0$, except for $n = 1$ and j integer, where, for $j \neq 0$, we will have $|2y| \leqq 1$ and $2y \in \frac{1}{2}\mathbb{Z}$, that is $y = 0$, or $y = \pm\frac{1}{4}$, or $y = \pm\frac{1}{2}$.

Thus, for the orthogonal series, the only $Sl(2)$ representation with non-zero $U(1)$ eigenvalues are those which appear in the fundamental representation as $n(\mathscr{D}_p(y) \oplus \mathscr{D}_p(-y))$ with $n = 1$ and p integer. Moreover, for $p \neq 0$, we have $|2y| = 0$, or $\frac{1}{2}$, or 1.

Note that these restrictions are necessary but not sufficient conditions on y: we still have to impose the non-degeneracy condition in the $\mathscr{G}$ adjoint. To be complete, let us add the formula:

$$\begin{aligned}&\{2[n\mathscr{D}_j(y) \oplus n\mathscr{D}_j(-y)] \times [p\mathscr{D}_k(y') \oplus p\mathscr{D}_k(-y')]\}_{\mathrm{A}}\\&\quad = (\mathscr{D}_n \times \mathscr{D}_p)(y + y') \oplus (\mathscr{D}_n \times \mathscr{D}_p)(-(y + y'))\\&\quad\quad \oplus (\mathscr{D}_n \times \mathscr{D}_p)(y - y') \oplus (\mathscr{D}_n \times \mathscr{D}_p)(-(y - y'))\,.\end{aligned} \tag{5.6}$$

As an example, we look at the principal $Sl(2)$ of $SO(2n-1)$ in $SO(2n+1)$:

$$\begin{aligned}&\underline{2n+1} = \mathscr{D}_{n-1}(0) \oplus \mathscr{D}_0(y) \oplus \mathscr{D}_0(-y) = \overline{2n+1}\,,\\&(\underline{2n+1} \times \underline{2n+1})_{\mathrm{A}} = (\mathscr{D}_{2n-3} \oplus \mathscr{D}_{2n-1} \oplus \cdots \oplus \mathscr{D}_1 \oplus \mathscr{D}_0)(0)\\&\qquad\qquad \oplus \mathscr{D}_{n-1}(y) \oplus \mathscr{D}_{n-1}(-y)\,,\end{aligned} \tag{5.7}$$

with the condition $|y| \leqq n - 1$.

5.3. $Sp(2n)$ Algebras. Finally, let us study the case $\mathscr{G} = C_n$. From the $SO(n)$ case, it is easy to deduce the rule:

$$\begin{aligned}&\{[n\mathscr{D}_j(y) \oplus n\mathscr{D}_j(-y)] \times [n\mathscr{D}_j(y) \oplus n\mathscr{D}_j(-y)]\}_{\mathrm{S}}\\&\quad = n^2(\mathscr{D}_j \times \mathscr{D}_j)(0) \oplus ((n\mathscr{D}_j) \times (n\mathscr{D}_j))_{\mathrm{S}}\,(2y)\\&\quad\quad \oplus ((n\mathscr{D}_j) \times (n\mathscr{D}_j))_{\mathrm{S}}\,(-2y) \quad \text{for } j \in \frac{1}{2}\mathbb{Z} \text{ and } n \in \mathbb{Z}\,,\end{aligned} \tag{5.8}$$

where $((n\mathscr{D}_j) \times (n\mathscr{D}_j))_{\mathrm{S}}$ is computed via (4.10).

Then, the $Sl(2) \oplus U(1)$ decomposition of C_n is deduced from the B_n one by exchanging integer and half-integer:

For the symplectic series, the only $Sl(2)$ representations with non-zero $U(1)$ eigenvalues are those which appear in the fundamental representation as $n(\mathscr{D}_{p+\frac{1}{2}}(y) \oplus \mathscr{D}_{p+\frac{1}{2}}(-y))$ with $n = 1$ and p integer. Moreover, the allowed eigenvalues for the $U(1)$ generator Y are $|2y| = 0$, or $\frac{1}{2}$, or 1.

We illustrate these results on the decomposition under the $Sl(2)$ of $Sl(2)^2 \oplus Sp(2n-2)$ in $Sp(2n+2)$:

$$\underline{2n+2} = \mathscr{D}_{n-(3/2)}(0) \oplus \mathscr{D}_{\frac{1}{2}}(y) \oplus \mathscr{D}_{\frac{1}{2}}(-y)\,, \tag{5.9}$$

$$\overline{2n+2} = \mathscr{D}_{n-(3/2)}(0) \oplus \mathscr{D}_{\frac{1}{2}}(-y) \oplus \mathscr{D}_{\frac{1}{2}}(y)\,, \tag{5.10}$$

$$\begin{aligned}(\underline{2n+2} \times \overline{2n+2})_S = {} & (\mathscr{D}_{2n-3} \oplus \mathscr{D}_{2n-5} \oplus \cdots \oplus \mathscr{D}_1)(0) \oplus (\mathscr{D}_1 \oplus \mathscr{D}_0)(0) \\ & \oplus \mathscr{D}_1(2y) \oplus \mathscr{D}_1(-2y) \oplus (\mathscr{D}_{n-1} \oplus \mathscr{D}_{n-2})(y) \\ & \oplus (\mathscr{D}_{n-1} \oplus \mathscr{D}_{n-2})(-y)\,,\end{aligned} \tag{5.11}$$

with $|2y| \leqq 1$.

6. Classification of (Half-)Integral Gradings

The decomposition of the adjoint of a simple Lie algebra $\mathscr{G}$ in terms of $Sl(2) \oplus U(1)$ representations gives an exhaustive classification of the different constrained WZW theory arising from a (half-)integral grading. Moreover, the different values of Y (at fixed $Sl(2)$ subalgebra) leads to the equivalent theories [7]. Thus, if we know how to reconstruct the gradation H from this decomposition, we will be able to give an explicit classification of gradations. This is the aim of this section.

6.1. Defining vectors. An $Sl(2)$ algebra in a simple Lie algebra $\mathscr{G}$ is specified [14] by its defining vector $(f_1, \ldots, f_r)$, itself defined from the relation

$$M_0 = \sum_{i=1}^{r} f_i H_i \quad f_i \text{ rational}\,, \tag{6.1}$$

where M_0 denotes the Cartan part of $Sl(2)$ and $\{H_1, \ldots, H_r\}$ a Cartan subalgebra of $\mathscr{G}$.

For the A, B, C, D algebras of rank up to 6, a defining vector for all $Sl(2)$ subalgebras has been explicitly computed in [15], and we will use the same normalization here, up to a global factor 2. We compute them in the general case.

We set for a while $Y = 0$, and look at the gradation produced by M_0, Cartan generator of a given $Sl(2)$ subalgebra of $\mathscr{G}$. This $Sl(2)$ subalgebra can always be seen as the principal embedding of a (regular or singular) subalgebra of $\mathscr{G}$.

First consider the case $\mathscr{G} = A_n$. The defining vector components are just the eigenvalues of M_0, since one can always diagonalize M_0 with hermitian matrices.

Then, we have the rules:

$$A_{2p} \subset A_n \to f = \left(p, p-1, \ldots, 1, \underbrace{0, \ldots, 0,}_{n+1-2p} -1, -2, \ldots, -p \right), \quad (6.2)$$

$$A_{2p+1} \subset A_n \to f = \left(p + \frac{1}{2}, p - \frac{1}{2}, \ldots, \frac{1}{2}, \underbrace{0, \ldots, 0,}_{n-2p-1} \frac{-1}{2}, \frac{-3}{2}, \ldots, -p - \frac{1}{2} \right). \quad (6.3)$$

For example, the defining vector of A_2 (resp. A_1) in A_4 is $(1, 0, 0, 0, -1)$ (resp. $(\frac{1}{2}, 0, 0, 0, -\frac{1}{2})$). The defining vector of $A_2 \oplus A_1$ is $(1, \frac{1}{2}, 0, -\frac{1}{2}, -1)$.

Let us now turn to the $SO(n)$ case. Because of the antisymmetry of the matrices in the fundamental representation, the Cartan generators cannot be diagonal. In fact, they are constructed with σ_2 matrices on the diagonal. Each σ_2 matrix possesses $+1$ and -1 as eigenvalues, so that one has only to specify the positive M_0-eigenvalues in the defining vector. The general rules are:

$$B_p \text{ or } D_{p+1} \subset B_n \to f = (p, p-1, \ldots, 1, 0, \ldots 0), \quad (6.4)$$

$$D_{p+1} \subset D_n \to f = (p, p-1, \ldots, 1, 0, \ldots 0), \quad (6.5)$$

$$A_{2p} \subset B_n \text{ or } D_n \to f = (p, p, p-1, p-1, \ldots, 1, 1, 0, \ldots 0), \quad (6.6)$$

$$A_{2p+1} \subset B_n \text{ or } D_n \to f = \left(p + \frac{1}{2}, p + \frac{1}{2}, p - \frac{1}{2}, p - \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, 0, \ldots 0 \right). \quad (6.7)$$

As there are some exceptional embeddings of $Sl(2)$ algebras in orthogonal ones, there will be also exceptions for the defining vectors. For $A_3 \equiv D_3$, they are two different defining vectors, one associated to "A_3," and the other one to "D_3":

$$\text{“}A_3\text{”} \subset B_n \text{ or } D_n \to f = \left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0 \right), \quad (6.8)$$

$$\text{“}D_3\text{”} \subset B_n \text{ or } D_n \to f = (2, 1, 0, \ldots, 0). \quad (6.9)$$

They are also two defining vectors for $2A_1 \subset SO(m)$,

$$\text{“}2A_1\text{”} \subset B_n \text{ or } D_n \to \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0 \right), \quad (6.10)$$

$$\text{“}D_2\text{”} \subset B_n \text{ or } D_n \to (1, 0, \ldots, 0). \quad (6.11)$$

Finally, for the short root of B_n, we have

$$A_1^2 \subset B_n \to (1, 0, \ldots, 0). \quad (6.12)$$

The defining vectors associated to the singular embeddings $(B_i \oplus B_j) \subset D_n$ (with $i + j = n - 1$, $i \neq j$) are computed with the above rules.

Finally, we study the case of $Sp(2n)$ algebras. The rules are similar to those of $SO(n)$ algebras:

$$A_1^1 \subset C_n \to f = \left(\frac{1}{2}, 0, \ldots, 0 \right), \quad (6.13)$$

$$A_{2p}^2 \subset C_n \to f = (p, p, p-1, p-1, \ldots, 1, 1, 0, \ldots 0), \quad (6.14)$$

$$A_{2p+1}^2 \subset C_n \rightarrow f = \left(p+\frac{1}{2}, p+\frac{1}{2}, p-\frac{1}{2}, p-\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, 0, \ldots 0\right), \tag{6.15}$$

$$C_p \subset C_n \rightarrow f = \left(p+\frac{1}{2}, p-\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots 0\right). \tag{6.16}$$

6.2. Case of $Sl(2) \oplus U(1)$ Decomposition. When $H \neq M_0$, we can no longer speak about defining vector for H, since H cannot be embedded in an $Sl(2)$ algebra. However, it is still possible to compute a vector $f = (f_1, \ldots, f_n)$ that defines H, using the relation (6.1). We give hereafter the rules to compute this vector associated to H.

Let us first look at the $SO(n)$ case, where Y appears, in the fundamental representation, only in combinations $\mathscr{D}_m(y) \oplus \mathscr{D}_m(-y)$ with m integer. The rule is then

$$\mathscr{D}_m(y) \oplus \mathscr{D}_m(-y) \text{ in Fund}^1\ (m \in \mathbb{Z}_+)$$
$$\rightarrow f = (m+y, m-y, m-1+y, m-1-y, \ldots, 1+y, 1-y, y, 0, \ldots, 0)\,. \tag{6.17}$$

For example, for $A_4 \subset D_6$, we have

$$\underline{12} = \mathscr{D}_2(y_2) \oplus \bar{\mathscr{D}}_2(-y_2) \oplus \mathscr{D}_0(y_0) \oplus \bar{\mathscr{D}}_0(-y_0)$$
$$f = (2+y_2, 2-y_2, 1+y_2, 1-y_2, y_2, y_0)\,. \tag{6.18}$$

For the case $\mathscr{G} = A_n$, the defining vector can be read directly in the fundamental decomposition: the piece corresponding to a representation $\mathscr{D}_i(y_i)$ in the fundamental is $(i+y_i, i-1+y_i, \ldots, -i+y_i)$. Note that the different eigenvalues y_i are related by a traceless condition:

$$\sum_i m_i(2i+1)y_i = 0 \quad \text{for} \quad \underline{n+1} = \bigoplus_i m_i \mathscr{D}_i(y_i)\,. \tag{6.19}$$

They are determined in the adjoint representation, by the usual condition $|y| \leqq j$ and $y \in \frac{1}{2}\mathbb{Z}$ for any representation $\mathscr{D}_j(y)$ in the adjoint.

For example, for the reduction of A_2 with respect to its regular A_1 algebra, we have

$$\underline{2} = \mathscr{D}_{\frac{1}{2}}(y) \oplus \mathscr{D}_0(-2y), \quad \text{thus} \quad f = \left(\frac{1}{2}+y, -\frac{1}{2}+y, -2y\right), \tag{6.20}$$

$$\underline{8} = (\mathscr{D}_1 \oplus \mathscr{D}_0)(0) \oplus \mathscr{D}_{\frac{1}{2}}(3y) \oplus \mathscr{D}_{\frac{1}{2}}(-3y)\,, \tag{6.21}$$

$$|\pm 3y| \leqq \frac{1}{3} \quad \text{and} \quad \pm 3y \in \frac{1}{2}\mathbb{Z} \Rightarrow y = 0, \pm\frac{1}{6}\,. \tag{6.22}$$

Finally, for the symplectic algebras, the rules are analogous to those of the B_n case, that is:

$$\mathscr{D}_{m+\frac{1}{2}}(y) \oplus \mathscr{D}_{m+\frac{1}{2}}(-y) \text{ in Fund}^1\ (m \in \mathbb{Z}_+)$$
$$\rightarrow f = \left(m+\frac{1}{2}+y, m+\frac{1}{2}-y, m-\frac{1}{2}+y, m-\frac{1}{2}-y, \ldots, \frac{1}{2}+y, \frac{1}{2}-y, 0, \ldots, 0\right). \tag{6.23}$$

7. Poisson Brackets of W Algebras

7.1. Generalities. The knowledge of the spin contents of a W algebra with the use of a $Sl(2) \oplus U_Y(1)$ decomposition, together with Proposition 4 of Sect. 2.2, allows us to determine many of the PB of this algebra, when Y exists. Indeed, let W_I be the W generators, $I \in \mathscr{I}$ indexing the generators. The theory possesses a grading operator H, and we suppose here that $H \neq M_0$. The spin content associated to the stress energy tensor T_H is then given by $s_I = 1 + j_I + y_I$. It is conserved through the PB, so that starting from the general form:

$$\begin{aligned}
&\{W_I(x), W_J(x')\}_{\mathrm{PB}} \\
&\quad = \sigma_0(I, J)\, \partial^{j_I + y_I + j_J + y_J + 1}\, \delta(x - x') \\
&\qquad + \sum_K \sum_{p,q} \sigma_1(I, J, K, p, q)\, (\partial^p W_K(x'))\, (\partial^q \delta(x - x')) \\
&\qquad + \sum_{K,L} \sum_{p,q,r} \sigma_2(I, J, K, L, p, q, r) (\partial^p W_K(x'))\, (\partial^r W_L(x')) (\partial^r \delta(x - x')) + \\
&\qquad \vdots\ ,
\end{aligned}$$

where the $\sigma_n(\ldots)$ are coefficients, the conformal invariance imposes the sums to satisfy the equalities

$$\begin{aligned}
&p, q, K \quad \text{such that } p + j_K + y_K + q = j_I + y_I + j_J + y_J\ , \\
&p, q, r, K, L \quad \text{such that } p + j_K + y_K + q + j_L + y_L + r + 1 = j_I + y_I + j_J + y_J \\
&\vdots\ .
\end{aligned} \tag{7.1}$$

But Proposition 4 ensures that this algebra is the same as the one obtained from the grading operator[3] M_0. The main change between tthese two algebras is the stress energy tensor (T_H or T_{M_0}). Then, the conformal invariance of the PB when the gradation is given by M_0 imposes:

$$\begin{aligned}
&p + j_K + q = j_I + j_J \\
&p + j_K + q + j_L + r + 1 = j_I + j_J \\
&\vdots\ .
\end{aligned} \tag{7.2}$$

Gathering (7.1) and (7.2) leads to:

$$\begin{aligned}
&p + j_K + q = j_I + j_J \quad \text{and} \quad y_K = y_I + y_J \\
&p + j_K + q + j_L + r + 1 = j_I + j_J \quad \text{and} \quad y_K + y_L = y_I + y_J \\
&\vdots\ .
\end{aligned}$$

For each line, the second equality shows that the charge associated to the $U(1)_Y$ generator is conserved. This severely limits the number of allowed fields in the r.h.s. of the PB, since not only the T_{M_0}-conformal spin (associated to $Sl(2)$) but also the "hypercharge" associated to Y is conserved. Note that in this context, T_{M_0} has a zero $U(1)_Y$ value.

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[3] This can be guessed if one remarks that the $Sl(2)$ highest weights are the same for $H = M_0$ and $H = M_0 + Y$

Finally, let us add that there may exist several independent Cartan generators Y_i which can be added to M_0 in such a way that $H_i = M_0 + Y_i$ is a non-degenerate gradation, the corresponding $Sl(2)$ subalgebra of which is still $(M_\pm, M_0)$. For example, in the decomposition of $SO(8)$ with respect to $Sl(3)$, we have

$$\begin{aligned}
\underline{8} &= \mathscr{D}_1(y_1) \oplus \mathscr{D}_1(-y_1) \oplus \mathscr{D}_0(y_0) \oplus \mathscr{D}_0(-y_0)\,, \\
(8 \times 8)_A &= (\mathscr{D}_2 \oplus \mathscr{D}_1 \oplus 2\mathscr{D}_0)\,(0) \oplus \mathscr{D}_1(2y_1) \oplus \mathscr{D}_1(-2y_1) \\
&\quad \oplus \mathscr{D}_1(y_1 + y_0) \oplus \mathscr{D}_1(-(y_1 + y_0)) \oplus \mathscr{D}_1(y_1 - y_0) \\
&\quad \oplus \mathscr{D}_1(-(y_1 - y_0))\,.
\end{aligned} \tag{7.3}$$

In the above decomposition of the adjoint representation, one sees that y_0 and y_1 can take the values 0, $\frac{1}{2}$, independently from one another, without violating the non-degeneracy condition. So, we can decompose Y in $Y_0 + Y_1$, Y_0 and Y_1 being defined by the vectors $f_0 = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ and $f_1 = (1, 1, \frac{1}{2}, 0)$.

Thus, we will now write the W generators as

$$W_{j+y+1} \equiv W_{j+1}^{(\vec{y})}\,, \tag{7.4}$$

$j + 1$ being the conformal spin in the basis where all the fields (but T) are primary, and $\vec{y}$ being the set of "hypercharges" associated to the different possible $U(1)_Y$.

For instance, in the case of $SO(8)$ reduced by $Sl(3)$, we will have as W generators:

$$\begin{aligned}
&T_{M_0}^{(0,0)}, \quad W_1^{Y_1}, \quad W_1^{Y_2}\,, \\
&W_3^{(0,0)}, \quad W_2^{(1,0)}, \quad W_2^{(-1,0)}\,, \\
&W_2^{(1/2,\,1/2)}, \quad W_2^{(-1/2,\,-1/2)}, \quad W_2^{(1/2,\,-1/2)}, \quad W_2^{(-1/2,\,1/2)}\,,
\end{aligned}$$

where the doublet superscript indicates the hypercharges of the W generator with respect to $W_1^{Y_1}$ and $W_1^{Y_2}$.

