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$\mathscr{W}$-Algebras, New Rational Models and Completeness of the $c = 1$ Classification

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Abstract. Two series of $\mathscr{W}$-algebras with two generators are constructed from chiral vertex operators of a free field representation. If $c = 1 - 24k$, there exists a $\mathscr{W}(2, 3k)$ algebra for $k \in \mathbb{Z}_+/2$ and a $\mathscr{W}(2, 8k)$ algebra for $k \in \mathbb{Z}_+/4$. All possible lowest-weight representations, their characters and fusion rules are calculated proving that these theories are rational. It is shown, that these non-unitary theories complete the classification of all rational theories with effective central charge $c_{\text{eff}} = 1$. The results are generalized to the case of extended supersymmetric conformal algebras.

1. Introduction

Since the fundamental work of Belavin, Polyakov and Zamolodchikov [3] one of the most exciting problems in theoretical physics is the classification of all possible conformal field theories (CFT). As is well known this outstanding question plays a central rôle in statistical physics as well as string theory and even in the mathematics of 3-manifolds due to its connection with topological quantum field theory [39, 32].

In the last years two, in some sense dual, concepts of classification were developped. One of them is the study of extended conformal symmetry algebras, the so-called $\mathscr{W}$-algebras, as introduced by Zamolodchikov [40]. In this approach one first explicitly constructs an algebra of local chiral fields and then gets insight into the CFT by the study of its irreducible representations. The other one deals with abstract properties of representations of conformally invariant operator algebras only, leaving the latter more or less unspecified. Here one tries to construct abstract fusion algebras [36]. The second approach is more restrictive since it only considers rational conformal theories (RCFT). In this case modular invariance of partition functions might be seen as a link between these two methods, since on the one hand they can be constructed from the characters of the irreducible representations of the symmetry algebras, on the other hand they assure the existence of a unitary and symmetric S-matrix yielding the fusion algebra via the famous Verlinde formula [38, 34].

$\mathscr{W}$-algebras describe the operator product expansion (OPE) of conformally invariant local chiral fields. The singular part of such an OPE yields a Lie bracket structure for the Fourier modes of the fields, the regular part an operation of forming normal ordered products. In the following we define a $\mathscr{W}$-algebra as generated by a finite set of primary fields $\phi_0, \phi_1, \ldots, \phi_n$ including the identity, whose modes yield an associative algebra closed under derivation ∂ and quasi-primary normal ordering $\mathscr{N}(\cdot,\cdot)$ (see [5]). In addition the fields ϕ_i are assumed to be simple, i.e. not composed from others by the operation $\mathscr{N}$. With the conformal dimensions $h(\phi_i) = d_i$ we denote such an algebra by $\mathscr{W}(2, d_1, \ldots, d_n)$, where all structure constants are left unspecified and $d_0 = 2$ stands for the Virasoro field instead of the identity.

Some of these $\mathscr{W}$-algebras were constructed in the last few years by different groups implementing the conformal bootstrap [23, 41, 18, 6]. Recently many new examples could be investigated using the Lie bracket approach [5, 27], which has beside others the great advantage of directly leading to a Lie algebra structure thus admitting the definition of lowest-weight representations [13, 37].

In this paper we establish a new class of RCFTs using both pictures. In the second chapter, starting from some explicitly constructed examples of $\mathscr{W}$-algebras, a whole series is extracted founded on general arguments from the theory of degenerate models. The structure constants of these $\mathscr{W}$-algebras are calculated in chapter three. In the fourth chapter we explain the explicit calculated irreducible representations of these examples by deriving general character formulae, the modular invariant partition function and finally the S-matrix and the fusion algebra, where details are shifted to two appendices. This leads to a new class of RCFTs for which the whole characterizing data is presented. It turns out that, while all these RCFTs are non-unitary, they fit in the frame of the classification of all theories with central charge $c = 1$. The completion of this classification towards the non-unitary case is the subject of chapter five. The sixth chapter gives the obvious generalization of the models to the supersymmetric case.

2. $\mathscr{W}$-Algebras and Degenerate Non-Minimal Models

This chapter mainly intends to explain the existence of series of $\mathscr{W}(2,\delta)$-algebras at the central charge values $c = 1 - 8\delta$ and $c = 1 - 3\delta$, which has been conjectured in [5]. Here the notation means a local chiral algebra with one simple generator in addition to the Virasoro field, whose Lie-algebra of modes is algebraically closed under commutators and normal ordered products in the closure of the envelopping algebra. For exact definitions and details we refer the reader to this paper and [35]. We just sketch the results obtained there, which motivated our work. The following table lists the explicitly constructed $\mathscr{W}$-algebras, which are conjectured to be members of two general series.

The c-values in brackets are extensions of the obtained results to the cases of generically, i.e. for all c-values up to finitely many exceptions, existing $\mathscr{W}$-algebras. Some explicit calculations concerning the representation theory of these generically existing algebras at the particular c-values of Table 2.1 may be found in [13], confirming the extensions of our list. The explicit result for $\mathscr{W}(2,9)$ has been obtained by [30]. In the last column of the table we list the squares of the self-coupling structure constant C_{WW}^{W} of the additional primary field W with dimension δ. The value for the $\mathscr{W}(2,2)$-algebra has been put in brackets, because this algebra exists for every central charge and every self-coupling independently, since it always can be linearly transformed into a copy of two commuting Virasoro-algebras. The particular given value will be justified later. Of course, the self-coupling vanishes necessarily for δ not even.

Table 2.1. Two sets of $\mathcal{W}$-algebras to rational c-values not contained in the minimal series

The series $\mathcal{W}(2,\delta)$ with $c = 1 - 8\delta$:		
$\mathcal{W}(2, \frac{3}{2})$	$(c = -11)$	$(C_{WW}^{W})^2 = 0$
$\mathcal{W}(2, 3)$	$(c = -23)$	$(C_{WW}^{W})^2 = 0$
$\mathcal{W}(2, \frac{9}{2})$	$c = -35$	$(C_{WW}^{W})^2 = 0$
$\mathcal{W}(2, 6)$	$c = -47$	$(C_{WW}^{W})^2 = 0$
$\mathcal{W}(2, \frac{15}{2})$	$c = -59$	$(C_{WW}^{W})^2 = 0$
$\mathcal{W}(2, 9)$	$c = -71$	$(C_{WW}^{W})^2 = 0$
The series $\mathcal{W}(2,\delta)$ with $c = 1 - 3\delta$:		
$\mathcal{W}(2, 2)$	$(c = -5)$	$((C_{WW}^{W})^2 = \frac{722}{33})$
$\mathcal{W}(2, 4)$	$(c = -11)$	$(C_{WW}^{W})^2 = -\frac{57434}{253}$
$\mathcal{W}(2, 6)$	$(c = -17)$	$(C_{WW}^{W})^2 = \frac{95922436000}{43340157}$
$\mathcal{W}(2, 8)$	$c = -23$	$(C_{WW}^{W})^2 = -\frac{127081705690919}{5974374591}$

The generically existing $\mathcal{W}$-algebras can be identified in the following way: $\mathcal{W}(2, \frac{3}{2})$ is nothing else than the Super-Virasoro-algebra, and $\mathcal{W}(2, 2)$ is the direct sum of two Virasoro-algebras. The latter and $\mathcal{W}(2, 3)$, $\mathcal{W}(2, 4)$, and $\mathcal{W}(2, 6)$ can be viewed as the "Casimir-algebras" of the affine Kac-Moody-algebras, or actually as the affinization of the Casimir-algebras, related to the semi-simple Lie-algebras $A_1 \oplus A_1$, A_2, B_2 or C_2, and G_2 respective.

In order to explain the existence of the series of $\mathcal{W}$-algebras of Table 2.1 we start to review very shortly the theory of the so-called Dotsenko-Fateev degenerate models and free-field construction [12]. The irreducible Virasoro lowest-weight modules of degenerate models are not identical with the Verma modules due to null states. In this case the operator algebra of the model is generated by primary fields which correspond to the lowest-weight states of dimensions

$$h_{r,s}(c) = \tfrac{1}{4}\left((r\alpha_- + s\alpha_+)^2 - (\alpha_- + \alpha_+)^2\right)\,, \tag{2.1}$$

where we parametrized the central charge as $c = 1 - 24\alpha_0^2$ and defined $\alpha_\pm = \alpha_0 \pm \sqrt{1 + \alpha_0^2}$. In these references the Virasoro algebra and their irreducible representation modules are constructed from Fock space representations of the Heisenberg algebra (the free field)

$$\left[j_m, j_n\right] = n\delta_{n+m,0}\,, \tag{2.2a}$$

built on the lowest-weight state $|\alpha, \alpha_0\rangle$ with

$$j_n|\alpha, \alpha_0\rangle = 0 \ \forall n < 0\,, \quad j_0|\alpha, \alpha_0\rangle = \sqrt{2}\alpha|\alpha, \alpha_0\rangle\,. \tag{2.2b}$$

These Fock spaces $\mathcal{F}_{\alpha,\alpha_0}$ obtain the structure of Virasoro modules if the Virasoro field is defined by

$$L(z) = \mathcal{N}(j, j)(z) + \sqrt{2}\alpha_0\partial_z j(z)\,, \tag{2.3a}$$

which has central charge $c = 1 - 24\alpha_0^2$. Here $\mathscr{N}(\cdot,\cdot)$ stands for the quasiprimary projection of a standard normal-ordered product. The Heisenberg lowest-weight states become Virasoro lowest-weight states with weights $h(\alpha) = \alpha^2 - 2\alpha\alpha_0$, i.e.

$$L_n|\alpha,\alpha_0\rangle = 0 \ \forall n < 0\,, \quad L_0|\alpha,\alpha_0\rangle = h(\alpha)|\alpha,\alpha_0\rangle\,. \tag{2.3b}$$

One also can construct chiral primary conformal fields of weight $h(\alpha)$, so-called vertex operators, which map Fock spaces of different charges into each other, $\psi_\alpha : \mathscr{F}_{\beta,\alpha_0} \longrightarrow \mathscr{F}_{\alpha+\beta,\alpha_0}$. They are given by the normal ordered expression

$$\psi_\alpha(z) = \exp\left(-\sum_{n>0}\sqrt{2}\alpha j_n\frac{z^n}{n}\right)\exp\left(-\sum_{n<0}\sqrt{2}\alpha j_n\frac{z^n}{n}\right)c(\alpha)z^{-\sqrt{2}\alpha j_0}\,, \tag{2.4}$$

where $c(\alpha)$ commutes with the j_n, $n \neq 0$, and maps groundstates to groundstates. If $\alpha = \alpha_{r,s} = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_+$, then $h(\alpha) = h_{r,s}(c)$ of (2.1). The so-called screening operators $Q_\pm^{(n)} = \oint dz_1 \ldots dz_n \psi_{\alpha\pm}(z_1)\ldots\psi_{\alpha\pm}(z_n)$ have particular importance with appropriately chosen integration paths. Since $h(\alpha_\pm) = 1$, they have conformal dimension zero but do change the charge of the Fock-space states to which they are applied. With the help of these operators both non-trivial n-point functions and conformal blocks can be constructed.

Products of para-fields of the form (2.4) only can be well-defined for radial ordered points, i.e. $\psi_\alpha(z_1)\psi_\beta(z_2)$ is only defined for $|z_1| > |z_2|$. One gets the other half by analytic continuation, and the chiral conformal blocks in general become multivalued functions $\psi_\alpha(z_1)\psi_\beta(z_2) = \varepsilon_{\alpha\beta}\psi_\beta(z_2)\psi_\alpha(z_1)$, where $\varepsilon_{\alpha\beta} = \exp(2\pi i\alpha\beta)$. Two fields ϕ,ψ are said to be local relative to each other, if their phase $\varepsilon_{\phi\psi} = \pm 1$ and their conformal dimensions differ by integers or half integers. Thus, chiral local fields necessarily must have integer or half integer weights.

Much work has been done to resolve the embedding structure of the Virasoro and Fock space Verma modules to irreducible lowest-weight modules, see e.g. [15, 16]. In fact, Felder showed that only on the Fock spaces $\mathscr{F}_{r,s}$ with charges $\alpha_{r,s}$ screening operators can act well-defined, and can be considered as the non-trivial coboundary operators (or so-called BRST operators) on the cohomology complex of the Fock spaces whose elements just are the irreducible Virasoro modules $\mathscr{H}_{r,s}$. Indeed the screened vertex operators

$$V_{(n'n)(m'm)}^{(l'l)}(z) = \psi_{n'n}(z)Q_-^{(r')}Q_+^{(r)} : \mathscr{F}_{m'm} \longrightarrow \mathscr{F}_{l'l}\,, \tag{2.5}$$

where $l = m + n - 2r - 1$, $l' = m' + n' - 2r' - 1$, are invariant (up to a phase) under the action of the screening charges (BRST-invariant), i.e. they map the cohomology spaces into each other.

In principle these facts contain all the information about the CFT such as fusion rules, braid matrices or OPE structure coefficients, see e.g. [17] for the case of symmetric theories. Let $\mathscr{W}$ denote the maximal extended symmetry algebra of the CFT (making the partition function diagonal). If the theory is rational, the symmetry algebra is a finitely generated $\mathscr{W}(d_0, d_1, \ldots, d_n)$-algebra, where $d_0 = 2$ denotes the always present Virasoro field. If we denote with $\mathscr{H}^{(\lambda)}$ an irreducible lowest-weight representation space of the $\mathscr{W}$-algebra, then this space can be decomposed with respect to the Virasoro-algebra. Thus the whole Hilbert space can be written as

$$\mathscr{H} = \bigoplus_{\lambda\in\Lambda}\left(\bigoplus_{(n'n)\in N_\lambda}\mathscr{H}_{n'n}^{(\lambda)} \otimes \bigoplus_{(n'n)\in N_\lambda}\mathscr{H}_{n'n}^{(\lambda)}\right)\,, \tag{2.6}$$

where Λ denotes the set of all irreducible $\mathscr{W}$-lowest-weight representations and N_λ the set of all irreducible Virasoro lowest-weight representations contained in $\mathscr{H}^{(\lambda)}$. Then the local fields of dimensions $\Delta = h_{n'n} + \bar{h}_{\bar{n}'\bar{n}}$ are glued together from the screened vertex operators,

$$\Phi_{n'n,\bar{n}'\bar{n}}(z,\bar{z}) = \sum_{m',m,l',l} \mathscr{D}^{(l'l)}_{(n'n)(m'm)} V^{(l'l)}_{(n'n)(m'm)}(z) \otimes \sum_{\bar{m}',\bar{m},\bar{l}',\bar{l}} \mathscr{D}^{(\bar{l}'\bar{l})}_{(\bar{n}'\bar{n})(\bar{m}'\bar{m})} V^{(\bar{l}'\bar{l})}_{(\bar{n}'\bar{n})(\bar{m}'\bar{m})}(\bar{z}), \tag{2.7}$$

where the coefficients $\mathscr{D}^{(l'l)}_{(n'n)(m'm)}$ are fixed up to normalization by the requirements of locality of the OPE and crossing-symmetry of the four-point-function. They are non-zero only, if $|n-m|+1 \le l \le m+n-1$, $l \equiv m+n-1 \bmod 2$, and similarily for l'. The situation simplifies drastically, if a chiral theory is considered, since then the locality condition is extremely restrictive. The OPE of two local chiral fields again only can contain local chiral fields on the right-hand side. Moreover, the chiral blocks have to be local for themselves. For a review on local chiral fields, their OPE and normal ordered products, see e.g. [21]. The consequences of these strong requirements will be worked out in the following.

