Topological Orbifold Models and Quantum Cohomology Rings

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Abstract. We discuss the topological sigma model on an orbifold target space. We describe the moduli space of classical minima for computing correlation functions involving twisted operators, and show, through a detailed computation of an orbifold of $\mathbf{CP}^1$ by the dihedral group D_4, how to compute the complete ring of observables. Through this procedure, we compute all the rings of dihedral $\mathbf{CP}^1$ orbifolds. We then consider $\mathbf{CP}^2/D_4$, and show how the techniques of topological-anti-topological fusion might be used to compute twist field correlation functions for nonabelian orbifolds.

1. Introduction and Summary

Orbifolds define consistent string vacua [11]. Therefore, we may wonder whether the string theories described by orbifolds have a simple topological description, or we may inquire about topological properties – for example Yukawa couplings of fermion generations – of string theories with orbifold compactifications. Such knowledge can also be applied to the non-topological theory as well. We consider topological sigma models on orbifolds of Kähler manifolds. These theories are defined by twisting the $N = 2$ supersymmetric sigma models, and have associated with them a ring of observables. This "quantum ring" is a generalization of the chiral primary ring to models which are not conformal field theories. The discussion of these rings – their characterization and product structure – for topological orbifold models is the focus of this paper.

The observables of the (untwisted) topological sigma model are described by cohomology classes of the target space. Interactions are treated by taking intersections of homology cycles in the moduli space of holomorphic maps (Sect. two). An orbifold is a possibly singular space, defined by equating points related by a group action. In order for the *orbifold* to have a sigma model description at the non-singular points, the metric and complex structure must be preserved by the action

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of the group on the target space. We show (Sect. three) that the observables of the
orbifold model are described in terms of the cohomology of the fixed point
manifolds of the group elements. At the singular points of the group action, there is
an identification of tangent space vectors. Since the fermions of the sigma model
have tangent space indices, the fermionic sector of a twisted state obeys twisted
boundary conditions. These conditions lead quite generally [20, 30] to a fractional
chiral fermion number assigned to the vacuum in this sector. Thus twisted states
have a shifted fermion number. By analogy with the familiar correspondence
between topological observables and cohomology elements (for non-orbifold the-
ories), we may assign Hodge labels based on the chiral fermion numbers of
observables. In this way, we describe the "cohomology" of the singular orbifold.
We show that Poincaré duality is preserved, and in the case of a Calabi–Yau
orbifold by a group action which preserves the unique $(d, 0)$ form, this "cohomol-
ogy" has the Hodge diamond we would expect from a Calabi–Yau manifold. In
fact, in several examples (Sect. four) we show that this cohomology is precisely that
of the manifold one gets by resolution of the singularities. Another check is
agreement with the appropriate Landau–Ginzburg orbifold theory, when the
manifold in question is a Calabi–Yau variety defined by a quasi-homogeneous
polynomial. We offer no general proof of this equivalence.

Computation of the product structure of the ring of observables involves
intersection numbers in an appropriate moduli space. For a correlation function
involving several g_i-twisted observables inserted at points p_i on a Riemann surface,
the moduli space is holomorphic maps having proper monodromies around these
points, or equivalently, holomorphic equivariant maps from an appropriate bran-
ched cover of the Riemann surface. We use this formalism in computing an explicit
example – a detailed computation of the complete chiral ring for the orbifold of
$\mathbf{CP}^1$ by the dihedral group D_4 (Sect. five). These findings can be the generalized to
the higher even dihedral groups D_{2k} and odd groups D_{2k+1} (Sect. six), or to a higher
dimensional target space (Sect. seven). With knowledge of these rings, and in
particular behavior under scale transformations, we can use recent techniques [6]
to try to compute the proper normalization of twist operators in the conformal
limit of large radius (Sect. eight). The $\mathbf{CP}^1$ orbifolds reduce to abelian orbifolds in
this limit, and the requirement of regularity fixes the boundary conditions, giving
the twist field correlations. For higher dimensional spaces, it is unclear whether
regularity is enough to determine the solution.

2. Topological Sigma Models and Quantum Rings

Let us briefly recall the topological sigma model on a Kähler manifold, K. In this
case, the action can be derived as a twisted $N = 2$ model. This twisting leads to an
isomorphism (as vector spaces) between local BRST observables and the states of
the chiral-primary ring. Specifically, we have [31]

$$S = 2t \int_\Sigma d^2z \, \frac{1}{2} g_{IJ} \partial_z \phi^I \partial_{\bar{z}} \phi^J + i\psi^{\bar{i}}_- D_z \psi^i_- g_{i\bar{i}} + i\psi^{\bar{i}}_+ D_{\bar{z}} \psi^i_+ g_{i\bar{i}}$$
$$+ R_{i\bar{i}j\bar{j}} \psi^i_+ \psi^{\bar{i}}_+ \psi^j_- \psi^{\bar{j}}_- \, . \tag{2.1}$$

Here Σ represents the Riemann surface, which, for our purpose will always be of
genus zero, g_{IJ} and $R_{i\bar{i}j\bar{j}}$ are respectively the metric and Riemann tensor of the

target space. D is the pull-back onto Σ of the connection under the map, Φ. The $N = 2$ structure implies a holomorphic $U(1)$ current, by which we may twist the energy-momentum tensor. Mathematically, this is equivalent to redefining the bundles in which the fields live. Specifically, we now take $\psi^i_+ \in \Phi^*(T^{1,0})$ and $\psi^{\bar{i}}_- \in \Phi^*(T^{0,1})$. And, we put $\psi^{\bar{i}}_+ \in \Omega^{1,0}(\Sigma; \Phi^*(T^{0,1}))$, and $\psi^i_- \in \Omega^{0,1}(\Sigma; \Phi^*(T^{1,0}))$, that is, they combine to form a one-form on Σ with values in the pull-back of the tangent space of K: call these components $\psi^{\bar{i}}_z$ and $\psi^i_{\bar{z}}$ respectively. These redefinitions correspond to shifting the spins of the fields by

$$T \to T - \frac{1}{2}\partial J , \quad \bar{T} \to \bar{T} + \frac{1}{2}\bar{\partial}\bar{J} , \tag{2.2}$$

which is equivalent to adding a background gauge field to the spin connection. To make this theory topological, we reinterpret the supersymmetry transformation as a BRST transformation associated to a topological symmetry (in order for this to close off-shell, more fields must be introduced) [31]. We make the replacement

$$Q_L + Q_R \to Q_{\text{BRST}} . \tag{2.3}$$

Thus the topological observables are precisely the chiral-chiral fields, and when the original model is a conformal field theory, i.e. when K is a Calabi–Yau manifold, the elements of the BRST cohomology correspond precisely with the chiral-primary ring of the conformal theory [23]. When the manifold is not Calabi–Yau, the topological theory is still well-defined, and the ring of observables generalizes the chiral primary ring; it can be thought of as a "quantum cohomology ring."

[Note that there is another "twist" we may perform, which, due to a global anomaly, is only defined on a manifold with vanishing first Chern class, i.e. a Calabi–Yau manifold. The observables in this theory have a different cohomological description [31].]

We have

$$S = it \int_{\Sigma} d^2z\{Q, V\} + t \int_{\Sigma} \Phi^*(k) , \tag{2.4}$$

where V is an appropriate pre-potential (see [31]) and $\Phi^*(k)$ is the pull-back of the Kähler form. The second term in (2.4) is a topological term, and for the moment, we restrict ourselves to maps Φ within a given component of the space of maps. That is, we take maps of a given instanton number, so that the second term in (2.4) is constant in this component of maps from Σ to K. By standard arguments based on the vanishing of all correlation functions with Q-exact terms, our calculations reduce to a semi-classical treatment. That is, we may take the large t limit, and restrict ourselves to the moduli space of classical minima, which occur when

$$\partial_{\bar{z}}\phi^i = \partial_z\phi^{\bar{i}} = 0 , \tag{2.5}$$

i.e. Φ is a holomorphic map. Thus the moduli space for this problem is

$$\mathcal{M} = \{\Phi: \Sigma \to K \,|\, \Phi \text{ holomorphic}\} = \bigoplus_i \mathcal{M}_i , \tag{2.6}$$

where i labels the instanton number.

The correspondence between the cohomology of the target space and the local observables (BRST cohomology) is described by replacing form components by the

fermion fields. Let $A = A_{I\bar{J}}dz^I d\bar{z}^J$ be a form on K, written in local coordinates, where I and J are multi-indices. The corresponding operator, $\mathcal{O}_A$, is obtained by replacing $dz \to \psi_z^{\bar{i}}$ and $d\bar{z} \to \psi_{\bar{z}}^i$. The isomorphism of cohomologies is described by the equation

$$\{Q, \mathcal{O}_A\} = -\mathcal{O}_{dA}. \qquad (2.7)$$

Since BRST trivial operators annihilate all correlators, expectation values only depend on BRST cohomology classes. If we label observables by their corresponding forms, this means we may choose the forms to have delta-function support on the cycle to which they are Poincaré-dual. This way of representing the observables clarifies the degree zero instanton sector contribution to observables: the correlation functions will have non-zero contribution only at the points of intersections of the representative manifolds. The degree zero holomorphic maps are simply constants, so the integral over $\mathcal{M}_0$ is just an integral over K. Thus, because of the cancellation of bosonic and fermionic determinants familiar to topological theories, each point of intersection contributes 1 to the correlator; so the degree zero correlations are precisely the intersection numbers of the cycles representing the observables.

Generally, the correlation function must be evaluated by considering the contribution from each component of moduli space. This is done as follows [31]. At a given component of moduli space $\mathcal{M}_i$, we define a manifold $L_{j,i} \subset \mathcal{M}_i$ for each observable $\mathcal{O}_j(p_j)$ to be the set of maps in $\mathcal{M}_i$ which take p_j to a point in the manifold representing the form corresponding to $\mathcal{O}_j$. Then the ith sector contribution to the correlation of any number of observables is given by the intersection number of the $L_{j,i}$. This is equivalent to integrating over the pullbacks of the forms by the evaluation maps at the points of insertions. In equations:

$$\left\langle \prod_{j=1}^n \mathcal{O}_{A_j}(p_j) \right\rangle = \sum_i \bigcap_{j=1}^n (\text{ev}_j^{-1} L_{j,i}) = \sum_i \int_{\mathcal{M}_i} \prod_{j=1}^n (\text{ev}_j^* A_j), \qquad (2.8)$$

where the evaluation map $\text{ev}_j: \mathcal{M} \to K$ is defined by $\text{ev}_j(\Phi) = \Phi(p_j)$. Here we have ignored the second term in (2.4). This is a topological term which has a constant value in each component of moduli space. Thus, if S_i represents the value in the ith component of moduli space, then the ith term in (2.8) must be weighted by e^{-S_i}. Note that the moduli space may need to be compactified in order to have a sensible intersection theory.