7.2. Use of the Stress-Energy Tensor. We know that the theory associated to H contains a stress-energy tensor T_H, and that all the fields but W_1^Y are primary. Moreover, from Eq. (2.6), it is clear that

$$T_H = T_{M_0} + \partial W_1^Y \quad \text{for } H = M_0 + Y\,. \tag{7.5}$$

Then, a generator $W_j^{(y)}$ being primary (we omit T and W_1^Y) with respect to T_H and T_{M_0}, we will have

$$\{\partial_x W_1^Y(x),\, W_j^{(y)}(x')\}_{\mathrm{PB}} = y\, W_j^{(y)}(x')\partial_x\delta(x - x')\,. \tag{7.6}$$

Note that although T_{M_0} is not an eigenvector of W_Y^1, we associated to it an "eigenvalue" 0. Of course, if there are several $U(1)$, each of them will satisfy this property.

Thus, the generator W_1^Y associated to $Y = H - M_0$ generates a conserved "hypercharge," and all the W generators except T are W_1^Y-eigenvectors:

$$\{W_1^Y(x),\, W_j^{(y)}(x')\}_{\mathrm{PB}} = y\, W_j^{(y)}(x')\,\delta(x - x')\,. \tag{7.7}$$

T possesses a zero hypercharge, but the PB *reads*:

$$\{T(x),\, W_1^Y(x')\}_{\mathrm{PB}} = -\,\partial W_1^Y(x')\delta(x - x') + W_1^Y(x')\partial\delta(x - x')\,. \tag{7.8}$$

Finally, let us remark that the set of spin 1 generators must be closed, because of the conservation of the conformal spin. This shows that we will have a KM algebra, corresponding to the part of $\mathscr{G}$ which has not been used for the definition of the $Sl(2)$ algebra, i.e. the commutant $\mathscr{C}$ of $Sl(2)$ in $\mathscr{G}$. About the position of Y in $\mathscr{C}$, please come back to Proposition 5 at the end of Sect. 2.2.

7.3. Example. As an example, let us look at the W algebra coming from non-Abelian Toda on $Sl(3)$. The W generators are

$$W_2,\ W_{3/2+y},\ W_{3/2-y},\ W_1 \quad \text{with } y = 0 \text{ or } \frac{1}{2}\,. \tag{7.9}$$

Applying the above procedure to the PB of this algebra, we can determine their structure. As a notation, we will write ∂ for ∂_x and ∂' for $\partial_{x'}$,

$$\begin{aligned}\{W_2(x), W_2(x')\}_{\mathrm{PB}} = (&a_1\partial' W_2(x') + a_3\partial'^2 W_1(x') + a_4\partial'(W_1 W_1)(x') \\ &+ a_2 W_{3/2+y} W_{3/2-y}(x'))\delta(x-x') \\ &+ (a_5 W_2(x') + a_6 W_1 W_1(x') + a_7 W_1(x'))\partial\delta(x-x') \\ &+ a_8 W_1(x')\partial^2\delta(x-x') + a_9\partial^3\delta(x-x')\,,\end{aligned} \tag{7.10}$$

$$\{W_2(x), W_{3/2\pm y}(x')\}_{\mathrm{PB}} = (a_{10}\partial' W_{3/2\pm y}(x')\delta(x-x') + a_{11} W_{3/2\pm y}(x')\partial\delta(x-x'') \tag{7.11}$$

$$\begin{aligned}\{W_2(x), W_1(x')\}_{\mathrm{PB}} = (&a_{12}\partial' W_1(x') + a_{13}\partial' W_2(x'))\delta(x-x') \\ &+ a_{14} W_1(x')\partial\delta(x-x') + a_{15}\partial^2\delta(x-x')\,,\end{aligned} \tag{7.12}$$

$$\{W_{3/2\pm y}(x), W_{3/2\pm y}(x')\}_{\mathrm{PB}} = 0\,, \tag{7.13}$$

$$\begin{aligned}\{W_{3/2+y}(x), W_{3/2-y}(x')\}_{\mathrm{PB}} = (&a_{16}\partial' W_1(x') + a_{17} W_1 W_1(x') \\ &+ a_{18} W_2(x'))\,\delta(x-x') \\ &+ a_{19} W_1(x')\partial\delta(x-x') + a_{20}\partial^2\delta(x-x')\,,\end{aligned} \tag{7.14}$$

$$\{W_1(x), W_{3/2\pm y}(x')\}_{\mathrm{PB}} = a_{21}^{\pm} W_{3/2\pm y}(x')\delta(x-x')\,, \tag{7.15}$$

$$\{W_1(x), W_1(x')\}_{\mathrm{PB}} = a_{22}\partial\delta(x-x')\,. \tag{7.16}$$

Now, assuming that $Y=0$, replacing W_2 by T the Virasoro tensor, and recognizing in W_1 the W_1^Y generator, we are led to the constrains:

$$a_1 = -1, \quad a_5 = 2, \quad a_2 = a_3 = a_4 = a_6 = a_7 = a_8 = 0\,, \tag{7.17}$$

$$a_{10} = -1, \quad a_{11} = \frac{3}{2}\,, \tag{7.18}$$

$$a_{12} = -1, \quad a_{14} = 1, \quad a_{13} = a_{15} = 0\,, \tag{7.19}$$

$$a_{21}^{\pm} = \pm 1\,. \tag{7.20}$$

Thus, the W algebra associated to the regular $Sl(2)$ in $Sl(3)$ must satisfy:

$$\begin{aligned}\{T(x), T(x')\}_{\mathrm{PB}} = &-\partial' T(x')\delta(x-x') + 2T(x')\partial\delta(x-x') \\ &+ c\partial^3\delta(x-x')\,,\end{aligned} \tag{7.21}$$

$$\{T(x),\ W^{\pm}_{3/2}(x')\}_{\mathrm{PB}} = -\partial' W^{\pm}_{3/2}(x')\delta(x-x') + \frac{3}{2} W^{\pm}_{3/2}(x')\partial\delta(x-x')\ , \quad (7.22)$$

$$\{T(x),\ W_1(x')\}_{\mathrm{PB}} = -\partial' W_1(x')\delta(x-x') + W_1(x')\partial\delta(x-x')\ , \quad (7.23)$$

$$\{W^{\pm}_{3/2}(x),\ W^{\pm}_{3/2}(x')\}_{\mathrm{PB}} = 0\ , \quad (7.24)$$

$$\{W^{+}_{3/2}(x),\ W^{-}_{3/2}(x')\}_{\mathrm{PB}} = (a_{16}\partial' W_1(x') + a_{17} W_1 W_1(x') + a_{18} T(x'))\,\delta(x-x') + a_{19} W_1(x')\partial\delta(x-x') + a_{20}\partial^2\delta(x-x')\ , \quad (7.25)$$

$$\{W_1(x),\ W^{\pm}_{3/2}(x')\}_{\mathrm{PB}} = \pm W^{\pm}_{3/2}(x')\delta x - x')\ , \quad (7.26)$$

$$\{W_1(x),\ W_1(x')\}_{\mathrm{PB}} = k\partial\delta(x-x')\ , \quad (7.27)$$

which has to be compared with the W algebra made explicit in [12]. Note that the Jacobi identities for the PB of the W algebra will also constrain the remaining structure constants.

8. The Exceptional Algebras G_2 and F_4

Let us first consider the algebra G_2. This (rank 2) algebra admits the system of roots:

$$\pm(e_i \pm e_j);\quad \pm(2e_i - e_j - e_k)\quad \text{with } i, j, k = 1, 2, 3 \text{ all different}\ . \quad (8.1)$$

The fundamental representation of G_2 is seven-dimensional, and its adjoint has the dimension 14. These representations are real. To simplify the discussion about $Sl(2) \oplus U(1)$ decomposition, we remark that G_2 can be embedded in $SO(7)$ (in a singular way). As a consequence, its adjoint representation will be present in the antisymmetric part of the product $\underline{7} \times \underline{7}$. Indeed, we have [17]:

$$(\underline{7} \times \underline{7})_{\mathrm{A}} = \underline{7} \oplus \underline{14}\ . \quad (8.2)$$

Thus, we can obtain the adjoint representation from the fundamental by $\underline{14} = (\underline{7} \times \underline{7})_{\mathrm{A}} - \underline{7}$. It is then sufficient to know the decomposition of the fundamental. This is done with the same rules as for the $SO(n)$ algebras (because of the embedding $G_2 \subset SO(7)$). Note that none of the $Sl(2)$ subalgebras of G_2 can be extended to a $Sl(2) \oplus U(1)$ subalgebra in such a way that (2.19) is still satisfied. The results are presented in Table 8. The defining vector is given in the Cartan basis of $SO(7)$, the Cartan generators of G_2 being given by $H_1 - H_2$ and $2H_2 - H_1 - H_3$ (see Sect. 6).

The exceptional algebra F_4 has rank 4 and dimension 52. Its fundamental representation has dimension 26, and F_4 can be (irregularily) embedded in $SO(26)$. However, one cannot directly obtain the adjoint representation from the fundamental one, since a new representation appears in the antisymmetric part of the product:

$$(26 \times 26)_{\mathrm{A}} = 52 + 273\ . \quad (8.3)$$

Thus, our general method cannot be applied to give the $U(1)$ dependence. The $Sl(2)$ algebras have already been studied in [14], where the decomposition of the fundamental representation was given: we recall in Table 9 this decomposition giving the conformal spin content.

9. W Algebras from Lie Algebras of Rank up to 4

As an application of the above formulation, we represent here an exhaustive classification of W algebras arising from constrained WZW models based on classical algebras of rank up to 4. For such a purpose, we follow the point of view developed in Sect. 2.4, using the results presented in Sects. 3–6. Although the algebras B_2 and C_2 on the one hand, and A_3 and D_3 on the other hand are isomorphic, we have separately considered these four algebras to show the differences in the calculations. The classification is listed in Tables 1–9, where the decomposition of the fundamental of $\mathscr{G}$ with respect to $Sl(2) \oplus U(1)$ is given. We give the minimal (i.e. the lowest dimensional) regular subalgebras containing the $Sl(2)$, when they exist. For the singular embedding associated to D_4, we mention the $SO(3) \oplus SO(5)$ subalgebra. Then, we give the conformal spin content $s = j + 1$, with the convention: $n{*}s$ means that the spin s appears n times. In the same column, we give under the spin s the hypercharge(s) y when it exists. Finally, we write the different gradations that lead to this W algebra.

Table 1. W algebras for Lie algebras of rank 1 and 2

$\mathscr{G}$	Sublag.	$Sl(2) \oplus U(1)$ decompos. (fundamental rep.)	Spin contents (Hypercharge)	Gradation
A_1	A_1	$\mathscr{D}_{1/2}$	2	$(\frac{1}{2}, -\frac{1}{2})$
A_2	A_1	$\mathscr{D}_{1/2}(y) \oplus \mathscr{D}_0(-2y)$	$2, \frac{3}{2}, \frac{3}{2}, 1$	$(\frac{1}{2}, 0, -\frac{1}{2})$
			$(0, 3y, -3y, 0)$	$(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$
	A_2	$\mathscr{D}_1$	3, 2	$(1, 0, -1)$
B_2	A_1	$2\mathscr{D}_{1/2} \oplus \mathscr{D}_0$	$2, 2{*}\frac{3}{2}, 3{*}1$	$(\frac{1}{2}, \frac{1}{2})$
	A_1^2, $2A_1$	$\mathscr{D}_1 \oplus \mathscr{D}_0(y) \oplus \mathscr{D}_0(-y)$	2, 2, 2, 1	$(1, 0)$
			$(0, y, -y, 0)$	$(1, \frac{1}{2})$
				$(1,1)$
	B_2	$\mathscr{D}_2$	4, 2	$(2, 1)$
C_2	A_1	$\mathscr{D}_{1/2} \oplus 2\mathscr{D}_0$	$2, 2{*}\frac{3}{2}, 3{*}1$	$(\frac{1}{2}, 0)$
	$2A_1$, A_1^2	$\mathscr{D}_{1/2}(y) \oplus \mathscr{D}_{1/2}(-y)$	2, 2, 2, 1	$(\frac{1}{2}, \frac{1}{2})$
			$(0, 2y, -2y, 0)$	$(\frac{3}{4}, \frac{1}{4})$
				$(1, 0)$
	C_2	$\mathscr{D}_{3/2}$	4, 2	$(\frac{3}{2}, \frac{1}{2})$

Table 2. W algebras for $A_3 \equiv D_3$

$\mathscr{G}$	Sublag.	$Sl(2) \oplus U(1)$ decompos. (fundamental rep.)	Spin contents (Hypercharge)	Gradation
A_3	A_1	$\mathscr{D}_{1/2}(y) \oplus 2\mathscr{D}_0(-y)$	$2, 4*\frac{3}{2}, 4*1$	$(\frac{1}{2}, 0, 0, \frac{-1}{2})$
			$(0, 2y, 2y, -2y, -2y, 4*0)$	$(\frac{3}{4}, \frac{-1}{4}, \frac{-1}{4}, \frac{-1}{4})$
	$2A_1$	$2\mathscr{D}_{1/2}$	$4*2, 3*1$	$(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$
	A_2	$\mathscr{D}_1(y) \oplus \mathscr{D}_0(-3y)$	$3, 2, 2, 2, 1$	$(1, 0, 0, -1)$
			$(0, 2y, -2y, 0, 0)$	$(\frac{5}{4}, \frac{1}{4}, \frac{-3}{4}, \frac{-3}{4})$
				$(\frac{9}{8}, \frac{1}{8}, \frac{-3}{8}, \frac{-7}{8})$
	A_3	$\mathscr{D}_{3/2}$	$4, 3, 2$	$(\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2})$
D_3	A_1	$2\mathscr{D}_{1/2} \oplus \mathscr{D}_0(y) \oplus \mathscr{D}_0(-y)$	$2, 4*\frac{3}{2}, 4*1$	$(\frac{1}{2}, \frac{1}{2}, 0)$
			$(0, y, y, -y, -y, 4*0)$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
	$2A_1$	$\mathscr{D}_1 \oplus 3\mathscr{D}_0$	$4*2, 3*1$	$(1, 0, 0)$
	A_2	$\mathscr{D}_1(y) \oplus \mathscr{D}_1(-y)$	$3, 2, 2, 2, 1$	$(1, 1, 0)$
			$(0, 2y, -2y, 0, 0)$	$(\frac{5}{4}, \frac{3}{4}, \frac{1}{4})$
				$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$
	D_3	$\mathscr{D}_2 \oplus \mathscr{D}_0$	$4, 3, 2$	$(2, 1, 0)$

Table 3. W algebras for B_3 and C_3

$\mathscr{G}$	Sublag.	$Sl(2) \oplus U(1)$ decompos. (fundamental rep.)	Spin contents (Hypercharge)	Gradation
B_3	A_1	$2\mathscr{D}_{1/2} \oplus 3\mathscr{D}_0$	$2, 6*\frac{3}{2}, 6*1$	$(\frac{1}{2}, \frac{1}{2}, 0)$
	A_1^2, $2A_1$	$\mathscr{D}_1 \oplus 4\mathscr{D}_0$	$5*2, 6*1$	$(1, 0, 0)$
	$A_1 \oplus A_1^2$	$\mathscr{D}_1 \oplus 2\mathscr{D}_{1/2}$	$\frac{5}{2}, \frac{5}{2}, 2, 2, \frac{3}{2}, \frac{3}{2}, 1, 1, 1$	$(1, \frac{1}{2}, \frac{1}{2})$
	A_2, $2A_1 \oplus A_1^2$	$\mathscr{D}_1(y) \oplus \mathscr{D}_1(-y) \oplus \mathscr{D}_0(0)$	$3, 5*2, 1$	$(1, 1, 0)$
			$(0, 2y, y, -y, -2y, 0, 0)$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$
	A_3, B_2	$\mathscr{D}_2(0) \oplus \mathscr{D}_0(y) \oplus \mathscr{D}_0(-y)$	$4, 3, 3, 2, 1$	$(2, 1, 0)$
			$(0, y, -y, 0, 0)$	$(2, 1, \frac{1}{2})$
				$(2, 1, 1)$
				$(2, \frac{3}{2}, 1)$
				$(2, 2, 1)$
	B_3	$\mathscr{D}_3$	$6, 4, 2$	$(3, 2, 1)$
C_3	A_1	$\mathscr{D}_{1/2} \oplus 4\mathscr{D}_0$	$2, 4*\frac{3}{2}, 10*1$	$(\frac{1}{2}, 0, 0)$
	A_1^2, $2A_1$	$\mathscr{D}_{1/2}(y) \oplus \mathscr{D}_{1/2}(-y) \oplus 2\mathscr{D}_0(0)$	$3*2, 4*\frac{3}{2}, 4*1$	$(\frac{1}{2}, \frac{1}{2}, 0)$
			$(0, 2y, -2y, 2*y, 2*(-y), 4*0)$	$(1, 0, 0)$
	A_2^2	$2\mathscr{D}_1$	$3*3, 2, 3*1$	$(1, 1, 0)$
	C_2	$\mathscr{D}_{3/2} \oplus 2\mathscr{D}_0$	$4, \frac{5}{2}, \frac{5}{2}, 2, 3*1$	$(\frac{3}{2}, \frac{1}{2}, 0)$
	$A_1 \oplus A_1^2$, $3A_1$	$3\mathscr{D}_{1/2}$	$6*2, 3*1$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
	$C_2 \oplus A_1$	$\mathscr{D}_{3/2} \oplus \mathscr{D}_{1/2}$	$4, 3, 3*2$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$
	C_3	$\mathscr{D}_{5/2}$	$6, 4, 2$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$

Table 4. *W* algebras for A_4

Sublag.	$Sl(2)\oplus U(1)$ decompos. (fundamental rep.)	Spin contents (Hypercharge)	Gradation
A_1	$\mathscr{D}_{1/2}(y)\oplus 3\mathscr{D}_0\left(\frac{-2y}{3}\right)$	$2,\ 6*\frac{3}{2},\ 9*1$	$(\frac{1}{2},\ 0,\ 0,\ 0,\ \frac{-1}{2})$
		$\left(0,\ 3*\frac{5y}{3},\ 3*\frac{-5y}{y},\ 9*0\right)$	$(\frac{4}{5},\ \frac{-1}{5},\ \frac{-1}{5},\ \frac{-1}{5},\ \frac{-1}{5})$
$2A_1$	$2\mathscr{D}_{1/2}(y)\oplus\mathscr{D}_0(-4y)$	$4*2,\ 4*\frac{3}{2},\ 4*1$	$(\frac{1}{2},\ \frac{1}{2},\ 0,\ \frac{-1}{2},\ \frac{-1}{2})$
		$(4*0,\ 2*5y,\ 2*(-5y),\ 4*0)$	$(\frac{3}{5},\ \frac{3}{5},\ \frac{-2}{5},\ \frac{-2}{5},\ \frac{-2}{5})$
A_2	$\mathscr{D}_1(y)\oplus 2\mathscr{D}_0\left(\frac{-3y}{2}\right)$	$3,\ 5*2,\ 4*1$	$(1,\ 0,\ 0,\ 0,\ -1)$
		$\left(0,\ 2*\frac{5y}{2},\ 0,\ 2*\frac{-5y}{2},\ 4*0\right)$	$(\frac{6}{5},\ \frac{1}{5},\ \frac{-3}{10},\ \frac{-3}{10},\ \frac{-4}{5})$
			$(\frac{7}{5},\ \frac{2}{5},\ -\frac{3}{5},\ -\frac{3}{5},\ -\frac{3}{5})$
$A_2\oplus A_1$	$\mathscr{D}_1(y)\oplus\mathscr{D}_{1/2}\left(\frac{-3y}{2}\right)$	$3,\ 2*\frac{5}{2},\ 2*2,\ 2*\frac{3}{2},\ 1$	$(1,\ \frac{1}{2},\ 0,\ \frac{-1}{2},\ -1)$
		$\left(0,\ \frac{5y}{3},\ \frac{-5y}{3},\ 0,\ 0,\ \frac{5y}{3},\ \frac{-5y}{3},\ 0\right)$	$(\frac{6}{5},\ \frac{1}{5},\ \frac{1}{5},\ \frac{-4}{5},\ \frac{-4}{5})$
A_3	$\mathscr{D}_{3/2}(y)\oplus\mathscr{D}_0(-4y)$	$4,\ 3,\ 2*\frac{5}{2},\ 2,\ 1$	$(\frac{3}{2},\ \frac{1}{2},\ 0,\ \frac{-1}{2},\ \frac{-3}{2})$
		$(0,\ 0,\ 5y,\ -5y,\ 2*0)$	$(\frac{9}{5},\ \frac{4}{5},\ \frac{-1}{5},\ \frac{-6}{5},\ \frac{-6}{5})$
			$(\frac{8}{5},\ \frac{3}{5},\ \frac{-2}{5},\ \frac{-2}{5},\ \frac{-7}{5})$
			$(\frac{17}{10},\ \frac{7}{10},\ \frac{-3}{10},\ \frac{-4}{5},\ \frac{-13}{10})$
A_4	$\mathscr{D}_2$	$5,\ 4,\ 3,\ 2$	$(2,\ 1,\ 0,\ -1,\ -2)$