Rational conformal field theories (RCFT) can be characterized by the fact that they have only a finite set of $\mathscr{W}$-primary fields or equivalently having only finitely many irreducible $\mathscr{W}$-lowest-weight representations, where $\mathscr{W}$ denotes the maximal extended symmetry algebra of the CFT. But this in return means the following: Firstly both the central charge as well as the conformal dimensions of all fields, which belong to the operator algebra, have to be rational numbers [2]. And secondly, infinitely many Virasoro primaries with rational dimensions are needed, because otherwise the characters will never be (*finite* linear combinations of elementary) modular functions.

For the degenerate models this means the following. If (2.1) is expressed in $k = \alpha_0^2$, thus $h_{r,s} = -k + \frac{1}{4}((2k+1)(r^2+s^2) + 2\sqrt{k(k+1)}(r^2-s^2) - 2rs)$, then we can distinguish three cases:

(i) $k, \sqrt{k(k+1)} \in \mathbb{Q}$. In this case necessarily k is of the form $\dfrac{(p-q)^2}{4pq}$ with $p,q \in \mathbb{N}$ coprime, thus c belongs to the minimal series (including the case $c = 1$). Moreover $h_{r,s} \in \mathbb{Q}\ \forall r,s \in \mathbb{Z}$.
(ii) $k \in \mathbb{Q}$, $\sqrt{k(k+1)} \in \mathbb{C} - \mathbb{Q}$. This yields all rational c-values not contained in the minimal series. In this case exactly the weights $h_{r,\pm r} \in \mathbb{Q}\ \forall r \in \mathbb{Z}$ only.
(iii) $k \in \mathbb{C} - \mathbb{Q}$. In this case neither c nor the h-values are rational (the latter up to the exception $h_{1,1} = 0$).

The proof of this statement is simple. First note, that the polynomials in r, s which have coefficients k or $\sqrt{k(k+1)}$ respectively are linearly independent. Thus case (i) is obtained by the requirement that all dimensions should be rational, i.e. the coefficients have to be rational. Put $\sqrt{k(k+1)} = \dfrac{n}{m} \in \mathbb{Q}$. Solving this for k and requiring rationality yields the diophantic equation $(2n)^2 + m^2 = l^2$ with the Pythagorian triples as their solutions. Parametrizing the coprime solutions yields c in the minimal series. Case (ii) simply comes out, if one looks for the zeros of the polynomials. Only $(r^2 - s^2)$ admits infinitely many solutions allowing rational h-values, while its coefficient is non-rational. The last case just covers the remaining possibilities. Note, that k and $\sqrt{k^2+k}$ are algebraically independent numbers over $\mathbb{Q}$ for all irrational k.

Of particular interest is case (ii). If one chooses k to be integer or half-integer, one finds that all weights

$$h_{r,r} = (r^2-1)k\,, \quad h_{r,-r} = (r^2-1)k + r^2 \tag{2.8}$$

are integer or half-integer. Moreover the phases $\varepsilon_{\alpha_{r,r}\alpha_{s,s}}$ all equal ± 1. If r is odd, then even $k \in \mathbb{Z}_+/4$ is possible. This shows that the "diagonal" set $\{V^{(l,l)}_{(n,n)(m,m)} \mid n, m, l \in \mathbb{Z}_+\,,\; l \equiv n+m-1 \bmod 2\}$ of BRST-invariant screened vertex operators is a local set, i.e. all operators are local to each other since the phases appearing by reordering the screening charges cancel as long as the number of reordered Q_+ charges equals the number of reordered Q_- charges, $Q^{(r)}_- Q^{(r)}_+ \psi_{\alpha_{n,n}}(z) = e^{4\pi i r \alpha_{n,n}\alpha_0}\psi_{\alpha_{n,n}}(z) Q^{(r)}_- Q^{(r)}_+$ and $4\alpha_{n,n}\alpha_0 = 4(1-n)\alpha_0^2 \in \mathbb{Z}$ for $\alpha_0^2 = k \in \mathbb{Z}_+/4$. From now on we consider this special case of "diagonal" operators, i.e. $n' = n$ in $\alpha_{n,n'}$. In the following we use the shorter notation $V^l_{n,m}(z) \equiv V^{(l,l)}_{(n,n)(m,m)}(z)$ for the "diagonal" BRST-invariant screened vertex operators (2.5) and $\mathscr{D}^l_{n,m} \equiv \mathscr{D}^{(l,l)}_{(n,n)(m,m)}$ for their coefficients in the chiral blocks $W^{(n)} \equiv \Phi_{n,n,1,1}$ according to (2.7).

Indeed, the chiral blocks must be glued together from the operators of the local set above. Otherwise they cannot represent chiral local fields. Locality also restricts the fusion rules for the chiral algebra, since chiral local operators map the spaces into each other such that the lowest weights differ by integers or half-integers [21]. Thus, the OPE of two of these fields acting on a lowest-weight module again only yields local fields acting on lowest-weight modules. This implies that the set of local chiral blocks,

$$\left\{ W^{(n)}(z) = \sum_{m\in\mathbb{Z}_+} \sum_{\substack{l\in\mathbb{Z}/(n+m)\mathbb{Z} \\ l+n+m\,\equiv\,1 \bmod 2}} \mathscr{D}^l_{n,m} V^l_{n,m}(z) \,\middle|\, n \in \mathbb{Z}_+ \right\} \tag{2.9a}$$

is a closed algebra with fusion rules

$$[W^{(n)}] \times [W^{(m)}] = \sum_{\substack{|n-m|+1\,\leq\, l\,\leq\, m+n+1 \\ l+m+n+1\,\equiv\,1 \bmod 2}} N^l_{n,m}[W^{(l)}]\,, \tag{2.9b}$$

where the fusion numbers $N^l_{n,m}$ are non negative integers. Moreover, the subset with n odd is a closed subalgebra, and will be called the odd sector of the algebra in the following.

It is important to notice that only (half-) integer k (or quarter-integer for the odd sector subalgebra) will lead to non-trivial RCFTs. One could think to take for k other rational numbers than these and to look for the subset of chiral vertex operators that are still local. But in this case it can happen that e.g. two operators $W^{(n)}$ and $W^{(m)}$ having (half-) integer dimensions are local with respect to each other, while one of them, say $W^{(n)}$, being not local to itself. Then it might happen that the conformal family of such an operator $W^{(n)}$ contributes to the right-hand side of the OPE of the other local field with itself. In this case, the simple field does not appear on the right-hand side but its normal ordered products, e.g. $\mathscr{N}(W^{(n)}, W^{(n)})$ which has to be understood as the chiral projection of the normal ordered product of the left-right symmetric field $W^{(n)}(z) \otimes W^{(n)}(\bar{z})$ with itself. As a consequence, no algebraically closed local algebra larger than the Virasoro algebra can be defined. Indeed, we will see later that the values $k \in \mathbb{Z}_+/4$ are the only possibilities to obtain non-minimal RCFTs from Dotsenko-Fateev models.

It remains to show that these local algebras are indeed $\mathcal{W}$-algebras, i.e. completely generated by the normal ordered products of (derivatives of) finitely many simple primary fields. From the fusion rules we learn that applying a field twice to a lowest-weight state will lead us to other lowest-weight states corresponding to other local primary fields. Indeed, the complete local system can be generated by application of $W^{(2)}$, the odd sector by using $W^{(3)}$. On the other hand we can consider the commutator of modes of two operators and use the truncation of the terms on the right-hand side by the conformal dimension rather than by the label of the Fock space charge. Writing the right-hand side symbolically in conformal families we find e.g.

$$[W_m^{(2)}, W_n^{(2)}] = [W_{m+n}^{(1)}] + [W_{m+n}^{(3)}], \tag{2.10}$$

where $W^{(1)}$ of course is the identity operator. For the dimensions we have $h_{3,3} = 8k > 2h_{2,2} - 1 = 2(3k) - 1$ with $k = (1-c)/24$. Thus no field of the conformal family $[W^{(3)}]$ will appear in the commutator of $W^{(2)}$ with itself showing that the modes of the latter field together with the Virasoro modes generate a Lie-algebra structure which closes in (the closure of) its envelopping algebra, actually they generate a $\mathcal{W}(2, 3k)$ algebra. The same argumentation applies to the odd sector algebras, where one can eliminate $W^{(5)}$ from the right-hand side of the commutator of $W^{(3)}$ with itself, yielding a $\mathcal{W}(2, 8k)$ algebra.

The associativity of the OPE is equivalent with the fact that the Jacobi identities are fulfilled. As was pointed out in [5] only the identities involving three simple fields have to be checked and there only the coefficients for the primary fields appearing on the right-hand side. This leaves us with one non-trivial condition in our case. If the simple field has dimension δ, then fields up to dimension $3\delta - 2$ will appear on the right-hand side of the Jacobi identity. Comparing again with the fusion rules we see for the odd sector algebras that $3h_{3,3} - 2 = 24k - 2 < h_{5,5} = 24k$, showing that no further primary field can contribute to the identity. In the other case we have $3h_{2,2} - 2 = 9k - 2 \geq h_{3,3} = 8k$ for $k \geq 2$ indicating that the field $W^{(3)}$ could contribute to the identity. But its coefficient must be zero because the self-coupling of $W^{(2)}$ vanishes due to the fusion rules. In fact, vanishing self-coupling means that no field can appear on the right-hand side, whose mode expansion has monomials involving more than one mode of $W^{(2)}$. But the composite primary field $W^{(3)}$, which is nothing more than the primary projection of $\mathcal{N}(W^{(2)}, \partial^{2k} W^{(2)})$, will have terms quadratic in $W^{(2)}$ in its mode expansion. Thus, if one weakens the assumptions in the definition of $\mathcal{W}$-algebras such that the generators need not be simple, then also a $\mathcal{W}(2, 3k, 8k)$ can be constructed, where $W^{(3)}$ is given as above. Note, that if $h_{2,2}$ is half-integer, so is k such that the statement above remains valid, since $2k$ is odd as it must be. This explains the existence of the $\mathcal{W}$-algebras listed in Table 2.1.

A $\mathcal{W}$-algebra is completely determined by the set of dimensions of the generators and a consistent choice of all free parameters. The dimension of the additional primary field and the central charge c are already fixed in our cases, the only free parameter left is the self-coupling structure constant of the primary field. In the case of the $\mathcal{W}(2, 3k)$-algebras it must vanish by symmetry, but for the case of $\mathcal{W}(2, 8k)$-algebras one might be interested in a formula expressing it by the only real input, the number k, which also paramterizes the central charge $c = 1 - 24k$ and the dimension $\delta = 8k$.

This and the still remaining determination of the coefficients in the chiral conformal blocks is done in the next chapter.

3. Structure Constants

We now come to the calculation of the structure constants. In particular, we show that the self-coupling structure constants C_{WW}^{W} of the $\mathscr{W}(2,8k)$-algebras can be derived from the structure constants of the Dotsenko-Fateev models. This will make our proof rigorous that these algebras can be represented by a free-field construction. In the following we use the notation as in [17].

As was pointed out in [5, 35], the commutators in a $\mathscr{W}$-algebra are fixed by SU(1,1)-invariance up to some structure constants. Furthermore, all structure constants for quasiprimary fields can be reduced by SU(1,1)-invariance to expressions involving only the central charge c and the structure constants connecting three simple primary fields. In the case of a $\mathscr{W}(2,\delta)$-algebra with only one additional primary field W, there is only one up to now free structure constant beside the central charge, the self-coupling constant C_{WW}^{W}. Its square is usually fixed by the validity of the Jacobi identity involving three times the field W, which in this case is the only identity one has to check.

Since every chiral local theory can be tensored with itself to yield a symmetric theory, we learn from (2.7) that the coefficients of the symmetric Dotsenko-Fateev models, as given in [17] and denoted there as D_{NM}^{L}, have to equal the squares of our $\mathscr{D}_{n,m}^{l}$ with $N=(n,n), M=(m,m)$ and $L=(l,l)$ (up to normalization). Felder, Fröhlich and Keller obtained the following result from the calculation of the braid matrices of the BRST-invariant vertex operators, which are proportional to the (quantum) $6j$-symbols of the quantum group $\mathrm{U}_q(\mathrm{SU}(2))$, and the crossing symmetry of the latter:

$$\begin{aligned}
\left(\mathscr{D}_{n,m}^{l}\right)^2 &= c\cdot\frac{h_{l,l}}{h_{n,n}h_{m,m}}D_{NM}^{L}\\
&= c\cdot\frac{h_{l,l}}{h_{n,n}h_{m,m}}\frac{N_{LL}^{(1,1)}}{N_{NN}^{(1,1)}N_{MM}^{(1,1)}}\Delta_{n,m}^{l}(x)\Delta_{n,m}^{l}(x')\,,\\
\Delta_{n,m}^{l}(x) &= (-1)^{\frac{1}{2}(n+m-l-1)}\left(\frac{[n]_x[m]_x[l]_x}{[1]_x}\right)^{\frac{1}{2}}\\
&\quad\times\prod_{j=(l+n-m+1)/2}^{n-1}[j]_x\prod_{j=(m+n-l+1)/2}^{n-1}[j]_x\prod_{j=(l+m-n+1)/2}^{(l+m+n-1)/2}\frac{1}{[j]_x}\,,
\end{aligned}\tag{3.1}$$

where the brackets are given by $[j]_x = x^{j/2}-x^{-j/2}$ with $x=\exp(2\pi i\alpha_+^2)$ and $x'=\exp(2\pi i\alpha_-^2)$. A prefactor $c\cdot h_{l,l}h_{n,n}^{-1}h_{m,m}^{-1}$ has been included to take care of our normalization of the two-point functions used in [5, 35], which is defined for chiral, simple, primary fields to take the value

$$\langle 0|W_{-h_{n,n}}^{(n)}W_{h_{m,m}}^{(m)}|0\rangle = \frac{c}{h_{n,n}}\delta_{n,m}\,. \tag{3.2}$$

The general normalization constants $N_{(n'n)(m'm)}^{(l'l)} = \langle h_{l'l}|V_{(n'n)(m'm)}^{(l'l)}(1)|h_{m'm}\rangle$ have been expressed by Felder in terms of Dotsenko-Fateev integrals [12] and are given here for completeness:

$$N^{(l'l)}_{(n'n)(m'm)} = (-1)^{\frac{1}{2}((2n'-1)r+(2n-1)r')} \alpha_+^{4rr'} \prod_{j'=1}^{r'} \frac{[m'-j']_{x'}[j']_{x'}}{[1]_{x'}} \prod_{j=1}^{r} \frac{[m-j]_x[j]_x}{[1]_x}$$
$$\times \prod_{j'=1}^{r'} \frac{\Gamma(j'\alpha_-^2)\Gamma(m+(j'-m')\alpha_-^2)\Gamma(n+(j'-n')\alpha_-^2)}{\Gamma(\alpha_-^2)\Gamma(m+n-2r+(r'-m'-n'+j')\alpha_-^2)}$$
$$\times \prod_{j=1}^{r} \frac{\Gamma(j\alpha_+^2-r')\Gamma(m'-r'+(j-m)\alpha_+^2)\Gamma(n'-r'+(j-n)\alpha_+^2)}{\Gamma(\alpha_+^2)\Gamma(m'-r'+n'+(r-m-n+j)\alpha_+^2)}, \tag{3.3}$$

where $l = n + m - 2r - 1$ and similar for l'. The structure constants of the OPE or equivalently of the Lie-algebra of the Fourier modes of the chiral local fields are then given by

$$C^l_{n,m} = \mathcal{D}^l_{n,m} N^L_{NM}\,. \tag{3.4}$$

Thus, in the case of our $\mathcal{W}(2, 8k)$-algebras we find that the square of the self-coupling of the additional simple primary field $W = W^{(3)}$ with dimension $\delta = 8k$ reads

$$\left(C^W_{WW}\right)^2 = \frac{c}{\delta} D^{(3,3)}_{(3,3)(3,3)} \left(N^{(3,3)}_{(3,3)(3,3)}\right)^2\,. \tag{3.5}$$

Note that only the square of the structure constant can be determined by (3.1). Since $N^{(2,2)}_{(2,2)(2,2)} = 0$ due to the fusion rules, the self-coupling of $W^{(2)}$ vanishes as expected. If one expresses the brackets as $[j]_x = 2i\sin(j\pi\alpha_+^2)$ and $[j]_{x'} = 2i\sin(j\pi\alpha_-^2)$, and reduces the Gamma-functions to terms of the form $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$, one finally arrives at the closed expressions

$$(C^W_{WW})^2 = \begin{cases} \dfrac{(1-24k)}{8k} \dfrac{\prod\limits_{j=1}^{8k}(j^2-64(k^2+k))^2 \prod\limits_{j=1}^{2k}(j^2-4(k^2+k))^3}{\prod\limits_{j=1}^{6k}(j^2-36(k^2+k)) \prod\limits_{j=1}^{4k}(j^2-16(k^2+k))^4} \\ \text{if } k \in \mathbb{Z}_+/2\,, \\ \dfrac{(1-24k)}{8k} \dfrac{\prod\limits_{j=1}^{8k}(j^2-64(k^2+k))^2 \prod\limits_{j=1}^{2k+1/2}((j-\frac{1}{2})^2-4(k^2+k))^3}{\prod\limits_{j=1}^{6k+1/2}((j-\frac{1}{2})^2-36(k^2+k)) \prod\limits_{j=1}^{4k}(j^2-16(k^2+k))^4} \dfrac{3}{4(k^2+k)} \\ \text{if } k \in \mathbb{Z}_+ + \frac{1}{4}\,. \end{cases} \tag{3.6}$$

This result agrees with the examples of Table 2.1. The value for the $\mathcal{W}(2,2)$ given there has been obtained from (3.6). While this algebra exists for every central charge and self-coupling independently, it is related to a Dotsenko-Fateev model for $c = -5$ only for this particular value of the structure constant. Other values could be obtained, if the Virasoro field (2.3) would be deformed.