It is instructive for us to discuss the $\mathbf{CP}^n$ model as an example [25]. Here we have $K = \mathbf{CP}^n$, which has $h_{ii} = 1, i = 0 \ldots n$, with all other Hodge numbers vanishing. The intersection theory of nontrivial cycles is very simple, then. The intersection number of homology cycles is one if the codimensions sum to n, zero otherwise. Basically, this is because L_i of codimension k can be taken to be the $\mathbf{CP}^{n-k}$ defined by setting k coordinates equal to zero, in an appropriate basis.

Consider $\mathcal{M}_k$, i.e. the holomorphic maps of degree k from $\mathbf{CP}^1$ to $\mathbf{CP}^n$ (we consider genus zero correlations, for these define the ring).[1] These are described by $(n + 1)$-tuples of homogeneous polynomials of degree k in two variables, which act

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[1] I thank S. Axelrod for explaining this to me

as shown below:

$$\Phi(X, Y) = \begin{pmatrix} \phi_{00} & \cdots & \phi_{0k} \\ \phi_{10} & \cdots & \phi_{1k} \\ . & . & . \\ . & . & . \\ . & . & . \\ \phi_{n0} & \cdots & \phi_{nk} \end{pmatrix} \begin{pmatrix} X^k \\ X^{k-1} Y \\ . \\ . \\ . \\ Y^k \end{pmatrix}. \tag{2.9}$$

The homogeneity property insures that scale changes on (X, Y), which are trivial on the $\mathbf{CP}^1$, only result in (trivial) scale changes on the $(n + 1)$-tuple. Now we should ensure that the polynomials, defined by the matrix rows, do not have common zeros. This would make the map Φ ill-defined. However, in the *compactified* moduli space, we allow such points, which can occur as limits of well-defined maps. Basically, if there is a common root, we can factor it out of the $(n + 1)$-tuple of polynomials and get a new, well-defined map (of a lower degree). Thus the only requirement we make on the matrix elements ϕ_{ij} is that they are not all zero. Of course, the matrix Φ is only defined modulo an overall scale. So we have shown

$$\mathcal{M}_k \cong \mathbf{CP}^{(n+1)(k+1)-1}. \tag{2.10}$$

The cohomology ring of K has a single generator X with $X^{n+1} = 0$. The quantum ring is defined by the correlation functions. Consider the correlator $\langle X^a X^b X^c \rangle$. This will be nonzero if there is a k such that $(n + 1)(k + 1) - 1 = a + b + c$. In this case, the instanton action is $e^{-kA} \equiv \beta^k$, where A is the one-instanton action. All these correlators derive simply from the relation

$$X^{n+1} = \beta, \tag{2.11}$$

which defines the chiral ring. Note that the chiral fermion number is conserved if we make the artificial assignment of $n + 1$ as the chiral fermion number of β.

3. The Orbifold Theory

We would like to study these theories when the target space is an orbifold, i.e. we consider the quotient K/G of a Kähler manifold under a group G, which acts on this manifold by isometry. Thus, the metric will be well-defined on the quotient space – the inner product of two vectors in K/G may be computed by choosing any lift of the vectors to K and using the metric on K; G-invariance guarantees independence of the particular lift. Furthermore, we will assume that the action of G preserves the complex structure. That is,

$$g_* \circ J = J \circ g_* \quad \text{for all } g \in G, \tag{3.1}$$

where the asterisk represents push-forward action on vectors, and J is the complex structure. When G acts with fixed points, the orbifold will have a set of singular points, though the string theory is not necessarily singular. If the manifold is not Calabi–Yau, then the quantum field theory is not conformal and not a string vacuum [5]; for K/G to be a "Calabi–Yau orbifold," we must have that G leaves

invariant the unique holomorphic $(d, 0)$ form under pull-back. In either case, though, the topological sigma model is well-defined.[2]

To properly consider the full orbifold theory, we must specify the action of the group G on every operator in the theory. In particular, if we were considering the orbifold as a space for string compactification, then we would need to specify the action of G on the fields representing the rank 16 gauge group. This can lead to phenomenologically desirable symmetry breaking. In any case, we see that the proper definition of a g-twisted state, Θ (where Θ may be any type of quantum field), is that

$$\Theta(\sigma + 2\pi) = g \circ \Theta(\sigma) , \qquad (3.2)$$

where σ is the coordinate along the string.

For our purposes, we will restrict attention to the twisted $N = 2$ theory at hand. The action of G on the bosonic fields Φ is the action considering the fields as coordinates on the manifold K. On the fermionic fields, which involve (pull-backs of) the tangent bundle TK, the action is induced from the coordinate action by the push-forward of vectors. At any point p on K, the tangent space T_p is identified with T_{gp}. However, at a fixed point f of g (i.e. $gf = f$), we must identify tangent vectors in T_f related by the action of g. More precisely, we must identify all tangent vectors related by the stabilizer group of elements fixing f: $S(f) = \{g \in G \mid gf = g\}$. Because g acts by isometry, each $g \in G$ defines an element of $SO(2d)$ at a fixed point, where d is the complex dimension of K ($SO(2d)$ may be replaced by some subgroup depending on the properties which G preserves). The tangent space for the orbifold (denoted T') at f is thus

$$T'_f = \mathbf{R}^{2d}/S(f) . \qquad (3.3)$$

On the fixed point sets, i.e. where $S(f)$ is non-trivial, the tangent space is not a vector space but the cone (3.3), so the orbifold is not a smooth manifold; it has a conical singularity.

3.1. Observables in the Orbifold Sigma Model. We have already discussed the isomorphism between local operators (BRST observables) and the cohomology classes of the target manifold. What, then, are the observables of the topological theory on the orbifold? To answer this question, we may begin by recalling the standard lore or orbifold theories [11]. For these theories, the Hillbert space of the theory splits into a direct sum of twisted sectors, one for each conjugacy class $\{g\}$ in the group G:

$$\mathcal{H} = \bigoplus_{\{g\}} \mathcal{H}_{\{g\}} . \qquad (3.4)$$

In each of these sectors, only the G-invariant states survive the group projection. A brief word on our notation is in order. Really, the Hilbert space splits into one sector per group element. However, the action of group elements not

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[2] As stated in [31], this follows from the positivity of the fermionic determinant, which allows us to define it as a function of the moduli. In general the fermionic determinant gives a line bundle over the moduli space of theories, which will lead to an anomaly. The anomaly cancellation condition for the topological theory of the inequivalent twist is that the manifold be Calabi–Yau

commuting with g permutes the sectors in the conjugacy class of g.[3] We thus define

$$\mathscr{H}_{\{g\}} = \bigoplus_{g \in \{g\}} \mathscr{H}_g = \bigoplus_{i=1}^{|\{g\}|} \mathscr{H}_{r_i g r_i^{-1}} \tag{3.5}$$

for an appropriate set $\{r_i\}$. We further take the projection onto $\{g\}$-invariant states. Now let

$$\zeta_{\{g\}} = (\zeta_g, \zeta_{r_1 g r_1^{-1}}, \ldots) \in \mathscr{H}_{\{g\}} . \tag{3.6}$$

The action of any h in the centralizer, $h \in C(g) \equiv \{k \,|\, kg = gk\}$, on $\zeta_{\{g\}}$ is defined by

$$h\zeta_{\{g\}} = (h\zeta_g, r_1 h r_1^{-1} \zeta_{r_1 g r_1^{-1}}, \ldots) . \tag{3.7}$$

This is still $\{g\}$-invariant. With these definitions, each $\mathscr{H}_{\{g\}}$ is invariant as a vector space under the action of any group element. Thus, the concept of G-invariant states now makes sense, and the state $\dfrac{1}{|C(G)|} \sum_{h \in C(g)} h\zeta_{\{g\}}$ is group invariant. In effect, we only have to take a $C(g)$ projection.

A similar description of the observables is found for the orbifold sigma model. Once again we will make notation simpler by eliminating the conjugacy class label, and only considering $C(g)$-invariant states. By the above procedure, in which g represents $\{g\}$, this suffices.

As always, we begin with the untwisted sector. Here we have all of the observables in the original theory (the cohomology classes of K), and must project onto those which are G-invariant. That is, we are interested in the differential forms A obeying $g^* A = A$. Let $\mathcal{O}_1^{p,q}$ represent the untwisted observables in the orbifold theory with fermion-anti-fermion number (p, q), where for simplicity in the following we have chosen the anti-chiral fermion number to be positive; thus the total fermion number is $p - q$. (Although the chiral fermion numbers will only be conserved for Calabi–Yau orbifolds, we will be able to make sense of chiral fermion number violation as we did following (2.11).) We see that we have

$$\mathcal{O}_1^{p,q} = H_G^{p,q}(K) , \tag{3.8}$$

where the subscript represents G-invariance. By considering the Poincaré duals of these forms, we may think of them as lying on the quotient K/G. In this way, we are able to see the equivalence between $H_G^*(K)$ and the simplicial cohomology $H_{\text{simp}}^*(K/G)$ of the coset space, which is well-defined even though K/G is not a smooth manifold. This interpretation allows us to show the familiar equivalence [11] between the untwisted $\text{Tr}(-1)^F$ and the Euler characteristic. Since (anti-)chiral fermion numbers correspond to (anti-)holomorphic form degree, we have

$$\text{Tr}_{\mathscr{H}_1}(-1)^F = \sum_{p,q=0}^{d} h_G^{p,q} = \chi_{\text{simp}}(K/G) . \tag{3.9}$$

In the above formula, $\mathscr{H}_1$ represents the untwisted, G-invariant Hilbert space, and the $h_G^{p,q}$ are the Betti numbers of the G-invariant simplicial cohomology. In fact the value of (3.9) may be calculated by considering the operator which projects to

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[3] For a string obeying $X(2\pi) = gX(0)$ we see that $hX(2\pi) = hgX(0) = (hgh^{-1})hX(0)$

group invariant states, $P = \dfrac{1}{|G|}\sum g$ (note $G = C(1)$). Now in the calculation of $\mathrm{Tr}_{\mathscr{H}_1}(-1)^F$ by the path integral, the presence of g (from the group projection) in the trace yields the Lefschetz number of g, which is the Euler number of the fixed point sets [32]. Hence [11],

$$\mathrm{Tr}_{\mathscr{H}_1}(-1)^F = \frac{1}{|G|}\sum \chi(M_g) \ . \tag{3.10}$$