Table 5. *W* algebras for B_4

Subalg.	$Sl(2)\oplus U(1)$ decompos. (fundamental rep.)	Spin contents (Hypercharge)	Gradation
A_1	$2\mathscr{D}_{1/2}\oplus 5\mathscr{D}_0$	$2,\ 10*\frac{3}{2},\ 13*1$	$(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0)$
A_1^2 $2A_1$	$\mathscr{D}_1\oplus 6\mathscr{D}_0$	$7*2,\ 15*1$	$(1,\ 0,\ 0,\ 0)$
$(2A_1)'$	$4\mathscr{D}_{1/2}\oplus\mathscr{D}_0$	$6*2,\ 4*\frac{3}{2},\ 10*1$	$(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2})$
$A_1\oplus A_1^2$ $3A_1$	$\mathscr{D}_1\oplus 2\mathscr{D}_{1/2}\oplus\mathscr{D}_0(y)$ $\oplus\mathscr{D}_0(-y)$	$\frac{5}{2},\ \frac{5}{2},\ 4*2,\ 6*\frac{3}{2},\ 4*1$	$(1,\ \frac{1}{2},\ \frac{1}{2},\ 0)$
		$(4*0,\ y,\ -y,\ 0,\ 0,\ y,\ y,$ $-y,\ -y,\ 4*0)$	$(1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2})$
A_2 $4A_1$ $2A_1\oplus A_1^2$	$\mathscr{D}_1(y)\oplus\mathscr{D}_1(-y)\oplus 3\mathscr{D}_0(0)$	$3,\ 9*2,\ 4*1$	$(1,\ 1,\ 0,\ 0)$
		$(0,\ 3*y,\ 3*(-y),\ 2y,\ -2y,\ 5*0)$	$(\frac{3}{2},\ \frac{1}{2},\ \frac{1}{2},\ 0)$
$A_2\oplus A_1^2$	$3\mathscr{D}_1$	$3*3,\ 6*2,\ 3*1$	$(1,\ 1,\ 1,\ 0)$
A_3	$2\mathscr{D}_{3/2}\oplus\mathscr{D}_0$	$4,\ 3*3,\ 2*\frac{5}{2},\ 2,\ 3*1$	$(\frac{3}{2},\ \frac{3}{2},\ \frac{1}{2},\ \frac{1}{2})$
B_2 A_3	$\mathscr{D}_2\oplus 4\mathscr{D}_0$	$4,\ 4*3,\ 2,\ 6*1$	$(2,\ 1,\ 0,\ 0)$

Table 5. (*continued*)

Subalg.	$Sl(2) \oplus U(1)$ decompos. (fundamental rep.)	Spin contents (Hypercharge)	Gradation
$B_2 \oplus A_1$	$\mathscr{D}_2 \oplus 2\mathscr{D}_{1/2}$	$4, 2*\frac{7}{2}, 2*\frac{5}{2}, 2, 2, 3*1$	$(2, 1, \frac{1}{2}, \frac{1}{2})$
$B_2 \oplus 2A_1$, $A_3 \oplus A_1^2$	$\mathscr{D}_2 \oplus \mathscr{D}_1 \oplus \mathscr{D}_0$	$4, 4, 3, 3, 4*2$	$(2, 1, 1, 0)$
B_3, D_4	$\mathscr{D}_3(0) \oplus \mathscr{D}_0(y) \oplus \mathscr{D}_0(-y)$	$6, 3*4, 2, 1$	$(3, 2, 1, 0)$
		$(0, y, -y, 3*0)$	$(3, 2, 1, y)$
B_4	$\mathscr{D}_4$	$8, 6, 4, 2$	$(4, 3, 2, 1)$

Table 6. *W* algebras for C_4

Subalg.	$Sl(2) \oplus U(1)$ decompos. (fundamental rep.)	Spin contents (Hypercharge)	Gradation
A_1	$\mathscr{D}_{1/2} \oplus 6\mathscr{D}_0$	$2, 6*\frac{3}{2}, 21*1$	$(\frac{1}{2}, 0, 0, 0)$
A_1^2, $2A_1$	$\mathscr{D}_{1/2}(y) \oplus \mathscr{D}_{1/2}(-y) \oplus 4\mathscr{D}_0(0)$	$3*2, 8*\frac{3}{2}, 11*1$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$
		$(0, 2y, -2y, 4*y, 4*(-y), 11*0)$	$(1, 0, 0, 0)$
$A_1 \oplus A_1^2$, $3A_1$	$3\mathscr{D}_{1/2} \oplus 2\mathscr{D}_0$	$6*2, 6*\frac{3}{2}, 6*1$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$
$2A_1^2$, $4A_1$, $2A_1 \oplus A_1^2$	$4\mathscr{D}_{1/2}$	$10*2, 6*1$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
A_2^2	$2\mathscr{D}_1 \oplus 2\mathscr{D}_0$	$3, 3, 3, 5*2, 6*1$	$(1, 1, 1, 0)$
$A_2^2 \oplus A_1$	$2\mathscr{D}_1 \oplus \mathscr{D}_{1/2}$	$3*3, 2*\frac{5}{2}, 2*2, 2*\frac{3}{2}, 3*1$	$(1, 1, \frac{1}{2}, 0)$
C_2	$\mathscr{D}_{3/2} \oplus 4\mathscr{D}_0$	$4, 4*\frac{5}{2}, 2, 10*1$	$(\frac{3}{2}, \frac{1}{2}, 0, 0)$
$C_2 \oplus A_1$	$\mathscr{D}_{3/2} \oplus \mathscr{D}_{1/2} \oplus 2\mathscr{D}_0$	$4, 3, 2*\frac{5}{2}, 3*2, 2*\frac{3}{2}, 3*1$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 0)$
$C_2 \oplus A_1^2$, $C_2 \oplus 2A_1$	$\mathscr{D}_{3/2}(0) \oplus \mathscr{D}_{1/2}(y) \oplus \mathscr{D}_{1/2}(-y)$	$4, 2*3, 6*2, 1$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
		$(0, y, -y, 2y, -2y, y, -y, 3*0)$	$(\frac{3}{2}, 1, \frac{1}{2}, 0)$
A_3^2, $2C_2$	$\mathscr{D}_{3/2}(y) \oplus \mathscr{D}_{3/2}(-y)$	$3*4, 3, 3*2, 1$	$(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$
		$(0, 2y, -2y, 0, 2y, -2y, 2*0)$	$(2, 1, 1, 0)$
C_3	$\mathscr{D}_{5/2} \oplus 2\mathscr{D}_0$	$6, 4, \frac{7}{2}, \frac{7}{2}, 2, 3*1$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, 0)$
$C_3 \oplus A_1$	$\mathscr{D}_{5/2} \oplus \mathscr{D}_{1/2}$	$6, 4, 4, 3, 2, 2$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$
C_4	$\mathscr{D}_{7/2}$	$8, 6, 4, 2$	$(\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$

Table 7. W algebras for D_4

Sublag.	$Sl(2) \oplus U(1)$ decompos. (fundamental rep.)	Spin contents (Hypercharge)	Gradation
A_1	$2\mathscr{D}_{1/2} \oplus 4\mathscr{D}_0$	$2,\ 8*\frac{3}{2},\ 9*1$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$
$2A_1$	$\mathscr{D}_1 \oplus 5\mathscr{D}_0$	$6*2,\ 10*1$	$(1, 0, 0, 0)$
$(2A_1)'$	$4\mathscr{D}_{1/2}$	$6*2,\ 10*1$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$3A_1$	$\mathscr{D}_1 \oplus 2\mathscr{D}_{1/2} \oplus \mathscr{D}_0$	$\frac{5}{2}, \frac{5}{2}, 3*2, 4*\frac{3}{2}, 3*1$	$(1, \frac{1}{2}, \frac{1}{2}, 0)$
$\left.\begin{matrix} A_2 \\ 4A_1 \end{matrix}\right\}$	$\mathscr{D}_1(y_1) \oplus \mathscr{D}_1(-y_1)$ $\oplus \mathscr{D}_0(y_0) \oplus \mathscr{D}_0(-y_0)$	$3,\ 7*2,\ 2*1$ $0,\ \pm y_1 \pm y_0,\ \pm 2y_1,\ 3*0)$	$(1, 1, 0, 0)$ $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ $(2, 1, 0, 0)$ $(1, 1, \frac{1}{2}, 0)$ $(1, 1, 1, 0)$ $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
A_3	$2\mathscr{D}_{3/2}$	$4,\ 3*3,\ 2,\ 3*1$	$(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$
D_3	$\mathscr{D}_2 \oplus 3\mathscr{D}_0$	$4,\ 3*3,\ 2,\ 3*1$	$(2, 1, 0, 0)$
$B_2 \oplus B_1$	$\mathscr{D}_2 \oplus \mathscr{D}_1$	$4,\ 4,\ 3,\ 3*2$	$(2, 1, 1, 0)$
D_4	$\mathscr{D}_3 \oplus \mathscr{D}_0$	$6,\ 4,\ 4,\ 2$	$(3, 2, 1, 0)$

Table 8. Classification for G_2

Minimal including regular subalgebra	$Sl(2)$ decomposition (fundamental rep.)	Spin contents	Defining vector
A_1	$2\mathscr{D}_{1/2} \oplus 3\mathscr{D}_0$	$2,\ 4*\frac{3}{2},\ 1,\ 1,\ 1$	$(\frac{1}{2}, \frac{1}{2}, 0)$
A_1^2	$\mathscr{D}_1 \oplus 2\mathscr{D}_{1/2}$	$\frac{5}{2}, \frac{5}{2}, 2, 1, 1, 1$	$(1, \frac{1}{2}, 0)$
$A_1 \oplus A_1^2$	$2\mathscr{D}_1 \oplus \mathscr{D}_0$	$3,\ 2,\ 2,\ 2$	$(1, 0, 0)$
G_2	$\mathscr{D}_3$	$6,\ 2$	$(2, \frac{3}{2}, \frac{1}{2})$

Table 9. Classification for F_4

Minimal including regular subalgebra	$Sl(2)$ decomposition (fundamental rep.)	Spin contents
A_1	$6\mathscr{D}_{1/2} \oplus 14\mathscr{D}_0$	$2,\ 14*\frac{3}{2},\ 21*1$
$\left.\begin{matrix} A_1^2 \\ 2A_1 \end{matrix}\right\}$	$\mathscr{D}_1 \oplus 8\mathscr{D}_{1/2} \oplus 7\mathscr{D}_0$	$7*2,\ 10*\frac{3}{2},\ 15*1$
$\left.\begin{matrix} A_1 \oplus A_1^2 \\ 3A_1 \end{matrix}\right\}$	$3\mathscr{D}_1 \oplus 6\mathscr{D}_{1/2} \oplus 5\mathscr{D}_0$	$2*\frac{5}{2},\ 6*2,\ 10*\frac{3}{2},\ 6*1$
$\left.\begin{matrix} 4A_1 \\ 2A_1 \oplus A_1^2 \\ A_2 \end{matrix}\right\}$	$6\mathscr{D}_1 \oplus 8\mathscr{D}_0$	$3,\ 13*2,\ 8*1$
A_2^2	$\mathscr{D}_2 \oplus 7\mathscr{D}_1$	$7*3,\ 2,\ 14*1$
$A_2 \oplus A_1^2$	$\mathscr{D}_2 \oplus 2\mathscr{D}_{3/2} \oplus 3\mathscr{D}_1 \oplus 2\mathscr{D}_{1/2}$	$2*4,\ 3*3,\ 6*2,\ 2*\frac{3}{2},\ 1$
$A_1 \oplus A_2^2$	$2\mathscr{D}_{3/2} \oplus 3\mathscr{D}_1 \oplus 4\mathscr{D}_{1/2} \oplus \mathscr{D}_0$	$3*3,\ 2*\frac{5}{2},\ 6*2,\ 4*\frac{3}{2},\ 3*1$

Table 9. (*continued*)

Minimal including regular subalgebra	$Sl(2)$ decomposition (fundamental rep.)	Spin contents
$A_2^2 \oplus A_2$ $A_3 \oplus A_1^2$ $B_2 \oplus A_1^2$ $B_2 \oplus 2A_1$	$3\mathscr{D}_2 \oplus 3\mathscr{D}_1 \oplus 2\mathscr{D}_0$	2*4, 4*3, 6*2
B_2 A_3	$\mathscr{D}_2 \oplus 4\mathscr{D}_{3/2} \oplus 5\mathscr{D}_0$	4, 4*3, 4*$\frac{5}{2}$, 2, 6*1
$B_2 \oplus A_1$	$2\mathscr{D}_2 \oplus 2\mathscr{D}_{3/2} \oplus \mathscr{D}_1 \oplus 2\mathscr{D}_{1/2} \oplus \mathscr{D}_1$	4, 2*$\frac{7}{2}$, 3, 4*$\frac{5}{2}$, 3*2, 3*1
B_3 D_4	$3\mathscr{D}_3 \oplus 5\mathscr{D}_0$	6, 5*4, 2, 3*1
B_4	$\mathscr{D}_5 \oplus \mathscr{D}_4 \oplus \mathscr{D}_2 \oplus \mathscr{D}_0$	8, 2*6, 4, 3, 2
C_3	$\mathscr{D}_4 \oplus 2\mathscr{D}_{5/2} \oplus \mathscr{D}_2$	6, 2*$\frac{11}{2}$, 4, 2*$\frac{5}{2}$, 2, 3*1
$C_3 \oplus A_1$	$\mathscr{D}_4 \oplus \mathscr{D}_3 \oplus 2\mathscr{D}_2$	2*6, 5, 4, 3, 3*2
F_4	$\mathscr{D}_8 \oplus \mathscr{D}_4$	12, 8, 6, 2

Part II. Super W Algebras Built on Lie Superalgebras

10. The $OSp(1|2)$ Subsuperalgebras of Simple Lie Superalgebras

The determination of the different $OSp(1|2)$ subalgebras in a simple Lie superalgebra $\mathscr{G} = \mathscr{G}_B \oplus \mathscr{G}_F$ is greatly simplified by the two following remarks:

1) The $Sl(2)$ part of $OSp(1|2)$ is in the (semi)simple bosonic part of the considered superalgebra. The knowledge of a method to classify the $Sl(2)$ subalgebras of a simple Lie algebra can be obviously generalized to the case of a direct sum of two (or three, cf. $D(2, 1; \alpha)$) simple algebras.
2) Any representation of $OSp(1|2)$ is completely irreducible, and any irreducible $OSp(1|2)$ representation $\mathscr{R}_j$ (j integer or half-integer) is the direct sum of two $Sl(2)$ representations $\mathscr{D}_j \oplus \mathscr{D}_{j-1/2}$ with an exception for the trivial one-dimensional representation $\mathscr{R}_0 = \mathscr{D}_0$. From the reduction of the fundamental representation of $\mathscr{G}$ into $Sl(2)$ ones, it is therefore easy to verify whether the $Sl(2)$ under consideration can be embedded into an $OSp(1|2)$ superalgebra.

Now, in the same way that the $Sl(2)$ subalgebras of a simple Lie algebra $\mathscr{G}$ are principal subalgebras of the $\mathscr{G}$ regular subalgebras (up to exceptions arising in the D_n case, see Sect. 3), it is rather clear that the $OSp(1|2)$ subsuperalgebras of a simple Lie superalgebra $\mathscr{G}$ are superprincipal in the $\mathscr{G}$ regular subsuperalgebras (up to exceptions arising in the $D(m, n)$ case). One recalls that the definition of a regular subsuperalgebra (SSA) is a direct generalization of that of an algebra, and such SSA can be obtained from the extended Dynkin diagrams for superalgebras, as for simple algebras [18]. Of course, since several Dynkin diagrams can be in general associated to the same superalgebra, one has to apply the method to each allowed Dynkin diagram specifying the superalgebra. A SSA of $\mathscr{G}$ which is not regular is called singular. An example of singular SSA of $\mathscr{G}$ is the superprincipal $OSp(1|2)$, when it exists. It is defined as

$$F_+ = \sum_{\alpha \in \Delta} E_\alpha \quad \text{and} \quad F_- = \sum_{\alpha \in \Delta} E_{-\alpha} , \tag{10.1}$$

$$E_+ = \{F_+, F_+\}, \quad E_- = \{F_-, F_-\} \quad \text{and} \quad H = \{F_+, F_-\} , \tag{10.2}$$

where Δ is a simple root system of $\mathcal{G}$.

Not all the simple Lie superalgebras admit a superprincipal embedding. Actually, it is clear from the expression of the $OSp(1|2)$ generators, that a superprincipal embedding can be defined only if the superalgebra under consideration has a completely fermionic simple root system Δ (which corresponds to a Dynkin diagram with only grey or/and black dots). Notice that this condition is necessary but not sufficient (the superalgebra $PSl(n|n)$ does not admit a superprincipal embedding although it has a completely fermionic simple root system). The simple superalgebras admitting a superprincipal $OSp(1|2)$ are the following: $Sl(n+1|n)$, $Sl(n|n+1)$, $OSp(2n \pm 1|2n)$, $OSp(2n|2n)$, $OSp(2n+2|2n)$ with $n \geqq 1$ and $D(2,1;\alpha)$ with $\alpha \neq 0, \pm 1$.

Finally, the method for classifying the $OSp(1|2)$ SSAs in a simple Lie superalgebra $\mathcal{G}$ can be summarized as follows:

Any $OSp(1|2)$ SSA *in a simple Lie superalgebra* $\mathcal{G}$ *can be considered as the superprincipal* $OSp(1|2)$ SSA *of a regular* SSA $\tilde{\mathcal{G}}$ *of* $\mathcal{G}$, *up to the following exceptions*:

i) *For* $\mathcal{G} = OSp(2n \pm 2|2n)$ *with* $n \geqq 2$, *besides the superprincipal* $OSp(1|2)$ SSA*s described above, there exist* $OSp(1|2)$ SSA*s associated to the singular embeddings* $OSp(2k \pm 1|2k) \oplus OSp(2n-2k \pm 1|2n-2k)$ *with* $1 \leqq k \leqq n-1$.

ii) *For* $\mathcal{G} = OSp(2n|2n)$ *with* $n \geqq 2$, *besides the* $OSp(1|2)$ *superprincipal embedding, there exist* $OSp(1|2)$ SSA*s associated to the singular embeddings* $OSp(2k \pm 1|2k) \oplus OSp(2n-2k \mp 1|2n-2k) \subset OSp(2n|2n)$ *with* $1 \leqq k \leqq n-1$.

11. $OSp(1|2)$ Decompositions of Simple Lie Superalgebras

Following the general method explained above, once the possible $OSp(1|2)$ embeddings are determined in the simple Lie superalgebra $\mathcal{G}$, one has to reduce the adjoint representation of $\mathcal{G}$ into $OSp(1|2)$ supermultiplets. Consider an $OSp(1|2)$ SSA of $\mathcal{G}$, and let $\tilde{\mathcal{G}}$ be the minimal including regular SSA of $\mathcal{G}$ having this $OSp(1|2)$ as superprincipal embedding. We will show on the example of $Sl(m|n)$ how to obtain the decomposition of a simple Lie superalgebra starting from the decompositions of its bosonic and fermionic parts with respect to the bosonic $Sl(2)$ subalgebra of the $OSp(1|2)$ under consideration. Moreover, we will see that such a decomposition can be obtained in a systematic way from the decomposition of the fundamental representation of the superalgebra with respect to the $OSp(1|2)$.