One remark is necessary here. The screened vertex operators $V^{(l'l)}_{(n'n)(m'm)}(z)$ in (2.5) carry a representation of the braid group, namely the braid matrices given by [17] and defined by the relation

$$V^J_{KL}(z)V^L_{NM}(w) = \sum_{L'} R(J,K,N,M)_{LL'}V^J_{NL'}(w)V^{L'}_{KM}(z) \tag{3.7}$$

valid for $|w| > |z|$. The ordering of the integration variables of the screening charges and the choice of their contours are of great importance for the calculation of the braid matrices. In our case, where $n' = n$ etc., one always has the same number of positive and negative screening charges. Thus, one can introduce similar vertex operators $\mathscr{V}^l_{n,m} = \psi_{n,n}(\widetilde{Q})^{(r)} : \mathscr{F}_{m,m} \to \mathscr{F}_{n,n}$ with $\widetilde{Q} = Q_-Q_+$, i.e. with a rearrangement of the ordering of the screening charges and a change of their contours: Applied to a vertex operator located at z, the new screening operator is given by $\widetilde{Q} = \oint\limits_z du \int\limits_z^u du' \psi_{\alpha_-}(u')\psi_{\alpha_+}(u)$, where the inner integration over u' follows the same contour as the outer one over u which encircles zero and starts at z. Then these operators look like fields of a thermal theory, i.e. a theory with $N = (1,n)$, $M = (1,m)$ etc., but with a double integration for every effective screening with $\widetilde{Q}$. Since the braid matrices are almost factorized in a left and right thermal part, their components connecting only "diagonal" operators are independent from the non-diagonal ones. Moreover, with the modified operators, the (thermal) braid matrices $r(j,k,n,m \mid x)$ in [17] simplify drastically by taking the limit $x \to 1$, since the effective phase of moving contours of the effective screening charge $\widetilde{Q}$ is $\alpha_+^2 + \alpha_-^2 = 2k+1 \in \mathbb{Z}_+/2$ for our particular models. In this limit $[m]_x/[n]_x \to m/n$, thus leaving us with simple rational numbers for the matrix elements and the $\mathscr{D}^l_{n,m}$ coefficients. On the other hand the behaviour of the analytic continuation of the normalization integrals also changes, actually simplifies, if they are defined in the modified vertex operators, since the latter have trivial monodromy properties. Of course both effects cancel out in formula (3.4) leaving the structure constants unchanged as it should be. But this remark shows the special rôle of the values $k \in \mathbb{Z}_+/4$ of the background charge: For these values the modified screened vertex operators form a very simple representation of the braid group.

In the next chapter the explicit calculation of all allowed lowest-weight representations together with a modular invariant partition function completes the description of these new CFTs and proves that they are indeed rational.

4. Representation Theory

In this chapter we discuss the representation theory of the $\mathscr{W}$-algebras established in the last chapter. The answering of this question yields the complete field content of $\mathscr{W}$-primary fields of the theory analogous to the case of the Virasoro algebra. Starting from the character of the vacuum representation of the $\mathscr{W}$-algebra we find all lowest-weight-representations by considering the behaviour of this character under modular transformations.

The case of the bosonic $\mathscr{W}$-algebras in the $1-8d$ series is treated in detail, for the other series only a brief discussion and the results are given.

In the previous chapter we have shown that for $c \in \mathbb{Q}$ but c not an element of the minimal series, only the degenerate conformal families with weights

$h_{n,n} = (n^2 - 1)\frac{1-c}{24}$ and $h_{n,-n} = (n^2 - 1)\frac{1-c}{24} + n^2$ have rational conformal weights. In particular for $c = 1 - 24k$, $k \in \mathbb{N}/2$ all these fields have integer or half-integer dimension, which is necessary for building chiral local symmetry algebras.

From now on let $c = 1 - 24k$ with $k \in \mathbb{N}/2$ fixed. Since the $\mathcal{W}$-algebra which is infinitely generated by the primary fields belonging to the weights $h_{n,n}$ for $c = 1-24k$ contains the $\mathcal{W}(2, 3k)$-algebra as finitely generated subalgebra, all primary fields with higher spin have to be composite. This follows using the isomorphism between the Hilbert space of the vacuum representation of the $\mathcal{W}(2, 3k)$-algebra, generated by the modes of the two simple fields, and the space of all quasiprimary fields, which can be generated by normal ordered products of (derivatives of) the simple fields. As deduced in the last chapter, the other primary fields then appear as primary projections of normal ordered products according to the fusion rules. For example the field $W = W^{(2)}$ obeys the fusion rule

$$[W^{(2)}] \times [W^{(2)}] = [\mathbb{1}] + [W^{(3)}], \tag{4.1}$$

where the weights are $h_{2,2} = 3k$ and $h_{3,3} = 8k$. This means that the conformal family $[W^{(3)}]$ does not occur in the commutator (or equivalently the singular part of the OPE) of W with itself, while the primary projection of $\mathcal{N}(W, \partial^{2k} W)$ is proportional to $W^{(3)}$.

Thus, remembering that all these primary fields belong to degenerate conformal families created by singular vectors, the $\mathcal{W}$-algebra character of the vacuum representation is given by summing up all the Virasoro characters of the lowest-weight representations $|h_{n,n}\rangle$. Following the work of Feigin and Fuks [15], every Virasoro lowest-weight module at level $h_{n,n}$ has exactly one null vector at level $h_{n,-n} = h_{n,n} + n^2$. Therefore, if $\chi_{h_{n,n}}(\tau)$ denotes the character of such a lowest-weight representation of the Virasoro algebra, it is given by

$$\chi_{h_{n,n}}(\tau) = \frac{q^{(1-c)/24}}{\eta(\tau)} \left(q^{h_{n,n}} - q^{h_{n,-n}}\right) = \frac{1}{\eta(\tau)} \left(q^{n^2 k} - q^{n^2(k+1)}\right), \tag{4.2}$$

where $q = e^{2\pi\imath\tau}$ as usual, τ being the modular parameter of the torus, and the Dedekind η-function is $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$. This implies that the $\mathcal{W}$-algebra character can be written in the form

$$\begin{aligned} \chi_0^{\mathcal{W}}(\tau) &= \sum_{n\in\mathbb{Z}_+} \chi_{h_{n,n}}(\tau) = \frac{1}{2\eta(\tau)} \sum_{n\in\mathbb{Z}} \left(q^{n^2 k} - q^{n^2(k+1)}\right) \\ &= \frac{1}{2\eta(\tau)} \left(\Theta_{0,k}(\tau) - \Theta_{0,k+1}(\tau)\right), \end{aligned} \tag{4.3}$$

where we have introduced the elliptic functions (modular functions of weight one-half)

$$\Theta_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{(2kn+\lambda)^2/4k}, \qquad \lambda \in \mathbb{Z}/2, \; k \in \mathbb{N}/2. \tag{4.4a}$$

We call λ the index and k the modulus of the function. Surprisingly, we can express our $\mathcal{W}$-algebra character by functions with well known properties under modular transformations, actually they will form a finite dimensional representation space of the modular group. Indeed we will show that these $\mathcal{W}$-algebras belong to a RCFT.

Note that in contrast to the known cases (e.g. minimal models, WZW models) elliptic functions of different moduli are involved.

Let us additionally introduce the alternating elliptic functions

$$\widetilde{\Theta}_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}}(-1)^n q^{(2kn+\lambda)^2/4k}\,, \qquad \lambda \in \mathbb{Z}/2\,,\ k \in \mathbb{N}/2\,. \tag{4.4b}$$

Then the modular properties are given by

$$\begin{aligned}
\Theta_{\lambda,k}(-\tfrac{1}{\tau}) &= \sqrt{\frac{-i\tau}{2k}}\sum_{\lambda'=0}^{2k-1} e^{i\pi\frac{\lambda\lambda'}{k}} \begin{cases} \Theta_{\lambda',k}(\tau) & \text{if } \lambda\in\mathbb{Z} \\ \widetilde{\Theta}_{\lambda',k}(\tau) & \text{if } \lambda\in\mathbb{Z}+\frac{1}{2}\,, \end{cases} \\
\widetilde{\Theta}_{\lambda,k}(-\tfrac{1}{\tau}) &= \sqrt{\frac{-i\tau}{2k}}\sum_{\lambda'=0}^{2k-1} e^{i\pi\frac{\lambda(\lambda'+\frac{1}{2})}{k}} \begin{cases} \Theta_{\lambda'+\frac{1}{2},k}(\tau) & \text{if } \lambda\in\mathbb{Z} \\ \widetilde{\Theta}_{\lambda'+\frac{1}{2},k}(\tau) & \text{if } \lambda\in\mathbb{Z}+\frac{1}{2}\,, \end{cases} \\
\Theta_{\lambda,k}(\tau+1) &= e^{i\pi\frac{\lambda^2}{2k}} \begin{cases} \Theta_{\lambda,k}(\tau) & \text{if } \lambda-k\in\mathbb{Z} \\ \widetilde{\Theta}_{\lambda,k}(\tau) & \text{if } \lambda-k\in\mathbb{Z}+\frac{1}{2}\,, \end{cases} \\
\widetilde{\Theta}_{\lambda,k}(\tau+1) &= e^{i\pi\frac{\lambda^2}{2k}} \begin{cases} \widetilde{\Theta}_{\lambda,k}(\tau) & \text{if } \lambda-k\in\mathbb{Z} \\ \Theta_{\lambda,k}(\tau) & \text{if } \lambda-k\in\mathbb{Z}+\frac{1}{2}\,, \end{cases} \\
\eta(-\tfrac{1}{\tau}) &= \sqrt{-i\tau}\,\eta(\tau)\,, \\
\eta(\tau+1) &= e^{\pi i/12}\,\eta(\tau)\,.
\end{aligned} \tag{4.5}$$

Thus, the functions $\Lambda_{\lambda,k}(\tau) = \Theta_{\lambda,k}(\tau)/\eta(\tau)$ are modular forms of weight zero to some $\Gamma(N) \subset \mathrm{PSL}(2,\mathbb{Z})$, e.g. N is the lowest common multiple of $4k$ and 24, if $\lambda - k \in \mathbb{Z}$. As was argued by Kiritsis [28] the Serre-Stark theorem assures the completeness of the set $\{\Lambda_{\lambda,k} \mid k \in \mathbb{N}/2,\ 0 \le \lambda \in \mathbb{Z}/2 \le k\}$ as a generating set for 1-singular modular forms as characters of RCFTs with $c_{\mathrm{eff}} \le 1$ are supposed to be (the *effective* central charge c_{eff} will be defined later).

We now have to identify the characters of the other representations, which are labelled by the pairs (λ,k), $0 \le \lambda \le k$ and $(\lambda, k+1)$, $0 \le \lambda \le k+1$. Obviously one can write

$$\begin{aligned}
\Lambda_{\lambda,k}(\tau) &= \frac{q^{(1-c)/24}}{\eta(\tau)}\sum_{n\in\mathbb{Z}} q^{h_{n+\frac{\lambda}{2k},n+\frac{\lambda}{2k}}}\,, \\
\Lambda_{\lambda,k+1}(\tau) &= \frac{q^{(1-c)/24}}{\eta(\tau)}\sum_{n\in\mathbb{Z}} q^{h_{n+\frac{\lambda}{2k+2},-n-\frac{\lambda}{2k+2}}}\,.
\end{aligned} \tag{4.6}$$

Here we have expressed the contributing lowest-weight values by formula (2.1) for numerating the degenerate weights by pairs of integers, but used rational non-integral numbers for the labeling except for the case $\lambda = 0$, which corresponds to the vacuum representation. Therefore, for $\lambda \neq 0$ there will be no null states in the corresponding Virasoro lowest-weight modules. Consequently, we identify the characters for the $\mathcal{W}$-algebra lowest-weight representations $|h_{\frac{\lambda}{2k},\frac{\lambda}{2k}}\rangle$ and $|h_{\frac{\lambda}{2k+2},-\frac{\lambda}{2k+2}}\rangle$ just to equal (see Table 4.1) $\chi^{\mathcal{W}}_{\lambda}(\tau) = \Lambda_{\lambda,k}(\tau)$, $1 \le \lambda < k$, $\chi^{\mathcal{W}}_{-\lambda}(\tau) = \Lambda_{\lambda,k+1}(\tau)$, $1 \le \lambda < k+1$ and $\chi^{\mathcal{W}}_{k}(\tau) = \frac{1}{2}\Lambda_{k,k}(\tau)$, $\chi^{\mathcal{W}}_{-k-1}(\tau) = \frac{1}{2}\Lambda_{k+1,k+1}(\tau)$, where the factor $1/2$ in the last two characters removes an unphysical double-counting of all states. Of course we still need one character which can be identified to be $\chi^{\mathcal{W}}_{k+1}(\tau) = \frac{1}{2}(\Lambda_{0,k}(\tau) + \Lambda_{0,k+1}(\tau))$. This can be seen as follows: It is clear that there must be one other linear combination of $\Lambda_{0,k}$ and $\Lambda_{0,k+1}$ except the one for the vacuum character. Other Λ-functions cannot be combined, because their q-powers will never differ by integers. Since the character is supposed not to involve degenerate multiplicities of the corresponding representation, its q-expansion has to start with the leading coefficient 1. This restricts the possibilities to the ansatz $\chi^{\mathcal{W}}_{k+1} = \mu\Lambda_{0,k} \pm (1-\mu)\Lambda_{0,k+1}$. The requirement of integer coefficients further restricts μ to the set $\{0, \frac{1}{2}, 1\}$. But the solutions $\mu = 0$ or $\mu = 1$ again yield an unphysical double-counting of all states with weights greater than zero, $\mu = \frac{1}{2}$ is left as the only possibility. This means physically that the vacuum representation lives on the invariant subspaces left after dividing out the modules generated by the null states (see [16]), while the other representation (which has the lowest-weight $h_{\text{min}} = -k < 0 = h_{\text{vac}}$ demonstrating the non-unitarity of the theory) lives on the direct sum of the whole Verma modules together with the modules generated by the singular vectors.