Note that (3.10) agrees with the right-hand side of (3.9), as it should.[4]

Consider now a g-twisted ground state, which corresponds to a string sitting at a point. If this state is twisted, the point must lie in the fixed point set of g. Let us call this manifold M_g.[5] These manifolds will play a crucial role in our analysis, so we pause here to consider the geometry of these spaces. It is important for us to show the complex structure of M_g. In fact, we may use the same J that we used for K, considering the tangent vectors on TM_g as vectors in the larger space TK (specifically, we use the push-forward under the inclusion map). Let $v \in T_f M_g$. We may express v as the "time" derivative of a path $Q(\tau)$ on M_g, i.e. $v = \dot{Q}(\tau_0)$. Now since the action of g is compatible with J, by (3.1) we have

$$g_* \circ J(v) = J \circ g_*(v) = J \circ g_*(\dot{Q}(\tau_0)) = J(\dot{Q}(\tau_0)) = J(v) \ , \tag{3.11}$$

where we have used the fact that $g_*(\dot{Q}) = \dot{Q}$, since Q lies entirely in the fixed point manifold M_g. So we see that $J(v)$ is fixed under g_*. But since

$$TK|_{M_g} = TM_g \oplus NM_g \ , \tag{3.12}$$

where NM_g is the normal bundle on M_g, on which g_* acts nontrivially, we see that $J(v) \in T_f M_g$, which shows that J is a complex structure on M_g. Similarly, one can show that M_g is Kähler. Therefore, it makes sense to speak of the Hodge numbers of the fixed point manifolds. The Dolbeault cohomology classes of these spaces will correspond to observables in the K/G theory.

Finally, we should consider the nature of the G-action on the normal bundle.[6] We know that G respects the metric, hence also the volume form. In a real basis $\{x^i\}$, the volume form is a multiple of $\eta = dx^1 \wedge \ldots \wedge dx^{2d}$. Now $g^*\eta = \eta$ means that at a fixed point, the pull-back action of g^* is represented by a matrix in $GL(2d, \mathbf{R})$ (g is invertible) satisfying

$$\overset{2d}{\bigwedge} g^* = 1 \ ; \tag{3.13}$$

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[4] This was proved, for example in [26]. The basic point is that we may take a simplicial decomposition of K on which G has a well-defined linear action on simplices of a given dimension. Then the simplices fixed under G form a decomposition of the fixed manifold. Now, when we sum over the group elements in the projection and take the trace over homology classes, we get zero from the other classes and the cardinality of the group for each fixed class

[5] To see that this space is a manifold, consider the linear g-action on the tangent space of K at a fixed point f of g. We denote this (push-forward) map by g_*. Then the exponential map $\exp: T_f K \to K$ will diffeomorphically map the linear subspace annihilated by dg onto the fixed point set of g. (Since G respects the metric and complex structure on a Kähler manifold, it commutes with the connection, and hence the exponential map as well.) This coordinatization shows why M_g is a manifold. Similar considerations reduce other questions about M_g to linear algebra

[6] G acts trivially on the tangent bundle since it fixes all possible paths in M_g and hence all vectors

but since the highest exterior power of a matrix is its determinant, we find that $g^* \in SL(2d, \mathbf{R})$. Note that the same is true of g_*, since $g_* = [(g^*)^T]^{-1}$ which can be easily seen by preservation of $\left\langle dz^i, \dfrac{\partial}{\partial z^j} \right\rangle = \delta^i_j$. The above reasoning extends to the other conditions we've placed on g. Since g preserves the complex structure, which means locally that it doesn't mix z's and $\bar{z}$'s, we can see that $g^* \in U(d)$ in a complex basis. We also require that g^* preserves the holomorphic top forms, which restricts the determinant to be unity, i.e. $g^*, g_* \in SU(d)$. But we already know that g_* decomposes into the identity on TM_g and a matrix which, by abuse of notation, we call g. That is, $g_* = 1 \oplus g$ in (3.12). Collecting this data, we have that g is *non-trivial* and

$$g \in SU(\mathrm{codim}_{\mathbf{C}} M_g) \,. \tag{3.14}$$

As an immediate corollary, we see that for group actions satisfying the Calabi–Yau conditions we have imposed, there are no fixed manifolds of codimension one. We will need this result.

　　To understand twisted observables one must first understand twisted interactions, a subject of the next subsection. Here we will need the following result, which is proved in that subsection. Essentially, twisted observables are also differential forms, but the only piece which matters in correlations is the value of the pull-back onto M_g by the inclusion map. Of course, for the untwisted case, this characterization is still true, since $M_1 = K$. Then, BRST cohomology corresponds to differential cohomology on M_g. Note, then, that the original form A need not be closed on K: if $i^* A$ is closed, then the non-closed part of A on K must contribute zero always. Hence, we have the twisted version of (3.8) for the g-twisted sector:

$$\mathcal{O}_g \cong H_G(M_g) \,, \tag{3.15}$$

where we have intentionally omitted the Hodge labels (p, q). Once again, we must keep in mind that each label represents a full conjugacy class. In this case, the different sectors within a conjugacy class are equivalent since $r : M_g \to M_{rgr^{-1}}$ is a holomorphic homeomorphism. For simplicity in labeling, we have dropped the conjugacy class symbol.

　　Recall now that the equivalence of (anti-)holomorphic form degree with (anti-)chiral fermion number was due to the construction of the observables with fermionic fields of definite chirality. Implicit in the above was that the vacuum had fermion number equal to zero. This reasoning breaks down in the twisted sector because of a shift in the fermion number of the vacuum [20, 30]. Although a constant bosonic field at a fixed point describes a vacuum, the fermionic fields, even though at a fixed point, cannot be constant – for to be twisted they must go from one tangent vector to the g-translated vector, and g acts nontrivially on all fields corresponding to normal directions. Thus the fermionic vacuum corresponds to a sector with generalized boundary conditions on the ends of the string; the shift in the chiral fermion number of the vacuum is a general phenomenon for fermions in one real spatial dimension obeying nontrivial boundary conditions (Originally, in [20], the fermions were in the presence of instantons). As we have previously discussed, g acts nontrivially on the normal bundle, NM_g, and trivially on the tangent bundle TM_g. Focusing on the chiral fermions, let us imagine just one chiral fermion in one spatial dimension obeying generalized boundary conditions. It was shown in [30] that when the (chiral) fermion number is properly regularized to

account for an infinite spectrum of energies, the more general boundary condition $\psi(\sigma + 2\pi) = e^{-2\pi i f}\psi(\sigma)$ leads to the non-zero result[7]

$$F = f \tag{3.16}$$

for the twisted fermion vacuum. We take $F = 0$ for the periodic case, corresponding to the unique Neveu–Schwartz vacuum (the fermionic fields are periodic after twisting). This argument extends simply to the anti-chiral and multiple-fermion cases. Choosing a basis for the tangent space so that g is diagonal, we can see that we have a separate shift for each of the chiral fermions. If the eigenvalues of g are $e^{-2\pi i f_j}$, $j = 1 \ldots n$, where $n = \text{codim}_{\mathbb{C}} M_g$, then the chiral fermion number of the vacuum shifts by

$$F_g = \sum_{j=1}^{n} f_j, \tag{3.17}$$

where we always take the f_j to satisfy $0 < f_j < 1$. The formula (3.17) looks like $F_g = (i/2\pi)\text{Tr}(\log g) = (-i/2\pi)\log\det g$, but is of course different (for example if α is a primitive third root of unity, then $\text{diag}(\alpha, \alpha, \alpha)$ yields $F_g = 1$ or 2). However, for Calabi–Yau orbifolds we do have $\det g = 1$, which means that F_g is integral. This gives us

$$0 < F_g < n \quad \text{for } g \neq 1, \quad F_g \in Z. \tag{3.18}$$

Some words are in order about the choice in defining fermion number. We have chosen the untwisted sector to have $F_1 = 0$, of course, and have $0 < f_j < 1$ for nontrivial g. The reasons for this choice are twofold. One way to set the fermion number is through interactions. Namely, the three-point functions on the sphere determine the ring structure of the observables (the chiral ring). For the vacua, these correlation functions correspond to twist field calculations. By requiring the twist fields to respect fermion grading (in the Calabi–Yau case this is possible), we are led to unambiguous assignments. We will encounter an example of this in Sect. five. The other way of determining the value of the shift is to consider the path of a twisted string with no oscillator modes. In one complex dimension, that path looks like $X(z) = z^{f_j}$, which is non-singular as $z \to 0$ for f_j positive, and is minimal for $f_j < 1$. (By this we mean that $X(z) = z^{1+f_j}$ could be thought of as the product of a twisted string and a closed untwisted string.) Finally, note that F_g is now well-defined and independent of which point on (the connected component of) M_g we choose to determine it; for g has a finite order, say m, so $f_j = k_j/m$, which is fixed, since it cannot vary continuously along a component of M_g (other components have forms corresponding to separate operators with different shifts).

We should also point out that the shift is the same in the anti-chiral sector: $F_a = F_c$, which yields the same shift of form degrees. (Remember, we choose the anti-fermion number to be $+1$ for an anti-fermion.) In the anti-chiral sector, g acts

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[7] The proof is straightforward [30]: the fermion number of the vacuum is the integral of the energy density for all energies less than zero. This is the filled fermi sea. We regularize this fermion number by subtracting the total number of fermionic states in the Hilbert space – a (perhaps infinite) constant – and inserting a convergence factor. We have $F = C - \frac{1}{2}\lim_{s\to 0}\int_{-\infty}^{\infty} dE\rho(E)\text{sgn}(E)\exp - s|E|$. We choose the constant C to be $-\frac{1}{2}$ by requiring the periodic fermionic vacuum to have zero fermion number. Using the plane wave solutions $\psi_n(\sigma) = \exp i(n - f)\sigma$ with $E_n = (n - f)$ yields the result for the boundary conditions stated above

by its complex conjugae (i.e. $f_j \to 1 - f_j$) but the change in the Hamiltonian compensates for this difference, yielding the desired result.

3.2. Twisted Interactions, Observables, and Poincaré Duality. In this section we will prove Eq. (3.15) by carefully considering interactions of twisted observables. We will also show that the interpretation of the observables as cohomology elements does not run counter to Poincaré or Hodge duality, or to the Calabi–Yau characterization. Namely, we show that the Hodge diamonds of Calabi–Yau orbifolds have all the properties one would require of Calabi–Yau manifolds.