11.1. The Unitary Superalgebras $Sl(m|n)$. The bosonic part of $\mathcal{G} = Sl(m|n)$ with $m \neq n$ is $\mathcal{G}_B = Sl(m) \oplus Sl(n) \oplus U(1)$ and the fermionic part $\mathcal{G}_F$ is the $(\underline{m}, \bar{\underline{n}}) \oplus (\bar{\underline{m}}, \underline{n})$ representation of $Sl(m) \oplus Sl(n)$. The regular SSAs of $Sl(m|n)$ which admit a superprincipal embedding are of the $Sl(p+1|p)$ or $Sl(p|p+1)$ type.

Consider an $OSp(1|2)$ SSA of $\mathcal{G}$ such that the minimal including regular SSA in $\mathcal{G}$ is $\tilde{\mathcal{G}} = Sl(p+1|p)$ with $p \leqq \inf(m-1, n)$. Under $Sl(2)$ (of $OSp(1|2)$), the representations $\underline{m}$ and $\underline{n}$ of $Sl(m)$ and $Sl(n)$ decompose as

$$\begin{aligned} \underline{m} &= \mathcal{D}_{p/2} \oplus (m-p-1)\mathcal{D}_0 , \\ \underline{n} &= \mathcal{D}_{(p-1)/2} \oplus (n-p)\mathcal{D}_0 . \end{aligned} \tag{11.1}$$

Therefore the fermionic part $\mathcal{G}_F$ reduces to

$$(\underline{m}, \bar{n}) \oplus (\bar{m}, \underline{n}) = 2(\mathcal{D}_{p/2} \oplus (m-p-1)\mathcal{D}_0) \times (\mathcal{D}_{(p-1)/2} \oplus (n-p)\mathcal{D}_0)$$
$$= 2\mathcal{D}_{p-1/2} \oplus 2\mathcal{D}_{p-3/2} \oplus \cdots \oplus 2\mathcal{D}_{1/2} \oplus 2(m-p-1)\mathcal{D}_{(p-1)/2}$$
$$\oplus 2(n-p)\mathcal{D}_{p/2} \oplus 2(m-p-1)(n-p)\mathcal{D}_0 \,. \tag{11.2}$$

The bosonic part $\mathcal{G}_B$ is decomposed as

$$\mathcal{G}_B = Sl(m) \oplus Sl(n) \oplus U(1)$$
$$= (\mathcal{D}_{p/2} \oplus (m-p-1)\mathcal{D}_0) \times (\mathcal{D}_{p/2} \oplus (m-p-1)\mathcal{D}_0)$$
$$\oplus (\mathcal{D}_{(p-1)/2} \oplus (n-p)\mathcal{D}_0) \times (\mathcal{D}_{(p-1)/2} \oplus (n-p)\mathcal{D}_0) - \mathcal{D}_0$$
$$= \mathcal{D}_p \oplus 2\mathcal{D}_{p-1} \oplus \cdots \oplus 2\mathcal{D}_1 \oplus 2(m-p-1)\mathcal{D}_{p/2}$$
$$\oplus 2(n-p)\mathcal{D}_{(p-1)/2} \oplus [(m-p-1)^2 + (n-p)^2 + 1]\mathcal{D}_0 \,. \tag{11.3}$$

Gathering the $Sl(2)$ representations $\mathcal{D}_j$ into $OSp(1|2)$ irreducible representations $\mathcal{R}_j$, one finds that the adjoint representation of $Sl(m|n)$ decomposes under the superprincipal $OSp(1|2)$ of $Sl(p+1|p) \subset Sl(m|n)$ as[4]:

$$\frac{\mathbf{Ad}[Sl(m|n)]}{Sl(p+1|p)} = \mathcal{R}_p \oplus \mathcal{R}_{p-1/2} \oplus \mathcal{R}_{p-1} \oplus \cdots \oplus \mathcal{R}_{1/2} \oplus 2(n-p)\mathcal{R}_{p/2}$$
$$\oplus 2(m-p-1)\mathcal{R}'_{p/2}$$
$$\oplus [(m-p-1)^2 + (n-p)^2]\mathcal{R}_0 \oplus 2(m-p-1)(n-p)\mathcal{R}'_0 \,. \tag{11.4}$$

Notice that the $W_{j+1/2}$ superfield corresponding to the representation $\mathcal{R}_j = \mathcal{D}_j \oplus \mathcal{D}_{j-1/2}$ has two component fields w_{j+1} and $w_{j+1/2}$ of spins $j+1$ and $j+1/2$ respectively. If the representation $\mathcal{D}_j$ comes from the bosonic (resp. fermionic) part, w_{j+1} is commuting (resp. anticommuting), whereas $w_{j+1/2}$ is anticommuting (resp. commuting). Therefore, if j is integer, the generators w_{j+1} and $w_{j+1/2}$ have the "right" statistics, whereas they have the "wrong" statistics if j is half-integer. The representations $\mathcal{R}_j$ denoted with a prime are used in the case of W superfields obeying the "wrong" statistics.

Actually, this decomposition (which was obtained above in a rather heavy way) can be derived directly from the decomposition of the fundamental representation of the superalgebra $Sl(m|n)$ with respect to the $OSp(1|2)$ under consideration. From (11.1), the fundamental representation of $Sl(m|n)$, of dimension $m+n$, decomposes as

$$\underline{m+n} = \mathcal{R}_{p/2} \oplus (m-p-1)\mathcal{R}_0 \oplus (n-p)\mathcal{R}_0^{\pi} \,, \tag{11.5}$$

where we have introduced two kinds of $OSp(1|2)$ representations. An $OSp(1|2)$ representation is denoted $\mathcal{R}_j$ if the representation $\mathcal{D}_j$ comes from the decomposition of the fundamental of $Sl(m)$ and $\mathcal{R}_j^{\pi}$ if $\mathcal{D}_j$ comes from the decomposition of the fundamental of $Sl(n)$.

Then the adjoint representation of $Sl(m|n)$ is obtained from the fundamental one by

$$\mathbf{Ad}[Sl(m|n)] = (\underline{m+n}) \times (\overline{m+n}) - \underline{1} \,. \tag{11.6}$$

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[4] In the following, we will use $\dfrac{\text{Ad}[\mathcal{G}]}{\tilde{\mathcal{G}}}$ to denote the decomposition of the adjoint representation of $\mathcal{G}$ with respect to the superprincipal $OSp(1|2)$ of $\tilde{\mathcal{G}} \subset \mathcal{G}$

Using the general formula giving the product of two $OSp(1|2)$ representations $\mathscr{R}_{q_1}$ and $\mathscr{R}_{q_2}$:

$$\mathscr{R}_{q_1} \times \mathscr{R}_{q_2} = \bigoplus_{q=|q_1-q_2|}^{q=q_1+q_2} \mathscr{R}_q \quad \text{with } q \text{ integer and half-integer} , \tag{11.7}$$

one recovers the decomposition of the adjoint representation of $Sl(m|n)$ under the superprincipal $OSp(1|2)$ of $Sl(p+1|p)$ given by (11.4).

Now, we consider the $OSp(1|2)$ superprincipal embedding of $Sl(p|p+1)$ in $\mathscr{G}$ with $p \leqq \inf(m, n-1)$. Then the decompositions of the representations $\underline{m}$ and $\underline{n}$ of $Sl(m)$ and $Sl(n)$ are:

$$\underline{m} = \mathscr{D}_{(p-1)/2} \oplus (m-p)\mathscr{D}_0 ,$$
$$\underline{n} = \mathscr{D}_{p/2} \oplus (n-p-1)\mathscr{D}_0 , \tag{11.8}$$

leading to the following decomposition of the fundamental representation $\underline{m+n}$ of $Sl(m|n)$:

$$\underline{m+n} = \mathscr{R}^{\pi}_{p/2} \oplus (m-p)\mathscr{R}_0 \oplus (n-p-1)\mathscr{R}^{\pi}_0 . \tag{11.9}$$

Therefore, the decomposition of the adjoint representation reads

$$\begin{aligned}\frac{\mathbf{Ad}[Sl(m|n)]}{Sl(p|p+1)} &= \mathscr{R}_p \oplus \mathscr{R}_{p-1/2} \oplus \mathscr{R}_{p-1} \oplus \cdots \oplus \mathscr{R}_{1/2} \oplus 2(m-p)\mathscr{R}_{p/2} \\ &\oplus 2(n-p-1)\mathscr{R}'_{p/2} \\ &\oplus [(m-p)^2 + (n-p-1)^2]\mathscr{R}_0 \oplus 2(m-p)(n-p-1)\mathscr{R}'_0 .\end{aligned} \tag{11.10}$$

More generally, if $\tilde{\mathscr{G}}$ is a sum of SSAs of $Sl(p+1|p)$ or $Sl(p|p+1)$ type, each factor $Sl(p+1|p)$ gives rise to an $OSp(1|2)$ representation $\mathscr{R}_{p/2}$ and each factor $Sl(p|p+1)$ to an $OSp(1|2)$ representation $\mathscr{R}^{\pi}_{p/2}$ in the decomposition of the fundamental $\underline{m+n}$ of $Sl(m|n)$, completed eventually by singlets $\mathscr{R}_0$ or $\mathscr{R}^{\pi}_0$. Then the decomposition of the adjoint representation of $Sl(m|n)$ is obtained by applying (11.6).

Finally, let us consider the case of the superalgebra $PSl(n|n)$ whose bosonic part is $Sl(n) \oplus Sl(n)$ and its fermionic part is the $(\underline{n}, \bar{n}) \oplus (\bar{n}, \underline{n})$ representation of the bosonic subalgebra. If the minimal including regular SSA is $Sl(p+1|p)$ with $p \leqq n-1$, the fundamental representation of $PSl(n|n)$ decomposes as

$$\underline{2n} = \mathscr{R}_{p/2} \oplus (n-p-1)\mathscr{R}_0 \oplus (n-p)\mathscr{R}^{\pi}_0 \tag{11.11}$$

and the adjoint representation of $PSl(n|n)$ is given by

$$\mathbf{Ad}[PSl(n|n)] = (\underline{2n}) \times (\overline{2n}) - 2\underline{1} . \tag{11.12}$$

One finds therefore

$$\begin{aligned}\frac{\mathbf{Ad}[PSl(n|n)]}{Sl(p+1|p)} &= \mathscr{R}_p \oplus \mathscr{R}_{p-1/2} \oplus \mathscr{R}_{p-1} \oplus \cdots \oplus \mathscr{R}_{1/2} \oplus 2(n-p)\mathscr{R}_{p/2} \\ &\oplus 2(n-p-1)\mathscr{R}'_{p/2} \\ &\oplus [(n-p-1)^2 + (n-p)^2 - 1]\mathscr{R}_0 \oplus 2(n-p-1)(n-p)\mathscr{R}'_0 .\end{aligned} \tag{11.13}$$

The computation is completely analogous if the minimal including regular SSA is $Sl(p|p+1)$.

11.2. The Orthosymplectic Superalgebras OSp(M|2n).

11.2.1. Products of OSp(1|2) Irreducible Representations. Consider on $OSp(1|2)$ SSA of $\mathscr{G} = OSp(M|2n)$ and let $\tilde{\mathscr{G}}$ be the minimal including SSA in $\mathscr{G}$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$, the fundamental representation of $\mathscr{G}$, of dimension $M + 2n$, decomposes in a sum of $OSp(1|2)$ representations, generically denoted as

$$\underline{M + 2n} = \left(\bigoplus_j \mathscr{R}_j\right) \oplus \left(\bigoplus_{j'} \mathscr{R}^{\pi}_{j'}\right), \tag{11.14}$$

where the representations $\mathscr{R}_j$ and $\mathscr{R}^{\pi}_{j'}$ have the same meaning as in the previous section: a representation $\mathscr{R}_j$ (resp. $\mathscr{R}^{\pi}_{j'}$) corresponds here to an $OSp(1|2)$ representation where the $\mathscr{D}_j$ comes from the decomposition of the $SO(M)$ (resp. $Sp(2n)$) part.

In order to know how to obtain the decomposition of the adjoint representation of $OSp(M|2n)$ from the decomposition of the fundamental one, we come back for a while to the Abelian case [10], specializing for the moment to the superalgebra $OSp(2m + 1|2m)$. In that case, the fundamental representation of $OSp(2m + 1|2m)$ of dimension $4m + 1$ decomposes under its superprincipal $OSp(1|2)$ as

$$\underline{4m + 1} = \mathscr{R}_m\,, \tag{11.15}$$

and thus the adjoint representation of $OSp(2m + 1|2m)$ decomposes as

$$\mathbf{Ad}[OSp(2m + 1|2m)] = (\mathscr{D}_m \times \mathscr{D}_m)_A \oplus (\mathscr{D}_{m-1/2} \times \mathscr{D}_{m-1/2})_S \oplus (\mathscr{D}_m \times \mathscr{D}_{m-1/2})\,. \tag{11.16}$$

The two first terms correspond to the adjoint representations of $SO(2m + 1)$ and $Sp(2m)$ respectively, and the last one to the fermionic representation $(\underline{2m + 1}, \underline{2m})$ of the bosonic part. Therefore, one has

$$\begin{aligned}
\mathbf{Ad}[OSp(2m + 1|2m)] &= \left(\bigoplus_{k=1}^{m} \mathscr{D}_{2k-1}\right) \oplus \left(\bigoplus_{k=1}^{m} \mathscr{D}_{2k-1}\right) \oplus \left(\bigoplus_{k=1/2}^{2m-1/2} \mathscr{D}_k\right) \\
&= \left(\bigoplus_{k=1}^{m} \mathscr{D}_{2k-1} \oplus \mathscr{D}_{2k-3/2}\right) \oplus \left(\bigoplus_{k=1}^{m} \mathscr{D}_{2k-1/2} \oplus \mathscr{D}_{2k-1}\right) \\
&= \bigoplus_{k=1}^{m} (\mathscr{R}_{2k-1} \oplus \mathscr{R}_{2k-1/2})\,.
\end{aligned} \tag{11.17}$$

By analogy with the bosonic $SO(2m)$ case (cf. 4.5), we set (with m integer)

$$(\mathscr{R}_m \times \mathscr{R}_m)_A = \bigoplus_{k=1}^{m} (\mathscr{R}_{2k-1} \oplus \mathscr{R}_{2k-1/2}) \quad \text{with } k \in \mathbb{Z}\,. \tag{11.18}$$

Now we specialize to the superalgebra $OSp(2m - 1|2m)$. In that case, the fundamental representation of $OSp(2m - 1|2m)$ of dimension $4m - 1$ decomposes under its superprincipal $OSp(1|2)$ as

$$\underline{4m - 1} = \mathscr{R}^{\pi}_{m-1/2} \tag{11.19}$$

and thus the adjoint representation of $OSp(2m - 1|2m)$ decomposes as

$$\begin{aligned}
\mathbf{Ad}[OSp(2m - 1|2m)] &= (\mathscr{D}_{m-1} \times \mathscr{D}_{m-1})_A \oplus (\mathscr{D}_{m-1/2} \times \mathscr{D}_{m-1/2})_S \\
&\quad \oplus (\mathscr{D}_{m-1} \times \mathscr{D}_{m-1/2})\,.
\end{aligned} \tag{11.20}$$

Therefore, one has

$$\begin{aligned}\mathbf{Ad}[OSp(2m-1|2m)] &= \left(\bigoplus_{k=1}^{m-1}\mathscr{D}_{2k-1}\right)\oplus\left(\bigoplus_{k=1}^{m}\mathscr{D}_{2k-1}\right)\oplus\left(\bigoplus_{k=1/2}^{2m-3/2}\mathscr{D}_k\right)\\ &= \left(\bigoplus_{k=1}^{m}\mathscr{D}_{2k-1}\oplus\mathscr{D}_{2k-3/2}\right)\oplus\left(\bigoplus_{k=1}^{m-1}\mathscr{D}_{2k-1/2}\oplus\mathscr{D}_{2k-1}\right)\\ &= \bigoplus_{k=1}^{m-1}(\mathscr{R}_{2k-1}\oplus\mathscr{R}_{2k-1/2})\oplus\mathscr{R}_{2m-1}\,.\end{aligned} \tag{11.21}$$

By analogy with the bosonic $Sp(2m)$ case (cf. 4.8), we set (with m integer)

$$(\mathscr{R}^{\pi}_{m-1/2}\times\mathscr{R}^{\pi}_{m-1/2})_{\mathrm{S}} = \bigoplus_{k=1}^{m-1}(\mathscr{R}_{2k-1}\oplus\mathscr{R}_{2k-1/2})\oplus\mathscr{R}_{2m-1} \quad \text{with } k\in\mathbb{Z}\,. \tag{11.22}$$

Using Eqs. (11.7), (11.18) and (11.22), one obtains also the useful formulae (with k and m integer)

$$(\mathscr{R}_{m-1/2}\times\mathscr{R}_{m-1/2})_{\mathrm{A}} = \bigoplus_{k=0}^{m-1}(\mathscr{R}_{2k}\oplus\mathscr{R}_{2k+1/2}) \tag{11.23}$$

and

$$(\mathscr{R}^{\pi}_{m}\times\mathscr{R}^{\pi}_{m})_{\mathrm{S}} = \bigoplus_{k=0}^{m-1}(\mathscr{R}_{2k}\oplus\mathscr{R}_{2k+1/2})\oplus\mathscr{R}_{2m}\,. \tag{11.24}$$

The products between $\mathscr{R}_j$ and $\mathscr{R}^{\pi}_j$ representations are given by

$$\begin{aligned}\mathscr{R}_{j_1}\times\mathscr{R}_{j_2} &= \begin{cases}\bigoplus\mathscr{R}_{j_3} & \text{if } j_1+j_2 \text{ is integer}\\ \bigoplus\mathscr{R}'_{j_3} & \text{if } j_1+j_2 \text{ is half-integer}\end{cases},\\ \mathscr{R}^{\pi}_{j_1}\times\mathscr{R}^{\pi}_{j_2} &= \begin{cases}\bigoplus\mathscr{R}_{j_3} & \text{if } j_1+j_2 \text{ is integer}\\ \bigoplus\mathscr{R}'_{j_3} & \text{if } j_1+j_2 \text{ is half-integer}\end{cases},\\ \mathscr{R}_{j_1}\times\mathscr{R}^{\pi}_{j_2} &= \begin{cases}\bigoplus\mathscr{R}'_{j_3} & \text{if } j_1+j_2 \text{ is integer}\\ \bigoplus\mathscr{R}_{j_3} & \text{if } j_1+j_2 \text{ is half-integer}\end{cases},\end{aligned} \tag{11.25}$$

where the representations $\mathscr{R}_{j_3}$ and $\mathscr{R}'_{j_3}$ correspond to W superfields which obey to "right" or "wrong" statistics respectively.

Finally, one has

$$(n\mathscr{R}_j\times n\mathscr{R}_j)_{\mathrm{A}} = \frac{n(n+1)}{2}(\mathscr{R}_j\times\mathscr{R}_j)_{\mathrm{A}}\oplus\frac{n(n-1)}{2}(\mathscr{R}_j\times\mathscr{R}_j)_{\mathrm{S}}\,, \tag{11.26}$$

$$(n\mathscr{R}_j\times n\mathscr{R}_j)_{\mathrm{S}} = \frac{n(n+1)}{2}(\mathscr{R}_j\times\mathscr{R}_j)_{\mathrm{S}}\oplus\frac{n(n-1)}{2}(\mathscr{R}_j\times\mathscr{R}_j)_{\mathrm{A}}\,, \tag{11.27}$$

and

$$((\mathscr{R}_{j_1}\oplus\mathscr{R}_{j_2})\times(\mathscr{R}_{j_1}\oplus\mathscr{R}_{j_2}))_{\mathrm{A}} = (\mathscr{R}_{j_1}\times\mathscr{R}_{j_1})_{\mathrm{A}}\oplus(\mathscr{R}_{j_2}\times\mathscr{R}_{j_2})_{\mathrm{A}}\oplus(\mathscr{R}_{j_1}\times\mathscr{R}_{j_2})\,, \tag{11.28}$$

$$((\mathscr{R}_{j_1}\oplus\mathscr{R}_{j_2})\times(\mathscr{R}_{j_1}\oplus\mathscr{R}_{j_2}))_{\mathrm{S}} = (\mathscr{R}_{j_1}\times\mathscr{R}_{j_1})_{\mathrm{S}}\oplus(\mathscr{R}_{j_2}\times\mathscr{R}_{j_2})_{\mathrm{S}}\oplus(\mathscr{R}_{j_1}\times\mathscr{R}_{j_2})\,. \tag{11.29}$$

Of course, the same formulae hold for $\mathscr{R}^{\pi}$ representations.