If one now expresses the modular properties in the basis of the characters found so far, one finds a S-matrix which is neither symmetric nor unitary. This comes from a hidden degeneracy of the representations on the lowest-weight states $|h_{\frac{1}{2},\pm\frac{1}{2}}\rangle$, as can be seen from the modular invariant partition function

$$\begin{aligned} Z(\tau,\bar{\tau}) &= \frac{1}{2}\sum_{\lambda=0}^{2k-1}|\Lambda_{\lambda,k}|^2 + \frac{1}{2}\sum_{\lambda'=0}^{2k}|\Lambda_{\lambda',k+1}|^2 \\ &= \sum_{\lambda=0}^{k-1}\left|\chi^{\mathcal{W}}_{\lambda}\right|^2 + \sum_{\lambda'=0}^{k}\left|\chi^{\mathcal{W}}_{-\lambda'}\right|^2 + 2\left|\chi^{\mathcal{W}}_{k}\right|^2 + 2\left|\chi^{\mathcal{W}}_{-k-1}\right|^2 \end{aligned} \tag{4.7}$$

which directly shows up the multiplicities. The reason for these degeneracies lies in the extended Cartan subalgebra. Indeed, in the explicitly computed examples [5, 13] of $\mathcal{W}$-algebras we found that exactly for these representations the W_0-eigenvalue is non-zero. Actually, since the selfcoupling of the W-field with itself is zero in all cases, only the value w^2 given by $W_0W_0|h,w\rangle = w^2|h,w\rangle$ can be computed by expressing $\mathcal{N}(W,W)$ in terms of (normal ordered products of) the Virasoro field yielding w^2 as a function in h and c if its zero mode is applied to the lowest-weight state. Of course, $w^2 \neq 0$ will give two representations $|h, \pm\sqrt{w^2}\rangle$.

In order to calculate the fusion algebra, one now either has to use the Verlinde formula [38, 34] in the modified form

$$N^k_{ij} = \frac{n_i n_j}{n_k}\sum_m \frac{S_{im}S_{jm}(S^{-1})_{mk}}{S_{0m}} \tag{4.8}$$

for a generalized diagonal modular invariant partition function

$$Z(\tau,\bar{\tau}) = \sum_m n_m\,|\chi_m|^2\,, \quad n_m \in \mathbb{Z}_+\,, \tag{4.9}$$

or one has to extend the S-matrix and the number of characters removing the degeneracies (see [33]). This second method means in our case, where we have two representations with multiplicities 2, that there is a doubling of the characters $\chi^{\mathcal{W}}_{k,+} = \chi^{\mathcal{W}}_{k,-} \equiv \chi^{\mathcal{W}}_{k}$ and $\chi^{\mathcal{W}}_{-k-1,+} = \chi^{\mathcal{W}}_{-k-1,-} \equiv \chi^{\mathcal{W}}_{-k-1}$. Thus, one has to extend the S-matrix by two rows and columns. The requirements symmetry, unitarity and S being at most of order 4, i.e.

$$S = S^t\,, \quad SS^+ = \mathbb{1}\,, \quad S^2 = C\,,$$

where C is the conjugation matrix, already fix S up to three free constants which can be uniquely determined from the N_{ij}^k, which should be non-negative integers. This leads to the S- and T-matrix given in Appendix A. The fusion algebra and a calculation of the three free parameters of the S-matrix are given in Appendix B. Here and in the sequel we use the following notation for the characters and representations

Table 4.1. Representations and their characters for the bosonic $\mathcal{W}(2,3k)$-algebras

h	w^2	$\chi_\lambda^{\mathcal{W}}$	Remark
$h_{1,1}$	0	$\chi_0^{\mathcal{W}} = \frac{1}{2}(\Lambda_{0,k} - \Lambda_{0,k+1})$	vacuum rep.
$h_{\frac{1}{2k},\frac{1}{2k}}$	0	$\chi_1^{\mathcal{W}} = \Lambda_{1,k}$	
$\vdots$	$\vdots$	$\vdots$	
$h_{\frac{k-1}{2k},\frac{k-1}{2k}}$	0	$\chi_{k-1}^{\mathcal{W}} = \Lambda_{k-1,k}$	
$h_{\frac{1}{2},\frac{1}{2}}$	$\neq 0$	$\chi_{k,+}^{\mathcal{W}} = \chi_{k,-}^{\mathcal{W}} = \frac{1}{2}\Lambda_{k,k}$	degenerate rep.
$h_{0,0}$	0	$\chi_{k+1}^{\mathcal{W}} = \frac{1}{2}(\Lambda_{0,k} + \Lambda_{0,k+1})$	rep. to $h_{\min}$
$h_{\frac{1}{2k+2},-\frac{1}{2k+2}}$	0	$\chi_{-1}^{\mathcal{W}} = \Lambda_{1,k+1}$	
$\vdots$	$\vdots$	$\vdots$	
$h_{\frac{k}{2k+2},-\frac{k}{2k+2}}$	0	$\chi_{-k}^{\mathcal{W}} = \Lambda_{k,k+1}$	
$h_{\frac{1}{2},-\frac{1}{2}}$	$\neq 0$	$\chi_{-k-1,+}^{\mathcal{W}} = \chi_{-k-1,-}^{\mathcal{W}} = \frac{1}{2}\Lambda_{k+1,k+1}$	degenerate rep.

which completely explains the representations of the $\mathcal{W}(2,3k)$-algebras for $c = 1 - 24k$, $k \in \mathbb{N}$ found in [13]. Note the change in the labelling of the representations due to the multiplicities. These characters diagonalize the modular invariant partition functions, while it is maximal non-diagonal expressed in terms of Virasoro characters. Thus these $\mathcal{W}(2,3k)$-algebras are very good examples that extending the symmetry algebra does make the modular invariant partition function more diagonal and can yield new RCFTs not related to minimal models or any coset construction. Moreover, these non-diagonal partition functions are not contained in the ADE-classification of Cappelli, Itzykson and Zuber [8] and probably not related to any other non-diagonal invariant coming from affine Lie algebras.

From (4.5) we learn that for $k, \lambda \in \mathbb{Z}$ the functions $\Lambda_{\lambda+\frac{1}{2},k}$ and $\Lambda_{\lambda+\frac{1}{2},k+1}$ built a space invariant under T^2 and S^2. These functions are the characters of the irreducible lowest-weight representations of the so-called twisted bosonic $\mathcal{W}(2,3k)$-algebra which is obtained by using half-integer fourier-modes, hence introducing antiperiodic boundary conditions. In the twisted sector of the bosonic $\mathcal{W}(2,3k)$-

algebras no linear combinations of these functions are necessary nor possible. Consequently the characters of the lowest-weight representations $|h_{\frac{2\lambda+1}{4k},\frac{2\lambda+1}{4k}}\rangle$ are simply $\chi^{\mathscr{W}}_{\lambda+\frac{1}{2}}(\tau) = \Lambda_{\lambda+\frac{1}{2},k}(\tau)$, $0 \leq \lambda < k$, and the ones of the lowest-weight representations $|h_{\frac{2\lambda+1}{4k+4},-\frac{2\lambda+1}{4k+4}}\rangle$ read $\chi^{\mathscr{W}}_{-\lambda-\frac{1}{2}}(\tau) = \Lambda_{\lambda+\frac{1}{2},k+1}(\tau)$, $0 \leq \lambda < k+1$.

Indeed, for some $\mathscr{W}$-algebras as examples these representations could be found by explicit calculations in [13].

Let us emphasize one point here. We show these theories to be RCFTs by constructing the S-Matrix and calculating the fusion rules. If one introduces the effective value of the central charge

$$c_{\text{eff}} = c - 24h_{\text{min}}\,, \tag{4.10}$$

one can compare non-unitary theories with unitary ones. Actually, the central charge c is nothing else than the mean expectation value of the Casimir effect contribution of the free energy due to the boundary conditions. As is well known, from the modular invariant partition function

$$Z(\tau,\bar{\tau}=\tau) = \operatorname{Tr} e^{2\pi i\tau(L_0-\frac{c}{24})} e^{2\pi i\bar{\tau}(\bar{L}_0-\frac{c}{24})}\,, \tag{4.11}$$

one easily derives the following expression for the central charge in dependency from the energy $L_0 + \bar{L}_0$ (the latter being defined up to an arbitrary additive constant)

$$c = 12\frac{\tau\langle L_0+\bar{L}_0\rangle_\tau - \frac{1}{\tau}\langle L_0+\bar{L}_0\rangle_{-\frac{1}{\tau}}}{\tau - \frac{1}{\tau}} \tag{4.12}$$

which simplifies for the fixed point of the S-transformation, $\tau = i$. Now we obtain with $\Delta_n = h_n + \bar{h}_n$

$$c = 12\frac{\sum\limits_n \Delta_n \exp(-2\pi\Delta_n)}{\sum\limits_n \exp(-2\pi\Delta_n)}\,, \tag{4.13}$$

where the sum extends over the weights of all states, both the conformal dimensions of the primary fields and the weights of all their descendents.

This formula shows that positive exponents will appear in non-unitary theories, since the state of lowest energy is not identical with the vacuum, thus violating the conservation of probability. Therefore it does make sense to redefine the energy by subtracting the energy of the ground state, $(L_0 + \bar{L}_0)_{\text{eff}} = L_0 + \bar{L}_0 - 12\Delta_{\text{min}}$, which in return forces to redefine the central charge

$$c_{\text{eff}} = 12\frac{\sum\limits_n(\Delta_n - \Delta_{\text{min}})\exp(-2\pi(\Delta_n - \Delta_{\text{min}}))}{\sum\limits_n \exp(-2\pi(\Delta_n - \Delta_{\text{min}}))}\,, \tag{4.14}$$

such that the characters will keep unchanged. This effective central charge measures the mean expectation value of the Casimir effect contribution of the free energy relative to the state of lowest energy. This is a physical observable which does conserve probability and can be used for both unitary and non-unitary theories as well. Consequentely $c_{\text{eff}} > 0$. Usually one considers symmetric theories with

$\bar{h} = h$ corresponding to diagonal modular invariant partition functions. In this case $\Delta_{\text{min}} = 2h_{\text{min}}$.

In particular our theories have $c_{\text{eff}} = 1$ and thus complete the classification of all rational theories with $c = 1$, given in [28, 20, 9], including the non-unitary case, since these theories exactly represent the only possible additional case found in [28] but rejected there due to the unnecessary restrictive assumption, the state of lowest energy would always be the vacuum. Note that most of the theorems used in the references given above are valid for the non-unitary case as well, one only has to distinguish carefully between the vacuum representation and the minimal representation. Quantum dimensions for example have to be defined with respect to the conformal $\mathscr{W}$-family to the Virasoro lowest weight h_{min} rather than to the identity family, if the theory is non-unitary. The next chapter is devoted to the proof of the statement concerning the $c_{\text{eff}} = 1$ models.

Now we will briefly discuss the case of the fermionic $\mathscr{W}(2,3k)$-algebras. Here $c = 1 - 24\frac{k}{2}$, again $k \in \mathbb{N}$. The vacuum representation belongs to the Neveu-Schwarz sector. For this sector we find that all the characters can be expressed in terms of the functions $\Lambda_{\lambda,\frac{k}{2}}(\tau)$ and $\Lambda_{\lambda,\frac{k}{2}+1}(\tau)$ with $\lambda \in \mathbb{Z}$. Conversely, for the Ramond sector we have $\lambda \in \mathbb{Z} + \frac{1}{2}$. Using the modular properties given in (4.5), it is easy to see that the Neveu-Schwarz sector is invariant under the transformations S and T^2, while the Ramond sector is under T and ST^2S, and that the transformation TST intertwines the two sectors. In particular, we obtain for the NS-sector

$$\Lambda_{\lambda,\frac{k}{2}}\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{k}} \sum_{\lambda'=0}^{k-1} e^{2i\pi\frac{\lambda\lambda'}{k}} \Lambda_{\lambda',\frac{k}{2}}(\tau)\,. \tag{4.15a}$$

Therefore we just can take the S-matrix for the bosonic case, as given in Appendix A, change every occurrence of k to $\frac{k}{2}$ and remove both the $(k,+)^{\text{th}}$ and $(k,-)^{\text{th}}$ row and column as well as the $(-k-1,+)^{\text{th}}$ and $(-k-1,-)^{\text{th}}$ ones. Note that degenerate representations do not appear in the NS-sector.

For the R-sector the situation is not as simple, because the appropriate transformation matrix is $\widetilde{S} = ST^2S$. Using again (4.5) and eliminating one summation by a Gauss-sum results in

$$\begin{aligned}\Lambda_{\lambda+\frac{1}{2},\frac{k}{2}}&\left(-\frac{\tau}{1-2\tau}\right) \\ &= \frac{1}{\sqrt{k}} \sum_{\lambda'=0}^{k-1} e^{-i\pi\frac{(\lambda+\lambda'+1)^2}{2k}} \frac{1+(-i)^{2(\lambda+\lambda'+1)+k}}{1+(-i)} \Lambda_{\lambda'+\frac{1}{2},\frac{k}{2}}(\tau)\,.\end{aligned} \tag{4.15b}$$

Now one might go through the same procedure as for the bosonic case with this matrix $\widetilde{S}$ and remove the degeneracies of two of the representations, but we will not go into further detail here, since it is not clear whether ST^2S can be used instead of S for calculating the fusion rules via the Verlinde formula, nor what should replace the identity representation.

Here we also briefly mention the $\mathscr{W}(2,8k)$-algebra series, which exists for all $k \in \mathbb{N}/4$. As has been explained in chapter 2 these algebras just represent the odd sector of the $\mathscr{W}(2,3k)$-algebras (note that there is no algebra built from the odd and even sector together for $k \in \mathbb{N}+\frac{1}{4}$ due to violation of locality). The vacuum character for the odd sector reads

$$\chi^{\mathscr{W}}_{0,\text{odd}}(\tau) = \frac{q^{(1-c)/24}}{\eta(\tau)} \sum_{\substack{r\in\mathbb{N}\\ r\equiv 1 \bmod 2}} \left(q^{h_{r,r}} - q^{h_{r,-r}}\right)$$
$$= \frac{1}{2\eta(\tau)} \left(\Theta_{4k,4k}(\tau) - \Theta_{4k+4,4k+4}(\tau)\right) . \tag{4.16}$$

The modular transformations involve the other functions $\Theta_{\lambda+4k+4\varepsilon,4k+4\varepsilon}$ where $\varepsilon = 0$ or 1. For $0 \leq \lambda \leq 4k + 4\varepsilon$ this is a complete set of linear independent Theta-functions. Rewriting the summations as sums over odd integers only, the label has to be multiplied by two, i.e. the lowest-weights are parametrized as $h_{\frac{2\lambda}{2(4k+4\varepsilon)},(-1)^{\varepsilon}\frac{2\lambda}{2(4k+4\varepsilon)}}$, since

$$\frac{1}{\eta(\tau)}\Theta_{\lambda+4k+4\varepsilon,4k+4\varepsilon}(\tau) = \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\frac{[2(4k+4\varepsilon)n+(\lambda+4k+4\varepsilon)]^2}{4(4k+4\varepsilon)}}$$
$$= \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\left[(2n+1)+\frac{2\lambda}{2(4k+4\varepsilon)}\right]^2 (k+\varepsilon)}$$
$$= \frac{q^{(1-c)/24}}{\eta(\tau)} \sum_{r\in(2\mathbb{Z}+1)+\frac{2\lambda}{2(4k+4\varepsilon)}} q^{(r^2-1)k+r^2\varepsilon} . \tag{4.17}$$

Some of the characters are linear combinations of elliptic functions to different moduli like in the bosonic case discussed in detail above. In particular we have

