Commun. Math. Phys. 157, 279–303 (1993)

Communications in
Mathematical
Physics

Hamiltonian BRST-anti-BRST Theory

Philippe Grégoire★ and Marc Henneaux★★
Faculté des Sciences, Université Libre de Bruxelles, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium

Received October 3, 1992

Abstract. The hamiltonian BRST-anti-BRST theory is developed in the general case of arbitrary reducible first class systems. This is done by extending the methods of homological perturbation theory, originally based on the use of a single resolution, to the case of a biresolution. The BRST and the anti-BRST generators are shown to exist. The respective links with the ordinary BRST formulation and with the $sp(2)$-covariant formalism are also established.

1. Introduction

It has been realized recently that the proper algebraic setting for the BRST theory is that of homological perturbation theory [1, 2]. Homological perturbation theory permits one not only to prove the existence of the BRST transformation, both in the lagrangian and the hamiltonian cases, but also establishes that the BRST cohomology at ghost number zero is given by the physical observables (the gauge invariant functions). These key properties, valid for irreducible or reducible gauge theories with closed or "open" algebras are what make the BRST formalism of physical interest [2, 3, 4, 5, 6].

The purpose of this paper is to extend the analysis of [2] to cover the anti-BRST transformation. The anti-BRST symmetry was formulated in the context of Yang–Mills theory immediately after the BRST symmetry was discovered [7, 8]. Although it does not play a role as fundamental as the BRST symmetry itself, it is a useful tool in the geometrical (superfield) description of the BRST transformation, in the investigation of perturbative renormalizability of Yang–Mills models, as well as in the understanding of the so-called non-minimal sector [9–15]. For all these reasons, it is of interest to develop the BRST-anti-BRST formalism in the general case of an arbitrary gauge system.

★ Chercheur IRSIA
★★ Maître de recherches au Fonds National de la Recherche Scientifique. Also at Centro de Estudios Cientificos de Santiago, Casilla 16443. Santiago 9, Chile
Email: henneaux@bbrbfu60

We show in this article that the methods of homological perturbation theory can be adapted to cover the anti-BRST transformation. This is done by duplicating each differential appearing in the BRST construction. In particular, the crucial Koszul–Tate complex [2, 16] is replaced by the Koszul–Tate bicomplex. The usual existence and uniqueness theorems for the BRST generator can then be extended without difficulty to the BRST-anti-BRST algebra by following the same lines as in the BRST case. Our results were announced in [17].

Although we consider here only the hamiltonian method, our approach can also be applied to the antifield formalism. However, the explicit form of the biresolution is then different, so that we shall reserve the discussion of the antifield anti-BRST theory for a separate publication [18][1].

Our paper is organized as follows. In the next section, we briefly review the salient facts of homological perturbation theory in the context of the BRST symmetry. We then introduce the concept of biresolution and develop its properties (Sect. 3). Section 4 is devoted to the proof of the main result of this paper, namely the existence of a Koszul–Tate biresolution associated with any constraint surface Σ embedded in phase space. In Sect. 5, we prove the existence and uniqueness of the BRST and the anti-BRST generators. We then establish some results about the BRST and the anti-BRST cohomologies (Sect. 6). Section 7 is devoted to the comparison between the BRST-anti-BRST formalism and the standard BRST theory; as a byproduct of this comparison, the equivalence between the two formulations is proven. In Sect. 8, we make the comparisons with the hamiltonian $Sp(2)$-formalism of references [27, 28].

2. Homological Perturbation Theory in Brief

2.1. Geometrical Ingredients of a Gauge Theory. In either the lagrangian or hamiltonian versions, the description of a gauge theory involves the following geometrical data:

1. A smooth manifold Γ with coordinates z^I. These are either the canonical coordinates of the hamiltonian formalism, or the "coordinates" of the histories of the fields in the lagrangian case.
2. A submanifold $\Sigma \subset \Gamma$ defined by implicit equations
$$G_{A_0}(z^I) \approx 0, \quad A_0 = 1, \ldots, M_0 \,. \tag{1}$$
These are the hamiltonian constraints or the Euler–Lagrange equations.
3. A distribution $\{\mathbf{X}_{\alpha_0};\, \alpha_0 = 1, \ldots, m_0\}$ tangent to Σ and in involution on it:
$$\mathbf{X}_{\alpha_0}[G_{A_0}] \approx 0 \,, \tag{2}$$
$$[\mathbf{X}_{\alpha_0}, \mathbf{X}_{\beta_0}] \approx C^{\gamma_0}_{\alpha_0\beta_0}\mathbf{X}_{\gamma_0} \,. \tag{3}$$
The vector fields $\mathbf{X}_{\alpha_0}$ generate the infinitesimal gauge transformations. These map Σ on itself (Eq. (2)) and are integrable on Σ (Eq. (3)). The corresponding integral submanifolds are the gauge orbits.

[1] The lagrangian BRST-anti-BRST formalism has been considered recently from different viewpoints in [19–26]

The observables of a gauge theory are the functions on Σ that are constant along the gauge orbits (gauge invariance). Thus, if we denote by $\Sigma/\mathcal{G}$ the "reduced" space obtained by taking the quotient of Σ by the gauge orbits, the algebra of observables is just $C^\infty(\Sigma/\mathcal{G})$. In principle, all the physical information about the gauge system is contained in $C^\infty(\Sigma/\mathcal{G})$.

2.2. BRST *Differential.* In practice, one cannot construct explicitly the algebra $C^\infty(\Sigma/\mathcal{G})$ of physical observables, either because one cannot solve the equations defining Σ, or because the integration of the gauge orbits is untractable. The BRST construction reformulates the concept of observables in an algebra that is more convenient, as the elements of the zeroth cohomology group of the BRST differential s,

$$s^2 = 0 . \tag{4}$$

Corresponding to the two ingredients contained in the definition of the observables, namely the restriction to Σ and the condition of gauge invariance, there are actually two differentials hidden in s. The first one is known as the Koszul–Tate differential δ and implements the restriction to Σ. More precisely, it yields a resolution of the algebra $C^\infty(\Sigma)$. The second one is (a model for) the longitudinal exterior derivative along the gauge orbits and is denoted by D. It imposes the condition of gauge invariance. One has [1, 2, 5, 6, 29]

$$s = \delta + D + \text{“more”} \tag{5}$$

and

$$H^0(s) \simeq C^\infty(\Sigma/\mathcal{G}) . \tag{6}$$

The existence of the additional terms in (5) necessary for the nilpotency (4) of s is a basic result of homological perturbation theory. It follows from the resolution property of the Koszul–Tate differential. We shall not reproduce the proof here but shall rather refer to the monograph [6].

Equation (6) provides the basic link between gauge invariance and BRST invariance. It explains why the BRST symmetry is physically relevant.

3. Biresolutions

3.1. Motivations. In the BRST-anti-BRST theory, the differential s is replaced by two differentials s_1 (BRST differential) and s_2 (anti-BRST differential) that anticommute,

$$s_1^2 = s_1 s_2 + s_2 s_1 = s_2^2 = 0 . \tag{7}$$

The relations (7) define the BRST-anti-BRST algebra. Furthermore, both s_1 and s_2 are such that

$$H^0(s_1) \simeq H^0(s_2) \simeq C^\infty(\Sigma/\mathcal{G}) \tag{8}$$

in a degree that will be made more precise below. This suggests that one should introduce two resolutions δ_1 and δ_2 of $C^\infty(\Sigma)$ that anticommute, instead of the single Koszul–Tate resolution δ of the BRST theory. Thus, we are led to the concept of biresolution.

3.2. Definition. Let $\mathscr{A}_0$ be an algebra and $\mathscr{A}$ be a bigraded algebra with bidegree called *resolution bidegree.* We set

$$\text{bires} = (\text{res}_1, \text{res}_2) \tag{9}$$

and

$$\text{res} = \text{res}_1 + \text{res}_2 \, . \tag{10}$$

We assume that both res_1 and res_2 are non-negative integers: $\text{res}_1 \geqq 0$ and $\text{res}_2 \geqq 0$.

Definition 3.1. *Let δ: $\mathscr{A} \to \mathscr{A}$ be a differential of resolution degree -1,*

$$\delta^2 = 0 \, , \tag{11}$$

$$\text{res}(\delta) = -1 \, , \tag{12}$$

i.e.

$$\text{res}(\delta a) = \text{res}(a) - 1 \quad \textit{when} \quad \text{res}(a) \geqq 1 \, , \tag{13}$$

$$= 0 \quad \textit{when} \quad \text{res}(a) = 0, \; (\textit{in which case } \delta a = 0) \, . \tag{14}$$

One says that the differential complex $(\mathscr{A}, \delta)$ is a biresolution of the algebra $\mathscr{A}_0$ if and only if:

1. *The differential δ splits as the sum of two derivations only*

$$\delta = \delta_1 + \delta_2 \tag{15}$$

with

$$\text{bires}(\delta_1) = (-1, 0), \quad \text{bires}(\delta_2) = (0, -1) \tag{16}$$

(*no extra piece, say, of resolution bidegree* $(-2, 1)$). *It follows from the nilpotency of δ that*

$$\delta_1^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = \delta_2^2 = 0 \, , \tag{17}$$

i.e. δ_1 *and* δ_2 *are differentials that anticommute.*

2. *One has*

$$H_{0,0}(\delta_1) = \mathscr{A}_0, \quad H_{0,k}(\delta_1) = 0 = H_{k,*}(\delta_1), \quad (k \neq 0) \tag{18}$$

$$H_{0,0}(\delta_2) = \mathscr{A}_0, \quad H_{k,0}(\delta_2) = 0 = H_{*,k}(\delta_2), \quad (k \neq 0) \tag{19}$$

$$H_0(\delta) = \mathscr{A}_0, \quad H_k(\delta) = 0, \quad (k \neq 0) \, . \tag{20}$$

Remark. The relation (20) is easily seen to be a consequence of (18) and (19).