It remains to obtain the decompositions of the adjoint representations of the simple Lie superalgebras from the decompositions of their fundamental representations for the different possible $OSp(1|2)$ embeddings in order to classify the super-Toda theories. The following subsections are devoted to the study of the superalgebras $OSp(2m|2n)$, $OSp(2m+1|2n)$, $OSp(2|2n)$ and to the irregular embeddings.

11.2.2. The Superalgebras $OSp(2m|2n)$. The regular SSAs of $\mathcal{G} = OSp(2m|2n)$ (with $m \geqq 2$) which admit a superprincipal embedding are of the type $OSp(2k|2k)$, $OSp(2k \pm 2|2k)$ and $Sl(p \pm 1|p)$.

Let $\tilde{\mathcal{G}} = OSp(2k|2k)$ with $1 \leqq k \leqq \inf(m, n)$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathcal{G}}$, the fundamental representation of $OSp(2m|2n)$ of dimension $2m + 2n$ decomposes as follows:

$$\underline{2m+2n} = \mathcal{R}^{\pi}_{k-1/2} \oplus (2m-2k+1)\mathcal{R}_0 \oplus (2n-2k)\mathcal{R}^{\pi}_0 \,. \tag{11.30}$$

The decomposition of the adjoint representation of $OSp(2m|2n)$ is obtained from the decomposition of the fundamental representation by taking the antisymmetric product of the orthogonal part and the symmetric product of the symplectic part; more precisely, one has

$$\begin{aligned}\frac{\mathbf{Ad}[OSp(2m|2n)]}{OSp(2k|2k)} &= ((2m-2k+1)\mathcal{R}_0) \times ((2m-2k+1)\mathcal{R}_0)|_{\mathrm{A}} \\ &\oplus (\mathcal{R}^{\pi}_{k-1/2} \oplus (2n-2k)\mathcal{R}^{\pi}_0) \times (\mathcal{R}^{\pi}_{k-1/2} \oplus (2n-2k)\mathcal{R}^{\pi}_0)|_{\mathrm{S}} \\ &\oplus ((2m-2k+1)\mathcal{R}_0) \times (\mathcal{R}^{\pi}_{k-1/2} \oplus (2n-2k)\mathcal{R}^{\pi}_0) \,. \end{aligned} \tag{11.31}$$

Using the formulae (11.18) and (11.22–11.29), one finds

$$\begin{aligned}\frac{\mathbf{Ad}[OSp(2m|2n)]}{OSp(2k|2k)} &= \mathcal{R}_{2k-1} \oplus \mathcal{R}_{2k-5/2} \oplus \mathcal{R}_{2k-3} \oplus \mathcal{R}_{2k-9/2} \oplus \cdots \oplus \mathcal{R}_{3/2} \oplus \mathcal{R}_1 \\ &\oplus (2m-2k+1)\mathcal{R}_{k-1/2} \oplus 2(n-k)\mathcal{R}'_{k-1/2} \\ &\oplus 2(2m-2k+1)(n-k)\mathcal{R}'_0 \\ &\oplus [(2m-2k+1)(m-k) + (2n-2k+1)(n-k)]\mathcal{R}_0 \,. \end{aligned} \tag{11.32}$$

Now, let $\tilde{\mathcal{G}} = OSp(2k+2|2k)$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathcal{G}}$, the fundamental representation of $OSp(2m|2n)$ decomposes as:

$$\underline{2m+2n} = \mathcal{R}_k \oplus (2m-2k-1)\mathcal{R}_0 \oplus (2n-2k)\mathcal{R}^{\pi}_0 \,. \tag{11.33}$$

Therefore, one has

$$\begin{aligned}\frac{\mathbf{Ad}[OSp(2m|2n)]}{OSp(2k+2|2k)} &= (\mathcal{R}_k \oplus (2m-2k-1)\mathcal{R}_0) \times (\mathcal{R}_k \oplus (2m-2k-1)\mathcal{R}_0)|_{\mathrm{A}} \\ &\oplus ((2n-2k)\mathcal{R}^{\pi}_0) \times ((2n-2k)\mathcal{R}^{\pi}_0)|_{\mathrm{S}} \\ &\oplus (\mathcal{R}_k \oplus (2m-2k-1)\mathcal{R}_0) \times ((2n-2k)\mathcal{R}^{\pi}_0) \,, \end{aligned} \tag{11.34}$$

and one obtains in that case

$$\frac{\mathbf{Ad}[OSp(2m|2n)]}{OSp(2k+2|2k)} = \mathscr{R}_{2k-1/2} \oplus \mathscr{R}_{2k-1} \oplus \mathscr{R}_{2k-5/2} \oplus \mathscr{R}_{2k-3} \oplus \cdots \oplus \mathscr{R}_{3/2} \oplus \mathscr{R}_1$$
$$\oplus (2m-2k-1)\mathscr{R}_k \oplus 2(n-k)\mathscr{R}'_k \oplus 2(2m-k-1)(n-k)\mathscr{R}'_0$$
$$\oplus [(2m-2k-1)(m-k-1) + (2n-2k+1)(n-k)]\,\mathscr{R}_0\,. \tag{11.35}$$

Finally, let us consider the case where $\tilde{\mathscr{G}}$ belongs to the unitary series. First, we study the case $\tilde{\mathscr{G}} = Sl(2k+1|2k)$ with $4k \leqq m+n-2$. The decomposition of the fundamental representation of $OSp(2m|2n)$ under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$ is given by

$$\underline{2m+2n} = 2\mathscr{R}_k \oplus 2(m-2k-1)\mathscr{R}_0 \oplus 2(n-2k)\mathscr{R}_0^{\pi}\,. \tag{11.36}$$

Therefore, one has

$$\frac{\mathbf{Ad}[OSp(2m|2n)]}{Sl(2k+1|2k)} = (2\mathscr{R}_k \oplus 2(m-2k-1)\mathscr{R}_0) \times (2\mathscr{R}_k \oplus 2(m-2k-1)\mathscr{R}_0)|_{\mathrm{A}}$$
$$\oplus (2(n-2k)\mathscr{R}_0^{\pi}) \times (2(n-2k)\mathscr{R}_0^{\pi})|_{\mathrm{S}}$$
$$\oplus (2\mathscr{R}_k \oplus 2(m-2k-1)\mathscr{R}_0) \times (2(n-2k)\mathscr{R}_0^{\pi})\,. \tag{11.37}$$

One obtains here

$$\frac{\mathbf{Ad}[OSp(2m|2n)]}{Sl(2k+1|2k)} = \mathscr{R}_{2k} \oplus 3\mathscr{R}_{2k-1} \oplus \mathscr{R}_{2k-2} \oplus \cdots \oplus \mathscr{R}_2 \oplus 3\mathscr{R}_1 \oplus \mathscr{R}_0$$
$$\oplus 3\mathscr{R}_{2k-1/2} \oplus \mathscr{R}_{2k-3/2} \oplus 3\mathscr{R}_{2k-5/2} \oplus \cdots \oplus 3\mathscr{R}_{3/2} \oplus \mathscr{R}_{1/2}$$
$$\oplus 4(m-2k-1)\mathscr{R}_k \oplus 4(n-2k)\mathscr{R}'_k$$
$$\oplus 4(m-2k-1)(n-2k)\mathscr{R}'_0$$
$$\oplus [(2m-4k-3)(m-2k-1) + (2n-4k+1)(n-2k)]\,\mathscr{R}_0\,. \tag{11.38}$$

The other cases are similar. One finds easily the following results. If $\tilde{\mathscr{G}} = Sl(2k-1|2k)$ with $4k \leqq m+n$, one has

$$\underline{2m+2n} = 2\mathscr{R}_{k-1/2}^{\pi} \oplus 2(m-2k+1)\mathscr{R}_0 \oplus 2(n-2k)\mathscr{R}_0^{\pi} \tag{11.39}$$

and

$$\frac{\mathbf{Ad}[OSp(2m|2n)]}{Sl(2k-1|2k)} = 3\mathscr{R}_{2k-1} \oplus \mathscr{R}_{2k-2} \oplus 3\mathscr{R}_{2k-3} \oplus \cdots \oplus \mathscr{R}_2 \oplus 3\mathscr{R}_1 \oplus \mathscr{R}_0$$
$$\oplus \mathscr{R}_{2k-3/2} \oplus 3\mathscr{R}_{2k-5/2} \oplus \mathscr{R}_{2k-7/2} \oplus \cdots \oplus 3\mathscr{R}_{3/2} \oplus \mathscr{R}_{1/2}$$
$$\oplus 4(m-2k+1)\mathscr{R}_{k-1/2} \oplus 4(n-2k)\,\mathscr{R}'_{k-1/2}$$
$$\oplus 4(m-2k+1)(n-2k)\mathscr{R}'_0$$
$$\oplus [(2m-4k+1)(m-2k+1) + (2n-4k+1)(n-2k)]\mathscr{R}_0\,. \tag{11.40}$$

If $\tilde{\mathscr{G}} = Sl(2k|2k+1)$, one has

$$\underline{2m+2n} = 2\mathscr{R}_k^\pi \oplus 2(m-2k)\mathscr{R}_0 \oplus 2(n-2k-1))\mathscr{R}_0^\pi \tag{11.41}$$

and

$$\begin{aligned}\frac{\mathbf{Ad}[OSp(2m|2n)]}{Sl(2k|2k+1)} &= 3\mathscr{R}_{2k} \oplus \mathscr{R}_{2k-1} \oplus 3\mathscr{R}_{2k-2} \oplus \cdots \oplus 3\mathscr{R}_2 \oplus \mathscr{R}_1 \oplus 3\mathscr{R}_0 \\ &\oplus \mathscr{R}_{2k-1/2} \oplus 3\mathscr{R}_{2k-3/2} \oplus \mathscr{R}_{2k-5/2} \oplus \cdots \oplus \mathscr{R}_{3/2} \oplus 3\mathscr{R}_{1/2} \\ &\oplus 4(m-2k)\mathscr{R}'_k \oplus 4(n-2k-1)\mathscr{R}_k \\ &\oplus 4(m-2k)(n-2k-1)\mathscr{R}'_0 \\ &\oplus [(2m-4k-1)(m-2k)+(2n-4k-1)(n-2k-1)]\mathscr{R}_0\,.\end{aligned} \tag{11.42}$$

Finally, if $\tilde{\mathscr{G}} = Sl(2k|2k-1)$, one has

$$\underline{2m+2n} = 2\mathscr{R}_{k-1/2} \oplus 2(m-2k)\mathscr{R}_0 \oplus 2(n-2k+1)\mathscr{R}_0^\pi \tag{11.43}$$

and

$$\begin{aligned}\frac{\mathbf{Ad}[OSp(2m|2n)]}{Sl(2k|2k-1)} &= \mathscr{R}_{2k-1} \oplus 3\mathscr{R}_{2k-2} \oplus \mathscr{R}_{2k-3} \oplus \cdots \oplus 3\mathscr{R}_2 \oplus \mathscr{R}_1 \oplus 3\mathscr{R}_0 \\ &\oplus 3\mathscr{R}_{2k-3/2} \oplus \mathscr{R}_{2k-5/2} \oplus 3\mathscr{R}_{2k-7/2} \oplus \cdots \oplus \mathscr{R}_{3/2} \oplus 3\mathscr{R}_{1/2} \\ &\oplus 4(m-2k)\mathscr{R}'_{k-1/2} \oplus 4(n-2k+1)\mathscr{R}_{k-1/2} \\ &\oplus 4(m-2k)(n-2k+1)\mathscr{R}'_0 \\ &\oplus [(2m-4k-1)(m-2k)+(2n-4k+3)(n-2k+1)]\mathscr{R}_0\,.\end{aligned} \tag{11.44}$$

11.2.3. The Superalgebras $OSp(2m+1|2n)$. The regular SSAs of $\mathscr{G} = OSp(2m+1|2n)$ which admit a superprincipal embedding are of the type $OSp(2k|2k)$, $OSp(2k+2|2k)$, $OSp(2k \pm 1|2k)$ and $Sl(p \pm 1|p)$.

Let $\tilde{\mathscr{G}} = OSp(2k|2k)$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$, the fundamental representation of $OSp(2m+1|2n)$, of dimension $2m+2n+1$, decomposes as follows:

$$\underline{2m+2n+1} = \mathscr{R}_{k-1/2}^\pi \oplus (2n-2k)\mathscr{R}_0^\pi \oplus (2m-2k+2)\mathscr{R}_0\,. \tag{11.45}$$

The decomposition of the adjoint representation is then

$$\begin{aligned}\frac{\mathbf{Ad}[OSp(2m+1|2n)]}{OSp(2k|2k)} &= ((2m-2k+2)\mathscr{R}_0)\times((2m-2k+2)\mathscr{R}_0)|_{\mathrm{A}} \\ &\oplus (\mathscr{R}_{k-1/2}^\pi \oplus (2n-2k)\mathscr{R}_0^\pi)\times(\mathscr{R}_{k-1/2}^\pi \oplus (2n-2k)\mathscr{R}_0^\pi)|_{\mathrm{S}} \\ &\oplus ((2m-2k+1)\mathscr{R}_0)\times(\mathscr{R}_{k-1/2}^\pi \oplus (2n-2k)\mathscr{R}_0^\pi)\,,\end{aligned} \tag{11.46}$$

i.e.

$$\frac{\mathbf{Ad}[OSp(2m+1|2n)]}{OSp(2k|2k)} = \mathscr{R}_{2k-1} \oplus \mathscr{R}_{2k-5/2} \oplus \mathscr{R}_{2k-3} \oplus \mathscr{R}_{2k-9/2} \oplus \cdots \oplus \mathscr{R}_{3/2} \oplus \mathscr{R}_1$$
$$\oplus 2(m-k+1)\mathscr{R}_{k-1/2} \oplus 2(n-k)\mathscr{R}'_{k-1/2}$$
$$\oplus 4(m-k+1)(n-k)\mathscr{R}'_0$$
$$\oplus [(2m-2k+1)(m-k+1) + (2n-2k+1)(n-k)]\mathscr{R}_0 \,. \tag{11.47}$$

Now, let $\tilde{\mathscr{G}} = OSp(2k+2|2k)$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$, the fundamental representation of $OSp(2m+1|2n)$ decomposes as:

$$\underline{2m+2n+1} = \mathscr{R}_k \oplus (2m-2k)\mathscr{R}_0 \oplus (2n-2k)\mathscr{R}_0^\pi \,. \tag{11.48}$$

Then one obtains

$$\frac{\mathbf{Ad}[OSp(2m+1|2n)]}{OSp(2k+2|2k)} = (\mathscr{R}_k \oplus (2m-2k)\mathscr{R}_0) \times (\mathscr{R}_k \oplus (2m-2k)\mathscr{R}_0)|_{\mathrm{A}}$$
$$\oplus ((2n-2k)\mathscr{R}_0^\pi) \times ((2n-2k)\mathscr{R}_0^\pi)|_{\mathrm{S}}$$
$$\oplus (\mathscr{R}_k \oplus (2m-2k)\mathscr{R}_0) \times ((2n-2k)\mathscr{R}_0^\pi) \,, \tag{11.49}$$

i.e.

$$\frac{\mathbf{Ad}[OSp(2m+1|2n)]}{OSp(2k+2|2k)} = \mathscr{R}_{2k-1/2} \oplus \mathscr{R}_{2k-1} \oplus \mathscr{R}_{2k-5/2} \oplus \mathscr{R}_{2k-3}$$
$$\oplus \cdots \oplus \mathscr{R}_{3/2} \oplus \mathscr{R}_1$$
$$\oplus 2(n-k)\mathscr{R}'_k \oplus 2(m-k)\mathscr{R}_k \oplus 4(m-k)(n-k)\mathscr{R}'_0$$
$$\oplus [(2m-2k-1)(m-k) + (2n-2k+1)(n-k)]\mathscr{R}_0 \,. \tag{11.50}$$

Finally, let $\tilde{\mathscr{G}} = OSp(2k-1|2k)$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$, the fundamental representation of $OSp(2m+1|2n)$ decomposes as

$$\underline{2m+2n+1} = \mathscr{R}_{k-1/2}^\pi \oplus (2n-2k)\mathscr{R}_0^\pi \oplus (2m-2k+2)\mathscr{R}_0 \,, \tag{11.51}$$

which is the same decomposition as the case $\tilde{\mathscr{G}} = OSp(2k|2k)$. Therefore, the two SSAs $OSp(2k|2k)$ and $OSp(2k-1|2k)$ (when both can be embedded in $\mathscr{G}$) lead to the same decomposition of the adjoint representation of $\mathscr{G}$ and consequently to the same theory. On the same lines, one finds that the two SSAs $OSp(2k+2|2k)$ and $OSp(2k+1|2k)$ lead to the same theory.

The last case is $\tilde{\mathscr{G}} = Sl(p \pm 1|p)$. We leave the different decompositions to the reader. The results are summarized in the table of Sect. 11.3.

11.2.4. The Irregular Embeddings. We will study now the irregular embeddings, which are present in $OSp(2n \pm 2|2n)$ and $OSp(2n|2n)$.