Let us briefly recall the procedure for computing interactions of orbifolds by the path integral method [17]. Consider a loop $X(\sigma)$ twisted such that $X(\sigma + 2\pi) = gX(\sigma)$. As a map from the Riemann surface, a configuration corresponding to a g-twisted state at $z = 0$ must satisfy $\Phi(e^{2\pi i}z) = g\Phi(z)$. Orbifold configurations involve multivalued maps $\Phi: \Sigma \to K$ with proper monodromies around points of insertion of twisted states. We can find an equivalent description with single-valued maps by choosing a cover $\tilde{\Sigma}$ of Σ on which G acts by automorphism (preserves metric, complex structure): $\Sigma \cong \tilde{\Sigma}/G$. Now for a g-twisted state at z_0, we choose our group action such that a small loop around z_0 (i.e., one not enclosing other points of insertion) will lift to a line from $\tilde{z}$ to $g\tilde{z}$, say. For an interaction involving observables twisted by $g_1, \ldots, g_n$ (with $\prod_i g_i = 1$ for $\Sigma \cong S^2$ the selection rule, viewing all states as incoming), at $p_1, \ldots, p_n$, we consider $\tilde{\Sigma}$, a G-cover of Σ, with loops around the p_i lifting to lines with endpoints separated by the action of g_i. In particular, continuity of the G action for very small loops means that the p_i descend from fixed points of g_i on $\tilde{\Sigma}$: $g_i \tilde{p}_i = \tilde{p}_i$. Now $\tilde{\Phi}: \tilde{\Sigma} \to K$ obeying

$$\tilde{\Phi}(g\tilde{z}) = g\tilde{\Phi}(\tilde{z}) \tag{3.19}$$

is a single-valued map with equivalent information. That is, instead of $S(\Phi; \Sigma)$ we consider the same theory on $\tilde{\Sigma}$ with $\tilde{\Phi}$ and the pull-back metric (under the projection from the cover), with the exception that we must divide by $N = |G|$,[8] since we have overcounted the area by the order of the covering. The genus of $\tilde{\Sigma}$ can be easily obtained from knowledge of G and the orders of the g_i [17] (see footnote following (5.9)); finding $\tilde{\Sigma}$ explicitly, however, may be very difficult. Of course the different thing about orbifold interactions is that each interaction requires a new $\tilde{\Sigma}$, and the functional integral will be taken only over equivariant maps, i.e. maps obeying (3.19).

Let us turn now to explaining (3.15). Generally, a candidate observable can be likened to a differential form (not necessarily a cohomology class) as discussed in Sect. two. There it was explained that exact forms should be set to zero, while the interest with BRST-compatible observables forced us to consider cohomology classes. This analysis must be reconsidered in the case of twisted observables. For example, suppose we were to consider a correlation function involving a g-twisted state at a point p on Σ. Then by the above, we would need to consider equivariant maps around $\tilde{p}$. But the equivariant condition (3.19), together with our observation that $\tilde{p}$ lies at a fixed point of g on $\tilde{\Sigma}$ means that at $\tilde{p}$ we have $\tilde{\Phi}(g\tilde{p}) = \tilde{\Phi}(\tilde{p}) = g\tilde{\Phi}(\tilde{p})$.

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[8] The cover need not be of order $|G|$, actually. The order of the cover can be chosen to be the order of the group generated by the g_i

Hence $\tilde{\Phi}(\tilde{p}) \in M_g$. This is an important observation! For instance, suppose we consider a differential form $A \neq 0$ on K. If, however, the restriction of A to M_g were equal to zero, i.e. $A|_{M_g} = 0$, then the observable corresponding to A would always be evaluated at some $\tilde{\Phi}(\tilde{p}) \equiv m \in M_g$ and hence always give zero. Thus as a g-twisted observable, we could set $\mathcal{O}_A = 0$; for every correlation function with $\mathcal{O}_A$ would be zero.

Of course the (not necessarily closed) form A must be invariant under the group action to be an observable in the orbifold theory. This gives us a condition for its value at m, which we just saw had to lie in the fixed manifold M_g. For suppose A had indices pointing in the direction normal to M_g. We know that g acts *nontrivially*, i.e. has nonunital eigenvalues, in these directions, so invariance at m – which is invariance under the differential matrix g – is impossible! Any normal components are projected out. Thus, we see

$$A|_{M_g} = i^*A + (\text{Noninvariant Terms} \to 0) , \tag{3.20}$$

where we have abused notation slightly by considering i^*A as a form on $TK|_{M_g}$ (this can be done because there is a 1-1 imbedding of TM_g into $TK|_{M_g}$.) Thus, since all values of A outside the manifold M_g do not contribute to correlation functions, and since the normal components of $A|_{M_g}$ are projected out, the observable A is completely determined by i^*A.

We are now eready to classify the g-twisted observables. Since M_g is imbedded in K, the map $i^*: \Omega^*(K) \to \Omega^*(M_g)$ is onto. Therefore, forms expressible as i^*A are isomorphic to all differential forms on M_g. Now, since i^* and h^* commute, where $h \in C(g)$, the invariant forms are just the $C(g)$-invariant forms on M_g (as always, g represents any element of the conjugacy class $\{g\}$). Finally, we must impose the BRST symmetry, which means only considering forms such that $di^*A = 0$ modulo all forms i^*dA. But pull-back commutes with exterior derivative, and so we must take all closed forms and mod out by all exact forms on M_g (remember i^* is onto). Thus we have shown (3.15).

We now know the observables associated to an orbifolded topological sigma model. Furthermore, by carefully keeping track of the fermion number shift associated to twisted boundary conditions on fermions, we were able to assign the correct fermion numbers to these observables. Now by analogy with the untwisted case, it is tempting to assert that these observables correspond to cohomology classes associated to the singular space K/G, with holomorphic form degrees given by the chiral fermion numbers. So let us consider what the Hodge diamond of such a space would be. In several examples in the next section, we show agreement with the Betti numbers of the resolutions of orbifolds.

In the g-twisted sector, the vacuum has chiral-anti-chiral fermion number (F_g, F_g), as we've defined it. We saw in (3.15) that the space of observables in this sector was isomorphic to the $C(g)$-invariant cohomology of M_g. These observables are built from untwisted fermion operators, which have their usual fermion numbers (form degrees), and the twist field part which shifts the vacuum. For example, the identity operator in the twisted sector is the actual twist field. Thus these operators have their degrees shifted by (F_g, F_g). We may thus *define* the twisted Hodge numbers of the orbifold K/G to be given by

$$H^{p,q}(K/G) \equiv \bigoplus_{\{g\}} H^{p - F_g, q - F_g}_{C(g)}(M_g) \tag{3.21}$$

for any g representing $\{g\}$.

Does this definition preserve the familiar structure of the Calabi–Yau Hodge diamond? Yes. This is easily seen by realizing that $M_g = M_{g^{-1}}$ and $C(g) = C(g^{-1})$. Thus if $\{k_j/m\}$ represents the f_j for the action of g on the normal bundle, then $\{(1 - k_j)/m\}$ represents g^{-1}. We conclude that

$$F_{g^{-1}} = n - F_g \,, \tag{3.22}$$

where $n = \text{codim}_{\mathbb{C}} M_g$. This shows us that there is indeed Poincaré duality. Namely, let θ be a $C(g)$-invariant (p, q) form on M_g. Then in the language of (3.21):

$$\theta \in H^{p + F_g, q + F_g}(K/G); \quad 0 \leqq p, q \leqq n \,. \tag{3.23}$$

Let $\tilde{\theta}$ be the Poincaré dual of θ in $M_{g^{-1}} = M_g$. As observables, the Poincaré dual of θ is $\tilde{\theta}$ in the sector twisted by g^{-1}. This is easily seen; for

$$\tilde{\theta} \in H^{(\dim_{\mathbb{C}} M_{g^{-1}} - p + F_{g^{-1}}, \dim_{\mathbb{C}} M_{g^{-1}} - p + F_{g^{-1}})}(K/G) \,. \tag{3.24}$$

Using (3.22), we see

$$\dim_{\mathbb{C}} M_{g^{-1}} - p + F_{g^{-1}} = d - (p + F_g) \tag{3.25}$$

and likewise for q. So Poincaré duality of K/G is shown. We note here that $\tilde{\theta}$ has the product structure of a Poincaré dual as well. That is, if we consider the correlation function $\langle \theta(p) \tilde{\theta}(q) \rangle$ on the sphere, then by considering θ and $\tilde{\theta}$ to have support only on their Poincaré duals, these will intersect only at a single point, call it x. So, going to the N-fold cover of the sphere with two fixed points, where N is the order of g (i.e. another sphere, with g acting as rotation by $2\pi/N$), we find a single equivariant holomorphic map of degree zero – the constant map x. Thus

$$\langle \theta(p) \tilde{\theta}(q) \rangle = 1 \,. \tag{3.26}$$

This suggests $\theta \tilde{\theta} = X$, where X represents the volume form on K (which is an untwisted observable). As we have discussed, however, nonabelian observables are composite operators. This complicates the product structure. The identity (3.26) requires the knowledge of the dimension of moduli space of equivariant holomorphic maps of given degree; we must show that there is no higher component of moduli space of dimension d containing an equivariant map with this property. This is easily seen in the examples we compute, as the dimension increases with instanton number.

We must also show that the $(p, 0)$ and $(0, q)$ cohomological structure of a Calabi–Yau manifold is preserved in the K/G theory. In fact the above proof suffices to show this. Since $F_g > 0$ for all non-trivial g, we see from (3.21) that no twisted sector can contribute to $H^{*, 0}(K/G)$ or $H^{0, *}(K/G)$. By the duality proven above, the same is true for $H^{*, d}(K/G)$ and $H^{d, *}(K/G)$. Finally, since the volume and holomorphic top forms are group invariant, the familiar structure of the Hodge diamond for Calabi–Yau orbifolds is preserved. As an example, let K be a three-fold with M_g codimension two (codimension one is impossible by (3.14)). Now (3.18) tells us that $F_g = 1$ and the Hodge diamond of M_g fits right in the center of the diamond $H^{**}(K/G)$.