Definition 3.2. *A biresolution is said to be symmetric if there exists an involution S ($S^2 = 1$) which* (i) *is an algebra isomorphism;* (ii) *maps an element of bidegree (a, b) on an element of bidegree (b, a) and* (iii) *maps δ_1 on δ_2 and vice-versa:*

$$S \delta_1 S = \delta_2 \, , \tag{21}$$

$$S \delta_2 S = \delta_1 \, . \tag{22}$$

Note that the relations (21) and (22) imply

$$S \delta S = \delta \, . \tag{23}$$

3.3. Basic Properties of Biresolutions.

Theorem 3.1. *Let* $(\mathscr{A}, \delta)$ *be a biresolution and* $\overset{(a,b)}{F} \in \mathscr{A}$, $\mathrm{bires}(\overset{(a,b)}{F}) = (a, b)$ *(with* $a + b > 0$*) be such that*

$$\begin{cases} \delta_1 \overset{(a,b)}{F} = 0 \\ \delta_2 \overset{(a,b)}{F} = 0 \end{cases} \Leftrightarrow \delta \overset{(a,b)}{F} = 0 \,. \tag{24}$$

Then

$$\overset{(a,b)}{F} = \delta_2 \delta_1 \overset{(a+1,b+1)}{M} \,. \tag{25}$$

Proof of Theorem 3.1. From $\delta_2 \overset{(a,b)}{F} = 0$, one gets

$$\overset{(a,b)}{F} = \delta_2 \overset{(a,b+1)}{R} \tag{26}$$

since $H_{a,b}(\delta_2) = 0$ for $a + b > 0$. But one has also $\delta_1 \overset{(a,b)}{F} = 0$, hence $\delta_2 \delta_1 \overset{(a,b+1)}{R} = 0$, i.e., there exists $\overset{(a-1,b+2)}{R}$ such that

$$\delta_1 \overset{(a,b+1)}{R} + \delta_2 \overset{(a-1,b+2)}{R} = 0 \,. \tag{27}$$

This leads to the *descent equations*

$$\delta_1 \overset{(a-1,b+2)}{R} + \delta_2 \overset{(a-2,b+3)}{R} = 0 \tag{28}$$

$$\vdots$$

$$\delta_1 \overset{(1,a+b)}{R} + \delta_2 \overset{(0,a+b+1)}{R} = 0 \tag{29}$$

$$\delta_1 \overset{(0,a+b+1)}{R} = 0 \,. \tag{30}$$

From the last equation and (18), one obtains

$$\overset{(0,a+b+1)}{R} = \delta_1 \overset{(1,a+b+1)}{M} \,. \tag{31}$$

Injecting this result in Eq. (29), one gets

$$\delta_1 \left(\overset{(1,a+b)}{R} - \delta_2 \overset{(1,a+b+1)}{M} \right) = 0 \,, \tag{32}$$

i.e., from (18)

$$\overset{(1,a+b)}{R} = \delta_2 \overset{(1,a+b+1)}{M} + \delta_1 \overset{(2,a+b)}{M} \,. \tag{33}$$

Going up the ladder in the same fashion, one finally gets for $\overset{(a,b+1)}{R}$,

$$\overset{(a,b+1)}{R} = \delta_2 \overset{(a,b+2)}{M} + \delta_1 \overset{(a+1,b+1)}{M} , \tag{34}$$

and thus, from (26)

$$\overset{(a,b)}{F} = \delta_2 \delta_1 \overset{(a+1,b+1)}{M} . \tag{35}$$

QED

Theorem 3.2. *Let* $\overset{(m)}{F} \in \mathscr{A}$, *with* $\operatorname{res}(\overset{(m)}{F}) = m > 0$, *be such that*

$$\overset{(m)}{F} = \sum_{p+q=m} \overset{(p,q)}{F} . \tag{36}$$

Assume that:

1. $\delta \overset{(m)}{F} = 0$,
2. *In the sum* (36), *only terms with* $p \leqq a$ *and* $q \leqq b$ *occur, for some* a *and* b *such that* $a + b > m$ *(strictly).*

Then,

$$\overset{(m)}{F} = \delta \overset{(m+1)}{P} , \tag{37}$$

where

$$\overset{(m+1)}{P} = \sum_{\bar{p}+\bar{q}=m+1} \overset{(\bar{p},\bar{q})}{P} \tag{38}$$

involves only terms $\overset{(\bar{p},\bar{q})}{P}$ *with* $\bar{p} \leqq a$ *and* $\bar{q} \leqq b$.

Proof of Theorem 3.2. One has

$$\overset{(m)}{F} = \overset{(a,m-a)}{F} + \cdots + \overset{(m-b,b)}{F} \tag{39}$$

(with $\overset{(i,j)}{F} = 0$ if $i < 0$ or $j < 0$). From $\delta \overset{(m)}{F} = 0$, one gets $\delta_2 \overset{(a,m-a)}{F} = 0$, i.e., using (19),

$$\overset{(a,m-a)}{F} = \delta_2 \overset{(a,m-a+1)}{P'} \tag{40}$$

(if $m - a < 0$, $\overset{(a,m-a)}{F} = 0$ and one takes $\overset{(a,m-a+1)}{P'} \equiv 0$). One has $m - a + 1 \leqq b$ because $m < a + b$. If one subtracts $\delta \overset{(a,m-a+1)}{P'}$ from $\overset{(m)}{F}$, one obtains

$$\overset{(m)}{F} - \delta \overset{(a,m-a+1)}{P'} = \overset{(a-1,m-a+1)}{F'} + \cdots + \overset{(m-b,b)}{F} . \tag{41}$$

One then keeps going (one removes $\overset{(a-1,m-a+1)}{F}$, etc. . . .) until one reaches the last step,

$$\overset{(m)}{F} - \delta\tilde{P} = \overset{(m-b,b)}{F'} \tag{42}$$

$$\delta \overset{(m-b,m)}{F'} = 0 \Leftrightarrow \begin{cases} \delta_1 \overset{(m-b,m)}{F'} = 0 \\ \delta_2 \overset{(m-b,m)}{F'} = 0 \,. \end{cases} \tag{43}$$

From Theorem 3.1, this implies that

$$\begin{aligned} \overset{(m-b,m)}{F'} &= \delta_1\delta_2 \overset{(m-b+1,m+1)}{S} \\ &= (\delta_1 + \delta_2)\delta_2 \overset{(m-b+1,m+1)}{S} \\ &= \delta \overset{(m-b+1,m)}{Q} \end{aligned} \tag{44}$$

with $\overset{(m-b+1,m)}{Q} = \delta_2 \overset{(m-b+1,m+1)}{S}$. One has $m - b + 1 \leqq a$ because $m < a + b$.

QED

Theorem 3.3. *Assume that in Theorem* 3.2, $\overset{(m)}{F}$ *is S-even*, i.e.,

$$S\overset{(m)}{F} = \overset{(m)}{F} \,. \tag{45}$$

Then $\overset{(m+1)}{P}$ *in* (37) *can also be chosen to be S-even*:

$$S \overset{(m+1)}{P} = \overset{(m+1)}{P} \,. \tag{46}$$

Similarly, if $\overset{(m)}{F}$ *is S-odd*, $\overset{(m+1)}{P}$ *can be chosen to be S-odd.*

Proof of Theorem 3.3. We treat only the case $\overset{(m)}{F}$ S-even. The case $\overset{(m)}{F}$ S-odd is treated in a similar fashion. Because $\overset{(m)}{F}$ is S-even, one can assume $a = b$ in the previous theorem. Now, from (45), (37) and (23), one gets

$$\overset{(m)}{F} = \delta\left[\frac{1}{2}\left(\overset{(m+1)}{P} + S\overset{(m+1)}{P}\right)\right]. \tag{47}$$

Both $\overset{(m+1)}{P}$ and $S\overset{(m+1)}{P}$ fulfill the conditions of Theorem 3.2 since $a = b$. Clearly, $1/2(\overset{(m+1)}{P} + S\overset{(m+1)}{P})$ is S-even. QED

4. Koszul–Tate Biresolution

4.1. Koszul–Tate Resolution. To warm up, we shall first recall a standard result on Koszul–Tate resolutions, which has been derived in the context of BRST theory [2, 5, 6]. To that end, we come back to the geometrical data of Sect. 2.1. Equations (2) defining the submanifold $\Sigma \subset \Gamma$,

$$G_{A_0} \approx 0 \tag{48}$$

may not be independent, i.e., there may be relations among the G_{A_0}'s:

$$Z^{A_0}_{A_1} G_{A_0} = 0 \quad \text{(identically)}\,. \tag{49}$$

The functions $Z^{A_1}_{A_0}$ are called the first order reducibility functions and provide a complete set of relations among the constraints. They may, in turn, be non-independent, i.e., there may be relations among them

$$Z^{A_1}_{A_2} Z^{A_1}_{A_0} \approx 0\,, \tag{50}$$

etc. There is thus a tower of reducibility identities of the form

$$Z^{A_{k-1}}_{A_k} Z^{A_{k-2}}_{A_{k-1}} \approx 0\,, \tag{51}$$

the last one being

$$Z^{A_{L-1}}_{A_L} Z^{A_{L-2}}_{A_{L-1}} \approx 0\,. \tag{52}$$

Definition 4.1. [2, 5] *The set* $\{G_{A_0},\ Z^{A_0}_{A_1}, \ldots, Z^{A_{L-1}}_{A_L}\}$ *provides a complete description of* Σ *if* $z^I \in \Sigma \Leftrightarrow G_{A_0}(z^I) = 0$, *and if*

$$\xi^{A_0} G_{A_0} = 0 \ \Leftrightarrow\ \xi^{A_0} = \xi^{A_1} Z^{A_0}_{A_1} + v^{A_0 B_0} G_{B_0}\,, \tag{53}$$

$$\vdots$$

$$\xi^{A_k} Z^{A_{k-1}}_{A_k} \approx 0 \ \Leftrightarrow\ \xi^{A_k} \approx \xi^{A_{k+1}} Z^{A_k}_{A_{k+1}}\,. \tag{54}$$

$$\vdots$$

$$\xi^{A_L} Z^{A_{L-1}}_{A_L} \approx 0 \ \Leftrightarrow\ \xi^{A_L} \approx 0\,, \tag{55}$$

where $v^{A_0 B_0} = (-)^{(\varepsilon_{A_0}+1)(\varepsilon_{B_0}+1)} v^{B_0 A_0}$.

Theorem 4.1. [2, 5] *To each complete description* $\{G_{A_0}, Z^{A_0}_{A_1}, \ldots, Z^{A_{L-1}}_{A_L}\}$ *of the surface* $\Sigma \subset \Gamma$, *one can associate a graded differential complex* (K_*, δ) *such that*

1. $K = \mathbf{C}[\mathscr{P}_{A_0}, \ldots, \mathscr{P}_{A_L}] \otimes C^\infty(\Gamma)$, *where*[2] $\mathrm{res}(\mathscr{P}_{A_n}) = n+1$.
2. *The operator* δ *is defined on the generators of the algebra* K *by*

$$\delta z^I = 0\,, \tag{56}$$

$$\delta \mathscr{P}_{A_0} = -G_{A_0}\,, \tag{57}$$

[2] Among the $\mathscr{P}_{A_0}, \ldots, \mathscr{P}_{A_L}$, some are commuting and some are anticommuting (see [2, 5]). We denote by $\mathbf{C}[\mathscr{P}_{A_0}, \ldots, \mathscr{P}_{A_L}]$ the algebra of polynomials in these variables with complex coefficients. For instance, for θ anticommuting, $\mathbf{C}[\theta] = \{\alpha + \beta\theta\}$, $\alpha, \beta \in \mathbf{C}$. Although we allow complex coefficients, the concept of *smoothness* is used in the real sense

$$\delta \mathscr{P}_{A_1} = -Z_{A_1}^{A_0} \mathscr{P}_{A_0}\,, \tag{58}$$

$$\vdots$$

$$\delta \mathscr{P}_{A_k} = -Z_{A_k}^{A_{k-1}} \mathscr{P}_{A_{k-1}} + M_{A_k}[\mathscr{P}_{A_0}, \dots, \mathscr{P}_{A_{k-2}}]\,, \tag{59}$$

$$\vdots$$

$$\delta \mathscr{P}_{A_L} = -Z_{A_L}^{A_{L-1}} \mathscr{P}_{A_{L-1}} + M_{A_L}[\mathscr{P}_{A_0}, \dots, \mathscr{P}_{A_{L-2}}]\,, \tag{60}$$

where the polynomials $M_{A_k} \in K$ are such that the Koszul–Tate operator δ is nilpotent, $\delta^2 = 0$.