Table 4.2. Representations and their characters for the bosonic $\mathscr{W}(2,8k)$-algebras

h	w	$\chi^{\mathscr{W}}_{\lambda,\text{odd}}$	Remark
$h_{1,1}$	0	$\chi^{\mathscr{W}}_{0} = \frac{1}{2}(\Lambda_{4k,4k} - \Lambda_{4k,4k+4})$	vacuum rep.
$h_{\frac{1}{4k},\frac{1}{4k}}$	$\neq 0$	$\chi^{\mathscr{W}}_{1} = \Lambda_{1+4k,4k}$	
$\vdots$	$\vdots$	$\vdots$	
$h_{\frac{4k-1}{4k},\frac{4k-1}{4k}}$	$\neq 0$	$\chi^{\mathscr{W}}_{4k-1} = \Lambda_{8k-1,4k} = \Lambda_{-1,4k}$	
$h_{2,2}$	$\neq 0$	$\chi^{\mathscr{W}}_{4k} = \frac{1}{2}(\Lambda_{0,4k} - \Lambda_{0,4k+4})$	rep. on $\lvert W^{(2)}\rangle$
$h_{0,0}$	$\neq 0$	$\chi^{\mathscr{W}}_{4k+1} = \frac{1}{2}(\Lambda_{0,4k} + \Lambda_{0,4k+4})$	rep. to $h_{\min}$
$h_{1,1}$	$\neq 0$	$\chi^{\mathscr{W}}_{4k+2} = \frac{1}{2}(\Lambda_{4k,4k} + \Lambda_{4k,4k+4})$	$\lvert h = 0, w \neq 0\rangle$ rep.
$h_{\frac{1}{4k+4},-\frac{1}{4k+4}}$	$\neq 0$	$\chi^{\mathscr{W}}_{-1} = \Lambda_{1+4k+4,4k+4}$	
$\vdots$	$\vdots$	$\vdots$	
$h_{\frac{4k+3}{4k+4},-\frac{4k+3}{4k+4}}$	$\neq 0$	$\chi^{\mathscr{W}}_{-4k-3} = \Lambda_{8k+7,4k+4} = \Lambda_{-1,4k+4}$	

Note that there is no need for degenerate representations. Indeed, all representations have multiplicity one. The reason is that the W_0 eigenvalue is uniquely expressable as a function in h, c, and the non-vanishing self-coupling C_{WW}^{W}. Again one considers the zero-modes of $\mathcal{N}(W, \partial^n W)$, $n = 0, 2$, applied to the vacuum and solves the resulting quadratic equations for w. These patterns also explain the existence of two representations with $h = 0$, only one of them being the vacuum representation, in [13]. Finally we note that the general addition law of the Jacobi-Riemann Θ-functions to moduli containing a square factor,

$$\sum_{\nu=0}^{n-1} \Lambda_{n\lambda+\nu,n^2k}(\tau) = \Lambda_{\lambda,k}(\tau)\,, \quad k \in \mathbb{Z}_+/2\,, \tag{4.18}$$

shows that the whole $\mathcal{W}(2, 3k)$-algebra can be regarded as built from their odd sector algebra $\mathcal{W}(2, 8k)$. In fact, the representation on a lowest weight $|h(\lambda)\rangle$ of the whole algebra is obtained by applying the odd sector to both, the lowest-weight state and the state $(\Phi_{2,2})_0|h(\lambda)\rangle$ since for the characters we have the relation $\Lambda_{2\lambda,4k}(\tau) + \Lambda_{2\lambda+1,4k}(\tau) = \Lambda_{\lambda,k}(\tau)$.

5. Classification of $c_{\text{eff}} = 1$ Theories

Now we come to the completion of the classification of all RCFTs with $c = 1$ by considering non-unitary models. As has been explained in the last chapter, one has to use $c_{\text{eff}} = c - 24h_{\text{min}}$ instead of the central charge for non-unitary theories. In the works of [28, 9, 20] all unitary models with $c = 1$ have been identified. But the proofs of the statements of these works are not affected by the assumption of unitarity as long as one keeps in mind that the vacuum representation is not necessarily the one with the minimal lowest weight. Since there are strong indications that modular forms to non-congruence subgroups of the modular group will have infinite denominators in their Fourier expansions, we assume that non-congruence subgroups do not lead to RCFTs. Thus, there is only one additional candidate for a $c = 1$ model (see [28]). It has the partition function

$$Z = \tfrac{1}{2}\left(Z(R_1) + Z(R_2)\right)\,, \tag{5.1}$$

where $Z(R)$ denotes the partition function of an U(1)-theory of mappings of the unit sphere $S^1 \to S^1$ with radius R, given by

$$Z(R) = \frac{1}{\eta(\tau)\eta(\bar\tau)} \sum_{m,n\in\mathbb{Z}} \left(q^{\frac{1}{8R^2}(n+2mR^2)^2} \bar q^{\frac{1}{8R^2}(n-2mR^2)^2} \right)\,. \tag{5.2}$$

If $2R^2 \in \mathbb{N}$, then this partition function can be expressed in the elliptic functions given by Eqs. (4.4), namely

$$Z(R) = \frac{1}{\eta(\tau)\eta(\bar\tau)} \sum_{1\le n\le 4R^2} \left|\Theta_{n,2R^2}\right|^2\,, \tag{5.3}$$

where the Theta-functions satisfy $\Theta_{n,2R^2} = \Theta_{-n,2R^2} = \Theta_{n+4R^2,2R^2}$ and $\Theta_{2R^2,2R^2}$ has only even integer coefficients, considered as power series in q. If now $2R^2 = \dfrac{P}{Q} \in \mathbb{Q}$

with P, Q coprime, then we can write

$$Z(R) = \frac{1}{\eta(\tau)\eta(\bar{\tau})} \sum_{n \bmod 2PQ} \Theta_{n,PQ}(\tau)\Theta_{n',PQ}(\bar{\tau}), \tag{5.4}$$

with n' given by $n' = QN + PM \bmod 2PQ$, if $n = QN - PM \bmod 2PQ$ for some integers N, M. Note that the integer case $Q = 1$ is correctly obtained from the general one.

It is now necessary for obtaining a RCFT from the partition function (5.1) to have $2R_i^2 = \frac{P_i}{Q_i}$, $i = 1, 2$. This yields the following two possibilities:

$$\begin{aligned} Z = (\eta\bar{\eta})^{-1} \Bigg(& \sum_{n=1}^{P_1Q_1-1} \Theta_{n,P_1Q_1}\bar{\Theta}_{n',P_1Q_1} + \sum_{m=1}^{P_2Q_2-1} \Theta_{m,P_2Q_2}\bar{\Theta}_{m',P_2Q_2} \\ & + \left|\frac{\Theta_{0,P_1Q_1} + \Theta_{0,P_2Q_2}}{2}\right|^2 + \left|\frac{\Theta_{0,P_1Q_1} - \Theta_{0,P_2Q_2}}{2}\right|^2 \\ & + \begin{cases} 2\left|\frac{\Theta_{P_1Q_1,P_1Q_1}}{2}\right|^2 + 2\left|\frac{\Theta_{P_2Q_2,P_2Q_2}}{2}\right|^2 \Bigg) \\ \left|\frac{\Theta_{P_1Q_1,P_1Q_1} + \Theta_{P_2Q_2,P_2Q_2}}{2}\right|^2 + \left|\frac{\Theta_{P_1Q_1,P_1Q_1} - \Theta_{P_2Q_2,P_2Q_2}}{2}\right|^2 \Bigg) . \end{cases} \end{aligned} \tag{5.5}$$

It is clear that only both linear combinations with a minus sign could be Virasoro vacuum characters, since the latter must have the form $(\eta\bar{\eta})^{-1}q^{h-c/24}(1 - q + \cdots)\bar{q}^{\bar{h}-c/24}(1 - \bar{q} + \cdots)$. Thus we can distinguish two cases:

$$\begin{aligned} &\text{(i)} & \frac{\Theta_{0,P_1Q_1} - \Theta_{0,P_2Q_2}}{2} &= q^{P_1Q_1} - q^{P_2Q_2} + q^{4P_1Q_1} + \cdots \\ &\text{(ii)} & \frac{\Theta_{P_1Q_1,P_1Q_1} - \Theta_{P_2Q_2,P_2Q_2}}{2} &= q^{\frac{P_1Q_1}{4}} - q^{\frac{P_2Q_2}{4}} + q^{\frac{9P_1Q_1}{4}} + \cdots . \end{aligned} \tag{5.6}$$

In case (i) we obtain the condition $P_2Q_2 = P_1Q_1 + 1$ and hence $c = 1 - 24P_1Q_1$, in case (ii) one has to satisfy $P_2Q_2 = P_1Q_1 + 4$ and hence $c = 1 - 6P_1Q_1$. These are exactly our series of c-values for the bosonic $\mathcal{W}(2, 3k)$-algebras with $k = P_1Q_1$ and for the odd-sector algebras $\mathcal{W}(2, 8k)$ with $k = P_1Q_1/4$. In fact, we have seen that under special assumptions on the radii R_i, $i = 1, 2$, Virasoro characters can be found in the partition function (5.1). The modular invariance of the latter and their well known decomposition shows that the theory is rational. Even more the extended symmetry algebra for this theory is known and can be identified with a certain $\mathcal{W}$-algebra.

Let us remark that there can be a lot of decompositions of the modulus in (5.4) into two coprime numbers P, Q. These different decompositions yield the non trivial automorphisms of the fusion numbers or equivalently the set of theories, which have related partition functions (5.5) with n' and m' given as described above. For details see Appendix B.

Finally we conjecture that the set of these theories lies dense in the set of all theories with partition function (5.1) to arbitrary radii $R_1, R_2 \in \mathbb{R}_+$. This conjecture is equivalent to the following problem: For every positive real numbers R_1, R_2 and every

$\varepsilon > 0$ find pairs of coprime integers P_1, Q_1 and P_2, Q_2 such that $\left|2R_i^2 - \frac{P_i}{Q_i}\right| < \varepsilon$, $i = 1, 2$, and $P_2Q_2 - P_1Q_1 = 1$ holds.

6. Generalization to the Supersymmetric Case

The theme of this chapter is a brief sketch of the rather straightforward generalization of the new RCFTs to the supersymmetric case. To fix the notation we set $c = \frac{3}{2} - 24k = \frac{3}{2}(1 - 16k) = \frac{3}{2}\hat{c}$ and again consider the case $k \in \mathbb{N}/4$, which will lead to theories with $c_{\text{eff}} = \frac{3}{2}$. With $\alpha_\pm = \sqrt{k} \pm \sqrt{k + \frac{1}{2}}$ we have the lowest-weight levels

$$h_{r,s}(c) = \tfrac{1}{4}(r\alpha_+ + s\alpha_-)^2 + \tfrac{1}{16}(\hat{c} - 1) + \tfrac{1}{32}(1 - (-1)^{r-s}), \qquad (6.1)$$

where for the NS-sector $r - s \equiv 0 \bmod 2$ and $\equiv 1 \bmod 2$ for the R-sector respective. As for the ordinary case, we first list the up to now known results, which have been obtained by explicit calculations [25, 31, 24, 19, 4, 14]. Here we used the common notation where the smaller dimension of the super-partners are denoted only, namely $\mathcal{SW}(\frac{3}{2}, \delta) = \mathcal{W}(2, \frac{3}{2}, \delta, \delta + \frac{1}{2})$. This is a supersymmetric conformal algebra extended by one additional covariant supersymmetric field $\Phi(z, \theta) = \phi(z) + \theta\psi(z)$ with dimension $(\delta, \delta + \frac{1}{2})$, where θ denotes a Grassman variable. In analogy to (2.9) we denote the super conformal blocks by $\Phi^{(n)}$. The $\mathcal{SW}(\frac{3}{2}, 3k)$-algebras are then formed by the field $\Phi^{(2)}$, the $\mathcal{SW}(\frac{3}{2}, 8k)$-algebras by $\Phi^{(3)}$, where we use similar arguments as those of the second chapter.

Table 6.1. Two sets of $\mathcal{SW}$-algebras to rational c-values not contained in the supersymmetric minimal series

The series $\mathcal{SW}(\frac{3}{2}, \delta)$ with $c = \frac{3}{2} - 8\delta$:		
$\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$	$(c = -\frac{21}{2})$	$((C^{\Phi}_{\Phi\Phi})^2 = 0)$
$\mathcal{SW}(\frac{3}{2}, 3)$	$c = -\frac{45}{2}$	$(C^{\Phi}_{\Phi\Phi})^2 = 0$
$\mathcal{SW}(\frac{3}{2}, \frac{9}{2})$	$c = -\frac{69}{2}$	$(C^{\Phi}_{\Phi\Phi})^2 = 0$
$\mathcal{SW}(\frac{3}{2}, 6)$	$c = -\frac{93}{2}$	$(C^{\Phi}_{\Phi\Phi})^2 = 0$
The series $\mathcal{SW}(\frac{3}{2}, \delta)$ with $c = \frac{3}{2} - 3\delta$:		
$\mathcal{SW}(\frac{3}{2}, 2)$	$(c = -\frac{9}{2})$	$(C^{\Phi}_{\Phi\Phi})^2 = \frac{242}{13}$
$\mathcal{SW}(\frac{3}{2}, 4)$	$c = -\frac{21}{2}$	$(C^{\Phi}_{\Phi\Phi})^2 = -\frac{508369}{2499}$
$\mathcal{SW}(\frac{3}{2}, 6)$	$c = -\frac{34}{2}$	$(C^{\Phi}_{\Phi\Phi})^2 = \frac{6309688448}{3137409}$

The c-value of some of the algebras has been put in brackets: The $\mathcal{SW}(\frac{3}{2}, \frac{3}{2})$-algebra does exist generically and for independently chosen self-coupling. It is the supersymmetric analogon to the $\mathcal{W}(2, 2)$, and thus nothing else than a direct sum of two supersymmetric Virasoro algebras. But only for vanishing self-coupling it is related to a supersymmetric free field construction due to the fusion rules of the latter.

The $\mathscr{SW}(\frac{3}{2}, 2)$-algebra exists for generic central charge. This seems natural since the classical counterpart of this algebra is the symmetry algebra of the Super-Toda theory corresponding to the Super-Lie-algebra $\mathfrak{osp}(3 \mid 2)$. These, and $\mathscr{SW}(\frac{3}{2}, \frac{1}{2})$ (Super-Kac-Moody algebra) and $\mathscr{SW}(\frac{3}{2}, 1)$ ($N = 2$ Super-Virasoro algebra) are the only known super-$\mathscr{W}$-algebras with two generators, which exist for generically chosen central charge.

Motivated by the analogy of these series to the conformal case we consider again the "diagonal" fields with weights $h_{r,r} = (r^2 - 1)k$ and $h_{r,-r} = (r^2 - 1)k + \frac{1}{2}r^2$ in the NS-sector. In the R-sector these weights have to be shifted, $h = h_{r,\pm r} + \frac{1}{16}$. Checking the locality conditions for chiral theories yields exactly the same pattern as in the non-supersymmetric case. $r = 2, k \in \mathbb{N}/2$ gives the analogon of the $\mathscr{W}(2, 3k)$-algebras, the $\mathscr{SW}(\frac{3}{2}, 3k)$-algebras, and $r = 2, k \in \mathbb{N}/4$ the analogon to the so-called odd sector subalgebras, $\mathscr{SW}(\frac{3}{2}, 8k)$.