To what extent can we show the equivalence of our cohomology with the standard cohomology of the resolved manifold? We know of no general proof (nor is there a complete understanding of the relationship between the Landau–Ginzburg models and geometry – see [26, 27, 2] and references therein). Let us thus concentrate on a less lofty equivalence – that of the Witten index or Euler

characteristic. As was shown quite generally in [11], the Witten index can be computed for the orbifold theory to be

$$\chi_{\text{orbifold}}(K/G) = \sum_{\{g\}} \frac{1}{|G|} \sum_{h \in C(g)} \chi(M_{g,h}) \,. \qquad (3.27)$$

Now by the footnote below (3.10), we can see that the h sum simply computes the Euler number of $C(g)$-invariant forms for each M_g. Thus,

$$\chi_{\text{orbifold}}(K/G) = \sum_{\{g\}} \chi_{C(g)}(M_g) \,. \qquad (3.28)$$

Since the fermion number shift of the observables always changes the form degree by $2F_g$, an *even* number for a Calabi–Yau orbifold, the "Euler number" of our orbifold, calculated directly from the counting of observables, agrees with expectations.

4. Hodge Numbers of Orbifolds: Some Examples

As a concrete example of a Calabi–Yau orbifold which is not expressible as a complete intersection (and hence has no simple Landau Ginzburg description – see [16]) we may consider the Z orbifold $Z = (T \times T \times T)/\mathbf{Z}_3$, where T is a torus with modular parameter $\tau = \lambda \equiv e^{2\pi i/3}$ and the $\mathbf{Z}_3$ group action is generated by diagonal multiplication by λ. Note that in this example $T \times T \times T$ is not Calabi–Yau, but the quotient gives a group-invariant cohomology with Calabi–Yau structure. There are 27 fixed points of this action, all of which have a Z_3 action on the (three-dimensional) normal bundle which is simply diagonal multiplication by α^{-1} (we must remember that tangent vectors transform contravariantly). Now let g be the generator of the $\mathbf{Z}_3$ action. We have $(f_1, f_2, f_3) = (1/3, 1/3, 1/3)$, which gives us $F_g = 1$. Thus, the 27 vacua in the g-twisted sectors all contribute to $H^{1,1}(Z)$. In the g^2 sector, $F_{g^2} = 2$, so we have a contribution of 27 elements to $H^{2,2}(Z)$. In the untwisted sector, the invariant forms contribute nine elements $(dz_i \, d\bar{z}_j)$ to $H^{1,1}(Z)$ and also nine (the duals) to $H^{2,2}(Z)$, in addition to the standard volume, identity, holomorphic and antiholomorphic forms. This analysis agrees with the Hodge structure of the resolution of Z [3].

Let us now compute another example which can be directly compared to a resolved manifold. We start with the quintic hypersurface K in $\mathbf{CP}^4$ defined by the zero locus of the homogeneous polynomial $W(X) = \sum_{i=1}^{5} X_i^5$. Now automorphisms of $\mathbf{CP}^4$ (given by $PGL(5)$) which leave $W(X)$ fixed will act on K. Let us consider the orbifold of K by $G = \mathbf{Z}_5$, where the generator g of G acts by

$$g : (X_1, X_2, X_3, X_4, X_5) \to (X_1, \alpha X_2, \alpha X_3, \alpha^4 X_4, \alpha^4 X_5) \,, \qquad (4.1)$$

where $\alpha = e^{2\pi i/5}$. If we recall [5] that the holomorphic three-form has a polynomial representation as $\prod_i X_i^3$ then we can easily see that this form is preserved (because the transformation acts analytically, G respects the complex structure as well).

Now it is simple to do our fixed point analysis. First, in the untwisted sector, we search for invariant forms. K has Hodge numbers $h^{1,1} = 1$, $h^{2,1} = 101$. The Kähler form (equivalent to complex structure) is preserved since G acts holomorphically, so it remains to calculate which of the 101 forms of $H^{2,1}$ are invariant. These have representatives as homogeneous polynomials of degree five, modulo the

polynomial ideal generated by X_i^4 [5]. Those monomials which remain invariant under the action (4.1) represent invariant forms [33]. It is not difficult to see that

$$X_1^3 AU \ (4), \ X_1 A^2 U^2 \ (4), \ A^3 B^2 \ (4), \ U^3 V^2 \ (4), \ X_1 ABUV \ (1) \qquad (4.2)$$

represent the seventeen invariant forms in $H_G^{2,1}$, where $A \neq B$ range over X_2, X_3 and $U \neq V$ range over X_3, X_4 (the numbers of such forms are in parentheses). First note that g has ten fixed points:

$$M_g = \{(0, 1, -\alpha^m, 0, 0), (0, 0, 0, 1, -\alpha^n): m, n = 0, \ldots, 4\} \qquad (4.3)$$

(this is the full set of fixed points – others are related by projective equivalence). Note that all group elements have the same fixed point sets; since these are discrete, we see that the vector bundle NM_g, which is trivial, has rank three. What is the action of g? Let us consider the point p, an element of the first set of five fixed points listed in (4.3). We can coordinatize the manifold K near p by $(\varepsilon_1, 1 + \varepsilon_2, -\alpha^m + \varepsilon_3, \varepsilon_4, \varepsilon_5)$. Now we may fix $\varepsilon_2 = 0$ by projective invariance, and use the defining quintic equation for K to determine ε_3. In this way $(\varepsilon_1, \varepsilon_4, \varepsilon_5)$ represent a basis for differentials near p. It is simple to see then that g acts by $\mathrm{diag}(\alpha^4, \alpha^3, \alpha^3)$ on these differentials. Since tangent vectors transform contravariantly to differentials, we find that $(f_1, f_2, f_3) = (4/5, 3/5, 3/5)$ (recall the hidden $(-)$ sign) and thus $F_g = 2$. Analyzing the second set of fixed points for g gives $F_g = 1$. Thus, in the g-twisted sector, we have $h_g^{2,2} = h_g^{1,1} = 5$, since each fixed point has a single invariant cohomology element. The same is true for g^2, g^3, g^4. We conclude that the orbifold observables have the structure exhibited in the figure below: $h^{1,1}(K/G) = h^{2,2}(K/G) = 21$; $h^{2,1}(K/G) = h^{1,2}(K/G) = 17$.

$$h_1^{**} \quad \begin{array}{|c|c|c|c|} \hline 1 & 0 & 0 & 1 \\ \hline 0 & 17 & 1 & 0 \\ \hline 0 & 1 & 17 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array}$$

$$\sum_{g \neq 1} h_g^{**} \quad \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 20 & 0 \\ \hline 0 & 20 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}$$

$$h^{**}(K/G) \quad \begin{array}{|c|c|c|c|} \hline 1 & 0 & 0 & 1 \\ \hline 0 & 17 & 21 & 0 \\ \hline 0 & 21 & 17 & 0 \\ \hline 1 & 0 & 0 & 1 \\ \hline \end{array}$$

Indeed the above numbers agree with the Hodge numbers of the resolved manifold of this singular space [15]. The same result can be obtained by considering an appropriate Landau–Ginzburg orbifold [16, 18, 29]. Namely, the topological

sigma model on K corresponds to the $N = 2$ superconformal Landau–Ginzburg model with superpotential $W = \sum_{i=1}^{5} \Phi_i^5$, orbifolded by the group j, which is generated by $j = e^{2\pi i J_0}$ [16]. If we consider the orbifold of this theory (i.e. we take $W/(j \times G)$), then a careful treatment of the $U(1)$ charges leads to this theory: one must identify the Hodge numbers (p, q) with $(J, 3 - \bar{J})$ of the NS sector of the $N = 2$ LG theory. Note that we are only interested in j cosets; for example, we consider all elements in the g, gj, gj^2, gj^3, gj^4 sectors to lie in the g-twisted sector, and of course only consider group-invariant states [18].

Let us consider a case involving a fixed manifold. Again we consider $W(X) = \sum_{i=1}^{5} X_i = 0$ in $\mathbf{CP}^4$. We now orbifold by the $\mathbf{Z}_5$ group generated by g:

$$g : (X_1, X_2, X_3, X_4, X_5) \rightarrow (\alpha X_1, \alpha^4 X_2, X_3, X_4, X_5) . \qquad (4.4)$$

Now $M_g = \{\mathbf{X} \in K : X_1 = X_2 = 0\}$. This is clearly a one (complex) dimensional space; we can compute its Euler number by a simple application of the adjunction formula for Chern classes (see, e.g., [16]). Since M_g is defined by the zero locus of the three polynomials W, X_1, X_2 of orders 5, 1, 1, we have

$$c(M_g) = \frac{(1 + J)^5}{(1 + 5J)(1 + J)(1 + J)} = 1 - 2J , \qquad (4.5)$$

where J is the Kähler form. This yields $\chi(M_g) = -10$. Of course all forms are group-invariant since they are invariant under g, the generator (this is true in all twisted sectors since the order of the group is prime). We may now use $(\varepsilon_1, \varepsilon_2)$ as infinitesimal coordinates normal to M_g. Then g^* acts by $\text{diag}(\alpha, \alpha^4)$, which gives $F_g = 1$, as it must for the Hodge diamond of M_g to fit into the orbifold cohomology without disturbing the Calabi–Yau properties. This same structure is repeated for each of the four non-trivial group elements. The results are summarized in the following tables:

1	0	0	1
0	25	1	0
0	1	25	0
1	0	0	1

h_1^{**}

0	0	0	0
0	24	4	0
0	4	24	0
0	0	0	0

$\sum_{g \neq 1} h_g^{**}$

1	0	0	1
0	49	5	0
0	5	49	0
1	0	0	1

$h^{**}(K/G)$

Once again, we find complete agreement with the appropriate orbifold of the corresponding Landau–Ginzburg model.

As a final example of computing the Hodge numbers of an orbifold, we consider the following mirror pair. Let

$$W = \sum_{k=1}^{5} X_k^5 - 5\psi \prod_{k=1}^{5} X_k \tag{4.6}$$

define a variety $M = (W = 0) \subset \mathbb{CP}^4$. This is a Calabi–Yau space, as is easily seen from the adjunction formula. Note that W is the most general quintic invariant under the $Z_5 \times Z_5 \times Z_5$ action generated by diagonal multiplication by

$$g_1 = (\alpha^1, 1, 1, 1, \alpha^4)$$

$$g_2 = (1, \alpha^1, 1, 1, \alpha^4)$$

$$g_3 = (1, 1, \alpha^1, 1, \alpha^4) . \tag{4.7}$$

Note $g_4 = (1, 1, 1, \alpha^1, \alpha^4) = (g_1 g_2 g_3)^{-1}$ is not independent.