3. $H_k(\delta) = 0$ *for* $k > 0$ *and* $H_0(\delta) = C^\infty(\Sigma)$, *that is,* (K_*, δ) *provides a homological resolution of the algebra* $C^\infty(\Sigma)$.

The graded differential complex (K_*, δ) is the Koszul–Tate differential complex and the associated resolution of $C^\infty(\Sigma)$ is called the Koszul–Tate resolution. Conversely, if a differential of the form (56)–(60) provides a homological resolution of $C^\infty(\Sigma)$, then, the functions $\{G_{A_0}, Z_{A_1}^{A_0}, \dots, Z_{A_L}^{A_{L-1}}\}$ appearing in (56)–(60) constitute a complete description of Σ.

Our purpose in this section is to show that for each complete description of the constraint surface, one can also associate a Koszul–Tate biresolution, by repeating an appropriate number of times the constraints and the reducibility functions.

4.2. Results. We have indicated in [17] the way in which one should proceed when the functions G_{A_0} defining Σ are independent (irreducible case). Rather than the single "ghost momentum" $\mathscr{P}_{A_0}$ of resolution degree one, one should introduce two ghosts momenta $\overset{(1,0)}{\mathscr{P}}_{A_0}$ and $\overset{(0,1)}{\mathscr{P}}_{A_0}$ at respective resolution bidegree (1, 0) and (0, 1). That is, one duplicates the constraints $G_{A_0} \approx 0$ by simply repeating them a second time. The description of Σ by means of the duplicated constraints is clearly no longer irreducible. One then introduces a ghost momentum $\overset{(1,1)}{\lambda}_{A_0}$ to compensate for the duplication and sets

$$\delta \overset{(1,0)}{\mathscr{P}}_{A_0} = -G_{A_0}\,, \tag{61}$$

$$\delta \overset{(0,1)}{\mathscr{P}}_{A_0} = -G_{A_0}\,, \tag{62}$$

$$\delta \overset{(1,1)}{\lambda}_{A_0} = \overset{(0,1)}{\mathscr{P}}_{A_0} - \overset{(1,0)}{\mathscr{P}}_{A_0}\,. \tag{63}$$

This defines the searched-for biresolution in the irreducible hamiltonian case. That biresolution is symmetric under the involution

$$S\overset{(1,0)}{\mathscr{P}}_{A_0} = \overset{(0,1)}{\mathscr{P}}_{A_0}, \quad S\overset{(0,1)}{\mathscr{P}}_{A_0} = \overset{(1,0)}{\mathscr{P}}_{A_0}\,, \tag{64}$$

$$S\overset{(1,1)}{\lambda}_{A_0} = -\overset{(1,1)}{\lambda}_{A_0}\,. \tag{65}$$

In the irreducible case, there are higher order ghost momenta $\mathscr{P}_{A_k}$ in the Koszul–Tate resolution, of resolution degree $k+1$. These should be replaced by $(k+2)$ ghost of ghost momenta $\overset{(i,j)}{\mathscr{P}}_{A_k}$, with $i+j=k+1, i \geqq 0, j \geqq 0$. This provides a spectrum symmetric for the interchange of i with j. This also amounts to repeating the reducibility functions $k+2$ times, increasing thereby the reducibility. One thus needs further ghosts of ghost momenta $\overset{(i+1,j+1)}{\lambda}_{A_k}$, with $i+j=k$, $i \geqq 0$, $j \geqq 0$, in order to compensate for that increase in reducibility.

Rather than trying to give a systematic, step-by-step derivation of the corresponding Koszul–Tate biresolution, we shall first state the results and then prove their correctness.

Theorem 4.2. *To each complete description* $\{G_{A_0}, Z^{A_0}_{A_1}, \ldots, Z^{A_{L-1}}_{A_L}\}$ *of the constraint surface* $\Sigma \subset \Gamma$, *one can associate a symmetric biresolution* $(K_*, \delta = \delta_1 + \delta_2)$ *of* $C^\infty(\Sigma)$ *defined as follows.*

1. *The graded algebra* K_* *is defined by*

$$K_* = \mathbf{C}[\overset{(i_0,j_0)}{\mathscr{P}}_{A_0}, \overset{(i_1,j_1)}{\mathscr{P}}_{A_1}, \ldots, \overset{(i_L,j_L)}{\mathscr{P}}_{A_L}, \overset{(i'_0+1,j'_0+1)}{\lambda}_{A_0}, \ldots, \overset{(i'_L+1,j'_L+1)}{\lambda}_{A_L}] \otimes C^\infty(\Gamma)\,, \tag{66}$$

with

$$i_k + j_k = k+1, \quad i_k \geqq 0, j_k \geqq 0\,, \tag{67}$$

$$i'_k + j'_k = k, \qquad i'_k \geqq 0, j'_k \geqq 0\,, \tag{68}$$

$$\mathrm{bires}(\overset{(i_k,j_k)}{\mathscr{P}}_{A_k}) = (i_k, j_k)\,, \tag{69}$$

$$\mathrm{bires}(\overset{(i'_k+1,j'_k+1)}{\lambda}_{A_k}) = (i'_k+1, j'_k+1)\,, \tag{70}$$

$$\varepsilon(\overset{(i_k,j_k)}{\mathscr{P}}_{A_k}) = \varepsilon_{A_k} + k + 1\,, \tag{71}$$

$$\varepsilon(\overset{(i'_k+1,j'_k+1)}{\lambda}_{A_k}) = \varepsilon_{A_k} + k \tag{72}$$

(where ε_{A_k} *is defined recursively through* $\varepsilon(G_{A_0}) = \varepsilon_{A_0}$, $\varepsilon(Z^{A_{k-1}}_{A_k}) = \varepsilon_{A_k} + \varepsilon_{A_{k-1}}$*).*

2. *The operator* $\delta = \delta_1 + \delta_2$ *acts on the generators as*

$$\begin{cases} \delta_1 z^I = 0\,, \\ \delta_2 z^I = 0\,, \end{cases} \tag{73}$$

$$\begin{cases} \delta_1 \overset{(k,0)}{\mathscr{P}}_{A_{k-1}} = -Z^{A_{k-2}}_{A_{k-1}} \overset{(k-1,0)}{\mathscr{P}}_{A_{k-2}} + \overset{(k-1,0)}{M}_{A_{k-1}} \\ \delta_2 \overset{(k,0)}{\mathscr{P}}_{A_{k-1}} = 0 \end{cases} \quad (k \geqq 0)\,, \tag{74}$$

$$\begin{cases} \delta_1 \overset{(0,k)}{\mathscr{P}}_{A_{k-1}} = 0 \\ \delta_2 \overset{(0,k)}{\mathscr{P}}_{A_{k-1}} = -Z^{A_{k-2}}_{A_{k-2}} \overset{(0,k-1)}{\mathscr{P}}_{A_{k-2}} + \overset{(0,k-1)}{\bar{M}}_{A_{k-1}} \end{cases} (k \geqq 0)\,, \tag{75}$$

$$\begin{cases} \delta_1 \overset{(i,j)}{\mathscr{P}}_{A_{k-1}} = -\frac{1}{2} Z^{A_{k-2}}_{A_{k-1}} \overset{(i-1,j)}{\mathscr{P}}_{A_{k-2}} + \overset{(i-1,j)}{M}_{A_{k-1}} \\ \delta_2 \overset{(i,j)}{\mathscr{P}}_{A_{k-1}} = -\frac{1}{2} Z^{A_{k-2}}_{A_{k-1}} \overset{(i,j-1)}{\mathscr{P}}_{A_{k-2}} + \overset{(i,j-1)}{\bar{M}}_{A_{k-1}} \end{cases} (i \neq 0 \neq j,\, i+j = k \geqq 0)\,, \tag{76}$$

$$\begin{cases} \delta_1 \overset{(i+1,j+1)}{\lambda}_{A_{k-2}} = -\overset{(i,j+1)}{\mathscr{P}}_{A_{k-2}} + \frac{1}{2} Z^{A_{k-3}}_{A_{k-2}} \overset{(i,j+1)}{\lambda}_{A_{k-3}} + \overset{(i,j+1)}{N}_{A_{k-2}} \quad (i \neq 0,\ k \geqq 2) \\ \delta_2 \overset{(i+1,j+1)}{\lambda}_{A_{k-2}} = -\overset{(i+1,j)}{\mathscr{P}}_{A_{k-2}} + \frac{1}{2} Z^{A_{k-3}}_{A_{k-2}} \overset{(i+1,j)}{\lambda}_{A_{k-3}} + \overset{(i+1,j)}{N}_{A_{k-2}} \quad (j \neq 0,\ k \geqq 2)\,, \end{cases} \tag{77}$$

$$\begin{cases} \delta_1 \overset{(1,j+1)}{\lambda}_{A_{k-2}} = -\overset{(0,j+1)}{\mathscr{P}}_{A_{k-2}} + \overset{(0,j+1)}{N}_{A_{k-2}} \\ \delta_2 \overset{(i+1,0)}{\lambda}_{A_{k-2}} = -\overset{(i+1,0)}{\mathscr{P}}_{A_{k-2}} + \overset{(i+1,0)}{\bar{N}}_{A_{k-2}} \end{cases} (k \geqq 2)\,. \tag{78}$$

The polynomials $M_{A_{k-1}}$, $\bar{M}_{A_{k-1}}$, $N_{A_{k-2}}$, $\bar{N}_{A_{k-2}}$ *depend only on* $\mathscr{P}_{A_u}$ *with* $u \leqq k-3$ *and* λ_{A_s} *with* $s \leqq k-4$. *They are determined recursively in such a way that* $\delta^2 = 0$ *(see below), and are such that*

$$S \overset{(i,j)}{M}_{A_{k-1}} = \overset{(j,i)}{\bar{M}}_{A_{k-1}}\,, \tag{79}$$

$$S \overset{(i,j)}{N}_{A_{k-2}} = \overset{(j,i)}{\bar{N}}_{A_{k-2}}\,, \tag{80}$$

where S is the symmetry

$$S \overset{(i,j)}{\mathscr{P}}_{A_{k-1}} = \overset{(j,i)}{\mathscr{P}}_{A_{k-1}}\,, \tag{81}$$

$$S \overset{(i+1,j+1)}{\lambda}_{A_{k-2}} = -\overset{(j+1,i+1)}{\lambda}_{A_{k-2}}\,. \tag{82}$$

4.3. Proof of Theorem 4.2. We define

$$K_k = \mathbf{C}[\mathscr{P}_{A_0}, \ldots, \mathscr{P}_{A_k}, \lambda_{A_0}, \ldots, \lambda_{A_{k-1}}] \otimes C^\infty \Gamma \tag{83}$$

and observe that $K_{L+1} = K_*$. The proof of Theorem 4.2 proceeds in steps.