Let us first consider the $\mathscr{SW}(\frac{3}{2}, 3k)$-theories. Again we start from the vacuum representation and get all other irreducible lowest-weight representations by modular transformations. The vacuum character is given by

$$\begin{aligned} \chi_0^{\mathscr{SW}}(\tau) &= \prod_{n\in\mathbb{N}} \frac{1 + q^{n-\frac{1}{2}}}{1 - q^n} q^{\frac{\hat{c}}{16}} \sum_{r\in\mathbb{Z}} \left(q^{h_{r,r}} - q^{h_{r,-r}}\right) \\ &= \frac{\eta(\frac{\tau+1}{2})}{\eta^2(\tau)} e^{-\frac{\pi i}{24}} \left(\Theta_{0,k}(\tau) - \Theta_{0,k+\frac{1}{2}}(\tau)\right) . \end{aligned} \tag{6.2}$$

The NS-sector turns out to be again invariant under S and T^2, using the $\Theta_{\lambda,k+\frac{\varepsilon}{2}}$-functions with $\lambda \in \mathbb{Z}$. The R-sector is a little bit more complicated. Here the combinatorial prefactor making the character a modular form of weight zero is $\prod \frac{1 + q^n}{1 - q^n} = \eta(2\tau)/\eta^2(\tau)$. From (4.5) we learn, that invariance under T enforces $\lambda - k + \frac{\varepsilon}{2} \in \mathbb{Z}$. Thus, the index λ is integer or half-integer, if the modulus k is integer or half-integer respective. Then the R-sector is invariant under T and ST^2S.

Note that we only consider the characters without fermion number counting $(-)^F$ since these are enough to classify the possible representations. Of course, in the modular invariant partition function the characters of the $\widetilde{\text{NS}}$-sector, given by $\text{tr}_{|h)}(-)^F q^{L_0 - c/24}$, have to be added to get invariance under the full modular group. But the latter are easy to obtain from the characters of the ordinary NS-sector without fermion number by applying the T-transformation to them, $\chi_{\lambda,\widetilde{\text{NS}}}^{\mathscr{SW}}(\tau) = \chi_{\lambda,\text{NS}}^{\mathscr{SW}}(\tau + 1)$. They involve the $\widetilde{\Theta}$-functions (4.4b) instead of the ordinary Θ-functions (4.4a) and get the prefactor $\prod \frac{1 - q^{n-\frac{1}{2}}}{1 - q^n} = \exp\left(-\frac{2}{16}\pi i\right)\eta\left(\frac{1}{2}\tau\right)\eta^{-2}(\tau)$. Thus, they are essentially given by the functions

$$\Lambda_{\lambda,k}^{\widetilde{\text{NS}}} = \frac{\eta(\frac{\tau}{2})}{\eta^2(\tau)} e^{-\frac{2\pi i}{16}} \widetilde{\Theta}_{\lambda,k}(\tau) , \tag{6.3}$$

but not considered further in the following. Since these two sectors are interchanged by T, the modular invariant partition function is forced to take the form

$$Z = a(Z^{\mathrm{NS}} + Z^{\widetilde{\mathrm{NS}}} + Z^{\mathrm{R}}) + bZ^{\widetilde{\mathrm{R}}}\,, \tag{6.4}$$

where Z^{A} denotes the diagonal partition function of the characters of the A-sector, i.e. $Z^{\mathrm{A}} = \sum_{\lambda} \left|\chi^{\mathcal{SW}}_{\lambda,\mathrm{A}}(\tau)\right|^2$. Here a, b are free constants up to normalization, and $Z^{\widetilde{\mathrm{R}}}$ is nothing else than $\mathrm{tr}(-)^F$. This particular ansatz cancels out the fermionic contributions in the NS-sector leaving us with the bosonic characters $\mathrm{tr}_{|h\rangle}(1+(-)^F)q^{L_0-c/24}$. Thus, partition function and characters are divided in the same sectors (of (anti-) periodic boundary conditions) as in the case of the ADE-classification of the minimal theories of the supersymmetric Virasoro-algebra by Cappelli [7, 8]. Note also that in the R-sector one has a non-trivial algebra of the zero modes of the fields, which can involve 2^n-dimensional additional representations of the Clifford algebra as for example the representation $(-)^F = \pm 1$ for the fermion number F.

Without loss of generality let us assume $k \in \mathbb{N}$ (this case we call the bosonic one in analogy to the $\mathcal{W}(2,3k)$-algebras). In this case the characters are generically – up to the appropriate linear combinations of the theta-functions to different moduli, if the q-powers differ by integers – given by the functions

$$\begin{aligned}
\Lambda^{\mathrm{NS}}_{\lambda,k+\frac{\varepsilon}{2}}(\tau) &= \frac{\eta(\frac{\tau+1}{2})}{\eta^2(\tau)} e^{-\frac{\pi i}{24}} \Theta_{\lambda,k+\frac{\varepsilon}{2}}(\tau) \quad \text{NS-sector}\,,\\
\Lambda^{\mathrm{R}}_{\lambda+\frac{\varepsilon}{2},k+\frac{\varepsilon}{2}}(\tau) &= \frac{\eta(2\tau)}{\eta^2(\tau)} \Theta_{\lambda+\frac{\varepsilon}{2},k+\frac{\varepsilon}{2}}(\tau) \quad \text{R-sector}\,,
\end{aligned} \tag{6.5}$$

where $\lambda \in \mathbb{Z}$ and again $\varepsilon = 0$ or 1. Note that TST intertwines both sectors. All weights in the R-sector have to be shifted by $\frac{1}{16}$. Last but not least one again has to deal with representations with multiplicities greater one, too, if the eigenvalue of the second element of the Cartan subalgebra does not vanish. The Table 6.2 sums up our results.

In the case of the fermionic $\mathcal{SW}$-algebras, i.e. $k \in \mathbb{Z}_+ + \frac{1}{2}$, the rôle of k and $k + \frac{1}{2}$ interchanges since k is now half-integer.

As in the ordinary case there exist the so-called odd sector algebras $\mathcal{SW}(\frac{3}{2}, 8k)$. The characters of the NS-sector are built up from the functions

$$\Lambda^{\mathrm{NS}}_{\lambda,4k+2\varepsilon}(\tau) = \frac{\eta\left(\frac{\tau+1}{1}\right)}{\eta^2(\tau)} e^{-\frac{\pi i}{24}} \Theta_{\lambda,4k+2\varepsilon}(\tau)\,, \tag{6.6}$$

where several linear combinations occur analogous to the characters of the $\mathcal{W}(2,8k)$-algebras. In the R-sector we have to use the functions

$$\Lambda^{\mathrm{R}}_{\lambda,4k+2\varepsilon}(\tau) = \frac{\eta(2\tau)}{\eta^2(\tau)} \Theta_{\lambda,4k+2\varepsilon}(\tau)\,, \tag{6.7}$$

where no linear combinations are possible due to the different parity of $h_{r,r}$ and $h_{r,-r}$ with respect to the $(-)^F$-operator representation appearing in the Ramond sector. But the lowest-weight representations to $h_{1,1}$, $h_{0,0}$ and $h_{1,-1}$ are now each twofold

Table 6.2. Representations and their characters for the bosonic $\mathcal{SW}(\frac{3}{2}, 3k)$-algebras

	h	w^2	$\chi_\lambda^{\mathcal{SW}}$	Remark
NS:	$h_{1,1}$	0	$\chi_{0,\mathrm{NS}}^{\mathcal{SW}} = \frac{1}{2}(\Lambda_{0,k}^{\mathrm{NS}} - \Lambda_{0,k+1/2}^{\mathrm{NS}})$	vacuum re
	$h_{\frac{1}{2k},\frac{1}{2k}}$	0	$\chi_{1,\mathrm{NS}}^{\mathcal{SW}} = \Lambda_{1,k}^{\mathrm{NS}}$	
	$\vdots$	$\vdots$	$\vdots$	
	$h_{\frac{k-1}{2k},\frac{k-1}{2k}}$	0	$\chi_{k-1,\mathrm{NS}}^{\mathcal{SW}} = \Lambda_{k-1,k}^{\mathrm{NS}}$	
	$h_{\frac{1}{2},\frac{1}{2}}$	$\neq 0$	$\chi_{k,+,\mathrm{NS}}^{\mathcal{SW}} = \chi_{k,-,\mathrm{NS}}^{\mathcal{SW}} = \frac{1}{2}\Lambda_{k,k}^{\mathrm{NS}}$	degenerate
	$h_{0,0}$	0	$\chi_{k+1,\mathrm{NS}}^{\mathcal{SW}} = \frac{1}{2}(\Lambda_{0,k}^{\mathrm{NS}} + \Lambda_{0,k+1/2}^{\mathrm{NS}})$	rep. to h_{mi}
	$h_{\frac{1}{2k+1},-\frac{1}{2k+1}}$	0	$\chi_{-1,\mathrm{NS}}^{\mathcal{SW}} = \Lambda_{1,k+1/2}^{\mathrm{NS}}$	
	$\vdots$	$\vdots$	$\vdots$	
	$h_{\frac{k}{2k+1},-\frac{k}{2k+1}}$	0	$\chi_{-k,\mathrm{NS}}^{\mathcal{SW}} = \Lambda_{k,k+1/2}^{\mathrm{NS}}$	
R:	$h_{0,0} + \frac{1}{16}$	0	$\chi_{0,\mathrm{R}}^{\mathcal{SW}} = \frac{1}{2}\Lambda_{0,k}^{\mathrm{R}}$	
	$h_{\frac{1}{2k},\frac{1}{2k}} + \frac{1}{16}$	0	$\chi_{1,\mathrm{R}}^{\mathcal{SW}} = \Lambda_{1,k}^{\mathrm{R}}$	
	$\vdots$	$\vdots$	$\vdots$	
	$h_{\frac{k-1}{2k},\frac{k-1}{2k}} + \frac{1}{16}$	0	$\chi_{k-1,\mathrm{R}}^{\mathcal{SW}} = \Lambda_{k-1,k}^{\mathrm{R}}$	
	$h_{\frac{1}{2},\frac{1}{2}} + \frac{1}{16}$	$\neq 0$	$\chi_{k,+,\mathrm{R}}^{\mathcal{SW}} = \chi_{k,-,\mathrm{R}}^{\mathcal{SW}} = \frac{1}{2}\Lambda_{k,k}^{\mathrm{R}}$	degenerate
	$h_{\frac{1}{4k+2},-\frac{1}{4k+2}} + \frac{1}{16}$	0	$\chi_{-1,\mathrm{R}}^{\mathcal{SW}} = \Lambda_{1/2,k+1/2}^{\mathrm{R}}$	
	$h_{\frac{3}{4k+2},-\frac{3}{4k+2}} + \frac{1}{16}$	0	$\chi_{-2,\mathrm{R}}^{\mathcal{SW}} = \Lambda_{3/2,k+1/2}^{\mathrm{R}}$	
	$\vdots$	$\vdots$	$\vdots$	
	$h_{\frac{2k-1}{4k+2},-\frac{2k-1}{4k+2}} + \frac{1}{16}$	0	$\chi_{-k,\mathrm{R}}^{\mathcal{SW}} = \Lambda_{k-1/2,k+1/2}^{\mathrm{R}}$	
	$h_{\frac{1}{2},-\frac{1}{2}} + \frac{1}{16}$	$\neq 0$	$\chi_{-k-1,+,\mathrm{R}}^{\mathcal{SW}} = \chi_{-k-1,-,\mathrm{R}}^{\mathcal{SW}} = \frac{1}{2}\Lambda_{k+1/2,k+1/2}^{\mathrm{R}}$	degenerate

degenerate, there are two values for the eigenvalue w of $\Phi_0 = \Phi_0^{(3)}$. In our case we have two representations at the ground level h_{min}. Some higher level representations can now be built up on either of these ground state representations by applying the mode $\Phi_{\frac{1}{2}}^{(2)}$ for $k \in \mathbb{Z}_+ + \frac{1}{2}$ or the mode $\Phi_1^{(2)}$ for $k \in \mathbb{Z}_+$ on these ground states. (Note that for $k \in \mathbb{Z}_+ + \frac{1}{4}$ this cannot be understood in the frame of chiral $\mathcal{SW}$-algebras, since this field is not local to itself, but to the other local fields of the $\mathcal{SW}$-algebra, $\mathbb{1}$, $\Phi^{(3)}$, etc. Therefore the action of the field $\Phi^{(2)}$ does not cause a real problem, as long as only one mode (symbolically notation!) $\Phi_{\frac{1}{4}}^{(2)}$ is allowed to appear in the monomials of the mode expansion of the resulting state.) The following table lists all irreducible lowest-weight representations:

Table 6.3. Representations and their characters for the odd sector $\mathcal{SW}(\frac{3}{2}, 8k)$-algebras

	h	w	$\chi^{\mathcal{SW}}_{\lambda,\mathrm{odd}}$	Remark
NS:	$h_{1,1}$	0	$\chi^{\mathcal{SW}}_{0,\mathrm{NS}} = \frac{1}{2}(\Lambda^{\mathrm{NS}}_{4k,4k} - \Lambda^{\mathrm{NS}}_{4k+2,4k+2})$	vacuum rep.
	$h_{\frac{1}{4k},\frac{1}{4k}}$	$\neq 0$	$\chi^{\mathcal{SW}}_{1,\mathrm{NS}} = \Lambda^{\mathrm{NS}}_{1+4k,4k}$	
	$\vdots$	$\vdots$	$\vdots$	
	$h_{\frac{4k-1}{4k},\frac{4k-1}{4k}}$	$\neq 0$	$\chi^{\mathcal{SW}}_{4k-1,\mathrm{NS}} = \Lambda^{\mathrm{NS}}_{8k-1,4k} = \Lambda^{\mathrm{NS}}_{-1,4k}$	
	$h_{2,2}$	$\neq 0$	$\chi^{\mathcal{SW}}_{4k,\mathrm{NS}} = \frac{1}{2}(\Lambda^{\mathrm{NS}}_{0,4k} - \Lambda^{\mathrm{NS}}_{0,4k+2})$	rep. on $\lvert\Phi^{(2)}\rangle$
	$h_{0,0}$	$\neq 0$	$\chi^{\mathcal{SW}}_{4k+1,\mathrm{NS}} = \frac{1}{2}(\Lambda^{\mathrm{NS}}_{0,4k} + \Lambda^{\mathrm{NS}}_{0,4k+2})$	rep. to $h_{\min}$
	$h_{1,1}$	$\neq 0$	$\chi^{\mathcal{SW}}_{4k+2,\mathrm{NS}} = \frac{1}{2}(\Lambda^{\mathrm{NS}}_{4k,4k} + \Lambda^{\mathrm{NS}}_{4k+2,4k+2})$	$\lvert h = 0, w \neq 0\rangle$ rep.
	$h_{\frac{1}{4k+2},-\frac{1}{4k+2}}$	$\neq 0$	$\chi^{\mathcal{SW}}_{-1,\mathrm{NS}} = \Lambda^{\mathrm{NS}}_{1+4k+2,4k+2}$	
	$\vdots$	$\vdots$	$\vdots$	
	$h_{\frac{4k+1}{4k+2},-\frac{4k+1}{4k+2}}$	$\neq 0$	$\chi^{\mathcal{SW}}_{-4k-1,\mathrm{NS}} = \Lambda^{\mathrm{NS}}_{8k+3,4k+2} = \Lambda^{\mathrm{NS}}_{-1,4k+2}$	
R:	$h_{1,1} + \frac{1}{16}$	$w_1 \neq w_2$	$\chi^{\mathcal{SW}}_{0,+,\mathrm{R}} = \frac{1}{2}\Lambda^{\mathrm{R}}_{4k,4k}$	degenerate rep.
	$h_{\frac{1}{4k},\frac{1}{4k}} + \frac{1}{16}$	$\neq 0$	$\chi^{\mathcal{SW}}_{1,\mathrm{R}} = \Lambda^{\mathrm{R}}_{1+4k,4k}$	
	$\vdots$	$\vdots$	$\vdots$	
	$h_{\frac{4k-1}{4k},\frac{4k-1}{4k}} + \frac{1}{16}$	$\neq 0$	$\chi^{\mathcal{SW}}_{4k-1,\mathrm{R}} = \Lambda^{\mathrm{R}}_{8k-1,4k} = \Lambda^{\mathrm{R}}_{-1,4k}$	
	$h_{0,0} + \frac{1}{16}$	$\neq 0$	$\chi^{\mathcal{SW}}_{4k,\mathrm{R}} = \frac{1}{\sqrt{2}}\Lambda^{\mathrm{R}}_{0,4k}$	first rep. to $h_{\min}$
	$h_{1,-1} + \frac{1}{16}$	$w_1 \neq w_2$	$\chi^{\mathcal{SW}}_{0,-,\mathrm{R}} = \frac{1}{2}\Lambda^{\mathrm{R}}_{4k+2,4k+2}$	degenerate rep.
	$h_{\frac{1}{4k+2},-\frac{1}{4k+2}} + \frac{1}{16}$	$\neq 0$	$\chi^{\mathcal{SW}}_{-1,\mathrm{R}} = \Lambda^{\mathrm{R}}_{1+4k+2,4k+2}$	
	$\vdots$	$\vdots$	$\vdots$	
	$h_{\frac{4k+1}{4k+2},-\frac{4k+1}{4k+2}} + \frac{1}{16}$	$\neq 0$	$\chi^{\mathcal{SW}}_{-4k-1,\mathrm{R}} = \Lambda^{\mathrm{R}}_{8k+3,4k+2} = \Lambda^{\mathrm{R}}_{-1,4k+2}$	
	$h_{0,0} + \frac{1}{16}$	$\neq 0$	$\chi^{\mathcal{SW}}_{-4k-2,\mathrm{R}} = \frac{1}{\sqrt{2}}\Lambda^{\mathrm{R}}_{0,4k+2}$	second rep. to $h_{\min}$