We must determine the fixed point structure of each of the 125 elements of the group. This is simplified by noting that whenever more than one homogeneous coordinate is multiplied by the same power of α, then there will be a fixed point set determined by setting all other coordinates to zero. The results are summarized in the following table, where we have denoted any (complex) curve by C, and a number indicates the number of discrete fixed points; group elements are denoted by the exponents of α: e.g. $g_1 = (1, 0, 0, 0, 4)$.

$\sharp$	g	Example	M_g	χ	χ_{inv}	
1	1	$(0, 0, 0, 0, 0)$	M	-200	0	
12	g_i^n	$(1, 0, 0, 0, 4)$	C	-10	2	
12	$g_i^{n_i} g_j^{n_j}$ $n_i + n_j = 5$	$(1, 4, 0, 0, 0)$	C	-10	2	
24	$g_i^{n_i} g_j^{n_j}$ $n_i + n_j \neq 5$	$(1, 2, 0, 0, 2)$	10	10	2	
12	$g_i^{n_i} g_j^{n_i}$	$(1, 1, 0, 0, 3)$	10	10	2	
24	$g_i^{n_i} g_j^{n_j} g_k^{n_k}$ $n_i \neq n_j \neq n_k \neq n_i$	$(1, 2, 3, 0, 4)$	0	0	0	(4.8)
12	$g_i^{n_i} g_j^{n_j} g_k^{n_k}$ $\sum n_i \in 5Z$	$(1, 1, 3, 0, 0)$	10	10	2	
12	$g_i^{n_i} g_j^{n_j} g_k^{n_k}$ $3n_i + n_k \in 5Z$	$(1, 1, 2, 0, 1)$	C	-10	2	
12	$g_i^{n_i} g_j^{n_j} g_k^{n_k}$ $2n_i + 2n_k \in 5Z$	$(1, 1, 4, 0, 4)$	10	10	2	
4	$(g_1 g_2 g_3)^n$	$(1, 1, 1, 0, 2)$	C	-10	2	

This table was calculated using the G-index theorem (or Lefschetz fixed point theorem) to compute group-invariant cohomology, as described below. For the elements with isolated fixed points, the group-invariant cohomology is just the

number of orbits under the action of the other group elements (things are simplified since this is an abelian orbifold). As an example, we consider $g_1 g_2 = (1, 1, 0, 0, 3)$. The fixed points are $(1, -\alpha^r, 0, 0, 0)$ and also $(0, 0, 1, -\alpha^s, 0), r, s = 0 \ldots 4$. We now concentrate on the first set of points. These points are also fixed under g_3. Now under g_1 or g_2 these points are mapped to $(1, -\alpha^{r-1}, 0, 0, 0)$ and $(1, -\alpha^{r+1}, 0, 0, 0)$ respectively. So the first class contains only one orbit under the group action. Similarly for the second. Thus $g_1 g_2$ has 2 orbits of fixed points: $\chi_{\text{inv}} = 2$. The analysis is similar for other group elements with isolated fixed points.

Now consider a fixed curve. All are of the form

$$C = \{X^5 + Y^5 + Z^5 = 0\} \subset \mathbb{CP}^2 , \tag{4.9}$$

and the adjunction formula tells us that $\chi = -10$; since this curve is a connected complex manifold, $h_{00} = h_{11} = 1, h_{10} = h_{01} = 6$. We also know that the volume form and trivial form 1 are invariant. So we have

$$h_{10,\text{inv}} = h_{01,\text{inv}} = \frac{2 - \chi_{\text{inv}}}{2} . \tag{4.10}$$

Now to figure out χ_{inv} we need to compute the alternating sum of invariant cohomology elements of various dimensions. This is much like the Euler characteristic, except we must insert a projection operator for $C(g)$ invariance, i.e.

$$\chi_{C(g)} = \sum_i (-1)^i \text{Tr} 1_{H^i_{C(g)}} = \frac{1}{|C(g)|} \sum_i (-1)^i \text{Tr} \, g|_H . \tag{4.11}$$

The above reduces to a sum over fixed points of g (when the fixed points are isolated [1]), where we have the formula

$$\sum_{j=0}^d (-1)^j \text{Tr} \, g|_{H^j} = \sum_{\text{fixed points}} \text{sgn}(\det(1 - dg)) , \tag{4.12}$$

where dg is the differential g-action on the cotangent space. In this example, the fixed point spaces are all one (complex) dimensional, so dg acts are a rotation by a phase. In the real sense, we see that $\det(1 - dg) = 2 - 2\cos(\theta) \geq 0$, with equality only for $g = 1$, in which case the g-index is just the Euler characteristic, $\chi = -10$ for any fixed curve. So we only have to count the number of fixed points for any $g \neq 1$.

Now how does G act on the fixed curve? Any fixed curve has the form of (4.9), with the action by the group equivalent to the group generated by the elements $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, where we have used the same notation as in (4.8). One of these elements is dependent, say $(0, 0, 1)$, so the action on a fixed curve is by $Z_5 \times Z_5$ generated by two elements a and b (it is obviously fixed under the Z_5 of the twisting element). This group has 25 elements, 12 of which are nontrivial and have fixed points. They are of the form $(i, 0, 0), (0, i, 0), (i, i, 0)$. Each of these elements has exactly five fixed points. Thus for any curve we insert the projection operator onto invariant states to get

$$\sum_{j=0}^2 (-1)^j \#\text{-inv}|_{H^j} = \frac{1}{25} \sum_{j=0}^2 \sum_{m=0}^4 \sum_{n=0}^4 (-1)^j \text{Tr} \, a^m b^n|_{H^j}$$

$$= \frac{1}{25}(-10, + 12 \cdot 5) = 2 . \tag{4.13}$$

Thus, for all fixed curves (4.10) tells us $h_G^{11} = h_G^{00} = 1, h_G^{10} = h_G^{01} = 0$.

Now to construct the Hodge diamond from the observables, we just need to shift by the appropriate amount. There are 101 elements with fixed points. The curves have codimension two and thus have a shift of one, fitting in the center of the Hodge diamond. A simple analysis shows that half of the 80 fixed point orbits have a shift of one, half by two. The invariant untwisted elements have no shift. Thus, the Hodge diamond is the same as that of the mirror manifold, obtained by resolving this orbifold [4]:

1	0	0	1
0	1	101	0
0	101	1	0
1	0	0	1

$h^{**}(M/G)$

This is the mirror orbifold of M.

5. A Dihedral Orbifold

Now that we know how to compute the "cohomology" of the orbifold, we would like to compute the ring structure as well. This involves computations of intersections on the moduli space of equivariant holomorphic maps from appropriate branched covers of the Riemann surface, depending on the interaction under consideration. In this section, we offer a detailed computation of this quantum ring for a nonabelian orbifold.

We wish to consider an orbifold of $\mathbf{CP}^1$ by the dihedral group D_4, the symmetry group of a square. Recall that $\mathbf{CP}^1$ is topologically a sphere, and that all the point groups act naturally on the sphere, since they are subgroups of the rotation group. The dihedral group D_N is generated by an order N rotation θ and a flip r, with the relations

$$r^2 = \theta^N = 1, \quad r\theta r^{-1} = \theta^{-1} . \tag{5.1}$$

We take the action on $\mathbf{CP}^1 \cong \mathbf{C} \cup \infty$ to be $r(z) = z^{-1}, \theta(z) = \alpha z$, with $\alpha = e^{2\pi i/N}$. Note that for the even dihedral groups there is a non-trivial center containing the element $\theta^{N/2}$. In homogeneous coordinates (X, Y) for $\mathbf{CP}^1$, this group has a representation in $PGL(2)$ given by[9]

$$r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{5.2}$$

Note that this is a projective representation – matrices are only defined modulo nonzero scale factors.

Let us first discuss the fixed point geometry. Each nontrivial group element g acts by a rotation of the sphere $\mathbf{CP}^1$, and thus has two fixed points, which we label A_g and B_g. Let us make the following definition for r:

$$A_r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B_r = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5.3}$$

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[9] We required action by a holomorphic isometry, hence the group must act as a subgroup of the automorphisms of $\mathbf{CP}^1$, i.e. of $PGL(2)$

Thus the cohomology of M_g is just $\mathbf{C} \oplus \mathbf{C}$. Now we know that we can only use $C(g)$-invariant forms. Consider the element $r \in \{r\}$. We have $C(r) = \{1, r, \theta^2, r\theta^2\}$, and thus

$$H_{C(r)}(M_r) = 1_{A_r} + 1_{B_r} \Rightarrow \mathbf{r} \tag{5.4}$$

since the two fixed points are related under $C(r)$ by θ^2 and $r\theta^2$. In this way, we can find all the observables of the theory.

Although $\mathbf{CP}^1$ is not a Calabi–Yau manifold, and thus the chiral fermion number is not conserved, we can still try to ascribe chiral fermion numbers to our observables using the methods described in this paper. This will then be conserved by assigning a chiral (and anti-chiral) fermion number $F_\beta = 2$ to the parameter β, representing the instanton action (recall $X^2 = \beta$ is the ring for the $\mathbf{CP}^1$ model, which is still true since X remains as an element in the untwisted sector). Quite generally, all elements g of order two in a one-dimensional complex space must have $F_g = \frac{1}{2}$, since in a neighborhood of a fixed point at $z = 0$, we have $g(z) = -z$, or $dg = -1$. For θ the action on a local coordinate at A_θ gives $d\theta = e^{2\pi i/4}$ and hence $F_{A_\theta} = \frac{1}{4}$. Conversely, $F_{B_\theta} = \frac{3}{4}$. These observations are tallied below.

Observable: $\mathcal{O}$	Sector	$F_{\mathcal{O}}$	
1	1	0	
X	1	1	
$\mathbf{r}$	$\{r\}$	$1/2$	
$\mathbf{g}$	$\{r\theta\}$	$1/2$	(5.5)
θ_A	$\{\theta\}$	$1/4$	
θ_2	$\{\theta^2\}$	$1/2$	
θ_B	$\{\theta\}$	$3/4$	

Before computing correlation functions, let us anticipate a symmetry of the chiral ring. The automorphisms of the group D_4 have a normal subgroup known as the inner automorphisms, given by conjugation by the various elements.[10] The outer automorphisms are those defined modulo inner automorphisms. Conjugation acts trivially on our ring elements by construction, but the outer automorphisms should survive in some form in our ring. The group of outer automorphisms of D_4 is easily seen to be Z_2 and is generated by σ, which is determined by its action on r and θ: $\sigma(r) = r\theta$, $\sigma(\theta) = \theta$.