Step 1. Assume that one has been able to find M_{A_i}, $\bar{M}_{A_i}$ up to $i = k-1$ and N_{A_i}, $\bar{N}_{A_i}$ up to $i = k-2$, in such a way that (i) $\delta^2 = 0$ on K_k; (ii) δ contains only pieces of bidegree $(-1, 0)$ and $(0, -1)$, that is, $\delta = \delta_1 + \delta_2$; and (iii) $S\delta S = \delta$. Then, it is easy to see that if the element $a \in K_i$ with $i < k$ fulfills both $\operatorname{res}(a) > 0$ and $\delta_\mu a = 0$, then $a = \delta_\mu b$ with $b \in K_{i+1}$. If $K_i = K_{L+1} = K_k$, then $a = \delta_\mu b$ with $b \in K_{L+1}$. Here δ_μ stands for either δ_1, δ_2 or δ.

Proof of Step 1. (a) We first consider the case $\delta_\mu = \delta_1$. Because δ_1 is $C^\infty(\Gamma)$-linear, one can proceed locally on Γ. Now, by a redefinition of the constraints and of the reducibility functions, one can assume that $Z^{A_j}_{A_{j+1}} Z^{A_{j-1}}_{A_j} = 0$ (strongly and not just weakly), at least locally. In that case, the operator δ_1 takes the simple form

$$\begin{cases} \delta_1 z^I = 0, \quad \delta_1 \overset{(1,0)}{\mathcal{P}}_{A_0} = -G_{A_0} \\ \delta_1 \overset{(j,0)}{\mathcal{P}}_{A_{j-1}} = -Z^{A_{j-2}}_{A_{j-1}} \overset{(j-1,0)}{\mathcal{P}}_{A_{j-2}} \quad (j \geqq 2), \end{cases} \tag{84}$$

$$\begin{cases} \delta_1 \overset{(0,j)}{\mathcal{P}}_{A_{j-1}} = 0 \\ \delta_1 \overset{(l,m)}{\mathcal{P}}_{A_{j-1}} = -\frac{1}{2} Z^{A_{j-2}}_{A_{j-1}} \overset{(l-1,m)}{\mathcal{P}}_{A_{j-2}} \quad (l \neq 0 \neq m,\ l+m=j) \\ \delta_1 \overset{(l+1,m+1)}{\lambda}_{A_{j-2}} = -\overset{(l,m+1)}{\mathcal{P}}_{A_{j-2}} + \frac{1}{2} Z^{A_{j-3}}_{A_{j-2}} \overset{(l,m+1)}{\lambda}_{A_{j-3}} \quad (l \neq 0,\ l+m=j-2) \\ \delta_1 \overset{(1,m+1)}{\lambda}_{A_{j-2}} = -\overset{(0,m+1)}{\mathcal{P}}_{A_{j-2}}\,. \end{cases} \tag{85}$$

If one redefines the variables $\overset{(l,m)}{\mathcal{P}}_{A_{j-2}}$, $m \neq 0$, as follows

$$\overset{(l,m+1)}{\mu}_{A_{j-2}} = -\overset{(l,m+1)}{\mathcal{P}}_{A_{j-2}} + \frac{1}{2} Z^{A_{j-3}}_{A_{j-2}} \overset{(l,m+1)}{\lambda}_{A_{j-3}} \quad (l \neq 0,\ l+m=j-2), \tag{86}$$

$$\overset{(0,j-1)}{\mu}_{A_{j-2}} = -\overset{(0,j-1)}{\mathcal{P}}_{A_{j-2}}, \tag{87}$$

one can rewrite δ_1 in the form

$$\begin{cases} \delta_1 z^I = 0, \quad \delta_1 \overset{(1,0)}{\mathcal{P}}_{A_0} = -G_{A_0} \\ \delta_1 \overset{(j,0)}{\mathcal{P}}_{A_{j-1}} = -Z^{A_{j-2}}_{A_{j-1}} \overset{(j-1,0)}{\mathcal{P}}_{A_{j-2}} \quad (j \geqq 2), \end{cases} \tag{88}$$

$$\left.\begin{cases} \delta_1 \overset{(l+1,m+1)}{\lambda}_{A_{j-2}} = \overset{(l,m+1)}{\mu}_{A_{j-2}} \\ \delta_1 \overset{(l,m+1)}{\mu}_{A_{j-2}} = 0 \end{cases}\right. \quad (l+m=j-2)\,. \tag{89}$$

Since μ is the δ_1-variation of λ, the $\lambda - \mu$ pairs cancel in δ_1-homology in K_k, except the unmatched variables $\overset{(l,m)}{\mu}_{A_k}$ $(l+m=k+1)$, for which the corresponding λ's do not live in K_k but in K_{k+1}. Furthermore, since (88) has the standard form of the resolution of $C^\infty(\Sigma)$ given in Theorem 4.1 (with $\mathcal{P}_{A_j} \to \overset{(j+1,0)}{\mathcal{P}}_{A_j}$), the non-trivial δ_1-cycles in K_i are all killed in K_{i+1} (or in K_i if $i \geqq L$). This proves step 1 for δ_1.
(b) Step 1 for δ_2 is proved similarly.
(c) The proof of Step 1 for $\delta = \delta_1 + \delta_2$ follows standard spectral sequence arguments. If $\delta a = 0$ with $\mathrm{res}(a) = j > 0$ and $a \in K_i$, then $a = \sum_t \overset{(t,j-t)}{a}$. The equation

$\delta a = 0$ implies $\delta_1 \overset{(t_{\min}, j - t_{\min})}{a} = 0$ for the component of a with smallest t. Then, $\overset{(t_{\min}, j - t_{\min})}{a} = \delta_1 \overset{(t_{\min}+1, j - t_{\min})}{b}$ with $\overset{(t_{\min}+1, j - t_{\min})}{b} \in K_{i+1}$ by (a), and the component with smallest t of $a - \delta \overset{(t_{\min}+1, j - t_{\min})}{b}$ has $t'_{\min} = t_{\min} + 1$. Going on recursively along the same line, one easily arrives at the desired result.

Step 2. It is clear if $a \in K_i$ fulfills $\delta a = 0$, $\mathrm{res}(a) > 0$ and the positivity properties of Theorem 3.2, then $b \in K_{i+1}$ (or K_{L+1} if $i = L + 1$) fulfills also the positivity properties of Theorem 3.2.

Step 3. δ is defined on K_0 and K_1 by

$$\delta z^I = 0 , \tag{90}$$

$$\delta \overset{(1,0)}{\mathscr{P}}_{A_0} = -G_{A_0}, \quad \delta \overset{(0,1)}{\mathscr{P}}_{A_0} = -G_{A_0} , \tag{91}$$

$$\delta \overset{(2,0)}{\mathscr{P}}_{A_1} = Z^{A_0}_{A_1} \overset{(1,0)}{\mathscr{P}}_{A_0}, \quad \delta \overset{(0,2)}{\mathscr{P}}_{A_1} = Z^{A_0}_{A_1} \overset{(0,1)}{\mathscr{P}}_{A_0} , \tag{92}$$

$$\delta \overset{(1,1)}{\mathscr{P}}_{A_1} = -\frac{1}{2} (\overset{(1,0)}{\mathscr{P}}_{A_0} + \overset{(0,1)}{\mathscr{P}}_{A_0}) , \tag{93}$$

$$\delta \overset{(1,1)}{\lambda}_{A_0} = \overset{(0,1)}{\mathscr{P}}_{A_0} - \overset{(1,0)}{\mathscr{P}}_{A_0} . \tag{94}$$

It is such that $\delta^2 = 0$, $\delta = \delta_1 + \delta_2$ and $S\delta S = \delta$. So let us assume that δ has been defined on K_i up to $i = k$, and let us show that one can extend δ to K_{k+1}, i.e., find $M_{A_{k+1}}, \bar{M}_{A_{k+1}}, N_{A_k}$ and $\bar{N}_{A_k} \in K_{k-1}$ such that $\delta^2 \mathscr{P}_{A_{k+1}} = \delta^2 \lambda_{A_k} = 0$, $\delta = \delta_1 + \delta_2$ and $S\delta S \mathscr{P}_{A_{k+1}} = \delta \mathscr{P}_{A_{k+1}}$, $S\delta S \lambda_{A_k} = \delta \lambda_{A_k}$. We shall only show how to define $\delta \overset{(i,j)}{\mathscr{P}}_{A_{k+1}}$ and $\delta \overset{(j,i)}{\mathscr{P}}_{A_{k+1}}$ $(i > 2,\ j > 2)$, with $i + j = k + 2$. One proceeds along identical lines for the other variables.

Let $M_{A_{k+1}}$ be the sum $\overset{(i-1,j)}{M}_{A_{k+1}} + \overset{(i,j-1)}{M}_{A_{k+1}}$ and let $\bar{M}_{A_{k+1}}$ be $\overset{(j-1,i)}{M}_{A_{k+1}} + \overset{(j,i-1)}{M}_{A_{k+1}}$. One must find $M_{A_{k+1}}$ and $\bar{M}_{A_{k+1}}$ in K_{k-1} such that the expressions

$$\delta \overset{(i,j)}{\mathscr{P}}_{A_{k+1}} = -\frac{1}{2} Z^{A_k}_{A_{k+1}} (\overset{(i-1,j)}{\mathscr{P}}_{A_k} + \overset{(i,j-1)}{\mathscr{P}}_{A_k}) + M_{A_{k+1}} , \tag{95}$$

$$\delta \overset{(j,i)}{\mathscr{P}}_{A_{k+1}} = -\frac{1}{2} Z^{A_k}_{A_{k+1}} (\overset{(j-1,i)}{\mathscr{P}}_{A_k} + \overset{(j,i-1)}{\mathscr{P}}_{A_k}) + \bar{M}_{A_{k+1}} \tag{96}$$

have vanishing δ. Furthermore, $M_{A_{k+1}}$ can contain only terms of bidegrees $(i-1, j)$ and $(i, j-1)$, and $\bar{M}_{A_{k+1}}$ can contain only terms of bidegree $(j-1, i)$ and $(j, i-1)$ (in order for δ to split as the sum of two differentials). Finally, one requires $SM_{A_{k+1}} = \bar{M}_{A_{k+1}}$.

