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Noncomputability Arising in Dynamical Triangulation Model of Four-Dimensional Quantum Gravity

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Abstract. Computations in dynamical triangulation models of four-dimensional Quantum Gravity involve weighted averaging over sets of all distinct triangulations of compact four-dimensional manifolds. In order to be able to perform such computations one needs an algorithm which for any given N and a given compact four-dimensional manifold M constructs all possible triangulations of M with $\leq N$ simplices. Our first result is that such algorithm does not exist. Then we discuss recursion-theoretic limitations of any algorithm designed to perform approximate calculations of sums over all possible triangulations of a compact four-dimensional manifold.

A well-known problem in physics is to unify Quantum Theory with General Relativity. One of the proposed approaches involves integration over the space of metrics on all compact four dimensional manifolds. The integration is done separately for any particular topological type of the four dimensional manifolds. Then one would like to sum over all topological types attributing an appropriate weight to every topological type.

Although there is no mathematically rigorous definition of a measure with the required properties on the (infinite dimensional) space of all metrics on a four dimensional manifold (but see [Po] for the two dimensional case), recently the following idea to do this computation was proposed. One considers a kind of grid in the space of all metrics. This grid is formed by metrics which are defined as follows: One starts from a triangulation of the smooth four dimensional manifold of interest. Then one considers all possible triangulations of the manifold combinatorially equivalent to the chosen initial triangulation. One does not distinguish between simplicially isomorphic triangulations. (From now on we will mean by a *triangulation* of a PL-manifold M a simplicial complex K such that the corresponding polyhedron $|K|$ is PL (piecewise-linearly) homeomorphic to M. This definition of triangulations somewhat differs from the standard one but is more convenient for our aims. Usually one requires only that $|K|$ be homeomorphic to M. We will not distinguish between simplicially isomorphic triangulations.) For any of these triangulations one assigns the unit length for any of its 1-dimensional simplices. In this way one gets for any triangulation a distance function on the manifold. The metrics corresponding to all possible triangulations are considered as a uniform grid in the space of all metrics. Afterwards one can approximate the necessary integrals by means of sums over all the nodes of the grid (see [BKKM, BD, M, ADF] for the two dimensional case and [AM, V, AJ] for the four dimensional case). To generate this grid, one uses the following method: One starts from a prescribed point (i.e. a given triangulation of the manifold). Then one makes elementary operations in order to move from a point to its neighboring points. (More precisely, one introduces a finite set of elementary operations. Operations from this set change any particular triangulation to some other triangulations which by definition are considered the neighboring triangulations of this particular triangulation.)

In practice the whole grid is not generated, but rather a probabilistic approximation is introduced. A Markov chain is defined, and the necessary sum is computed by means of a Monte-Carlo method. In order to validate this procedure one needs to fulfill in particular the following requirement of *ergodicity*: Using the set of elementary moves one should be able to get any triangulation from any other triangulation.

From the computational point of view it is more sensible to impose the following stronger constraint of *computational ergodicity* on the considered set of elementary moves: There exists a recursive function r such that for any N and any two prescribed triangulations with $\leq N$ simplices there exists a sequence of not more than $r(N)$ elementary moves which transforms one triangulation to the other.

The first main point of this note is the observation that in the dynamical triangulation model for four dimensional quantum gravity there is no finite set of elementary operations such that the *computational ergodicity* will hold (although, as it was shown in [GV], the set of elementary moves considered in [AM, V, AJ] satisfies the ergodicity requirement). That is, for every finite set of elementary operations and for every recursive function $r(N)$, there always exist some N and two triangulations with N simplices such that the number of operations needed to transform one triangulation to the other exceeds $r(N)$.

This observation is based on the classical result of Markov on algorithmic unrecognizability of a specific four dimensional manifold (cf. [BHP]). (As it follows from the proof of this result given in [F], Theorem 14.1, this manifold can be taken, for example, diffeomorphic to the four dimensional sphere with 46 attached handles of index two. Denote this manifold by S_0). The proof of this result of Markov is based on results of Rabin and Adyan on algorithmic unsolvability of the triviality problem for finitely presented groups. More precisely, Markov proved that there is no algorithm which for a given, finitely presented, four dimensional manifold verifies whether or not it is homeomorphic (or diffeomorphic, or PL-homeomorphic) to S_0. Furthermore we would like to mention the related result of S.P. Novikov (published as Ch. 10 of [VKF]), which states the algorithmic unrecognizability of spheres S^n for any $n \geq 5$. Although it is not known now whether or not the four dimensional sphere can be recognized in the class of all PL (or smooth) four dimensional manifolds, it seems very plausible that the four dimensional sphere is also algorithmically unrecognizable. In all these unrecognizability results for PL-manifolds all PL-manifolds can be assumed presented by some triangulation.