Consider first the superalgebra $\mathscr{G} = OSp(2n+2|2n)$ and take the $OSp(1|2)$ SSA of $\mathscr{G}$ such that the minimal including SSA in $\mathscr{G}$ (which is now singular) is $\tilde{\mathscr{G}} = OSp(2k+1|2k) \oplus OSp(2n-2k+1|2n-2k)$ and $1 \leq k \leq \left[\frac{n-1}{2}\right]$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$, the fundamental representation of $\mathscr{G}$, of dimension $4n+2$, decomposes as

$$\underline{4n+2} = \mathscr{R}_k \oplus \mathscr{R}_{n-k} \,, \tag{11.52}$$

and we get for the $OSp(2n+2|2n)$ adjoint representation

$$\frac{\mathbf{Ad}[OSp(2n+2|2n)]}{OSp(2k+1|2k) \oplus OSp(2n-2k+1|2n-2k)} = (\mathscr{R}_k \oplus \mathscr{R}_{n-k}) \times (\mathscr{R}_k \oplus \mathscr{R}_{n-k})|_{\mathrm{A}} \,, \tag{11.53}$$

which leads to the following decomposition:

$$\begin{aligned} &\frac{\mathbf{Ad}[OSp(2n+2|2n)]}{OSp(2k+1|2k) \oplus OSp(2n-2k+1|2n-2k)} \\ &= \mathscr{R}_{2n-2k-1} \oplus \mathscr{R}_{2n-2k-3} \oplus \cdots \oplus \mathscr{R}_1 \\ &\quad \oplus \mathscr{R}_{2n-2k-1/2} \oplus \mathscr{R}_{2n-2k-3/2} \oplus \cdots \oplus \mathscr{R}_{3/2} \oplus \mathscr{R}_{2k-1} \oplus \mathscr{R}_{2k-3} \oplus \cdots \oplus \mathscr{R}_1 \\ &\quad \oplus \mathscr{R}_{2k-1/2} \oplus \mathscr{R}_{2k-3/2} \oplus \cdots \oplus \mathscr{R}_{3/2} \oplus \mathscr{R}_n \oplus \mathscr{R}_{n-1} \oplus \cdots \oplus \mathscr{R}_{n-2k} \\ &\quad \oplus \mathscr{R}_{n-1/2} \oplus \mathscr{R}_{n-3/2} \oplus \cdots \oplus \mathscr{R}_{n-2k+1/2} \,. \end{aligned} \tag{11.54}$$

Consider then the superalgebra $\mathscr{G} = OSp(2n-2|2n)$ with $\tilde{\mathscr{G}} = OSp(2k-1|2k) \oplus OSp(2n-2k-1|2n-2k)$ and $1 \leqq k \leqq \left[\frac{n-3}{2}\right]$. The fundamental representation of $\mathscr{G}$, of dimension $4n-2$, decomposes under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$ as

$$\underline{4n-2} = \mathscr{R}^{\pi}_{k-1/2} \oplus \mathscr{R}^{\pi}_{n-k-1/2} \,. \tag{11.55}$$

The adjoint representation of $OSp(2n-2|2n)$ is given by

$$\begin{aligned} \frac{\mathbf{Ad}[OSp(2n-2|2n)]}{OSp(2k-1|2k) \oplus OSp(2n-2k-1|2n-2k)} &= (\mathscr{R}^{\pi}_{k-1/2} \oplus \mathscr{R}^{\pi}_{n-k-1/2}) \\ &\quad \times (\mathscr{R}^{\pi}_{k-1/2} \oplus \mathscr{R}^{\pi}_{n-k-1/2})|_{\mathrm{S}} \,, \end{aligned} \tag{11.56}$$

i.e.

$$\begin{aligned} &\frac{\mathbf{Ad}[OSp(2n-2|2n)]}{OSp(2k-1|2k) \oplus OSp(2n-2k-1|2n-2k)} \\ &= \mathscr{R}_{2n-2k-1} \oplus \mathscr{R}_{2n-2k-3} \oplus \cdots \oplus \mathscr{R}_1 \\ &\quad \oplus \mathscr{R}_{2n-2k-5/2} \oplus \mathscr{R}_{2n-2k-7/2} \oplus \cdots \oplus \mathscr{R}_{3/2} \oplus \mathscr{R}_{2k-1} \oplus \mathscr{R}_{2k-3} \oplus \cdots \oplus \mathscr{R}_1 \\ &\quad \oplus \mathscr{R}_{2k-5/2} \oplus \mathscr{R}_{2k-7/2} \oplus \cdots \oplus \mathscr{R}_{3/2} \oplus \mathscr{R}_{n-1} \oplus \mathscr{R}_{n-2} \oplus \cdots \oplus \mathscr{R}_{n-2k} \\ &\quad \oplus \mathscr{R}_{n-3/2} \oplus \mathscr{R}_{n-5/2} \oplus \cdots \oplus \mathscr{R}_{n-2k+1/2} \,. \end{aligned} \tag{11.57}$$

Consider finally the superalgebra $\mathscr{G} = OSp(2n|2n)$ with $\tilde{\mathscr{G}} = OSp(2k+1|2k) \oplus OSp(2n-2k-1|2n-2k)$ and $1 \leqq k \leqq \left[\frac{n-1}{2}\right]$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$, the fundamental representation of $\mathscr{G}$, of dimension $4n$, decomposes as

$$\underline{4n} = \mathscr{R}_k \oplus \mathscr{R}^{\pi}_{n-k-1/2} \tag{11.58}$$

and we get for the $OSp(2n|2n)$ adjoint representation

$$\frac{\mathbf{Ad}[OSp(2n|2n)]}{OSp(2k+1|2k)\oplus OSp(2n-2k-1|2n-2k)}$$
$$= (\mathscr{R}_k\times\mathscr{R}_k)_A\oplus(\mathscr{R}^\pi_{n-k-1/2}\times\mathscr{R}^\pi_{n-k-1/2})_S\oplus(\mathscr{R}_k\times\mathscr{R}^\pi_{n-k-1/2}) \tag{11.59}$$

which leads to

$$\begin{aligned}&\frac{\mathbf{Ad}[OSp(2n|2n)]}{OSp(2k+1|2k)\oplus OSp(2n-2k-1|2n-2k)}\\ &= \mathscr{R}_{2n-2k-1}\oplus\mathscr{R}_{2n-2k-3}\oplus\cdots\oplus\mathscr{R}_1\\ &\quad\oplus\mathscr{R}_{2n-2k-5/2}\oplus\mathscr{R}_{2n-2k-7/2}\oplus\cdots\oplus\mathscr{R}_{3/2}\oplus\mathscr{R}_{2k-1}\oplus\mathscr{R}_{2k-3}\oplus\cdots\oplus\mathscr{R}_1\\ &\quad\oplus\mathscr{R}_{2k-1/2}\oplus\mathscr{R}_{2k-3/2}\oplus\cdots\oplus\mathscr{R}_{3/2}\oplus\mathscr{R}_{n-1}\oplus\mathscr{R}_{n-2}\oplus\cdots\oplus\mathscr{R}_{n-2k}\\ &\quad\oplus\mathscr{R}_{n-1/2}\oplus\mathscr{R}_{n-3/2}\oplus\cdots\oplus\mathscr{R}_{n-2k-1/2}\,.\end{aligned} \tag{11.60}$$

If $\tilde{\mathscr{G}} = OSp(2k-1|2k)\oplus OSp(2n-2k+1|2n-2k)$ with $1\leqq k\leqq\left[\frac{n-2}{2}\right]$, the fundamental representation of $\mathscr{G}$, of dimension $4n$, decomposes under the superprincipal $OSp(1|2)$ *of* $\tilde{\mathscr{G}}$ as

$$\underline{4n} = \mathscr{R}_{n-k}\oplus\mathscr{R}^\pi_{k-1/2}\,, \tag{11.61}$$

and we have the following decomposition of the adjoint representation of $OSp(2n|2n)$:

$$\begin{aligned}&\frac{\mathbf{Ad}[OSp(2n|2n)]}{OSp(2k-1|2k)\oplus OSp(2n-2k+1|2n-2k)}\\ &= \mathscr{R}_{2n-2k-1}\oplus\mathscr{R}_{2n-2k-3}\oplus\cdots\oplus\mathscr{R}_1\\ &\quad\oplus\mathscr{R}_{2n-2k-1/2}\oplus\mathscr{R}_{2n-2k-3/2}\oplus\cdots\oplus\mathscr{R}_{3/2}\oplus\mathscr{R}_{2k-1}\oplus\mathscr{R}_{2k-3}\oplus\cdots\oplus\mathscr{R}_1\\ &\quad\oplus\mathscr{R}_{2k-5/2}\oplus\mathscr{R}_{2k-7/2}\oplus\cdots\oplus\mathscr{R}_{3/2}\oplus\mathscr{R}_{n-1}\oplus\mathscr{R}_{n-2}\oplus\cdots\oplus\mathscr{R}_{n-2k+1}\\ &\quad\oplus\mathscr{R}_{n-1/2}\oplus\mathscr{R}_{n-3/2}\oplus\cdots\oplus\mathscr{R}_{n-2k+1/2}\,.\end{aligned} \tag{11.62}$$

11.2.5. The Superalgebras $OSp(2|2n)$. The superalgebra $OSp(2|2n)$ requires special attention. Actually, the regular SSAs of $\mathscr{G} = OSp(2|2n)$ which admit a superprincipal embedding are only $OSp(2|2)$ and $Sl(1|2)$.

Let $\tilde{\mathscr{G}} = OSp(2|2)$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$, the fundamental representation of $OSp(2|2n)$, of dimension $2n+2$, decomposes as follows:

$$\underline{2n+2} = \mathscr{R}^\pi_{1/2}\oplus\mathscr{R}_0\oplus(2n-2)\mathscr{R}^\pi_0\,. \tag{11.63}$$

Therefore, the decomposition of the adjoint representation of $OSp(2|2n)$ under the superprincipal $OSp(1|2)$ of $OSp(2|2)\subset OSp(2|2n)$ is

$$\frac{\mathbf{Ad}[OSp(2|2n)]}{OSp(2|2)} = \mathscr{R}_1\oplus\mathscr{R}_{1/2}\oplus(2n-2)\mathscr{R}'_{1/2}\oplus(2n-1)(n-1)\mathscr{R}_0\oplus(2n-2)\mathscr{R}'_0\,. \tag{11.64}$$

Now, let $\tilde{\mathscr{G}} = Sl(1|2)$. Under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}}$, the fundamental representation of $OSp(2|2n)$ decomposes as:

$$\underline{2n} = 2\mathscr{R}^{\pi}_{1/2} \oplus (2n-4)\mathscr{R}^{\pi}_{0} \,. \tag{11.65}$$

In that case, the decomposition of the adjoint representation is

$$\frac{\mathbf{Ad}[OSp(2|2n)]}{Sl(1|2)} = 3\mathscr{R}_1 \oplus \mathscr{R}_{1/2} \oplus (4n-8)\mathscr{R}'_{1/2} \oplus [(2n-3)(n-2)+1]\mathscr{R}_0 \,. \tag{11.66}$$

11.3. Summary of the Results. The previous results can be easily extended to the case of sums of simple Lie SSAs. The decomposition of the fundamental representation is obtained by taking the corresponding $OSp(1|2)$ representation for each factor of the sum, which can be read in the following tableau. Then, starting from a decomposition of the fundamental representation of the form

$$\mathbf{F} = \left(\bigoplus_i \mathscr{R}_i\right) \oplus \left(\bigoplus_j \mathscr{R}^{\pi}_j\right) \tag{11.67}$$

the decomposition of the adjoint is given, in the orthosymplectic series, by

$$\mathbf{Ad} = \left(\bigoplus_i \mathscr{R}_i\right) \times \left(\bigoplus_i \mathscr{R}_i\right)\Bigg|_{\mathrm{A}} \oplus \left(\bigoplus_j \mathscr{R}^{\pi}_j\right) \times \left(\bigoplus_j \mathscr{R}^{\pi}_j\right)\Bigg|_{\mathrm{S}} \oplus \left(\bigoplus_i \mathscr{R}_i\right) \times \left(\bigoplus_j \mathscr{R}^{\pi}_j\right), \tag{11.68}$$

and in the unitary series, by

$$\mathbf{Ad} = \left(\bigoplus_i \mathscr{R}_i \bigoplus_j \mathscr{R}^{\pi}_j\right) \times \left(\bigoplus_i \mathscr{R}_i \bigoplus_j \mathscr{R}^{\pi}_j\right) - \mathscr{R}_0 \quad \text{for } SL(m|n)\ m \neq n \,, \tag{11.69}$$

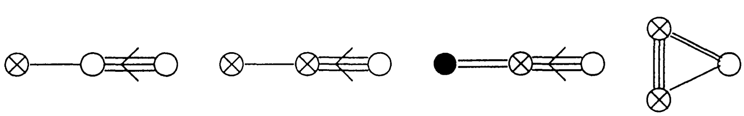
$$\mathbf{Ad} = \left(\bigoplus_i \mathscr{R}_i \bigoplus_j \mathscr{R}^{\pi}_j\right) \times \left(\bigoplus_i \mathscr{R}_i \bigoplus_i \mathscr{R}^{\pi}_j\right) - 2\mathscr{R}_0 \quad \text{for } PSl(m|m) \,. \tag{11.70}$$

For explicit formulae, one has to apply the product rules given in (11.18) and (11.22–11.29).

$\mathscr{G}$	$\tilde{\mathscr{G}}$	Fund. Rep. of $\mathscr{G}$
$Sl(m\|n)$	$Sl(p+1\|p)$	$\mathscr{R}_{p/2} \oplus (m-p-1)\mathscr{R}_0 \oplus (n-p)\mathscr{R}^{\pi}_0$
	$Sl(p\|p+1)$	$\mathscr{R}^{\pi}_{p/2} \oplus (m-p)\mathscr{R}_0 \oplus (n-p-1)\mathscr{R}^{\pi}_0$
$OSp(2m\|2n)$	$OSp(2k\|2k)$	$\mathscr{R}^{\pi}_{k-1/2} \oplus (2n-2k)\mathscr{R}^{\pi}_0 \oplus (2m-2k+1)\mathscr{R}_0$
	$OSp(2k+2\|2k)$	$\mathscr{R}_k \oplus (2m-2k-1)\mathscr{R}_0 \oplus (2n-2k)\mathscr{R}^{\pi}_0$
	$Sl(p+1\|p)$	$2\mathscr{R}_{p/2} \oplus 2(m-p-1)\mathscr{R}_0 \oplus 2(n-p)\mathscr{R}^{\pi}_0$
	$Sl(p\|p+1)$	$2\mathscr{R}^{\pi}_{p/2} \oplus 2(n-p-1)\mathscr{R}^{\pi}_0 \oplus 2(m-p)\mathscr{R}_0$

$\mathcal{G}$	$\tilde{\mathcal{G}}$	Fund. Rep. of $\mathcal{G}$
$OSp(2m+1\|2n)$	$\left.\begin{matrix} OSp(2k\|2k) \\ OSp(2k-1\|2k) \end{matrix}\right\}$	$\mathcal{R}^{\pi}_{k-1/2} \oplus (2n-2k)\mathcal{R}^{\pi}_0 \oplus (2m-2k+2)\mathcal{R}_0$
	$\left.\begin{matrix} OSp(2k+2\|2k) \\ OSp(2k+1\|2k) \end{matrix}\right\}$	$\mathcal{R}_k \oplus (2m-2k)\mathcal{R}_0 \oplus (2n-2k)\mathcal{R}^{\pi}_0$
	$Sl(p+1\|p)$	$2\mathcal{R}_{p/2} \oplus 2(m-p-1)\mathcal{R}_0 \oplus \mathcal{R}_0 \oplus 2(n-p)\mathcal{R}^{\pi}_0$
	$Sl(p\|p+1)$	$2\mathcal{R}^{\pi}_{p/2} \oplus 2(n-p-1)\mathcal{R}^{\pi}_0 \oplus \mathcal{R}_0 \oplus 2(m-p)\mathcal{R}_0$
$OSp(2\|2n)$	$OSp(2\|2)$	$\mathcal{R}^{\pi}_{1/2} \oplus \mathcal{R}_0 \oplus (2n-2)\mathcal{R}^{\pi}_0$
	$Sl(1\|2)$	$2\mathcal{R}^{\pi}_{1/2} \oplus (2n-4)\mathcal{R}^{\pi}_0$
$OSp(2n+2\|2n)$	$OSp(2k+1\|2k) \oplus OSp(2n-2k+1\|2n-2k)$	$\mathcal{R}_k \oplus \mathcal{R}_{n-k}$
$OSp(2n-2\|2n)$	$OSp(2k-1\|2k) \oplus OSp(2n-2k-1\|2n-2k)$	$\mathcal{R}^{\pi}_{k-1/2} \oplus \mathcal{R}^{\pi}_{n-k-1/2}$
$OSp(2n\|2n)$	$OSp(2k+1\|2k) \oplus OSp(2n-2k-1\|2n-2k)$	$\mathcal{R}_k \oplus \mathcal{R}^{\pi}_{n-k-1/2}$
	$OSp(2k-1\|2k) \oplus OSp(2n-2k+1\|2n-2k)$	$\mathcal{R}_{n-k} \oplus \mathcal{R}^{\pi}_{k-1/2}$

11.4. The Exceptional Superalgebra $G(3)$. The superalgebra $G(3)$ has dimension 31 and rank 3, with $\mathcal{G}_B = G_2 \oplus Sl(2)$ as bosonic part and the representation $(\underline{7}, \underline{2})$ of $\mathcal{G}_B$ as fermionic part. The Dynkin diagrams of $G(3)$ are



leading to the following regular sub(super)algebras:

$$G_2 \oplus A_1,\ G_2,\ A_2,\ A_1$$
$$B(1,1) \oplus A_1,\ B(1,1),\ C(2),\ B(0,1),\ A_2 \oplus B(0,1)$$
$$A(0,2),\ A(0,1),\ A(1,0),\ D(2,1;3),\ G(3)\,. \tag{11.71}$$

Only the superalgebras $B(0,1)$, $C(2)$, $B(1,1)$, $A(0,1)$, $A(1,0)$ and $D(2,1;3)$ admit a super-principal embedding. As an example, we will treat the case of $B(1,1) \equiv OSp(3|2)$. From the results of Sect. 8, the bosonic part $G_2 \oplus Sl(2)$ decomposes under the principal $Sl(2)$ of $SO(3) \oplus Sl(2)$ as

$$\mathbf{Ad}[G_2 \oplus Sl(2)] = \mathcal{D}_{3/2} \oplus \bar{\mathcal{D}}_{3/2} \oplus 2\mathcal{D}_1 \oplus 3\mathcal{D}_0\,, \tag{11.72}$$

and the fermionic part $(\underline{7}, \underline{2})$ as

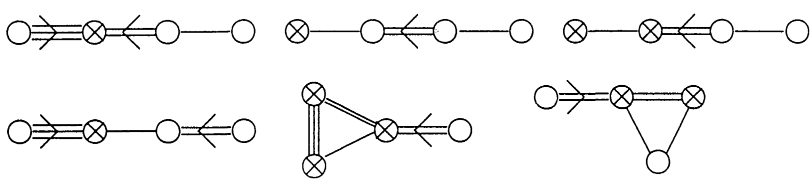
$$(\underline{7}, \underline{2}) = \mathscr{D}_{3/2} \oplus 2\mathscr{D}_1 \oplus \mathscr{D}_{1/2} \oplus 2\mathscr{D}_0 \,. \tag{11.73}$$

Putting together the $Sl(2)$ representations into $OSp(1|2)$ ones, one obtains the following decomposition under the superprincipal $OSp(1|2)$ of $OSp(3|2) \subset G(3)$:

$$\frac{\mathbf{Ad}[G(3)]}{OSp(3|2)} = \mathscr{R}_{3/2} \oplus 2\mathscr{R}'_{3/2} \oplus \mathscr{R}_1 \oplus 3\mathscr{R}_0 \oplus 2\mathscr{R}'_0 \,. \tag{11.74}$$

The other cases are similar and are summarized in Table 15.

11.5. The Exceptional Superalgebra $F(4)$. The superalgebra $F(4)$ has dimension 40 and rank 4, with $\mathscr{G}_B = Sl(2) \oplus O(7)$ as bosonic part and the representation $(\underline{2}, \underline{8})$ of $\mathscr{G}_B$ as fermionic part. Its Dynkin diagrams are:



The SSAs of $F(4)$ which admit a superprincipal embedding are $A(0, 1)$, $A(1, 0)$, $C(2)$ and $D(2, 1; 2)$ (the extended Dynkin diagrams of $F(4)$ can be found in [18]). As an example, we will treat the case of $C(2) \equiv OSp(2|2)$. The bosonic part $Sl(2) \oplus O(7)$ decomposes then as

$$\mathbf{Ad}[Sl(2) \oplus O(7)] = 5\mathscr{D}_1 \oplus 9\mathscr{D}_0 \tag{11.75}$$

and the fermionic part $(\underline{2}, \underline{8})$ as

$$(\underline{2}, \underline{8}) = 8\mathscr{D}_{1/2} \,. \tag{11.76}$$

Putting together the $Sl(2)$ representations into $OSp(1|2)$ ones, one obtains the following decomposition under the superprincipal $OSp(1|2)$ of $OSp(2|2) \subset F(4)$:

$$\frac{\mathbf{Ad}[F(4)]}{OSp(2|2)} = 5\mathscr{R}_1 \oplus 3\mathscr{R}_{1/2} \oplus 6\mathscr{R}_0 \,. \tag{11.77}$$

The other cases are analogous and are summarized in Table 16.

12. $OSp(1|2) \oplus U(1)$ Decompositions of Simple Lie Superalgebras

12.1. Introduction of the $U(1)$. Now, we are in position to introduce the $U(1)$ factor. In the case of the unitary superalgebras, since the formulae for $Sl(p+1|p)$ are completely analogous to those of $Sl(n)$ (the $\mathscr{R}_j$ representations replacing the $\mathscr{D}_j$ ones), one can write the following statement.