Let $X_{A_{k+1}} \in K_{k-1}$ be

$$X_{A_{k+1}} = +(-)^k C^{A_{k-1}A_0}_{A_{k+1}} \left[\frac{1}{4} \overset{(1,0)}{\mathscr{P}}_{A_0} \overset{(i-2,j)}{\mathscr{P}}_{A_{k-1}} + \frac{1}{4} \big(\overset{(1,0)}{\mathscr{P}}_{A_0} + \overset{(0,1)}{\mathscr{P}}_{A_0}\big) \overset{(i-1,j-1)}{\mathscr{P}}_{A_{k-1}} + \frac{1}{4} \overset{(0,1)}{\mathscr{P}}_{A_0} \overset{(i,j-2)}{\mathscr{P}}_{A_{k-1}} \right], \tag{97}$$

where $C^{A_{k-1}A_0}_{A_{k+1}}$ are the structure functions appearing in the identity

$$Z^{A_k}_{A_{k+1}} Z^{A_{k-1}}_{A_k} = (-)^{\varepsilon_{A_{k-1}}} C^{A_{k-1}A_0}_{A_{k+1}} G_{A_0} \,. \tag{98}$$

Set

$$M_{A_{k+1}} = M'_{A_{k+1}} + X_{A_{k+1}} \,, \tag{99}$$

$$\bar{M}_{A_{k+1}} = \bar{M}'_{A_{k+1}} + SX_{A_{k+1}} \,. \tag{100}$$

The unknown functions $M'_{A_{k+1}}$ and $\bar{M}'_{A_{k+1}} \in K_{k-1}$ are subject to the equations:

$$\delta M'_{A_{k+1}} = \delta D_{A_{k+1}} \,, \tag{101}$$

$$\delta \bar{M}'_{A_{k+1}} = \delta S D_{A_{k+1}} \tag{102}$$

where

$$D_{A_{k+1}} = -X_{A_{k+1}} + \frac{1}{2} Z^{A_k}_{A_{k+1}} \big(\overset{(i-1,j)}{\mathscr{P}}_{A_k} + \overset{(i,j-1)}{\mathscr{P}}_{A_k} \big) \tag{103}$$

belongs to K_k. Because δ is nilpotent in (the already constructed) K_k, one has $\delta(\delta D_{A_{k+1}}) = 0$. Furthermore, a straightforward calculation using identity (98) shows that $\delta D_{A_{k+1}} \in K_{k-2}$. Hence, there exists $M'_{A_{k+1}} \in K_{k-1}$ such that Eq. (101) is satisfied (see Step 1). Note that $M'_{A_{k+1}} \neq D_{A_{k+1}}$ because $D_{A_{k+1}} \in K_k$. Since $\delta D_{A_{k+1}}$ contains only terms of bidegrees $(i-2,j)$, $(i-1,j-1)$ and $(i,j-2)$, one infers, using Step 2 and Theorem 3.2, that $M'_{A_{k+1}}$ can be taken to contain only terms of bidegrees $(i-1,j)$ and $(i,j-1)$, as required. Finally, one solves the second equation (102) by taking $\bar{M}'_{A_{k+1}} = SM'_{A_{k+1}}$. This is acceptable because $S\delta = \delta S$ in the already constructed K_k.

This completes the definition of $\delta \overset{(i,j)}{\mathscr{P}}_{A_{k+1}}$ and $\delta \overset{(j,i)}{\mathscr{P}}_{A_{k+1}}$. By a similar reasoning, one defines δ on all the other new generators of K_{k+1}, with the required properties. Step 3 and the proof of Theorem 4.2 are thereby finished.

5. BRST and Anti-BRST Generators

5.1. Review of Results from the BRST *Theory.* The existence of the Koszul–Tate biresolution is the hard core of the BRST-anti-BRST theory. The rest of this paper merely takes advantage of this result by applying it in the context of standard BRST theory.

We recall that in the hamiltonian formulation of gauge theories, the manifold Γ is the phase space, with canonical coordinates (q^i, p_i). The functions G_{A_0} defining the constraint surface are first class,

$$[G_{A_0}, G_{B_0}] = C^{C_0}_{A_0 B_0} G_{C_0} \tag{104}$$

(after all the second class have been eliminated, e.g. through the Dirac bracket method). The observables are the equivalence classes of first class phase space functions F_0 that coincide on the constraint surface

$$[F_0, G_{A_0}] = F_{A_0}^{B_0} G_{B_0} , \tag{105}$$

$$F_0 \sim F_0 + \lambda^{A_0} G_{A_0} . \tag{106}$$

One then has the important theorem [1, 2, 5, 6, 29].

Theorem 5.1. *To any homological resolution (K_*, δ) of the constraint surface, one can associate a nilpotent function in an extended phase space:*

$$[\Omega, \Omega] = 0, \quad \varepsilon(\Omega) = 1 , \tag{107}$$

which has the form

$$\Omega = -\sum \eta(\delta \mathscr{P}) + \text{“more .”} \tag{108}$$

The BRST generator Ω *generates the* BRST *transformation through*

$$s\cdot = [\cdot, \Omega] . \tag{109}$$

Equation (107) *is equivalent to*

$$s^2 = 0 \tag{110}$$

and one has

$$H^0(s) \simeq C^\infty(\Sigma/\mathscr{G}) = \{\text{observables}\} . \tag{111}$$

Actually, $H^*(s) \simeq H^*(d)$, where d is the exterior longitudinal derivative along the gauge orbits on Σ for non-negative degree and $H^*(s) = 0$ for negative degree.

The variables η appearing in (108) are conjugate to the generators $\mathscr{P}$ of the given resolution of $C^\infty(\Sigma)$. They are called *ghosts.* The variables η^{A_k} with $k > 0$ are also called *ghosts of ghosts.*

5.2. Extended Phase Space. The BRST-anti-BRST algebra (7) implies

$$s^2 \equiv (s_1 + s_2)^2 = 0 \tag{112}$$

and conversely, (112) implies (7) provided s splits as a sum of two differentials and no more (if s were to split into more pieces, s_1 and s_2 would obey equations involving the extra derivations contained in s). We shall use the previous theorem and the biresolution of Sect. 4 to establish the existence of the BRST and anti-BRST generators. The idea is the same as that exposed in [17] for the irreducible case. Namely, one constructs directly the generator Ω of the sum $s_1 + s_2$ by using the ordinary BRST theory, i.e. Theorem 5.1, but applied to the description of Σ associated with the differential δ of the previous section. And one controls that Ω splits as a sum of two terms only by means of Theorem 3.2.

The extended phase space is obtained by associated wth each $\overset{(i,j)}{\mathscr{P}_{A_k}}$ and $\overset{(i+1,j+1)}{\lambda_{A_k}}$ of the previous section a conjugate ghost, denoted by $\overset{(i,j)}{\eta^{A_k}}$ or $\overset{(i+1,j+1)}{\pi^{A_k}}$:

$$[\overset{(-i,-j)}{\mathscr{P}_{A_k}}, \overset{(i',j')}{\eta^{A'_{k'}}}] = \delta^{ii'}\delta^{jj'}\delta^{A'_{k'}}_{A_k} , \tag{113}$$

$$[\overset{(-(i+1),-(j+1))}{\lambda_{A_k}}, \overset{(i'+1,j'+1)}{\pi^{A'_{k'}}}] = \delta^{ii'}\delta^{jj'}\delta^{A'_{k'}}_{A_k} , \tag{114}$$

All the other brackets involving the ghosts or the ghost momenta vanish. The ghosts $\overset{(i,j)}{\eta^{A_k}}$ and $\overset{(i+1,j+1)}{\pi^{A_k}}$ can be seen as the generators of a model for the longitudinal exterior differential complex (L^*, d) [2, 5, 6]. Actually, this model (K^*, D) has a bicomplex structure and $D = D_1 + D_2$. The double differential complex $(K^{*,*}; D_1, D_2)$ is bigraded by the *pure ghost bidegree*, denoted bipgh and defined by:

$$\mathrm{bipgh}(\overset{(i,j)}{\eta^{A_k}}) = (i,j) , \tag{115}$$

$$\mathrm{bipgh}(\overset{(i+1,j+1)}{\pi^{A_k}}) = (i+1, j+1) . \tag{116}$$

The original canonical variables have zero bipgh. In so far as this does not play an important role in our construction, we will not elaborate more here on this aspect of the geometrical interpretation of the BRST-anti-BRST theory.

Following what is done in the usual BRST context, we also define *ghost bidegree*, denoted bigh to be

$$\mathrm{bigh} = \mathrm{bipgh} - \mathrm{bires} = (\mathrm{gh}_1, \mathrm{gh}_2) . \tag{117}$$

It is such that one has $\mathrm{gh}_1(s_1) = 1 = \mathrm{gh}_2(s_2)$ and $\mathrm{gh}_1(s_2) = 0 = \mathrm{gh}_1(s_2)$. Also, one defines the *ghost degree* $\mathrm{gh} = \mathrm{gh}_1 + \mathrm{gh}_2$. From now on the superscript (i,j) will always indicates the ghost bidegree. So $\overset{(i,j)}{\mathscr{P}_{A_k}}$ becomes $\overset{(-i,-j)}{\mathscr{P}_{A_k}}$ as already anticipated in (113). We denote the bigraded polynomial algebra of polynomials in the ghosts and the ghosts momenta with coefficients that are functions of the original canonical variables by $\mathscr{K}^{*,*}$. One extends the definition of δ on $\mathscr{K}^{*,*}$ by requiring that $\delta\eta = 0 = \delta\pi$; with this definition of δ, one has that $\mathrm{bigh}(\delta_1) = (1, 0)$ and $\mathrm{bigh}(\delta_2) = (0, 1)$. From the point of view of the BRST theory based on δ, the variables η^{A_k} with $k > 0$ and π^{A_k} are the ghosts of ghosts associated with the reducible description of Σ defined by δ. The degrees res and gh are respectively the corresponding resolution degree and ghost number.

5.3. *A Positivity Theorem.*

Definition 5.1. *Let* $F \in \mathscr{K}^{*,*}$. *If the polynomial* F *satisfied* $\mathrm{gh}(F) = k > 0$ (*respectively* $\mathrm{gh}(F) = k \geqq 0$), *then* F *is said to be of* positive ghost bidegree (*respectively* non-negative ghost bidegree) *if it can be decomposed as*

$$F = \sum_{i+j=k} \overset{(i,j)}{F} , \tag{118}$$

where $i \geqq 0, j \geqq 0$ *and* $\mathrm{bigh}(\overset{(i,j)}{F}) = (i, j)$. *The algebra of polynomials of positive ghost bidegree (respectively non-negative ghost bidegree) is denoted by* $\mathscr{K}^{*,*}_{+,+}$ *(respectively* $\mathscr{K}^{*,*}_{+}$*). In particular, one has* $\mathscr{K}^{*,*}_{+,+} \subset \mathscr{K}^{*,*}_{+}$.