Proposition 1. *Let M_0 be a* PL*-manifold that can not be recognized by any algorithm. Then there is no finite set of elementary moves on the set of all triangulations of M_0 satisfying the requirement of computational ergodicity.*

Proof. One can prove this proposition by contradiction using an argument very similar to the argument used in [ABB] to prove Theorem 5b) there. Indeed, suppose that a finite set of elementary moves on the set of all triangulations of M_0 satisfying the requirement of computational ergodicity exists. Then, applying all possible chains of the moves to any fixed given triangulation of M_0 one can generate the list of all possible triangulations of M_0 with $\leq N$ simplices in time recursively depending on N. Hence given a manifold M presented by some its triangulation T with $N(T)$ simplices one can decide whether or not M is PL-homeomorphic to M_0 as follows: First, it is necessary to generate the list of all triangulations of M_0 with $\leq N(T)$ simplices. Then, for any triangulation on this list one checks whether or not it is simplicially isomorphic to T. (This step evidently can be effectively done; cf. [ABB, Lemma 2.16].) Hence, the assumption that there exists a computationally ergodic finite set of elementary moves on the set of triangulations of M_0 implies the existence of an algorithm checking for any manifold M presented by some triangulation whether or not M is PL-homeomorphic to M_0. Q.E.D.

Thus, at least for the mentioned manifold S_0 there is no computationally ergodic finite set of elementary moves on the set of its triangulations. Moreover, if S^4 cannot be effectively recognized in the class of all 4-dimensional manifolds, then there is no computationally ergodic finite set of elementary moves on the set of triangulations of S^4.

Hence, in general, it is natural to expect that a computation of integrals within to accuracy ε using the described above approach will require an amount of time growing non-recursively fast with $[1/\varepsilon]$. (Of course, if the integrated function is of a special form this can be not the case.)

Consider now an ergodic finite set of elementary moves on the space of triangulations of a compact four-dimensional manifold M_0 which cannot be recognized by any algorithm. Let T_0 be an arbitrary triangulation of M_0. Proposition 1 immediately implies that for any recursive function $t(N)$ there exist arbitrary large N such that some triangulations of M_0 with N simplices cannot be obtained from T_0 by less than $t(N)$ elementary moves. Taking, for example, $t(N) = [\exp(\exp(\ldots \exp(N)))]$ (the exponentiation is performed N times), we see that from the practical computational point of view some part of the grid will be out of reach for large N. Now the following question naturally arises:

Which part of the grid will be out of reach assymptotically, when N tends to infinity? More precisely, let $A(N)$ be an algorithm which for any N produces some amount of distinct triangulations of M_0 with $\leq N$ simplices. Denote the number of triangulations with $\leq N$ simplices produced by A by $s_A(N)$. Denote by $s(N)$ the total number of triangulations of M_0 with $\leq N$ simplices. Now our problem can be formulated as determining the value of S defined by the following formula:

$$S = \sup_{A} \lim_{N\to\infty} \sup s_A(N)/s(N) . \tag{1}$$

(The maximum is taken over the set of all possible algorithms.)

First, note that $s_A(N)$ is a recursive function. Now let us prove the following proposition:

Proposition 2. *$s(N)$ is a non-recursive function.*

Proof. In order to see that $s(N)$ is not recursive assume the opposite. Then we have the following algorithm recognizing for a given polyhedron P whether or not it is PL-homeomorphic to M_0 (providing M_0 and P are presented by some triangulations T_0 of M and T of P):

Put N to be equal to the number of simplices in T. Compute $s(N)$. Apply successively all possible finite combinations of the Alexander move and its inverse (see [A], [GV]) to T_0, keeping record of the number of distinct triangulations of M_0, which are already obtained. If this number is equal to $s(N)$, then stop. Since the Alexander move and its inverse form an ergodic set of moves on the set of triangulations of any compact manifold ([A]), eventually we shall obtain the list of all distinct triangulations of M_0 with $\leq N$ simplices. The last step of the algorithm will be to compare T with all $s(N)$ triangulations from the obtained list of triangulations of M_0 with N simplices. The existence of this algorithm provides the desired contradiction which proves the non-recursiveness of $s(N)$. Q.E.D.