A decomposition of the fundamental representation $\mathbf{F}$ of $\mathscr{G} = Sl(m|n)$ under the superprincipal $OSp(1|2)$ of $\tilde{\mathscr{G}} \subset \mathscr{G}$ being given,

$$\mathbf{F} = \left(\bigoplus_i n_i \mathscr{R}_i \right) \oplus \left(\bigoplus_j n_j \mathscr{R}_j^{\pi} \right), \tag{12.1}$$

the corresponding decomposition under $OSp(1|2) \oplus U(1)$ has the form

$$\mathbf{F} = \left(\bigoplus_i n_i \mathscr{R}_i(y_i) \right) \oplus \left(\bigoplus_j n_j \mathscr{R}_j^{\pi}(y_j) \right), \tag{12.2}$$

identical representations (i.e. labelled by the same index i or j) having the same value of y. Moreover, one has to impose the supertraceless condition

$$\sum_i n_i y_i - \sum_j n_j y_j = 0 . \tag{12.3}$$

Then the decomposition of the adjoint is given by

$$\mathbf{Ad} = \left(\bigoplus_i n_i \mathscr{R}_i(y_i) \bigoplus_j n_j \mathscr{R}_j^{\pi}(y_j) \right) \times \left(\bigoplus_i n_i \mathscr{R}_i(-y_i) \bigoplus_j n_j \mathscr{R}_j^{\pi}(-y_j) \right) - \mathscr{R}_0(0) . \tag{12.4}$$

For an explicit calculation of this expression, one uses the fact that

$$(n_i \mathscr{R}_i(y_i)) \times (n_j \mathscr{R}_j(y_j)) = n_i n_j \bigoplus_{k=|i=j|}^{i+j} \mathscr{R}_k(y_i + y_j) \quad \text{with } k \text{ integer and half-integer} \tag{12.5}$$

and the same formula for $\mathscr{R}^{\pi}$ representations.

In the case of the orthosymplectic superalgebras, one considers the following decomposition of the $OSp(M|2n)$ fundamental representation:

$$\mathbf{F} = \left(\bigoplus_i n_i \mathscr{R}_i \right) \oplus \left(\bigoplus_i n_j \mathscr{R}_j^{\pi} \right) \tag{12.6}$$

which implies for the fundamental representations of $SO(M)$ and $Sp(2n)$:

$$\begin{aligned} \underline{M} &= \left(\bigoplus_i n_i \mathscr{D}_i \right) \oplus \left(\bigoplus_i n_j \mathscr{D}_{j-1/2} \right), \\ \underline{2n} &= \left(\bigoplus_i n_i \mathscr{D}_{i-1/2} \right) \oplus \left(\bigoplus_j n_j \mathscr{D}_j \right). \end{aligned} \tag{12.7}$$

For the $SO(M)$ part, one can introduce a non-zero $U(1)$ eigenvalue y_i only for representations $\mathscr{D}_i$ with i integer, which appear twice and only twice. For the $Sp(2n)$ part, a non-zero $U(1)$ eigenvalue y_i is allowed only for representations $\mathscr{D}_i$ with i half-integer, which appear twice and only twice.

For the superalgebra $\mathscr{G}$ itself, one has to group the $Sl(2) \oplus U(1)$ representations $\mathscr{D}_i(y_i)$ into $OSp(1|2) \oplus U(1)$ representations $\mathscr{R}_j(y_j) = D_j(y_j) \oplus \mathscr{D}_{j-1/2}(y_j)$. Therefore, if the decomposition of the $OSp(M|2n)$ fundamental representation $\mathbf{F}$ under a certain $OSp(1|2)$ is given by (12.6), non-zero values y of the $U(1)$ factor are allowed for the following combinations:

- the representation $\mathscr{R}_i$ appears twice and only twice ($n_i = 2$), and i is integer,
- the representation $\mathscr{R}_i^{\pi}$ appears twice and only twice ($n_i = 2$), and i is half-integer.

Moreover, y can only take the values 0, 1/4 or 1/2 if $i \neq 0$ (which lead to the values 0, $\pm 1/2$ or ± 1 for the $U(1)$ factor in the adjoint representation of $\mathscr{G}$). Finally, starting from a decomposition of the fundamental representation of $OSp(M|2n)$

under $OSp(1|2) \oplus U(1)$ of the form

$$\mathbf{F} = \left(\bigoplus_i \mathscr{R}_i(y_i) \oplus \mathscr{R}_i(-y_i)\right) \oplus \left(\bigoplus_j \mathscr{R}_j^\pi(y_j) \oplus \mathscr{R}_j^\pi(-y_j)\right) \oplus \left(\bigoplus_{i,n_i \neq 2} n_i \mathscr{R}_i(0)\right)$$
$$\oplus \left(\bigoplus_{j,n_j \neq 2} n_j \mathscr{R}_j^\pi(0)\right), \tag{12.8}$$

the decomposition of the adjoint is given by

$$\begin{aligned}\mathbf{Ad} = &\left(\bigoplus_i \mathscr{R}_i(y_i) \oplus \mathscr{R}_i(-y_i) \bigoplus_{i,n_i \neq 2} n_i \mathscr{R}_i(0)\right)\\ &\times \left(\bigoplus_i \mathscr{R}_i(y_i) \oplus \mathscr{R}_i(-y_i) \bigoplus_{i,n_i \neq 2} n_i \mathscr{R}_i(0)\right)\Bigg|_{\mathrm{A}}\\ &\oplus \left(\bigoplus_j \mathscr{R}_j^\pi(y_j) \oplus \mathscr{R}_j^\pi(-y_i) \bigoplus_{j,n_j \neq 2} n_j \mathscr{R}_j^\pi(0)\right)\\ &\times \left(\bigoplus_j \mathscr{R}_j^\pi(y_j) \oplus \mathscr{R}_j^\pi(-y_j) \bigoplus_{j,n_j \neq 2} n_j \mathscr{R}_j^\pi(0)\right)\Bigg|_{\mathrm{S}}\\ &\oplus \left(\bigoplus_i \mathscr{R}_i(y_i) \oplus \mathscr{R}_i(-y_j) \bigoplus_{i,n_i \neq 2} n_j \mathscr{R}_i(0)\right)\\ &\times \left(\bigoplus_j \mathscr{R}_j^\pi(y_j) \oplus \mathscr{R}_j^\pi(-y_j) \bigoplus_{j,n_j \neq 2} n_j \mathscr{R}_j^\pi(0)\right).\end{aligned} \tag{12.9}$$

The (anti)symmetric products of $\mathscr{R}$ representations are given by the formulae (11.18) and (11.22–11.29) modulo the following modifications due to the $U(1)$ eigenvalue:

$$\begin{aligned}&(\mathscr{R}_i(y_i) \oplus \mathscr{R}_i(-y_i)) \times (\mathscr{R}_i(y_i) \oplus \mathscr{R}_i(-y_i))|_{\mathrm{A}}\\ &= (\mathscr{R}_i \times \mathscr{R}_i)_{\mathrm{A}}\,(2y_i) \oplus (\mathscr{R}_i \times \mathscr{R}_i)_{\mathrm{A}}(-2y_i) \oplus (\mathscr{R}_i \times \mathscr{R}_i)(0)\end{aligned} \tag{12.10}$$

and

$$\begin{aligned}&(\mathscr{R}_i^\pi(y_i) \oplus \mathscr{R}_i^\pi(-y_i)) \times (\mathscr{R}_i^\pi(y_i) \oplus \mathscr{R}_i^\pi(-y_i))|_{\mathrm{S}}\\ &= (\mathscr{R}_i^\pi \times \mathscr{R}_i^\pi)_{\mathrm{S}}\,(2y_i) \oplus (\mathscr{R}_i^\pi \times \mathscr{R}_i^\pi)_{\mathrm{S}}(-2y_i) \oplus (\mathscr{R}_i^\pi \times \mathscr{R}_i^\pi)(0)\,.\end{aligned} \tag{12.11}$$

Finally, considering $D(2,1;\alpha)$, $G(3)$ and $F(4)$, a direct calculation shows that no $U(1)_Y$ can be added to any of the $OSp(1|2)$ subsuperalgebras of these exceptional superalgebras.

12.2. Superdefining Vector. The determination of the grading H from the $OSp(1|2) \oplus U(1)$ decomposition of the fundamental representation is strictly the same as for the algebras case. One just has to "double the calculation" since the bosonic part of $\mathscr{G}$ is in general the direct sum of two simple algebras. Using the same basis for the Cartan algebras (see Sect. 6.1), we will denote the defining vector as

$$f = (f_1, \ldots, f_n; f'_1, \ldots, f'_{n'})\,, \tag{12.12}$$

where f_i refers to the first simple algebra and the f_i' to the second. For example, for the case of $Sl(m|n)$ superalgebras, the contribution of a representation is:

$$\mathscr{R}_j(y) \to \Big(j + y, j - 1 + y, \ldots, -j + y, 0, \ldots, 0; j - \frac{1}{2} + y,$$
$$j - \frac{3}{2} + y, \ldots, -j + \frac{1}{2} + y, 0, \ldots, 0 \Big),$$
$$\mathscr{R}_j^\pi(y) \to \Big(j - \frac{1}{2} + y, j - \frac{3}{2} + y, \ldots, -j + \frac{1}{2} + y, 0, \ldots, 0;$$
$$j + y, j - 1 + y, \ldots, -j + y, 0, \ldots, 0 \Big).$$

The other cases are analogous.

13. *W* Superalgebras from Lie Superalgebras of Rank up to 4

In the following tables, we present an exhaustive classification of super W algebras arising from super Toda models based on classical superalgebras of rank up to 4. The classification is listed in Tables 10 to 17.

For the infinite series $\mathscr{G} = A(m, n) = Sl(m + 1|n + 1)$ with $m \neq n$, $A(n, n) = Sl(n + 1|n + 1)/U(1)$, $B(m, n) = OSp(2m + 1|2n)$, $C(n + 1) = OSp(2|2n)$ and $D(m, n) = OSp(2m|2n)$, We give the decomposition of the fundamental representation of $\mathscr{G}$ with respect to $OSp(1|2) \oplus U(1)$, the minimal (i.e. the lowest dimensional) regular SSAs containing the $OSp(1|2)$ or (for the irregular cases) the corresponding

Table 10. $A(m, n)$ superalgebras up to rank 4

$\mathscr{G}$	SSA in $\mathscr{G}$	Decomposition of the fundamental of $\mathscr{G}$	Superconformal spin of the W superfields (Hypercharge)
$A(0, 1)$	$A(0, 1)$	$\mathscr{R}^\pi_{1/2}$	$\frac{3}{2}, 1$
$A(0, 2)$	$A(0, 1)$	$\mathscr{R}^\pi_{1/2}(y) \oplus \mathscr{R}^\pi_0(-y)$	$\frac{3}{2}, 1, 1', 1', \frac{1}{2}$ $(0, 0, 2y, -2y, 0)$
$A(1, 1)$	$A(0, 1)$	$\mathscr{R}^\pi_{1/2} \oplus \mathscr{R}_0$	$\frac{3}{2}, 1, 1, 1$
$A(0, 3)$	$A(0, 1)$	$\mathscr{R}^\pi_{1/2}(y) \oplus 2\mathscr{R}^\pi_0(-y/2)$	$\frac{3}{2}, 1, 4{*}1', 4{*}\frac{1}{2}$ $\left(0, 0, \frac{3y}{2}, \frac{3y}{2}, \frac{-3y}{2}, \frac{-3y}{2}, 0, 0, 0, 0\right)$
$A(1, 2)$	$A(1, 2)$	$\mathscr{R}^\pi_1$	$\frac{5}{2}, 2, \frac{3}{2}, 1$
	$A(0, 1)$	$\mathscr{R}^\pi_{1/2}(0)$ $\oplus \mathscr{R}_0(y) \oplus \mathscr{R}^\pi_0(y)$	$\frac{3}{2}, 1, 1, 1, 1', 1', \frac{1}{2}, \frac{1}{2}, \frac{1'}{2}, \frac{1'}{2}$ $\left(0, 0, \frac{y}{2}, \frac{-y}{2}, \frac{y}{2}, \frac{-y}{2}, 0, 0, 0, 0\right)$
	$A(1, 0)$	$\mathscr{R}_{1/2}(y) \oplus 2\mathscr{R}^\pi_0(y/2)$	$\frac{3}{2}, 5{*}1, 4{*}\frac{1}{2}$ $\left(0, 0, \frac{y}{2}, \frac{y}{2}, \frac{-y}{2}, \frac{-y}{2}, 0, 0, 0, 0, 0\right)$

Table 11. $B(m, n)$ superalgebras of rank 2 and 3

$\mathscr{G}$	SSA in $\mathscr{G}$	Decomposition of the fundamental of $\mathscr{G}$	Superconformal spin of the W superfields (Hypercharge)
$B(0,2)$	$B(0,1)$	$\mathscr{R}^{\pi}_{1/2} \oplus 2\mathscr{R}^{\pi}_{0}$	$\frac{3}{2}, 1', 1', \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
$B(1,1)$	$B(1,1)$	$\mathscr{R}_1$	$2, \frac{3}{2}$
	$\left.\begin{matrix} C(2) \\ B(0,1) \end{matrix}\right\}$	$\mathscr{R}^{\pi}_{1/2} \oplus \mathscr{R}_0(y) \oplus \mathscr{R}_0(-y)$	$\frac{3}{2}, 1, 1, \frac{1}{2}$ $(0, y, -y, 0)$
$B(0,3)$	$B(0,1)$	$\mathscr{R}^{\pi}_{1/2} \oplus 4\mathscr{R}^{\pi}_{0}$	$\frac{3}{2}, 1', 1', 1', 1', 10*\frac{1}{2}$
$B(1,2)$	$B(1,2)$	$\mathscr{R}^{\pi}_{3/2}$	$\frac{7}{2}, 2, \frac{3}{2}$
	$B(1,1)$	$\mathscr{R}_1 \oplus 2\mathscr{R}^{\pi}_{0}$	$2, \frac{3}{2}, \frac{3}{2}', \frac{3}{2}', \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
	$\left.\begin{matrix} C(2) \\ B(0,1) \end{matrix}\right\}$	$\mathscr{R}^{\pi}_{1/2} \oplus \mathscr{R}_0(y) \oplus \mathscr{R}_0(-y)$ $\oplus 2\mathscr{R}^{\pi}_{0}$	$\frac{3}{2}, 1, 1, 1', 1', 4*\frac{1}{2}, 4*\frac{1}{2}'$ $(0, y, -y, 6*0, y, y, -y, -y)$
	$\left.\begin{matrix} C(2) \oplus B(0,1) \\ A(0,1) \end{matrix}\right\}$	$\mathscr{R}^{\pi}_{1/2}(y) \oplus \mathscr{R}^{\pi}_{1/2}(-y) \oplus \mathscr{R}_0$	$\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1, 1, \frac{1}{2}$ $(2y, -2y, 0, y, -y, 0, 0)$
$B(2,1)$	$\left.\begin{matrix} D(2,1) \\ B(1,1) \end{matrix}\right\}$	$\mathscr{R}_1 \oplus \mathscr{R}_0(y) \oplus \mathscr{R}_0(-y)$	$2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}$ $(0, 2y, -2y, 0, 0)$
	$\left.\begin{matrix} C(2) \\ B(0,1) \end{matrix}\right\}$	$\mathscr{R}^{\pi}_{1/2} \oplus 4\mathscr{R}_0$	$\frac{3}{2}, 4*1, 6*\frac{1}{2}$
	$A(1,0)$	$2\mathscr{R}_{1/2} \oplus \mathscr{R}_0$	$\frac{3}{2}, 1, 1, 1, 1', 1', \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

Table 12. $B(m, n)$ superalgebras of rank 4

$\mathscr{G}$	SSA in $\mathscr{G}$	Decomposition of the fundamental of $\mathscr{G}$	Superconformal spin of the W superfields (Hypercharge)
$B(0,4)$	$B(0,1)$	$\mathscr{R}^{\pi}_{1/2} \oplus 6\mathscr{R}^{\pi}_{0}$	$\frac{3}{2}, 6*1', 21*\frac{1}{2}$
$B(1,3)$	$B(1,2)$	$\mathscr{R}^{\pi}_{3/2} \oplus 2\mathscr{R}^{\pi}_{0}$	$\frac{7}{2}, 2, 2', 2', \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
	$B(1,1)$	$\mathscr{R}_1 \oplus 4\mathscr{R}^{\pi}_{0}$	$2, \frac{3}{2}, \frac{3}{2}', \frac{3}{2}', \frac{3}{2}', \frac{3}{2}', 10*\frac{1}{2}$
	$\left.\begin{matrix} C(2) \oplus B(0,1) \\ A(0,1) \end{matrix}\right\}$	$\mathscr{R}^{\pi}_{1/2}(y) \oplus \mathscr{R}^{\pi}_{1/2}(-y)$ $\oplus \mathscr{R}_0 \oplus 2\mathscr{R}^{\pi}_{0}$	$3*\frac{3}{2}, 1, 1', 1' 1, 1', 1', 1, 4*\frac{1}{2}, 2*\frac{1}{2}'$ $(2y, -2y, 0, 3*y, 3*-y, 7*0)$
	$\left.\begin{matrix} C(2) \\ B(0,1) \end{matrix}\right\}$	$\mathscr{R}^{\pi}_{1/2} \oplus 4\mathscr{R}^{\pi}_{0}$ $\oplus \mathscr{R}_0(y) \oplus \mathscr{R}_0(-y)$	$\frac{3}{2}, 2*1, 4*1', 11*\frac{1}{2}, 9*\frac{1}{2}'$ $(0, y, -y, 15*0, 4*y, 4*-y, 0)$
$B(2,2)$	$B(2,2)$	$\mathscr{R}_2$	$4, \frac{7}{2}, 2, \frac{3}{2}$
	$\left.\begin{matrix} D(2,2) \\ B(1,2) \end{matrix}\right\}$	$\mathscr{R}^{\pi}_{3/2} \oplus \mathscr{R}_0(y) \oplus \mathscr{R}_0(-y)$	$\frac{7}{2}, 2, 2, 2, \frac{3}{2}, \frac{1}{2}$ $(0, \mathrm{y}, -y, 0, 0, 0)$
	$\left.\begin{matrix} D(2,1) \\ B(1,1) \end{matrix}\right\}$	$\mathscr{R}_1 \oplus 2\mathscr{R}^{\pi}_{0}$ $\oplus \mathscr{R}_0(y) \oplus \mathscr{R}_0(-y)$	$2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}', \frac{3}{2}', 4*\frac{1}{2}, 4*\frac{1}{2}'$ $(0, 2y, -2y, 0, 6*0, y, y, -y, -y)$

Table 12. (*continued*)

$\mathscr{G}$	SSA in $\mathscr{G}$	Decomposition of the fundamental of $\mathscr{G}$	Superconformal spin of the W superfields (Hypercharge)
	$D(2,1)\oplus B(0,1)$ $B(1,1)\oplus C(2)$	$\mathscr{R}_1\oplus\mathscr{R}^\pi_{1/2}\oplus\mathscr{R}_0$	$2, 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1$
	$C(2)\oplus C(2)$ $A(0,1)$	$\mathscr{R}^\pi_{1/2}(y)\oplus\mathscr{R}^\pi_{1/2}(-y)$ $\oplus 3\mathscr{R}_0$	$\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 7*1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ $(2y, -2y, 0, 3*y, 3*-y, 0, 4*0)$
	$C(2)$ $B(0,1)$	$\mathscr{R}^\pi_{1/2}\oplus 4\mathscr{R}_0\oplus 2\mathscr{R}^\pi_0$	$\frac{3}{2}, 4*1, 2*1', 9*\frac{1}{2}, 8*\frac{1}{2}'$
	$A(1,0)$	$2\mathscr{R}_{1/2}\oplus\mathscr{R}_0\oplus 2\mathscr{R}^\pi_0$	$\frac{3}{2}, 7*1, 1', 1', 6*\frac{1}{2}, 2*\frac{1}{2}'$
$B(3,1)$	$D(2,1)$ $B(1,1)$	$\mathscr{R}_1\oplus 4\mathscr{R}_0$	$2, 5*\frac{3}{2}, 6*\frac{1}{2}$
	$C(2)$ $B(0,1)$	$\mathscr{R}^\pi_{1/2}\oplus 6\mathscr{R}_0$	$\frac{3}{2}, 6*1, 15*\frac{1}{2}$
	$A(1,0)$	$2\mathscr{R}_{1/2}\oplus 3\mathscr{R}_0$	$\frac{3}{2}, 3*1, 6*1', 6*\frac{1}{2}$