We have the following important theorem

Theorem 5.2. *Let* $F \in \mathscr{K}^{*,*}_{+,+}$ *be such that* (i) $\mathrm{res}(F) = m > 0$ *and* (ii) $\delta F = 0$. *Then,* $\exists P \in \mathscr{K}^{*,*}_{+}$ *such that* $\delta P = F$.

Proof of Theorem 5.2. Let $F[\alpha, \beta; r, s]$ be the component of F satisfying $\mathrm{bipgh}(F[\alpha, \beta; r, s]) = (\alpha, \beta)$ and $\mathrm{bires}(F[\alpha, \beta; r, s]) = (r, s)$. The condition $\delta F = 0$ implies $\delta \sum_{r+s=k} F[\alpha, \beta; r, s] = 0$, while the condition $F \in \mathscr{K}^{*,*}_{+}$ implies $r \leqq \alpha$ and $s \leqq \beta$ with $\alpha + \beta > m$. Applying Theorem 3.2, one obtains that there exists $P[\alpha, \beta] = \sum_{\bar{r}+\bar{s}=m+1} P[\alpha, \beta; \bar{r}, \bar{s}]$ such that $\bar{r} \leqq \alpha$ and $\bar{s} \leqq \beta$. Thus $P = \sum_{\alpha,\beta} P[\alpha, \beta] \in \mathscr{K}^{*,*}_{+}$. QED

5.4. BRST *and Anti-*BRST *Generators.* Let us consider the homological resolution $\delta = \delta_1 + \delta_2$ of Theorem 4.2. By Theorem 5.1, we know that there exists a *total* BRST *charge* Ω, that starts as

$$\Omega = G_{A_0}(\overset{(1,0)}{\eta^{A_0}} + \overset{(0,1)}{\eta^{A_0}}) + (\overset{(0,-1)}{\mathscr{P}_{A_0}} - \overset{(-1,0)}{\mathscr{P}_{A_0}})\overset{(1,1)}{\pi^{A_0}} + \cdots . \tag{119}$$

However, we want more than just a mere solution of $[\Omega, \Omega] = 0$. We want this solution to incorporate the full BRST-anti-BRST algebra. As stressed already above, this means that the total BRST transformation $s = [\cdot, \Omega]$ must split in two pieces s_1 and s_2 of different degrees. Accordingly, we require the total BRST generator Ω itself to split also in two pieces Ω_1 and Ω_2 with $\mathrm{bigh}(\Omega_1) = (1, 0)$ and $\mathrm{bigh}(\Omega_2) = (0, 1)$. If this is the case, then the differentials s_1 and s_2 defined by $s_1 = [\cdot, \Omega_1]$ and $s_2 = [\cdot, \Omega_2]$ fulfill (7).

Theorem 5.3. *Suppose that* $\Omega = \Omega_1 + \Omega_2$ *with* $\mathrm{bigh}(\Omega_1) = (1, 0)$ *and* $\mathrm{bigh}(\Omega_2) = (0, 1)$, *then* $[\Omega, \Omega] = 0$ *if and only if* $[\Omega_1, \Omega_1] = 0 = [\Omega_2, \Omega_2]$ *and* $[\Omega_1, \Omega_2] = 0$.

Proof of Theorem 5.3. Obvious by degree counting arguments:

$$\begin{aligned}[\Omega, \Omega] &= [\Omega_1 + \Omega_2, \Omega_1 + \Omega_2] \\ &= [\Omega_1, \Omega_1] + [\Omega_2, \Omega_2] + 2[\Omega_1, \Omega_2] .\end{aligned} \tag{120}$$

Clearly, $\mathrm{bigh}[\Omega_1, \Omega_1]) = (2, 0)$, $\mathrm{bigh}([\Omega_2, \Omega_2]) = (0, 1)$ and $\mathrm{bigh}([\Omega_1, \Omega_2]) = (1, 1)$. Thus, $[\Omega, \Omega] = 0$ if and only if each term of the right-hand side of (120) vanishes, that is, if and only if $[\Omega_1, \Omega_1] = 0 = [\Omega_2, \Omega_2]$ and $[\Omega_1, \Omega_2] = 0$. QED

We now prove that the total BRST charge can be split in just two pieces Ω_1 and Ω_2.

Theorem 5.4. *One can choose the extra terms in* (119) *such that* (i) Ω *splits as a sum of two terms of definite ghost bidegree,* $\Omega = \Omega_1 + \Omega_2$ *with* $\mathrm{bigh}(\Omega_1) = (1, 0)$ *and* $\mathrm{bigh}(\Omega_2) = (0, 1)$ *and* (ii) $[\Omega, \Omega] = 0$.

Proof of Theorem 5.4. Using homological perturbation theory, the equation $[\Omega, \Omega] = 0$ is equivalent to the family

$$\delta\overset{(n)}{\Omega} = \overset{(n-1)}{D}[\overset{(0)}{\Omega}, \dots, \overset{(n-1)}{\Omega}], \quad n > 0 , \tag{121}$$

where res($\overset{(n)}{\Omega}$) $= n$. The explicit form of $\overset{(n-1)}{D}$, given in [2], is

$$\overset{(n-1)}{D}[\overset{(0)}{\Omega},\dots,\overset{(n-1)}{\Omega}] = \frac{1}{2}\left\{\sum_{m=0}^{n-1}[\overset{(n-m-1)}{\Omega},\overset{(m)}{\Omega}]_{\mathrm{orig}} + \sum_{m=1}^{n-1}\sum_{k=0}^{m-1}\left\{[\overset{(n-m+k)}{\Omega},\overset{(m)}{\Omega}]_{(\mathscr{P}_{A_k},\eta^{A_k})} + [\overset{(n+1-m+k)}{\Omega},\overset{(m)}{\Omega}]_{(\lambda_{A_k},\pi^{A_k})}\right\}\right\}, \tag{122}$$

where $[\cdot,\cdot]_{\mathrm{orig}}$ refers to the original Poisson bracket not involving the ghosts, $[\cdot,\cdot]_{(\mathscr{P}_{A_k},\eta^{A_k})}$ and $[\cdot,\cdot]_{(\lambda_{A_k},\pi^{A_k})}$ denote respectively the Poisson bracket with respect to the ghost pairs $(\mathscr{P}_{A_k},\eta^{A_k})$ and $(\lambda_{A_k},\pi^{A_k})$. Clearly, one has $\overset{(0)}{\Omega} = \overset{(0)}{\Omega}_1 + \overset{(0)}{\Omega}_2$. Suppose now that $\overset{(j)}{\Omega} = \overset{(j)}{\Omega}_1 + \overset{(j)}{\Omega}_2$, for $j < n$, then let us prove that $\overset{(n)}{\Omega}$ can be chosen such that $\overset{(n)}{\Omega} = \overset{(n)}{\Omega}_1 + \overset{(n)}{\Omega}_2$ with bigh$(\Omega_1) = (1,0)$ and bigh$(\Omega_2) = (0,1)$. Actually, using Definition 5.3, one can reformulate this properly as follows. Suppose that $\overset{(j)}{\Omega}$ is of positive ghost bidegree for $j < n$, then, we must show that $\overset{(n)}{\Omega}$ may be chosen to be of positive ghost bidegree.

Lemma 5.5. *Suppose that $\overset{(j)}{\Omega}$ is of positive ghost bidegree for $j < n$, then $\overset{(n-1)}{D}$ is of positive ghost bidegree.*

Proof of Lemma 5.5. We observe that $\overset{(n-1)}{D}$ is as follows:

$$\begin{aligned}\overset{(n-1)}{D}[\overset{(0)}{\Omega},\dots,\overset{(n-1)}{\Omega}] &= \overset{(n-1)}{D_{11}}[\overset{(0)}{\Omega}_1,\dots,\overset{(n-1)}{\Omega}_1] \\ &\quad + \overset{(n-1)}{D_{22}}[\overset{(0)}{\Omega}_2,\dots,\overset{(n-1)}{\Omega}_2] \\ &\quad + \overset{(n-1)}{D_{12}}[\overset{(0)}{\Omega}_1,\dots,\overset{(n-1)}{\Omega}_1;\overset{(0)}{\Omega}_2,\dots,\overset{(n-1)}{\Omega}_2],\end{aligned} \tag{123}$$

where $\overset{(n-1)}{D_{11}}$ stands for the terms computed from the sole $\overset{(j)}{\Omega}_1, j < n$, $\overset{(n-1)}{D_{22}}$ from the sole $\overset{(j)}{\Omega}_2, j < n$ and $\overset{(n-1)}{D_{12}}$ for the mixed terms. Using (122), it is then easy to see that

$$\mathrm{bigh}(\overset{(n-1)}{D_{11}}) = (2,0)\,, \tag{124}$$

$$\mathrm{bigh}(\overset{(n-1)}{D_{22}}) = (0,2)\,, \tag{125}$$

$$\mathrm{bigh}(\overset{(n-1)}{D_{12}}) = (1,1)\,. \tag{126}$$

This clearly shows that $\overset{(n-1)}{D}$ is of positive ghost bidegree. $\lhd$

Thus, in equation $\delta\overset{(n)}{\Omega} = \overset{(n-1)}{D}[\overset{(0)}{\Omega}, \ldots, \overset{(n-1)}{\Omega}]$, the right-hand side is of positive ghost bidegree and because $\delta\overset{(n-1)}{D} = 0$, there exists $\overset{(n)}{\Omega}$ such that (i) $\delta\overset{(n)}{\Omega} = \overset{(n-1)}{D}$ and (ii) $\overset{(n)}{\Omega}$ is of postive ghost bidegree (by Theorem 5.2). So, we have proven by induction on the resolution degree that Ω is of positive ghost bidegree and this, in turn, implies that

$$\Omega = \Omega_1 + \Omega_2 \tag{127}$$

with $\mathrm{bigh}(\Omega_1) = (1, 0)$ and $\mathrm{bigh}(\Omega_2) = (0, 1)$. QED

A nice consequence of this theorem is that the family of Eqs. (121) can be decomposed in three pieces:

$$\delta_1\overset{(n)}{\Omega_1} = \overset{(n-1)}{D_{11}}[\overset{(0)}{\Omega_1}, \ldots, \overset{(n-1)}{\Omega_1}], \tag{128}$$

$$\delta_2\overset{(n)}{\Omega_2} = \overset{(n-1)}{D_{22}}[\overset{(0)}{\Omega_2}, \ldots, \overset{(n-1)}{\Omega_2}], \tag{129}$$

$$\delta_1\overset{(n)}{\Omega_2} + \delta_2\overset{(n)}{\Omega_1} = \overset{(n-1)}{D_{12}}[\overset{(0)}{\Omega_1}, \ldots, \overset{(n-1)}{\Omega_1}; \overset{(0)}{\Omega_2}, \ldots, \overset{(n-1)}{\Omega_2}] \tag{130}$$

which are equivalent to the three equations

$$[\Omega_1, \Omega_1] = 0 = [\Omega_2, \Omega_2] \quad \text{and} \quad [\Omega_1, \Omega_2] = 0\,. \tag{131}$$

As mentioned above, these last equations are equivalent to the BRST-anti-BRST defining equations for the derivations $s_1 = [\,\cdot\,, \Omega_1]$ and $s_2 = [\,\cdot\,, \Omega_2]$. Thus, we have proved the existence of the BRST-anti-BRST transformation for any complete description of the constraint surface Σ. This was done not by trying to solve directly (128–130), but rather by solving the sum (121) and controlling that it splits appropriately.