In complete analogy to chapter 3 the structure constants and decomposition coefficients into chiral BRST-invariant vertex operators can be calculated. In the papers [1, 29] the supersymmetric versions of the normalization integrals of Dotsenko-Fateev type are calculated. There it is shown that the monodromy coefficients are given by the same formulae as in the conformal case, only α_+^2 has to be substituted by $\dfrac{\alpha_+^2 - 1}{2}$. But this means that the same is true for the braid matrices of [17] and consequently for the Δ-coefficients in (3.2). Using this and the normalization constants

$$N^{(p'p)}_{(k'k)(l'l)} = \widehat{I}_{r',r}(\alpha_+\alpha_{k',k}, \alpha_+\alpha_{l',l}, \alpha_+^2) \tag{6.8}$$

taken from [1], where $p = k + l - 2r - 1$ and similar for p', we finally obtain in our case ($r = r'$, thus in particular $r + r'$ even) for the self-coupling structure constants (actually their square)

$$(C^{\Phi}_{\Phi\Phi})^2 = \begin{cases} \dfrac{(\frac{3}{2} - 24k)}{8k} \dfrac{\prod\limits_{j=1}^{8k} (j^2 - 64(k^2 + \frac{k}{2}))^2 \prod\limits_{j=1}^{2k} (j^2 - 4(k^2 + \frac{k}{2}))^3}{\prod\limits_{j=1}^{6k} (j^2 - 36(k^2 + \frac{k}{2})) \prod\limits_{j=1}^{4k} (j^2 - 16(k^2 + \frac{k}{2}))^4} \\ \text{if } k \in \mathbb{Z}_+/2\,, \\ \dfrac{(\frac{3}{2} - 24k)}{8k} \dfrac{\prod\limits_{j=1}^{8k} (j^2 - 64(k^2 + \frac{k}{2}))^2 \prod\limits_{j=1}^{2k+1/2} ((j - \frac{1}{2})^2 - 4(k^2 + \frac{k}{2}))^3}{\prod\limits_{j=1}^{6k+1/2} ((j - \frac{1}{2})^2 - 36(k^2 + \frac{k}{2})) \prod\limits_{j=1}^{4k} (j^2 - 16(k^2 + \frac{k}{2}))^4} \dfrac{3}{4(k^2 + \frac{k}{2})} \\ \text{if } k \in \mathbb{Z}_+ + \frac{1}{4}\,. \end{cases} \tag{6.9}$$

Needless to say that these results explain all explicit calculations of these algebras obtained so far. The supersymmetric case shows a structure, which is closely related to the conformal case, the only surprise coming from the Ramond sector. Hence we do not want to be more detailed here.

7. Summary and Conclusion

With this paper we established a whole class of new RCFTs which are not related to minimal models or any coset constructions.

First, starting from some explicitly calculated examples [5] we constructed a class of extended chiral symmetry algebras related to the non-minimal Dotsenko-Fateev models [12]. In these models for the special values $c = 1 - 24k$, $k \in \mathbb{N}/4$ for the central charge the requirement of locality for the chiral symmetry algebra enables one to determine the field content of the latter, which turns out to be a finitely generated $\mathcal{W}$-algebra, and to give abstract fusion rules.

Secondly, we were able to calculate the non-trivial structure constants of these $\mathcal{W}$-algebras, namely the self-coupling of the additional primary field, and, as a byproduct, the decomposition coefficients of this field into its chiral BRST-invariant vertex operators. The results are strongly related to the expressions obtained by Felder, Fröhlich and Keller [17] from the braiding properties of the chiral vertex operators. By a redefinition of the chiral vertex operators and of the screening charges the algebra gets a thermal structure simplifying the braid group representation and the analytical behaviour of the Dotsenko-Fateev integrals.

Thirdly, we calculated the characters of the vacuum representations of these chiral algebras and then, via modular transformations, the indeed finite set of all representations yielding the complete CFT which turns out to be rational. Thus, the chiral algebra already is maximally extended since it diagonalizes the modular invariant partition function. In particular, we worked out the explicit form of the S-matrix and the structure constants of the fusion algebra for the subclass of bosonic

$\mathscr{W}$-algebras. The fermionic case and the case of the odd sector subalgebras were briefly discussed. All results are in complete agreement with the explicitly calculated examples in [13].

Next, we completed the proof of the classification of all RCFTs with central charge $c = 1$, given by Kiritsis [28], towards the non-unitary case. For this we used the effective central charge $c_{\text{eff}} = c - 24h_{\text{min}}$. It turned out that the models described in this work are the only possible non-unitary ones, who have $c_{\text{eff}} = 1$. They fit in the only case of a modular invariant partition function, which had been rejected by Kiritsis due to his restriction to unitary theories.

In the last part of the paper we outlined a generalization of our results to the supersymmetric case where also some examples of $\mathscr{S}\mathscr{W}$-algebras are now available, see [4] and [14] for their representations. The structure of the results is very similar to the non supersymmetric case.

Our arguments cover the complete set of possible chiral extended symmetry algebras and thus RCFTs coming from degenerate models, since these are either minimal models and coset constructions or the models discussed in this work. In particular, the classification of all RCFTs with $c_{\text{eff}} = 1$ is completed including the non-unitary case.

Still, a lot of questions remain open. The most exciting one in the frame of this work might be, what the other possible combinations of theta-functions with moduli say k and $k + k'$, $k' \notin \{0, \frac{1}{2}, 1, 2, 4\}$ physically could mean. In our case the combinations were necessary due to the embedding structure of the Virasoro Verma-modules coming from null states. Let us again stress the point of c rational but not contained in the minimal series nor in the set $c = 1 - 24k$, $k \in \mathbb{N}/4$. From Eq. (4.18) one might think that there should be at least RCFTs for $k = \frac{p}{\alpha^2}$. With this ansatz one obtains (again $\varepsilon = 0$ or 1)

$$\begin{aligned} \Lambda_{\lambda,\alpha^2(k+\varepsilon)}(\tau) &= \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{(\alpha n + \frac{\alpha\lambda}{2\alpha^2(k+\varepsilon)})^2(k+\varepsilon)} \\ &= \frac{1}{\eta(\tau)} q^k \sum_{r\in\alpha\mathbb{Z}+\frac{\alpha\lambda}{2\alpha^2(k+\varepsilon)}} q^{h_{r,(-)^\varepsilon r}}, \end{aligned} \tag{7.1}$$

which yields a condition on r or equivalently on λ in order to get integer or half-integer weights

$$h_{r,(-)^\varepsilon r} = \left(\left(\alpha n + \frac{\alpha\lambda}{2\alpha^2(k+\varepsilon)}\right)^2 - 1\right)\frac{p}{\alpha^2} = pn^2 + n\lambda + \frac{\lambda^2}{4p} - \frac{p}{\alpha^2}. \tag{7.2}$$

Therefore we must put $\lambda = p = \alpha^2(k + \varepsilon)$ resulting in the condition $\frac{p}{4} - \frac{p}{\alpha^2} \in \mathbb{Z}$ which can only be fulfilled for $\alpha = 2$ corresponding to our odd sector subalgebras. Thus, these algebras are the only ones which can be extracted out of a larger set of not necessarily chiral local operators. This is the case for $k \in \mathbb{N}/4$, where the even operators are not local to themselves and hence cannot be added to the chiral algebra.

Another question might be whether the labeling of the lowest-weight levels h with rational indices has some physical meaning in the frame of $\mathscr{W}$-gravity, where e.g. rational powers of screening charges are used [22, 11].

Finally, the classification of all RCFTs, in particular for $c > 1$, is still far away from being completed. But a big step towards the classification of all $\mathscr{W}$-algebras with one additional generator could be achieved. The situation is now the following: Several classes of such $\mathscr{W}(2, \delta)$-algebras have been established.

(i) The generically existing algebras $\mathscr{W}(2, \delta)$ with $\delta \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, 3, 4, 6\}$. All these algebras have well known classical counterparts as the algebra of Casimir operators of a Lie-algebra.

(ii) The algebras, which exist for c an element of the minimal series $c = 1 - 6\frac{(p-q)^2}{pq}$, $p, q \in \mathbb{N}$ coprime. These algebras are related to the ADE-classification of modular invariant partition functions of Virasoro minimal models [8], as has been worked out in [37] for the fermionic case.

(iii) $\mathscr{W}(2, 2q-1)$-algebras to $c = 1 - 6\frac{(1-q)^2}{q}$, $q \in \mathbb{N}$, (minimal series with $p = 1$). These algebras have been studied in [26]. They are not extended symmetry algebras of a RCFT.

(iv) The $\mathscr{W}(2, 3k)$ and $\mathscr{W}(2, 8k)$ algebras with central charge $c = 1 - 24k$, $k \in \mathbb{N}/4$ as discussed in this paper. These are the only algebras related to non-minimal degenerate Virasoro models.

(v) $\mathscr{W}$-algebras to isolated irrational c-values. Following [2], these algebras cannot belong to RCFTs.

(vi) Some exceptional $\mathscr{W}$-algebras, mainly the $\mathscr{W}(2, 8)$ for all the values of the central charge not covered by (i) to (v), have been found. Probably they are related to other finite groups which can be represented by the modular group, see [13]. In all these cases the self-coupling is non zero.

A very similar pattern is valid for the $\mathscr{S}\mathscr{W}(\frac{3}{2}, \delta)$-algebras (but without solutions of type (v), i.e. without isolated irrational solutions). Since all known examples of set (vi) have non-vanishing self-coupling, we conjecture that the classification of the $\mathscr{W}(2, \delta/2)$-algebras to rational central charge, $\delta \in \mathbb{N}$, is complete.

We want to conclude with one very speculative remark. We have found RCFTs for the central charges of the form $c = 1 - x$ with x having divisor 24 (bosonic case) or 12 (fermionic case). There is an interesting work by Goddard [10] in which nice RCFTs, related to selfdual even lattices, with central charges $c = x$, x having divisor 24 (bosonic case) or 12 (fermionic case) are found. Is there a relation between these theories or even in general between theories with c and $1 - c$?

Appendix A. The S and T matrix for $\mathscr{W}(2, 3k)$

This Appendix presents the general form of the S-matrix for the case of bosonic $\mathscr{W}(2, 3k)$ theories, i.e. $k \in \mathbb{N}$. For the Neveu-Schwarz sector of the fermionic case ($k \in \mathbb{N} + \frac{1}{2}$) the S-matrix is exactly the same, if one removes the rows and columns belonging to the characters $\chi^{\mathscr{W}}_{k,\pm}$ and $\chi^{\mathscr{W}}_{-k-1,\pm}$, i.e. to the degenerate representations, and if one substitutes k by $\frac{k}{2}$. So, let $k \in \mathbb{N}$. Define the functions

$\mathscr{C}_\alpha(x) = \dfrac{1}{\sqrt{2\alpha}} \cos\left(\pi \dfrac{x}{\alpha}\right)$. Then S is given by

$$\left(\begin{array}{ccccc}
\frac{1}{2}(\mathscr{C}_k(0)+\mathscr{C}_{k+1}(0)) & \mathscr{C}_k(0) \;\cdots & \mathscr{C}_k(0) & \frac{1}{2}\mathscr{C}_k(0) & \frac{1}{2}\mathscr{C}_k(0) \\
\mathscr{C}_k(0) & 2\mathscr{C}_k(1) \;\cdots & 2\mathscr{C}_k(k-1) & \mathscr{C}_k(k) & \mathscr{C}_k(k) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathscr{C}_k(0) & 2\mathscr{C}_k(k-1) \;\cdots & 2\mathscr{C}_k((k-1)^2) & \mathscr{C}_k(k(k-1)) & \mathscr{C}_k(k(k-1)) \\
\frac{1}{2}\mathscr{C}_k(0) & \mathscr{C}_k(k) \;\cdots & \mathscr{C}_k(k(k-1)) & A & \mathscr{C}_k(k^2)-A \\
\frac{1}{2}\mathscr{C}_k(0) & \mathscr{C}_k(k) \;\cdots & \mathscr{C}_k(k(k-1)) & \mathscr{C}_k(k^2)-A & A \\
\frac{1}{2}(\mathscr{C}_k(0)-\mathscr{C}_{k+1}(0)) & \mathscr{C}_k(0) \;\cdots & \mathscr{C}_k(0) & \frac{1}{2}\mathscr{C}_k(0) & \frac{1}{2}\mathscr{C}_k(0) \\
-\mathscr{C}_{k+1}(0) & 0 \;\cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-\mathscr{C}_{k+1}(0) & 0 \;\cdots & 0 & 0 & 0 \\
-\frac{1}{2}\mathscr{C}_{k+1}(0) & 0 \;\cdots & 0 & C & -C \\
-\frac{1}{2}\mathscr{C}_{k+1}(0) & 0 \;\cdots & 0 & -C & C
\end{array}\right.$$

$$\left.\begin{array}{ccccc}
\frac{1}{2}(\mathscr{C}_k(0)-\mathscr{C}_{k+1}(0)) & -\mathscr{C}_{k+1}(0) \;\cdots & -\mathscr{C}_{k+1}(0) & -\frac{1}{2}\mathscr{C}_{k+1}(0) & -\frac{1}{2}\mathscr{C}_{k+1}(0) \\
\mathscr{C}_k(0) & 0 \;\cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathscr{C}_k(0) & 0 \;\cdots & 0 & 0 & 0 \\
\frac{1}{2}\mathscr{C}_k(0) & 0 \;\cdots & 0 & C & -C \\
\frac{1}{2}\mathscr{C}_k(0) & 0 \;\cdots & 0 & -C & C \\
\frac{1}{2}(\mathscr{C}_k(0)+\mathscr{C}_{k+1}(0)) & \mathscr{C}_{k+1}(0) \;\cdots & \mathscr{C}_{k+1}(0) & \frac{1}{2}\mathscr{C}_{k+1}(0) & \frac{1}{2}\mathscr{C}_{k+1}(0) \\
\mathscr{C}_{k+1}(0) & 2\mathscr{C}_{k+1}(1) \;\cdots & 2\mathscr{C}_{k+1}(k) & \mathscr{C}_{k+1}(k+1) & \mathscr{C}_{k+1}(k+1) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathscr{C}_{k+1}(0) & 2\mathscr{C}_{k+1}(k) \;\cdots & 2\mathscr{C}_{k+1}(k^2) & \mathscr{C}_{k+1}(k(k+1)) & \mathscr{C}_{k+1}(k(k+1)) \\
\frac{1}{2}\mathscr{C}_{k+1}(0) & \mathscr{C}_{k+1}(k+1) \;\cdots & \mathscr{C}_{k+1}(k(k+1)) & B & \mathscr{C}_{k+1}((k+1)^2)-B \\
\frac{1}{2}\mathscr{C}_{k+1}(0) & \mathscr{C}_{k+1}(k+1) \;\cdots & \mathscr{C}_{k+1}(k(k+1)) & \mathscr{C}_{k+1}((k+1)^2)-B & B
\end{array}\right) \tag{A.1}$$