In order to derive the chiral ring, we must compute all the three point functions of the theory. There is a subtlety, though. When we write the observable $\mathbf{r}$ we really mean a sum of terms related by conjugation. In the case of r, for example, we have a nontrivial centralizer which includes the element θ^2, relating A_r and B_r. Thus, we have

$$\mathbf{r} = A_r + B_r + A_{r\theta^2} + B_{r\theta^2} . \tag{5.6}$$

In order to compute a correlation function involving an r-twisted operator at p, we have to choose an appropriate cover $\tilde{\Sigma}$ over Σ. (We will always take Σ to be a sphere, since the genus zero correlation functions determine the ring of

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[10] $N \subset G$ is normal if $aN = Na \;\; \forall a \in G$. Let I be the inner automorphisms, $i_g \in I$ represents conjugation by g. Let ρ be an automorphism. I is normal because $\rho \circ i_g(x) = (\rho(g))\rho(x)(\rho(g))^{-1}$ $= i_{\rho(g)} \circ \rho(x)$. So $\rho I = I\rho$

observables.) The different choices of points $\tilde{p}$ covering p are related by the group elements and correspond to different twistings in the conjugacy class. By the way we constructed our operator $\mathbf{r}$, our results will be independent of this choice. However, to compute correlations involving $\mathbf{r}$ we must choose a particular lift.

We begin by considering some simple correlation functions involving two twist fields. An explicit computation will show us how to generalize our procedure for the more complicated three-point functions. In genus zero, the selection rule states that the product of all twists is the identity (we consider all states as incoming). Let us compute $\langle A_\theta(p) B_{\theta^3}(q) X^r \rangle$, for example; the "pre-operators" in this correlation function are pieces of a full-fledged observable – they are only defined for a particular choice of lift. The first thing we notice is that the two twisting elements commute. In fact they generate an abelian Z_4 subgroup, which means that our cover need only be a Z_4 cover of the sphere; the other elements of G will act redundantly. One can compute general Z_n orbifolds of the sphere by a similar calculation [8]. Since it is a twice-twisted correlator, we need a cover of the sphere, branched by θ and θ^3 over p and q. Since we can choose an automorphism of the sphere which takes p to the south pole and q to the north pole, we may choose p and q to be the points $z = 0$ and $z = \infty$. The covering surface $\tilde{\Sigma}$, is also a sphere, and the Z_4 acts by rotation. If w is the coordinate on $\tilde{\Sigma}$, then $\theta(w) = iw$. The covering map is $w \mapsto w^4$, or in other words $z = w^4$, so the lifts of a point z are given by the four points $w = z^{1/4}$. At $z = 0$, a branching point, there is only one w, and we note that a small circle around the origin lifts to one whose endpoints are separated by the action of θ. Now we need to find the equivariant maps from $\tilde{\Sigma} \cong \mathbf{CP}^1$ to the target space $K \cong \mathbf{CP}^1$. We know the (compactified) moduli space $\mathcal{M} \equiv \{\Phi: \mathbf{CP}^1 \to \mathbf{CP}^1 | \Phi \text{ holomorphic}\}$ decomposes into maps of degree k, with $\mathcal{M}_k \cong \mathbf{CP}^{(1+1)(k+1)-1}$. We need to find equivariant maps. Consider the general degree k holomorphic map given by (see (2.9))

$$\Phi: (X, Y) \mapsto \left(\sum \phi_{0l} X^{k-l} Y^l, \sum \phi_{1l} X^{k-l} Y^l \right). \tag{5.7}$$

Now θ acts by $Y \mapsto iY$, so recalling that there is an overall scale ambiguity, we see that $\Phi = \phi_{lm}$ is equivariant if $m \equiv l + 1 \bmod 4$ and l has ranges over a *fixed* value mod 4. The four values of m mod 4 represent the four components of $\mathcal{M}_k$, which we label $\mathcal{M}_{k,m}$. For example, we have $\mathcal{M}_{9,1} = \{(a_1 X^8 Y^1 + a_5 X^4 Y^5 + Y^9, b_2 X^7 Y^2 + b_6 X^3 Y^6)\}$. The astute reader will recognize from the form of (2.9) that equivalence of Φ means that it commutes with the projective group action, and so the different sectors of $\mathcal{M}_k$ correspond to different spaces of intertwiners of projective representations of the dihedral group with various multipliers [24]. The group action on the space of homogeneous polynomials of degree k is obtained by the symmetric tensor product of the representation on (X, Y).

We need the maps which take $p = (1, 0) \mapsto (1, 0) = A_\theta$ and $q = (0, 1) \mapsto (0, 1) = B_\theta$. The maps will be ill-defined unless there are terms like $X^k Y^0$ and $X^0 Y^k$. So we require the X^k term to be in the first coordinate, and the Y^k term in the second. Thus we must have $l \equiv 0$ and $k \equiv 1$. Let us write $k = 4q + 1$. Counting a's and b's, we see that $\dim \mathcal{M}_{4q+1,0} = (q+1) + (q+1) - 1$, where we must subtract one for global rescaling of a's and b's. The minimum dimension is one, so we must add the observable X, representing the volume form, to our correlation function in order to get a non-zero correlation number (i.e., to have finite intersection of the cycles in the moduli space): $r = 1$. Since X will require maps from a given point to a single point in the target space, X is a linear condition on the as and bs, and so defines

a cycle of codimension one. Thus, we see that there is a unique map of degree $4a + 1$ for the correlation function $\langle A_\theta(p) B_{\theta^3}(q) X^{2a+1} \rangle$. Since we need the three-point function, we take $a = 0$. Then $k = 1$ and the equivariant maps are (aX, bY). If we take the point of insertion for the observable X to be $(1, 1)$, say, and we represent the (dual of the) volume form by the point (c, d), then the unique map is just $\Phi(X, Y) = (cX, dY)$. Now since the degree just counts the instanton number, let $\beta = e^{-A}$ represent the contribution of instanton number one (A is the area of $\mathbf{CP}^1$). We recall again that we must rescale the action (and the area) by $S \to S/N$ for an N-fold cover. We find:

$$\langle A_\theta B_{\theta^3} X \rangle = \beta^{1/4} . \tag{5.8}$$

Although we should really only consider three point functions to define the ring, we note here that $\langle A_\theta B_{\theta^3} X^{2a+1} \rangle = \beta^a \beta^{1/4}$ is consistent with the known relation $X^2 = \beta$. From this we can see how some ring relations are derived. For example, from the above, with the knowledge that X is the only observable with $\langle X \rangle = 1$, we can guess that

$$A_\theta \cdot B_{\theta^3} = \beta^{1/4} , \tag{5.9}$$

although this product could conceivably contain other untwisted elements like X – further analysis shows it does not. Again let us stress that we are deriving these relationships for a particular lift to $\tilde{\Sigma}$. The full ring of observables (**r**, etc.) is independent of this choice.

The procedure is similar for the three-point functions. We briefly consider the correlation function $\langle A_r(p_1) B_\theta(p_2) A_{r\theta}(p_3) \rangle$. One can apply the Riemann–Hurewicz formula to find the genus of the appropriate covering space.[11] The cover is once again a sphere, where we take the group action to be the same as for the target space, namely that of (5.2). We take the lift of p_1 to be $p_r \equiv (1, 1)$ (not $(1, -1)$), with $p_\theta \equiv (1, 0)$ and $p_{r\theta} \equiv (\alpha, 1)$.

We begin by considering the equivariant map

$$\phi_l : (X, Y) \mapsto (X^{k-l} Y^l, \varepsilon X^l Y^{k-l}) , \tag{5.10}$$

where equivariance under θ and r (and hence all of D_4) requires

$$k \equiv 2l + 1 \bmod 4, \quad \varepsilon = \pm 1 . \tag{5.11}$$

The general equivariant map will be a sum of the ϕ_l of fixed values of ε and ($l \bmod 4$). Therefore, there are eight sectors of equivariant maps of a fixed degree. As before, the different sectors have different properties, sending p_r, p_θ, and $p_{r\theta}$ to different fixed points in $\mathbf{CP}^1$. Some sectors drop out, all maps being multiples by XY of other maps (of two degrees less), and hence equivalent. For example, to compute the correlation function $\langle B_r A_\theta B_{r\theta} \rangle$, we find that there is a unique map of degree one, namely $(X, -Y)$, which gives

$$\langle B_r A_\theta B_{r\theta} \rangle = \beta^{1/8} . \tag{5.12}$$

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[11] This formula [17, 13] gives the genus $\tilde{g}$ of the covering space in terms of the orders v_i of the twisting elements, and the cardinality N of the group they generate: $2 - 2\tilde{g}$
$= N(2 - 2g) - N \sum \left(1 - \dfrac{1}{v_i} \right)$

Note that the chiral fermion number is always violated mod 2 in correlations. This allows the ring structure to preserve F as long as we take β to have $F_\beta = 2$, as in the untwisted $\mathbf{CP}^1$ case. Similarly, one must compute *all* three-point functions for pre-operators. These include the abelian ones involving $(r)(\theta^2)(r\theta^2)$, which only require a four-fold cover of the sphere (by a sphere), since the three group elements only generate a $Z_2 \times Z_2$ subgroup.