5.5. Uniqueness of the BRST *and* Anti-BRST *Generators.* By the standard BRST theory, the total BRST generator is unique up to canonical transformation in the extended phase space. In its infinitesimal form, this result states that if Ω and Ω' are two nilpotent generators satisfying the same boundary conditions, then $\Omega' = \Omega + [M, \Omega]$, where the function M is of ghost number zero [2]. More explicitly, if one decomposes M according to the resolution degree, one has $\overset{(r)}{\Omega'} = \overset{(r)}{\Omega} + \delta\overset{(r+1)}{M}$. Actually, one can assume that the function M is of homogeneous ghost bidegree (0, 0), $\mathrm{bigh}(M) = (0, 0)$. Indeed, suppose that one has $\Omega_1 + \Omega_2 = \Omega'_1 + \Omega'_2$, with the same boundary conditions. Suppose that until resolution degree p,

$$\overset{(r)}{\Omega'_1} = \overset{(r)}{\Omega_1}, \tag{132}$$

$$\overset{(r)}{\Omega'_2} = \overset{(r)}{\Omega_2}, \quad r \leqq p\,. \tag{133}$$

Let us prove that there exist a canonical transformation

$$\Omega \to \Omega + [\overset{(p+2)}{M}, \Omega] \tag{134}$$

such that $\overset{(p+1)}{\Omega'_1} = \overset{(p+1)}{\Omega_1}$ and $\overset{(p+1)}{\Omega'_2} = \overset{(p+1)}{\Omega_2}$. By construction, one has

$$\delta \overset{(p+1)}{\Omega} = \overset{(p)}{D} = \delta \overset{(p+1)}{\Omega'} . \tag{135}$$

Thus, there exist $\overset{(p+2)}{M}$ such that

$$\overset{(p+1)}{\Omega'} = \overset{(p+1)}{\Omega} + \delta \overset{(p+2)}{M} . \tag{136}$$

Furthermore, because $(\overset{(p+1)}{\Omega'} - \overset{(p+1)}{\Omega}) \in \mathscr{K}^{*;*}_{+;+}$, one can take $\overset{(p+2)}{M}$ in $\mathscr{K}^{*;*}_{+}$, that is, $\mathrm{bigh}(\overset{(p+2)}{M}) = (0, 0)$. The canonical transformation (134) with that solution $\overset{(p+2)}{M}$ of (136) is the searched-for canonical transformation. Equation (134) splits as

$$\Omega_1 \to \Omega_1 + [\overset{(p+2)}{M}, \Omega_1] = \Omega_1 + s_1 \overset{(p+2)}{M} , \tag{137}$$

$$\Omega_2 \to \Omega_2 + [\overset{(p+2)}{M}, \Omega_2] = \Omega_2 + s_2 \overset{(p+2)}{M} . \tag{138}$$

6. Classical BRST Cohomology

In order to construct a gauge fixed (hamiltonian) action, it is necessary to define the total BRST extension H of the canonical (gauge invariant) hamiltonian H_0. That is, one must find a function H with $\mathrm{gh}(H) = 0$ such that $H = H_0 + \cdots$ and $[H, \Omega] = 0$. If one decomposes H according to the resolution degree

$$H = \sum_{r=0} \overset{(r)}{H}, \quad \mathrm{res}(\overset{(r)}{H}) = r , \tag{139}$$

then, the equation $[H, \Omega] = 0$ is equivalent to the family of equations

$$\delta \overset{(p+1)}{H} = \overset{(p)}{M}[\overset{(0)}{H}, \ldots, \overset{(p)}{H}] , \tag{140}$$

where the function $\overset{(p)}{M}$ is defined by [6, 30]

$$\begin{aligned} \overset{(p)}{M} = & \sum_{k=0}^{p} [\overset{(p-k)}{H}, \overset{(k)}{\Omega}]_{\mathrm{orig}} \\ & + \sum_{k=0}^{p} \sum_{s=0}^{k+p-1} \left\{ [\overset{(k)}{H}, \overset{(p+s+1-k)}{\Omega}]_{(\mathscr{P}_{A_s}, \eta^{A_s})} + [\overset{(k)}{H}, \overset{(p+s+2-k)}{\Omega}]_{(\lambda_{A_s}, \pi^{A_s})} \right\} . \end{aligned} \tag{141}$$

The general theorems of BRST theory guarantee the existence of H. Again, one has here a stronger result, namely, H can be chosen to be of ghost bidegree (0, 0).

Clearly, one has $\overset{(0)}{H} = H_0$. It is also easy to see that[3] $\overset{(1)}{H} = \overset{(-1,0)}{\mathscr{P}_{A_0}} V^{A_0}_{B_0} \overset{(1,0)}{\eta^{A_0}} + \overset{(0,-1)}{\mathscr{P}_{A_0}} V^{A_0}_{B_0} \overset{(0,1)}{\eta^{A_0}}$. This shows that $\mathrm{bigh}(\overset{(0)}{H}) = \mathrm{bigh}(\overset{(1)}{H}) = (0, 0)$. As in Lemma 5.5, one can conclude that $\overset{(1)}{M}$ in (141) belongs to $\mathscr{K}^{*,*}_{+,+}$. Because $\delta \overset{(1)}{M} = 0$, by Theorem 5.2, there exists $\overset{(2)}{H}$ such that $\mathrm{bigh}(\overset{(2)}{H}) = (0, 0)$ and $\delta \overset{(2)}{H} = \overset{(1)}{M}$. Continuing in the same fashion, one finally obtains the following theorem

Theorem 6.1. *The total* BRST *invariant extension H of the canonical hamiltonian H_0 may be chosen in such a way that* $\mathrm{bigh}(H) = (0, 0)$. *The equation $[H, \Omega] = 0$ imply then that H is both* BRST *and* anti-BRST *invariant, that is, $s_1 H = 0 = s_2 H$.*

By standard BRST arguments one also obtains easily the

Theorem 6.2. *The total* BRST *extension H of the canonical hamiltonian H_0 is unique up to* BRST-*exact term: the equations $[H, \Omega] = 0 = [H', \Omega]$, with $\overset{(0)}{H} = \overset{(0)}{H'} = H_0$ imply the existence of a function K such that $H = H' + [K, \Omega]$.*

The *gauge fixed hamiltonian* $H_\Psi = H + s\Psi$ is simply a choice of a representant in the equivalence class of BRST invariant extensions of the canonical hamiltonian H_0.

7. Comparison with the Standard BRST Formalism

It is clear that the above approach yields the same physical results as the standard BRST formalism. Indeed, it is known that these physical results do not depend on the particular resolution of $C^\infty(\Sigma)$ that is adopted. However, it is of interest to make a more explicit contact with the standard BRST construction. To that end, we observe that the BRST generator Ω_1 given here starts as

$$\Omega_1 = -\sum_{k=0} \overset{(k+1,0)}{\eta^{A_k}} \; \delta_1 \; \overset{(-(k+1),0)}{\mathscr{P}_{A_k}} + \text{“more”}, \quad [\Omega_1, \Omega_1] = 0\,, \tag{142}$$

where the operator δ_1 provides a homological resolution of the algebra $C^\infty(\Sigma)$. Equations (142) precisely define the standard BRST of the standard theory charge with a non-minimal sector: besides the minimal variables $\overset{(-(k+1),0)}{\mathscr{P}_{A_k}}$ and $\overset{(k+1,0)}{\eta^{A_k}}$, there are extra non-minimal variables (all the others). Hence, we can indeed identify Ω_1 with the standard (non-minimal) BRST generator. The ghosts $\overset{(1,1)}{\pi^{A_0}}$, which appear in our approach as ghosts of ghosts related to the duplication of the constraints, are viewed as non-minimal variables in the standard BRST context. Note that this non-minimal sector turns out to be the non-minimal sector required

[3] As usual, we define $[H_0, G_{A_0}] = V^{B_0}_{A_0} G_{B_0}$; that is the first class condition on the canonical hamiltonian H_0

for convenient gauge fixing (for instance, the Feynman gauge for the Yang–Mills action).

The ghost number of the standard BRST formalism can be expressed as

$$\mathrm{gh}_{\mathrm{standard}} = \mathrm{gh}_1 - \mathrm{gh}_2 \,. \tag{143}$$

Hence, one has $\mathrm{gh}_{\mathrm{standard}}(\Omega_1) = +1$ and $\mathrm{gh}_{\mathrm{standard}}(\Omega_2) = -1$. Moreover, the total BRST extension of the canonical hamiltonian is also a standard BRST extension for the standard BRST charge Ω_1: $[H, \Omega_1] = 0$. The ambiguity in H explained in Theorem 6.2 may be rewritten as $H \to H + [K', \Omega_1]$, where K' is such that $[K', \Omega_1]$ is anti-BRST invariant. Thus, it is of the standard form from the BRST point of view based on Ω_1. Indeed, because $\mathrm{bigh}(sK) = (0, 0)$, one has that sK is BRST and anti-BRST invariant. On the other hand, $sK|_{\mathscr{P}=\lambda=G=0} = 0$. Thus, sK is (i) s_1-closed and (ii) an extension of zero. Hence, it is s_1-exact (see [6]), $sK = s_1 K'$ for some K' with $s_1 K'$ anti-BRST invariant.

Actually, from the standard BRST viewpoint, one only requires the standard ghost number of the BRST extension of H_0 to be zero, i.e., H may contain also terms of bidegree (k, k) with $k \neq 0$. We have the following general theorem that allows one to make the link between the standard BRST theory and the BRST-anti-BRST theory at the gauge fixing level:

Theorem 7.1. *Let Ψ be a fermionic function such that $s\Psi$ contains only terms of ghost bidegree of the form (k, k). Then $s\Psi$ is* BRST *and* anti-BRST *invariant and $s\Psi = s_1 \Psi'$ for some fermionic function Ψ'. Conversely, if $s_1 \Psi'$ is* anti-BRST *invariant and contains only terms of ghost bidegree of the form (k, k), then it can be written as $s\Psi$ for some fermionic function Ψ.*

Proof of Theorem 7.1. Let us expand the function Ψ according to the standard ghost number: $\Psi = \sum_n \Psi_n$, where $\mathrm{gh}_{\mathrm{standard}}(\Psi_n) = n$. The requirement that $\mathrm{gh}_{\mathrm{standard}}((s_1 + s_2)\Psi) = 0$ translates into the following family of equations:

$$(s_1 + s_2)\Psi = s_1 \Psi_{-1} + s_2 \Psi_1 \,, \tag{144}$$

$$s_2 \Psi_{-1} + s_1 \Psi_{-3} = 0 \,, \tag{145}$$

$$s_1 \Psi_1 + s_2 \Psi_3 = 0 \,, \tag{146}$$

$$\vdots$$

Hence, we have $s_1(s_1 \Psi_{-1} + s_2 \Psi_1) = s_1 s_2 \Psi = -s_2 s_1 \Psi_1 = s_2^2 \Psi_3 = 0$, and similarly, one can see that $s_2(s_1 \Psi_{-1} + s_2 \Psi_1) = s_1^2 \Psi_{-3} = 0$. One can also see that $s_2 \Psi_1$ is s_1-exact, because it is s_1-closed, of standard ghost number zero and it vanishes when $\mathscr{P} = \lambda = G = 0$. Thus, one finds that

$$(s_1 + s_2)\Psi = s\Psi' \,, \tag{147}$$

where the function Ψ' is of standard ghost number minus one and such that $s_2 s_1 \Psi' = 0$.