Here the three free parameters are determined by the requirement that the fusion algebra structure constants are non-negative integers. This is carried out in Appendix B. The result is

$$\begin{aligned}
C &= (i)^k \frac{1}{\sqrt{8}}\,, \\
A &= (-1)^k \left(\frac{1}{2\sqrt{2k}} \pm C\right)\,, \\
B &= (-1)^{k+1} \left(\frac{1}{2\sqrt{2k+2}} \pm C\right)\,.
\end{aligned} \tag{A.2}$$

Note that $\dfrac{1}{\sqrt{2k}}(-1)^k - A_\pm = (-1)^k\left(\dfrac{1}{2\sqrt{2k}} \mp C\right) = A_\mp$ and similarly for B. Thus the two solutions for A and B just mean an interchange or a reordering of the degenerate representations. The T-matrix is much simpler and given by

$$\begin{aligned}\operatorname{diag}\Bigg(&\exp\left(\pi i\left(-\frac{1}{12}\right)\right), \exp\left(\pi i\left(\frac{1}{2k}-\frac{1}{12}\right)\right), \ldots, \exp\left(\pi i\left(\frac{k}{2k}-\frac{1}{12}\right)\right),\\ &\exp\left(\pi i\left(\frac{k}{2k}-\frac{1}{12}\right)\right), \exp\left(\pi i\left(-\frac{1}{12}\right)\right), \exp\left(\pi i\left(\frac{1}{2k+2}-\frac{1}{12}\right)\right), \ldots,\\ &\exp\left(\pi i\left(\frac{k+1}{2k+2}-\frac{1}{12}\right)\right), \exp\left(\pi i\left(\frac{k+1}{2k+2}-\frac{1}{12}\right)\right)\Bigg). \end{aligned} \tag{A.3}$$

The S and T matrices for the odd sector algebras are easy to obtain from Eqs. (4.5) and Table 4.2, which shows how the characters can be expressed in Λ-functions. Therefore we do not go into further details here.

Appendix B. The Fusion Algebra for $\mathscr{W}(2, 3k)$

Finally, in this appendix we calculate the fusion algebra for the bosonic $\mathscr{W}(2, 3k)$ and show explicitly that all structure constants are indeed non-negative integers. This also determines the free parameters of the extended S-matrix uniquely completing the proof of rationality of the theories.

One starts with $A, B, C \in \mathbb{C}$ as arbitrary free complex numbers. With the Verlinde formula one calculates some particular structure constants $N_{\alpha\beta}^{\gamma}$. For example

$$\begin{aligned} N_{k,+;k,+}^{-j} &= \frac{1}{2} - (-1)^j 4C^2\,, \qquad N_{k,+;k,-}^{-j} = \frac{1}{2} + (-1)^j 4C^2\,,\\ N_{k,+;-j}^{k,+} &= \frac{1}{2} - (-1)^j 4|C|^2\,, \quad N_{k,+;-j}^{k,-} = \frac{1}{2} + (-1)^j 4|C|^2\,, \end{aligned} \tag{B.1}$$

where $-k \leq -j \leq -1$. Since all these numbers should be non-negative integral ones, the only solutions are either 0 or 1. Hence we get $|C|^2 = \pm C^2$, i.e. C purely real or purely imaginary. Furthermore, the absolute value is fixed to be $|C| = \frac{1}{\sqrt{8}}$. This leaves us with the ansatz

$$C = (i)^{\alpha_C} \frac{1}{\sqrt{8}}\,. \tag{B.2a}$$

Next we look at the structure constants

$$N_{k,+;k,-}^{-k-1,+} = N_{k,+;k,-}^{-k-1,-} = \frac{1}{4} + (-1)^{k+1} 2C^2\,. \tag{B.3}$$

With Eq. (B.2a) we see that 0 is the only allowed solution and therefore we need $\alpha_C \equiv k \bmod 2$. In the following we put without loss of generality

$$C = (i)^k \frac{1}{\sqrt{8}}\,. \tag{B.2b}$$

Thirdly, we can consider the following structure constants:

$$
\begin{aligned}
N^{j}_{k,+;k,-} &= \frac{1}{2} + (-1)^{j}\left(4A^2 - \frac{4}{\sqrt{2k}}(-1)^k A + \frac{1}{2k}\right), \\
N^{j}_{k,+;k,+} &= \frac{1}{2} - (-1)^{j}\left(4A^2 - \frac{4}{\sqrt{2k}}(-1)^k A + \frac{1}{2k}\right), \\
N^{-j'}_{-k-1,+;-k-1,+} &= \frac{1}{2} - (-1)^{j'}\left(4B^2 - \frac{4}{\sqrt{2k+2}}(-1)^{k+1} B + \frac{1}{2k+2}\right), \\
N^{-j'}_{-k-1,+;-k-1,-} &= \frac{1}{2} + (-1)^{j'}\left(4B^2 - \frac{4}{\sqrt{2k+2}}(-1)^{k+1} B + \frac{1}{2k+2}\right),
\end{aligned}
\tag{B.4}
$$

where $1 \leq j \leq k-1$ and $-k \leq -j' \leq 1$. These pairs of equations again have as only allowed solutions that one equation of a pair is 1 and the other 0. With this information one can solve the quadratic equations and gets as ansatz

$$
\begin{aligned}
A &= (-1)^k\left(\frac{1}{2\sqrt{2k}} + (i)^{\alpha_A}\frac{1}{\sqrt{8}}\right), \\
B &= (-1)^{k+1}\left(\frac{1}{2\sqrt{2k+2}} + (i)^{\alpha_B}\frac{1}{\sqrt{8}}\right).
\end{aligned}
\tag{B.5}
$$

In order to determine the free powers α_A and α_B one looks at the constants

$$
\begin{aligned}
N^{k,+}_{k,-;-k-1,+} &= \frac{1}{4} - (-1)^k 2\left((A-A^*)C + |C|^2\right), \\
N^{k,+}_{k,-;-k-1,-} &= \frac{1}{4} + (-1)^k 2\left((A-A^*)C - |C|^2\right), \\
N^{-k-1,+}_{k,+;-k-1,-} &= \frac{1}{4} + (-1)^{k+1} 2\left((B-B^*)C + |C|^2\right), \\
N^{-k-1,+}_{k,-;-k-1,-} &= \frac{1}{4} - (-1)^{k+1} 2\left((B-B^*)C - |C|^2\right).
\end{aligned}
\tag{B.6}
$$

With Eqs. (B.2) and (B.5) one now sees that if C is real then $\operatorname{Im} A = \operatorname{Im} B = 0$, or if C is imaginary then $\operatorname{Im} A = -\operatorname{Im} B = \frac{1}{\sqrt{8}}$. Therefore we can choose $\alpha_A = \alpha_B = k$ obtaining Eq. (A.2).

Finally we want to list the whole set of the fusion coefficients to show that indeed they all are non-negative integers. To save space we only list the not obviously related constants. The others can be obtained by one of the following formulae: Let $\varepsilon = k \bmod 2$. Then we have $S^{2(1+\varepsilon)} = C^{1+\varepsilon} = \mathbb{1}$. Thus, the charge conjugation, denoted by $C : \phi_\alpha \longmapsto \phi_{\bar{\alpha}}$ is trivial for k even. Nonetheless we denote by $E : \phi_\alpha \longmapsto \phi_{\widehat{\alpha}}$ the exchange of the degenerate representations, i.e. $\widehat{(k,\pm)} = (k,\mp)$, analogously for $(-k-1,\pm)$ and $\widehat{\alpha} = \alpha$ else. Then we have, using the conjugation matrix to raise or lower indices,

$$
\begin{aligned}
N^{\gamma}_{\alpha\beta} &= N^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}} = N^{\widehat{\gamma}}_{\widehat{\alpha}\widehat{\beta}}, \\
N^{\gamma}_{\alpha\beta} &= N^{\bar{\beta}}_{\alpha\bar{\gamma}} = N^{\bar{\alpha}}_{\bar{\gamma}\beta}, \\
N^{\gamma}_{\alpha\beta} &= N^{\gamma}_{\beta\alpha}.
\end{aligned}
\tag{B.7}
$$

With these relations and the following set of fusion numbers it is straightforward to calculate all the $N_{\alpha\beta}^{\gamma}$. All sums of indices in the Kronecker symbols are understood to be taken modulo $2k$ for positive indices or modulo $2k+2$ for negative ones. Here always $j, j', j'' \in \{1, \dots, k-1\}$ and $-j, -j', -j'' \in \{-1, \dots, -k\}$. Also note that we only distinguish the degenerate representations by the usage of E. Thus, normally the choice of one of the degenerate representations is arbitrary, only if two or three indices belong to degenerate representations the hat ($\widehat{\ }$) indicates, where the relatively other choice has to be made. Finally, we write $N(\alpha, \beta; \gamma)$ instead of $N_{\alpha\beta}^{\gamma}$ for better readability.

$$\begin{aligned}
N(\alpha, \beta; 0) &= \delta_{\alpha+\bar{\beta},0} = C_{\alpha\beta} \\
N(j, j'; j'') &= 2 + \delta_{j+j'-j'',0} + \delta_{j'+j''-j,0} + \delta_{j''+j-j',0} \\
&\quad +\delta_{j+j'+j'',0} \\
N(j, k; j'') &= \tfrac{1}{2}(2 + \delta_{j+k-j'',0} + \delta_{k+j''-j,0} + \delta_{j''+j-k,0} \\
&\quad +\delta_{j+k+j'',0}) \\
N(j, k+1; j'') &= 2 + \delta_{j-j'',0} \\
N(j, -j'; j'') &= 2 \\
N(j, -k-1; j'') &= 1 \\
N(k, k; j'') &= \tfrac{1}{2}(1 + (-1)^{j''+k}) \\
N(k, \widehat{k}; j'') &= \tfrac{1}{2}(1 - (-1)^{j''+k}) \\
N(k, k+1; j'') &= 1 \\
N(k, -j'; j'') &= 1 \\
N(k, -k-1; j'') &= \tfrac{1}{2}(1 + (-1)^{j''}) \\
N(k, -\widehat{k-1}; j'') &= \tfrac{1}{2}(1 - (-1)^{j''}) \\
N(k+1, k+1; j'') &= 2 \\
N(k+1, -j'; j'') &= 2 \\
N(k+1, -k-1; j'') &= 1 \\
N(-j, -j'; j'') &= 2 \\
N(-j, -k-1, +; j'') &= 1 \\
N(-k-1, -k-1; j'') &= \tfrac{1}{2}(1 + (-1)^{j''+k}) \\
N(-k-1, -\widehat{k-1}; j'') &= \tfrac{1}{2}(1 - (-1)^{j''+k}) \\
N(k, k; k) &= \tfrac{1}{2}(1 + (-1)^{k} \\
N(k, \widehat{k}; k) &= 0 \\
N(k, k+1; k) &= 1 \\
N(k, -j'; k) &= \tfrac{1}{2}(1 - (-1)^{j'}) \\
N(k, -k-1; k) &= \tfrac{1}{2}(1 + (-1)^{k}) \\
N(k, -\widehat{k-1}; k) &= 0 \\
N(\widehat{k}, \widehat{k}; k) &= \tfrac{1}{2}(1 - (-1)^{k}) \\
N(\widehat{k}, k+1; k) &= 1 \\
N(\widehat{k}, -j'; k) &= \tfrac{1}{2}(1 + (-1)^{j'}) \\
N(\widehat{k}, -k-1; k) &= \tfrac{1}{2}(1 - (-1)^{k}) \\
N(\widehat{k}, -\widehat{k-1}; k) &= 0
\end{aligned}$$

$$
\begin{aligned}
&N(k+1,k+1;k) &&= 0\\
&N(k+1,-j';k) &&= 1\\
&N(k+1,-k-1;k) &&= \tfrac{1}{2}(1+(-1)^k)\\
&N(k+1,-\widehat{k-1};k) &&= \tfrac{1}{2}(1-(-1)^k)\\
&N(-j,-j';k) &&= 1\\
&N(-j,-k-1,+;k) &&= \tfrac{1}{2}(1+(-1)^{j+k})\\
&N(-j,-\widehat{k-1};k) &&= \tfrac{1}{2}(1-(-1)^{j+k})\\
&N(-k-1,-k-1;k) &&= 0\\
&N(-k-1,-\widehat{k-1};k) &&= 0\\
&N(-\widehat{k-1},-\widehat{k-1};k) &&= 1\\
&N(k+1,k+1;k+1) &&= 2\\
&N(k+1,-j';k+1) &&= 2\\
&N(k+1,-k-1;k+1) &&= 1\\
&N(-j,-j';k+1) &&= 2-\delta_{j-j',0}\\
&N(-j,-k-1;k+1) &&= 1\\
&N(-k-1,-k-1;k+1) &&= 0\\
&N(-k-1,-\widehat{k-1};k+1) &&= 0\\
&N(-j,-j';-j'') &&= 2-\delta_{j+j'-j'',0}-\delta_{j'+j''-j,0}-\delta_{j''+j-j',0}\\
& &&\quad -\delta_{j+j'+j'',0}\\
&N(-j,-k-1;-j'') &&= \tfrac{1}{2}(2-\delta_{j+(k+1)-j'',0}-\delta_{(k+1)+j''-j,0}\\
& &&\quad -\delta_{j''+(k+1)-j',0}-\delta_{j+(k+1)+j'',0})\\
&N(-k-1,-k-1;-j'') &&= \tfrac{1}{2}(1-(-1)^{j''+k})\\
&N(-k-1,-\widehat{k-1};-j'') &&= \tfrac{1}{2}(1+(-1)^{j''+k})\\
&N(-k-1,-k-1;-k-1) &&= \tfrac{1}{2}(1+(-1)^k)\\
&N(-k-1,-\widehat{k-1};-k-1) &&= 0\\
&N(-\widehat{k-1},-\widehat{k-1};-k-1) &&= \tfrac{1}{2}(1-(-1)^k)
\end{aligned}
\tag{B.8}
$$

The automorphisms of the fusion rules can be read off from the decompositions of k into two coprime factors and similarily for $k+1$. Without loss of generality let us assume that $k = pq$ with p, q coprime. Then we have an automorphism of the fusion algebra, namely $j \mapsto pj \bmod 2q$, $j \in \{0, 1, \ldots, k\}$ and all other labels are left unchanged (in particular the label $k+1$ has to be considered as zero). In particular $N_{pj,pj'}^{pj''} = N_{j,j'}^{j''}$ where all indices are taken modulo $2q$ and $j, j', j'' \in \{1, \ldots, k-1\}$, as can be seen directly from the explicit form (B.8) of these fusion numbers. If $k+1$ has such a decomposition, $k+1 = pq$, then there is an automorphism $-j \mapsto -pj \bmod 2q$, $-j \in \{0, -1, \ldots, -k-1\}$ and again all other labels have to be left unchanged.

The one-one correspondence of theories and automorphisms, together with our arguments of Sect. 5, assure that there are no other non-trivial automorphisms. In fact, a theory with $2R_1^2 = \dfrac{P_1}{Q_1}$ and $2R_2^2 = \dfrac{P_2}{Q_2}$ in partition function (5.1) such that $P_2Q_2 - P_1Q_1 = 1$ yields an automorphism, as described above, and the set of these theories is complete.

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