(This non-recursiveness result has the following direct physical interpretation. The partition function defined in [AM] for Quantum Gravity is:

$$\mathscr{Z}(\hat{\lambda}_4, \hat{\lambda}_0) = \sum_{\text{triangulations of a manifold}} \exp\big(- \hat{\lambda}_4 N_4 - \hat{\lambda}_0 N_0 - \Delta\lambda_4 (N_4 - \hat{N}_4)^2 - \Delta\lambda_0 (R - \hat{R})^2 \big), \tag{2}$$

where N_0, N_1, N_2, N_3, N_4 are the numbers of 0, 1, 2, 3, 4-simplexes in the simplicial complex respectively, $\hat{\lambda}_4$, $\hat{\lambda}_0$, $\Delta\lambda_4$, $\Delta\lambda_0$, $\hat{N}_4$ and $\hat{R}$ are the parameters of the model and R is defined by:

$$R = 4\pi(N_0 + N_4 - 2) - 10\alpha N_4\,, \quad \alpha = \arccos\left(\frac{1}{\text{Dimension}}\right). \tag{3}$$

This partition function can be defined for any topological type of compact smooth four-dimensional manifolds. In [AM] it was considered for S^4. Consider it for the manifold S_0. Note that $s(N)$ equals the partition function defined above, where $\hat{\lambda}_0 = \hat{\lambda}_4 = \Delta\lambda_0 = 0$ and $\Delta\lambda_4 = \infty$, $\hat{N}_4 = N$).

Our question about the value of S is in a sense equivalent to asking how asymptotically closely one can minorize the non-recursive function $s(N)$ by a recursive function. Our conjecture is that the value of S defined by (1) is strictly less than one.

The argument in favour of this conjecture is the following one. Note that the non-recursiveness of $s(N)$ is a corollary of the unsolvability of the halting problem for Turing machines. Moreover, the mentioned proofs of unrecognizability of S_0 and S^n for $n \geq 5$ rely on the existence of an effective procedure which for a given Turing machine τ and its arbitrary input w constructs a triangulation T of a smooth manifold M in such a manner that τ starting its work from w will eventually halt if and only if M will be homeomorphic (or PL-homeomorphic, or diffeomorphic) to S_0 (or, correspondingly, to S^n). Thus, it seems reasonable to check whether or not the recursion-theoretic analogue of the conjecture $S < 1$ will hold. This analogue can be formulated as follows:

Let U be a universal Turing machine. Let $H_U(N)$ denote the number of its inputs w of length $\leq N$ such that U starting to work from w will eventually halt.

The unsolvability of the halting problem implies that $H_U(N)$ will be a non-recursive function. Let A be an algorithm which answers for any input w of U whether or not U starting to work from w eventually halts, but which is permitted to say "do not know" concerning some inputs. (Such algorithms obviously exist.) Let $H_{UA}(N)$ denote the number of inputs of length $\leq N$ for which A tells that U eventually halts. We are interested in $H(U)$ which is defined as $\sup_A \limsup_{N\to\infty} \frac{H_{UA}(N)}{H_U(N)}$. Moreover, we are interested in "natural" universal Turing machines for which all inputs are meaningful and which have no redundancy. (Examples of such machines are given in the book [C1].) For "natural" Turing machines $H(U) < 1$. The proof of this fact was given by Chaitin ([C3]) using algorithmic information theory. (Good expositions of algorithmic information theory can be found in the book [C1] and reviews [LV] and [ZL]). Thus, G. Chaitin proved a recursion-theoretic analogue of our conjecture "$S < 1$." We would like to mention a paper [S] which also contains a related result in logic but uses a different approach.

It should be noted that a possible absence of computability in Quantum Gravity theories was mentioned in the Geroch and Hartle paper [GH]. In [GH] it was observed that if one needs to perform a summation over all possible topological types of four dimensional PL-manifolds then this can turn out to be impossible due to the fact that the identification of the topological type of a given manifold is an unsolvable problem. In this paper we discuss a Markov process for *one fixed* topological type of manifolds. Thus, the absence of computability discussed in the present paper is essentially different from the result of Geroch and Hartle. The possibility of non-computability arising in Statistical Mechanics models was also dicussed in [K].

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Note added in proof. Recently Prof. O. Viro informed us that the elementary moves used in [AM] were studied by U. Pachner (Arch. Math. **30**, 89 (1978); Europ. J. Combinatorics **12**, 129 (1991)). Pachner proved, in particular, the ergodicity of this set of elementary moves on the set of all triangulations of any compact PL-manifold. We wish to thank Prof. O. Viro for this information.

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