Table 13. $D(m, n)$ superalgebras up to rank 4

$\mathscr{G}$	SSA in $\mathscr{G}$	Decomposition of the fundamental of $\mathscr{G}$	Superconformal spin of the W superfields (Hypercharge)
$D(2,1)$	$D(2,1)$	$\mathscr{R}_1\oplus\mathscr{R}_0$	$2, \frac{3}{2}, \frac{3}{2}$
	$C(2)$	$\mathscr{R}^\pi_{1/2}\oplus 3\mathscr{R}_0$	$\frac{3}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
	$A(1,0)$	$2\mathscr{R}_{1/2}$	$\frac{3}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
$D(2,2)$	$D(2,2)$	$\mathscr{R}^\pi_{3/2}\oplus\mathscr{R}_0$	$\frac{7}{2}, 2, 2, \frac{3}{2}$
	$D(2,1)$	$\mathscr{R}_1\oplus\mathscr{R}_0\oplus 2\mathscr{R}^\pi_0$	$2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}', \frac{3}{2}', 5*\frac{1}{2}$
	$C(2)$	$\mathscr{R}^\pi_{1/2}\oplus 3\mathscr{R}_0\oplus 2\mathscr{R}^\pi_0$	$\frac{3}{2}, 1, 1, 1, 1', 1', 6*\frac{1}{2}, 6*\frac{1}{2}'$
	$C(2)\oplus C(2)$ $A(0,1)$	$\mathscr{R}^\pi_{1/2}\oplus\mathscr{R}_0(y)\oplus\mathscr{R}_0(-y)$	$\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 5*1, \frac{1}{2}, \frac{1}{2}$ $(3*0, y, -y, 5*0)$
	$B(1,1)\oplus B(0,1)$	$\mathscr{R}_1\oplus\mathscr{R}^\pi_{1/2}$	$2, 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1$
	$A(1,0)$	$2\mathscr{R}_{1/2}\oplus 2\mathscr{R}^\pi_0$	$\frac{3}{2}, 7*1, 6*\frac{1}{2}$
$D(3,1)$	$D(2,1)$	$\mathscr{R}_1\oplus 3\mathscr{R}_0$	$2, 4*\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
	$C(2)$	$\mathscr{R}^\pi_{1/2}\oplus 5\mathscr{R}_0$	$\frac{3}{2}, 5*1, 10*\frac{1}{2}$
	$A(1,0)$	$2\mathscr{R}_{1/2}\oplus 2\mathscr{R}_0$	$\frac{3}{2}, 1, 1, 1, 4*1', 4*\frac{1}{2}$

singular embedding. Then, we give the superspin content with the same convention as for the bosonic tables. We recall that to a W_s superfield correspond two fields w_s and $w_{s+1/2}$. When the superspin is marked with a prime ($'$), the corresponding superfield W_s has the "wrong" statistics (commuting fermions and anticommuting

Table 14. $C(n+1)$ superalgebras up to rank 4

$\mathscr{G}$	SSA in $\mathscr{G}$	Decomposition of the fundamental of $\mathscr{G}$	Superconformal spin of the W superfields (Hypercharge)
$C(3)$	$A(0,1)$	$\mathscr{R}^{\pi}_{1/2}(y) \oplus \mathscr{R}^{\pi}_{1/2}(-y)$	$\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, \frac{1}{2}$
			$(0,\ 2y,\ -2y,\ 0,\ 0)$
	$C(2)$	$\mathscr{R}^{\pi}_{1/2} \oplus \mathscr{R}_0 \oplus 2\mathscr{R}^{\pi}_0$	$\frac{3}{2}, 1, 1', 1', \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}', \frac{1}{2}'$
$C(4)$	$A(0,1)$	$\mathscr{R}^{\pi}_{1/2}(y) \oplus \mathscr{R}^{\pi}_{1/2}(-y) \oplus 2\mathscr{R}^{\pi}_0$	$3{*}\frac{3}{2}, 1, 4{*}1', 4{*}\frac{1}{2}$
			$(0, 2y, -2y, 0, y, y, -y, -y, 4{*}0)$
	$C(2)$	$\mathscr{R}^{\pi}_{1/2} \oplus \mathscr{R}_0 \oplus 4\mathscr{R}^{\pi}_0$	$\frac{3}{2}, 1, 4{*}1', 10{*}\frac{1}{2}, 4{*}\frac{1}{2}'$

Table 15. The exceptional superalgebra $G(3)$

SSA	$OSp(1\|2)$ decomposition of $G(3)$	Superconformal spin of the W superfields
$A(1,0)$	$\mathscr{R}_1 \oplus 3\mathscr{R}_{1/2} \oplus 4\mathscr{R}'_{1/2} \oplus 3\mathscr{R}_0 \oplus 2\mathscr{R}'_0$	$\frac{3}{2}, 1, 1, 1, 4{*}1', \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}', \frac{1}{2}'$
$A(1,0)'$	$2\mathscr{R}'_{3/2} \oplus \mathscr{R}_1 \oplus 3\mathscr{R}_{1/2} \oplus 3\mathscr{R}_0$	$2', 2', \frac{3}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
$B(0,1)$	$\mathscr{R}_0 \oplus 6\mathscr{R}_{1/2} \oplus 8\mathscr{R}_0$	$\frac{3}{2}, 6{*}1, 8{*}\frac{1}{2}$
$B(1,1)$	$\mathscr{R}_{3/2} \oplus 2\mathscr{R}'_{3/2} \oplus R_1 \oplus 3\mathscr{R}_0 \oplus 2\mathscr{R}'_0$	$2, 2', 2', \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}', \frac{1}{2}'$
$D(2,1;3)$	$\mathscr{R}_2 \oplus \mathscr{R}_{3/2} \oplus 3\mathscr{R}_1$	$\frac{5}{2}, 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}$

Table 16. The exceptional superalgebra $F(4)$

SSA	$OSp(1\|2)$ decomposition of $F(4)$	Superconformal spin of the W superfields
$A(1,0)$	$\mathscr{R}_1 \oplus 7\mathscr{R}_{1/2} \oplus 14\mathscr{R}_0$	$\frac{3}{2}, 7{*}1, 14{*}\frac{1}{2}$
$A(0,1)$	$\mathscr{R}_1 \oplus 3\mathscr{R}_{1/2} \oplus 6\mathscr{R}'_{1/2} \oplus 6\mathscr{R}_0 \oplus 2\mathscr{R}'_0$	$\frac{3}{2}, 3{*}1, 6{*}1', 6{*}\frac{1}{2}, \frac{1}{2}', \frac{1}{2}'$
$C(2)$	$5\mathscr{R}_1 \oplus 3\mathscr{R}_{1/2} \oplus 6\mathscr{R}_0$	$5{*}\frac{3}{2}, 3{*}1, 6{*}\frac{1}{2}$
$D(2,1;2)$	$\mathscr{R}_{3/2} \oplus 2\mathscr{R}'_{3/2} \oplus 2\mathscr{R}_1 \oplus 2\mathscr{R}'_{1/2} \oplus 3\mathscr{R}_0$	$2, 2', 2', \frac{3}{2}, \frac{3}{2}, 1', 1', \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

Table 17. The exceptional superalgebra $D(2,1;\alpha)$

SSA	Decomposition of the fundamental of $D(2,1;\alpha)$	Superconformal spin of the W superfields
$D(2,1)$	$\mathscr{R}_1 \oplus \mathscr{R}_0$	$2, \frac{3}{2}, \frac{3}{2}$
$C(2)$	$\mathscr{R}^{\pi}_{1/2} \oplus 3\mathscr{R}_0$	$\frac{3}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
$A(1,0)$	$2\mathscr{R}_{1/2}$	$\frac{3}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

bosons). In the same column, we give under the superspin s the hypercharge(s) y when they exist.

For the two exceptional superalgebras $\mathscr{G} = G(3)$ and $F(4)$, we give the minimal regular SSA containing the $OSp(1|2)$ embedding, the decomposition of the adjoint representation of $\mathscr{G}$, and the superspin content.

14. Quadratic-, Quasi- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-Superconformal Algebras

We have a natural framework to study superconformal algebras. Let us first recall that a quadratic-superconformal algebra is a Zamolodchikov superalgebra made of one spin 2 field corresponding to $T(x)$ (and forming a Virasoro algebra), N fermionic supersymmetry charges $G^\alpha(x)$ which are spin $\frac{3}{2}$ primary fields with respect to $T(x)$, and a Kac–Moody (KM) algebra (i.e. spin 1 primary fields). The spin $\frac{3}{2}$ generators are required to form a representation of the KM algebra, but the quadratic-superconformal superalgebra is not (in general) a Lie superalgebra in the sense that the PB $\{G^\alpha(x), G^\beta(x')\}_{\text{PB}}$ contains quadratic terms in the KM currents [16, 19].

The "usual" superconformal algebras, i.e. the Ademollo et al. algebras [20] and the one parameter algebra found in [21], are the only closed Lie superconformal algebras we know. We will refer to them as *Lie* superconformal algebras and call the corresponding supersymmetries "true" supersymmetries.

The same definition holds for a quasi-superconformal algebra [16], except that its spin $\frac{3}{2}$ fields $G^\alpha(x)$ are bosonic ("wrong" statistics). As an example, the algebra made explicit in Sect. 7.3, possessing two spin-$\frac{3}{2}$ and one spin-1 fields, is quasi-superconformal.

An algebra with both bosonic and fermionic spin $\frac{3}{2}$ currents is called $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebra. In that case, spin 1 fermions may also appear.

It should be clear to the reader that Part I contains all the tools necessary for the determination of the quasi-superconformal algebras, whereas the quadratic and $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebras can be obtained from Part II. Note however, that the supersymmetric treatment we have used (and which naturally makes appear a $N = 1$ Lie superconformal algebra) leads to the emergence of spin $\frac{1}{2}$ fields. As it is now well-known, to avoid these fermions, one can factorize them [22]. These algebras (without spin $\frac{1}{2}$ fermions) have already been classified at the quantum level in [16]. We show hereafter that all the algebras of [16] can be realized at the

Table 18. Classification of quasi-superconformal algebras

Algebra $\mathcal{G}$	Decomposition of the fundamental of $\mathcal{G}$	Conformal spin of the W generators	Residual Kac–Moody algebra
$Sl(n)$	$\underline{n} = \mathcal{D}_{1/2} \oplus (n-2)\mathcal{D}_0$	$2,\ 2(n-2)*\frac{3}{2},\ (n-2)^2*1$	$Sl(n-2) \oplus U(1)$
$SO(n)$	$\underline{n} = 2\mathcal{D}_{1/2} + (n-4)\mathcal{D}_0$	$2,\ 2(n-4)*\frac{3}{2},\ \left[\frac{(n-4)(n-5)}{2} + 3\right]*1$	$SO(n-4) \oplus Sl(2)$
$Sp(2n)$	$\underline{2n} = \mathcal{D}_{1/2} + (2n-2)$	$2,\ (2n-4)*\frac{3}{2},\ (n-2)(2n-3)*1$	$Sp(2n-2)$
G_2	$\underline{7} = 2\mathcal{D}_{1/2} + 3\mathcal{D}_0$	$2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1, 1$	$Sl(2)$
F_4	$\underline{26} = 6\mathcal{D}_{1/2} + 14\mathcal{D}_0$	$2,\ 14*\frac{3}{2},\ 21*1$	$Sp(6)$
E_6	$\underline{27} = 6\mathcal{D}_{1/2} + 15\mathcal{D}_0$	$2,\ 20*\frac{3}{2},\ 35*1$	$Sl(6)$
E_7	$\underline{56} = 12\mathcal{D}_{1/2} + 32\mathcal{D}_0$	$2,\ 32*\frac{3}{2},\ 66*1$	$SO(12)$
E_8	$\underline{248} = \mathcal{D}_1 + 56\mathcal{D}_{1/2} + 133\mathcal{D}_0$	$2,\ 56*\frac{3}{2},\ 133*1$	E_7

classical level as symmetries of Toda models. Moreover, two new (with respect to [16]) $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebras can be identified from the study of $G(3)$ and $F(4)$.

14.1. Quasi-Superconformal Algebras. From the study of Part I, we can see that such algebras, with only one spin 2 and no spin $s > 2$, are obtained when the fundamental representation of $Sl(n)$ and $Sp(2n)$ (resp. $SO(n)$) algebras contains only one (resp. two) $\mathscr{D}_{1/2}$ representation(s). This means that we are reducing these Lie algebras with respect to a regular A_1. Using the results of Part I and [17] for the exceptional algebras $E_{6,7,8}$, we obtain the classification of Table 18.

14.2. Quadratic-Superconformal Algebras. They are obtained from the reduction of a superalgebra with respect to an $OSp(1|2)$ SSA. Note that "wrong" statistic superfields may appear and lead to $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebras. From the rules given in Sect. 11.2.1, relating $\mathscr{R}'$ representations of the adjoint, to $\mathscr{R}$ and $\mathscr{R}^\pi$ representations of the fundamental, it is easy to compute the allowed reductions. As an example, let us study the $Sl(m|n)$ algebras: the reduction with respect to $Sl(1|2)$ reads $\underline{n+m} = \mathscr{R}^\pi_{1/2} + (m-1)\mathscr{R}_0 + (n-2)\mathscr{R}^\pi_0$, so that we must set $n = 2$ to avoid "wrong" statistics. Thus, only the $Sl(n|2)$ (or $Sl(2|n)$) algebra leads to quadratic-superconformal algebras. The same calculation leads to the list:

$$Sl(n|2), \quad OSp(4|2n), \quad OSp(n|2), \quad F(4), \quad G(3)\,. \tag{14.1}$$

We summarize the results in Table 19. Note that the regular superalgebra which characterizes the $OSp(1|2)$, provides the number N_0 of "true" supersymmetries of the W algebra: $N_0 = 1$ for a regular $OSp(1|2)$, $N_0 = 2$ for the superprincipal $OSp(1|2)$ of $Sl(1|2)$ and $OSp(2|2)$, $N_0 = 3$ if the previous $Sl(1|2)$ or $OSp(2|2)$ can be

Table 19. Quadratic-superconformal algebras

$\mathscr{G}$	Min. includ. regular SSA	N_0	Superconformal spin of the W generators	Super KM algebra
$A(1,n)$	$A(1,0)$	2	$\frac{3}{2}$, $(2n+1)*1$, $n^2*\frac{1}{2}$	$A_{n-1} \oplus U(1)$
$D(2,n)$	$A(1,0)$	4	$\frac{3}{2}$, $(4n-1)*1$, $[(n-1)(2n-1)+3]*\frac{1}{2}$	$C_{n-1} \oplus 3U(1)$
$D(m,1)$	$C(2)$	4	$\frac{3}{2}$, $(2m-1)*1$, $(m-1)(2m-1)*\frac{1}{2}$	B_{m-1}
$B(m,1)$	$C(2)$	$\begin{cases} 3 (m=1) \\ 4 (m>1) \end{cases}$	$\frac{3}{2}$, $2m*1$, $m(2m-1)*\frac{1}{2}$	D_m
$G(3)$	$B(0,1)$	1	$\frac{3}{2}$, $6*1$, $8*\frac{1}{2}$	A_2
$F(4)$	$A(1,0)$	2	$\frac{3}{2}$, $7*1$, $14*\frac{1}{2}$	G_2

Table 20. $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebras (no superspin $\frac{1}{2}$ bosonic superfield)

$\mathscr{G}$	Min. includ. regular SSA	N_0	Superconformal spin of the W generators	Super KM algebra
$D(m,1)$	$A(1,0)$	4	$\frac{3}{2}$, $3*1$, $4(m-2)*1'$, $[(m-2)(2m-5)+3]*\frac{1}{2}$	$D_m \oplus 3U(1)$
$B(m,1)$	$A(1,0)$	4	$\frac{3}{2}$, $3*1$, $2(2m-3)*1'$, $[(m-2)(2m-3)+3]*\frac{1}{2}$	$B_{m-2} \oplus 3U(1)$
$B(0,n)$	$B(0,1)$	1	$\frac{3}{2}$, $(2n-2)*1'$, $(n-1)(2n-1)*\frac{1}{2}$	B_{n-1}

Table 21. $\mathbb{Z}_2 \times \mathbb{Z}_2$ superalgebras (with superspin $\frac{1}{2}$ bosons)

$\mathcal{G}$	Min. includ. regular SSA	N_0	Superconformal spin of the W generators
$A(m, n)$	$A(1, 0)$	2	$\frac{3}{2}$, $(2n+1)*1$, $2(m-1)*1'$, $[(m-1)^2+n^2]*\frac{1}{2}$, $2(m-1)n*\frac{1}{2}'$
$D(m, n)$	$C(2)$	4	$\frac{3}{2}$, $(2m-1)*1$, $(2n-2)*1'$, $[(m-1)(2m-1)+(n-1)(2n-1)]*\frac{1}{2}$, $(2m-1)(2n-2)*\frac{1}{2}$
	$A(1, 0)$	4	$\frac{3}{2}$, $(4n-1)*1$, $4(m-2)*1'$, $[(m-2)(2m-5)+(n-1)(2n-1)+3]*\frac{1}{2}$, $4(m-2)(n-1)*\frac{1}{2}'$
$B(m, n)$	$C(2)$	$\begin{cases} 3 (m=1) \\ 4 (m>1) \end{cases}$	$\frac{3}{2}$, $2m*1$, $(2n-2)*1'$ $[m(2m-1)+(n-1)(2n-1)]*\frac{1}{2}$, $4m(n-1)*\frac{1}{2}'$
	$A(1, 0)$	4	$\frac{3}{2}$, $(4n-1)*1$, $2(2m-3)*1'$, $[(m-2)(2m-3)+(n-1)(2n-1)+3]*\frac{1}{2}$, $2(2m-3)(n-1)*\frac{1}{2}'$
$C(n+1)$	$C(2)$	2	$\frac{3}{2}$, 1, $(2n-2)*1'$ $(n-1)(2n-1)*\frac{1}{2}$, $(2n-2)*\frac{1}{2}'$
$G(3)$	$A(1, 0)$	4	$\frac{3}{2}$, $3*1$, $4*1'$, $3*\frac{1}{2}$, $2*\frac{1}{2}'$
$F(4)$	$A(0, 1)$	4	$\frac{3}{2}$, $3*1$, $6*1'$, $6*\frac{1}{2}$, $2*\frac{1}{2}'$

embedded in an $OSp(3|2)$ SSA, and $N_0 = 4$ if the $Sl(2|1)$ or $OSp(2|2)$ is contained in $OSp(4|2)$ or $D(2, 1; \alpha)$ SSAs.

14.3. $\mathbb{Z}_2 \times \mathbb{Z}_2$ Superconformal Algebras. Their classification is easily deduced from the previous section. We begin with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebras that do not contain superspin $\frac{1}{2}$ bosons, so that we can define a (right statistic) super-KM algebra). These algebras are listed in Table 20.

If now one introduces the superspin $\frac{1}{2}$ bosons, the number of allowed superalgebras is much larger. In fact, in accordance with [16], we find one (resp. two) $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebras from each $A(m, n)$ and $C(n+1)$ (resp. $B(m, n)$ and $D(m, n)$) superalgebras. However, for $F(4)$ and $G(3)$, we find two new $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebras, different from the two quadratic-superconformal algebras of [16], already listed in Table 19. This seems to indicate that these two algebras exist only at the classical level. The results are summarized in Table 21.

15. Conclusion

In the classification we have obtained, each W(super)algebra is characterized by its (super) conformal spin content and the couple $(Sl(2), \mathcal{G})$ if $\mathcal{G}$ is a simple Lie algebra, respectively $(OSp(1|2), \mathcal{G})$ if $\mathcal{G}$ is a Lie superalgebra. The PB of the corresponding W(super)algebra can then be determined via the general method recalled in Sect. 2.1. However, rather important simplifications occur when the $U(1)$ factor commuting with $Sl(2)$, resp. $OSp(1|2)$, exists: the admitted Y values are also provided in our tables.

It has seemed to us necessary to reconsider in a first step the problem of the $Sl(2)$ subalgebras in a simple Lie algebra $\mathscr{G}$, in order to make explicit our results in the algebraic case, and also to propose the generalization we have obtained for the supersymmetric one. We hope that the tables in which our results are gathered are presented in a convenient enough way to allow direct use. This has been at least the case for us to easily recognize the superconformal algebras of [16].

Among the different problems one can immediately think of, an urgent one is of course the quantum case. Some interesting works [19, 23–26] already exist, but a general treatment would be necessary. Another question we wish could answer is how large is the class of W (super)algebras which are symmetries of Toda theories, in the complete set of W algebras.

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