Conversely, suppose that one has $s_1 \Psi'$ with $s_2 s_1 \Psi' = 0$ and $\mathrm{gh}_{\mathrm{standard}}(\Psi') = -1$. Then, one can find Ψ such that $s_1 \Psi' = (s_1 + s_2)\Psi' - s_2 \Psi'$. But $s_1 s_2 \Psi' = 0$ and $s_2 \Psi'$ is of standard ghost number -2. Hence, $s_2 \Psi'$ is s_1-trivial (no s_1-cohomology at standard negative ghost degrees): $s_2 \Psi' = -s_1 \Psi_{-3}$ and so one obtains $\Psi' = (s_1 + s_2)(\Psi' + \Psi_{-3}) - s_2 \Psi_{-3}$. Going on recursively in the same fashion at lower standard ghost numbers, one concludes that $s_1 \Psi' = (s_1 + s_2)\Psi$. QED

Those gauge fixed hamiltonians are to be used in the path integral in order to quantize gauge systems. The fact that the path integral does not depend on the choice of the fermionic function Ψ follows from the Fradkin and Vilkovisky theorem [31]. On the other hand, the path integral obtained by applying the BRST-anti-BRST formalism is of the form of the standard BRST path integral, since $s_1 \Psi' = s\Psi$. Hence, the equivalence of the BRST-anti-BRST formalism with the standard BRST formalism (at the path integral level) is obvious.

8. Comparison with the $sp(2)$ Formalism

The $sp(2)$ formalism has attracted considerable attention in connection with string field theory, see [20, 21, 22, 27, 28, 32, 33, 34]. Our BRST-anti-BRST formulation reproduces the $sp(2)$ formulation of [27, 28] when the ambiguity in Ω is appropriately handled. This can be seen as follows. First of all, the spectra of ghosts and of ghost momenta are the same. Using the notations of [28], one has the following correspondence for the ghost momenta:

$$\begin{cases} \overset{(-1,0)}{\mathscr{P}}_{A_0} \leftrightarrow \mathscr{P}_{A_0|1} \\ \overset{(0,-1)}{\mathscr{P}}_{A_0} \leftrightarrow \mathscr{P}_{A_0|2} \end{cases} \tag{148}$$

$$\overset{(-1,-1)}{\lambda}_{A_0} \leftrightarrow \lambda_{A_0} \tag{149}$$

$$\begin{cases} \overset{(-2,0)}{\mathscr{P}}_{A_1} \leftrightarrow \mathscr{P}_{A_1|11} \\ \overset{(-1,-1)}{\mathscr{P}}_{A_1} \leftrightarrow \mathscr{P}_{A_1|12} \equiv \mathscr{P}_{A_0|21} \\ \overset{(0,-2)}{\mathscr{P}}_{A_1} \leftrightarrow \mathscr{P}_{A_1|22} \end{cases} \tag{150}$$

$$\vdots$$

$$\overset{(-i,-j)}{\mathscr{P}}_{A_k} \leftrightarrow \mathscr{P}_{A_k|\underbrace{1\ldots1}_{i}\,\underbrace{2\ldots2}_{j}} \tag{151}$$

$$\vdots$$

$$\overset{(-i,-j)}{\lambda}_{A_k} \leftrightarrow \lambda_{A_k|\underbrace{1\ldots1}_{i}\,\underbrace{2\ldots2}_{j}} \tag{152}$$

$$\vdots$$

where $\mathscr{P}_{A_k|a_1\ldots a_{k+1}}$ and $\lambda_{A_k|a_1\ldots a_k}$ are symmetric $sp(2)$ tensors. The identification for the ghosts are then obvious. Second, the ghost number gh introduced in the present paper is exactly the *new ghost number* of [27, 28]. Finally, a close inspection of Eqs. (128), (129) and (130) shows that one can make the choice $S\Omega_1 = \Omega_2$ and $S\Omega_2 = \Omega_1$ (this follows from the fact that $SD_{11} = D_{22}, SD_{22} = D_{11} SD_{12} = D_{21}$ and

Theorem 4.2). Then one has

$$Ss_1S = s_2 \tag{153}$$

$$Ss_2S = s_1 \tag{154}$$

and

$$SsS = s\,. \tag{155}$$

With that choice, there is a complete symmetry between the BRST and the anti-BRST generators, as in the $sp(2)$ theory and the generators Ω_1 and Ω_2 coincide with the generators Ω^a ($a = 1, 2$) of references [27, 28].

9. Conclusions

In this paper we have explored the algebraic structure of the BRST-anti-BRST formalism. We have proven the existence of the BRST-anti-BRST transformation for an arbitrary gauge system. To that end, it was found necessary to enlarge the ghost system and to introduce a Koszul–Tate *biresolution* of the algebra of smooth functions defined on the constraint surface. One can then apply the *standard* BRST *techniques* to the corresponding reducible description of the constraint surface, to get directly the generator Ω of the sum of the BRST and the anti-BRST transformations. A crucial positivity theorem controls that Ω indeed splits as a sum of just two terms ($\Omega^{\mathrm{BRST}} = \Omega_1$ and $\Omega^{\text{anti-BRST}} = \Omega_2$), and no more. This positivity theorem, in turn, is a consequence of the algebraic properties of the Koszul–Tate biresolution. Our approach clearly explains the complexity of the ghost-antighost spectrum necessary for the BRST-anti-BRST formulation and also shows in a straightforward way the equivalence between the standard BRST formalism and the BRST-anti-BRST one. The arguments developed in this article can be applied, with some modifications, to the extended antifield-antibracket formalism. We shall return to this question in a separate publication [18].

References

1. Hirsch, G.: Bull. Soc. Math. Bel. **6**, 79 (1953); Stasheff, J.D.: Trans. Am. Math. Soc. **108**, 215, 293 (1963); Gugenheim, V.K.A.M.: J. Pure Appl. Alg. **25**, 197 (1982); Gugenheim, V.K.A.M., May, J.P.: Mem. AMS **142** (1974); Gugenheim, V.K.A.M., Stasheff, J.D.: Bull. Soc. Math. Bel. **38**, 237 (1986); Lambe, L., Stasheff, J.D.: Manus. Math. **58**, 363 (187); Stasheff, J.D.: Bull. Am. Soc. **19**, 287 (1988)
2. Fisch, J., Henneaux, M., Stasheff, J., Teitelboim, C.: Commun. Math. Phys. **120**, 379 (1989)
3. Fisch, J.M.L., Henneaux, M.: Commun. Math. Phys. **128**, 627 (1990)
4. Henneaux, M., Teitelboim, C.: Commun. Math. Phys. **115**, 213 (1988)
5. Dubois-Violette, M.: Ann. Inst. Fourier **37**, 45 (1987)
6. Henneaux, M., Teitelboim, C.: Quantization of Gauge Systems. Princeton, NJ: Princeton U.P. 1992
7. Curci, G., Ferrari, R.: Phys. Lett. **B68**, 91 (1976); Nuovo Cimento **A32**, 151
8. Ojima, I.: Prog. Theor. Phys. **64**, 625 (1980)
9. Hwang, S.: Nucl. Phys. **B231**, 386 (1984)
10. Bonora, L., Tonin, M.: Phys. Lett. **B98**, 83 (1981)
11. Bonora, L., Pasti, P., Tonin, M.: J. Math. Phys. **23**, 83 (1982)
12. Baulieu, L., Thierry-Mieg, J.: Nucl. Phys. **B197**, 477 (1982)

13. Ore, F.R., van Nieuwenhuizen, P.: Nucl. Phys. **B204**, 317 (1982)
14. Alvarez-Gaumé, L., Baulieu, L.: Nucl. Phys. **B212**, 255 (1983)
15. Hwang, S.: Nucl. Phys. **B322**, 107 (1989)
16. McMullan, D.: J. Math. Phys. **28**, 428 (1987)
17. Grégoire, P., Henneaux, M.: Phys. Lett. **B277**, 459 (1992)
18. Grégoire, P., Henneaux, M.: to appear in J. Phys A. Math. Gen.
19. Hoyos, J., Quiros, M., Ramirez Mittelbrunn, J., de Urries, F.J.: Nucl. Phys. **B218**, 159 (1983)
20. Spiridonov, V.P.: Nucl. Phys. **B308**, 257 (1988)
21. Batalin, I.A., Lavrov, P.M., Tyutin, I.V.: J. Math. Phys. **31**, 1487 (1990)
22. Batalin, I.A., Lavrov, P.M., Tyutin, I.V.: J. Math. Phys. **32**, 532 (1991)
23. Gomis, J., Roca, J.: Nucl. Phys. **B343**, 152 (1990)
24. Dur, E., Gates, Jr. S.J.: Nucl. Phys. **B343**, 622 (1990)
25. Hull, C.M., Spence, B., Vasquez-Bello, J.L.: Nucl. Phys. **B348**, 108 (1991)
26. Hull, C.M.: Mod. Phys. Lett. **A5**, 1871 (1990)
27. Batalin, I.A., Lavrov, P.M., Tyutin, I.V.: J. Math. Phys. **31**, 6 (1990)
28. Batalin, I.A., Lavrov, P.M., Tyutin, I.V.: J. Math. Phys. **31**, 2708 (1990)
29. Forger, M., Kellendonk, J.: Commun. Math. Phys. **143**, 235 (1992)
30. Batalin, I.A., Fradkin, E.S.: Phys. Lett. **B122**, 157 (1983)
31. Fradkin, E.S., Vilkovisky, G.A.: CERN Report TH-2332 (1977)
32. Aratyn, H., Ingermanson, R., Niemi, A.J.: Phys. Lett. **B189**, 427 (1987); Nucl. Phys. **B307**, 157 (1988)
33. Baulieu, L., Siegel, W., Zwiebach, B.: Nucl. Phys. **B287**, 93 (1987)
34. Siegel, W., Zwiebach, B.: Nucl. Phys. **B288**, 332 (1987)

Communicated by R.H. Dijkgraaf