Once we have solved for the (now commutative) chiral ring, we try to find an economical way of presenting it. It turns out that all the ring relations are generated by the following:

$$\mathbf{r} \cdot (\theta_A)^2 = 4\beta^{1/4}\mathbf{r} \ ,$$

$$\mathbf{r} \cdot \mathbf{r} = 4X + 4\beta^{1/2} - 4\beta^{1/4}(\theta_A)^2 \ ,$$

$$(\theta_A)^4 = 2X - 2\beta^{1/2} + 4\beta^{1/4}(\theta_A)^2 \ ,$$

$$X \cdot \theta_A = \beta^{1/4}(\theta_A)^3 - 3\beta^{1/2}\theta_A \ . \tag{5.13}$$

The other observables are expressable in terms of $\mathbf{r}$ and θ_A (for example, the right-hand side of the last equation is just $\beta^{1/4}\theta_B$). In fact, using the second and third equations in (5.13) we can eliminate X, and make the ring "dimensionless." We also normalize the variables in a way which is most suitable to more general even dihedral group orbifolds. We define:

$$\rho \equiv \frac{1}{(4\beta^{1/2})^{1/2}}\mathbf{r} \ , \quad \phi \equiv \frac{1}{(\beta^{1/4})^{1/2}}\theta_A \ . \tag{5.14}$$

In terms of these generators, the ring of observables is defined by

$$\rho\phi^2 = 4\rho \ , \quad 2\rho^2 = \phi^4 - 2\phi^2 \ , \quad \phi^5 = 6\phi^3 - 8\phi \ . \tag{5.15}$$

This ring contains all the information of the topological theory. We use it to define higher genus amplitudes through factorization. Note too that the single outer automorphism survives as an automorphism of the *ring* of observables. In the variables of (5.15) it has the form

$$\rho \to \frac{1}{2}\rho\phi \ , \quad \phi \to \phi \ . \tag{5.16}$$

The ring (5.15) is the ring of observables of a topological sigma model orbifold on the space $\mathbf{CP}^1$. This space is not a Calabi–Yau manifold. However, as we let the area of the space go to infinity, the curvature must go to zero, giving us a Ricci-flat manifold – the plane. Thus, as in [8], the limit $\beta \to 0$ should give the chiral primary ring of a conformal field theory. In order to take the $\beta \to 0$ limit, we should use (5.14) to recover the β dependence of the ring. In doing so, we easily obtain the following chiral-primary ring:

$$\rho\phi^2 = 0 \ , \quad 2\rho^2 = \phi^4 \ , \quad \phi^5 = 0 \ . \tag{5.17}$$

We may ask whether this ring is familiar. Is it the ring of a Landau–Ginzburg model in two variables? The anaser is no. In fact, it is quite easy to see that no such superpotential could give rise to this ring. However, the ring (5.15) contains an interesting subring. Let us consider the ring generated by the elements ρ and ϕ^2. In

terms of these generators, the last relation in (5.15) becomes dependent on the others. It only enters as $\phi^6 = 6\phi^4 - 8\phi^2$, a simple consequence of the other equations. Let us define $x \equiv \phi^2$, $y \equiv \rho$ (do not confuse x with the observable X). This subring is then described by the relations

$$x^2 = 2y^2 + 2x , \quad xy = 4y . \tag{5.18}$$

Now *this* ring has a simple Landau–Ginzburg description. It is the same as the ring derived from the superpotential

$$W = \frac{x^3}{3} - 2xy^2 - x^2 + 8y^2 . \tag{5.19}$$

The last two terms in (5.19) are the β-dependent perturbations, which vanish as $\beta \to 0$. In this limit, we recover the superpotential

$$W_{D_4} = \frac{x^3}{3} - 2xy^2 , \tag{5.20}$$

which is none other than the superpotential corresponding to D_4 in the $A - D - E$ classification of $N = 2$ minimal models. So a subring of the dihedral $\mathbf{CP}^1$ orbifold is the same as the ring of the corresponding dihedral Landau–Ginzburg series!? There is no obvious connection. In fact, we will show in the next section that this relationship is somewhat general: the chiral ring of the D_{2k} orbifold has a subring described by a perturbation of the D_{k+2} superpotential $W = x^{k+1} + xy^2$ (up to normalization). It is a coincidence that $2 \cdot 2 = 2 + 2$.

6. $\mathbf{CP}^1/D_N$

In this section, we will outline the generalization to orbifolds of $\mathbf{CP}^1$ by an arbitrary dihedral group D_N. Let us first consider the even case $N = 2k$. The features of the previous section are quite general, so we will be brief. The dihedral group is defined by (5.1). When $N = 2k$, there are two "flip" conjugacy classes, $\{r\}$ and $\{r\theta\}$, as before. We also have the trivial class 1, the central element θ^k, and $k - 1$ conjugacy classes $\{\theta^i\}$, $i = 1 \ldots (k - 1)$ (here $\{\theta^i\} = \{\theta^i, \theta^{-i}\}$).

Now to determine the ring, we must compute many correlation functions involving the twists $(r)(\theta^l)(r\theta^l)$. These turn out to be very similar to the ones we just computed. The main difference is in the factors of β in the ring coefficients. However, by F conservation, we can always determine the correct β-dependence from the "dimensionless" operators ρ and ϕ. Once again, these generate the ring, though the relations between them are a bit more complicated.

Consider the l-twisted sector, by which we mean the conjugacy class of $\{\theta^l\}$. There are two observables in this sector, which we will label ϕ_l and ϕ_{2k-l}. Here we define

$$\phi_l \equiv (\beta^{-1/4k})(A_{\theta^l} + B_{\theta^{-l}}) . \tag{6.1}$$

We use the convention $\phi_0 \equiv 2$, and the abelian result $(A_\theta)^{2k} = X$ gives us that $\phi_{2k} = (\beta^{-1/2})2X \equiv 2\chi$ (χ is the dimensionless version of X). We also have the generalization of (5.9):

$$A_\theta \cdot B_{\theta^{-1}} = \beta^{1/2k} . \tag{6.2}$$

This, combined with another abelian result, $A_\theta A_\theta = A_{\theta^2}$, allows us to compute all products of the ϕ_l in terms of $\phi_1 \equiv \phi$. The trick is to derive a recursion relationship for the ϕ_l. Note, for example, that

$$\phi \cdot \phi_1 = \phi_1 \cdot \phi_1$$
$$= \beta^{-1/2k}(A_\theta + B_{\theta-1})(A_\theta + B_{\theta-1})$$
$$= \beta^{-1/2k}(A_{\theta^2} + B_{\theta-2}) + 2\beta^{1/2k}$$
$$= \phi_2 + \phi_0 . \tag{6.3}$$

More generally, we find the following recursion relation among the ϕ_i:

$$\phi \cdot \phi_n = \phi_{n+1} + \phi_{n-1} . \tag{6.4}$$

This is a difference relation which can be solved as follows. First, assume that ϕ acts as a constant (which it is not); let's call it A. Then, as for a second order differential equation, we say that $\phi_n \sim t^n$, solve for t and impose boundary conditions. We easily see that we must have

$$t^2 - At + 1 = 0 \tag{6.5}$$

which gives

$$t_\pm = (A/2) \pm i\sqrt{1 - (A/2)^2} . \tag{6.6}$$

The general solution is $\phi_n = c_+ t_+^n + c_- t_-^n$. We must have that $\phi_0 = 2$ and $\phi_1 = A$. This gives

$$\phi_n = t_+^n + t_-^n . \tag{6.7}$$

If we formally put $A = 2\cos(z)$, then $t_\pm = e^{\pm iz}$, and we can easily see that $\phi_n = 2\cos(nz)$. The Chebyshev polynomial $W_n(X)$ is a degree n polynomial in X defined by (conventions vary)

$$W_n(X = 2\cos(z)) = 2\cos(nz) . \tag{6.8}$$

We thus have derived

$$\phi_n = W_n(\phi) . \tag{6.9}$$

Although the recursion relation did not have constant coefficients, the ultimate justification of this method is that it works!

Now the generalization of (5.8), along with (6.1), tells us that

$$\chi\phi_l = \phi_{2k-l} . \tag{6.10}$$

Of course $\chi = (1/2)\phi_{2k}$, so, in particular

$$\phi W_{2k}(\phi) = 2W_{2k-1}(\phi) . \tag{6.11}$$

In fact this relation generates all of the equations in (6.10). For example,

$$\chi\phi_2 = \chi(\phi^2 - 2) = (\chi\phi_1)\phi - \phi_{2k} = \phi_{2k-1}\phi - \phi_{2k} = \phi_{2k-2} , \tag{6.12}$$

where we have made use of (6.11) and the recursion relation (6.4).

The ring relations involving r can now be made simpler by defining

$$\rho \equiv \frac{\mathbf{r}}{(2k\beta^{1/2})^{1/2}} , \tag{6.13}$$

where $\mathbf{r}$ is the conjugacy class operator and contains $2k$ terms, exactly analogously to (5.6). The simple relations $A_r A_r = X$ and $A_r B_r = \beta^{1/2}$, along with their generalizations for the other flips, are helpful in deriving

$$\rho^2 = 1 + \chi + \sum_{l=1}^{k-1} \phi_{2l} \,, \tag{6.14}$$

which we can rewrite as

$$\rho^2 = 1 + \frac{1}{2} W_{2k}(\phi) + \sum_{l=1}^{k-1} W_{2l}(\phi) \,. \tag{6.15}$$

Finally, the first relation in (5.15) survives unchanged with our present definitions. This relation exactly parallels the multiplication of conjugacy classes in the group ring. Summarizing, the general ring of observables for the topological orbifold $\mathbf{CP}^1/D_{2k}$ as:

$$\rho \phi^2 = 4\rho \,,$$

$$\rho^2 = 1 + \frac{1}{2} W_{2k}(\phi) + \sum_{l=1}^{k-1} W_{2l}(\phi) \,,$$

$$\phi W_{2k}(\phi) = 2 W_{2k-1}(\phi) \,. \tag{6.16}$$

The group outer automorphism survives in the ring as before, and we have defined our generators so that (5.16) is valid as written.

Once again, our ring has a subring generated by $x \equiv \phi^2$ and $y \equiv \rho$, and the last relation in (6.16) becomes redundant. Note that $W_{2l}(\phi)$ is a degree l polynomial in x alone, so we can define a degree $k+1$ polynomial $F(x)$ such that the right-hand side of the second equation in (6.16) is given by $F'(x)$. We can write this subring as the chiral ring associated to the superpotential

$$W = F(x) - xy^2 + 4y^2 \,. \tag{6.17}$$

This is a perturbation of the D_{k+2} Landau–Ginzburg potential. The perturbation involves the Chebyshev polynomials, which have been shown to be integrable [10, 12], though we don't know whether this model is integrable. This is reminiscent of the $\mathbf{CP}^1/Z_n$ case, where the ring was found to be that of a perturbed A_{2n} minimal Landau–Ginzburg model. (For recent work on the relationship of orbifolds to Landau–Ginzburg models, see [9].) Work in progress shows a connection between $\mathbf{CP}^1/D_k$ orbifolds and extended Dynkin diagrams of $\hat{D}_{k+2}$ [9, 35].

In the odd case, $N = 2k + 1$, there is perhaps only one subtlety. In considering the covering surface of the sphere for the three point function, one must be careful in choosing the lift. For example, the sphere covers the sphere with the usual action, but if we are considering a $(r)(\theta)(r\theta)$ correlation, we should make sure the points representing r and $r\theta$ do not lie on the same orbit (or else they represent the same point on the underlying sphere). For the odd orbifolds, there is only one "flip" conjugacy class, but there are two operators associated to it, since the two fixed points are not related by any element. Proceeding in much the same way as for the even case, we find the following ring:

$$\rho \phi^2 = 4\rho \,,$$

$$\rho^2 = \frac{1}{2} W_{2k+1}(\phi) + \sum_{l=1}^{k} W_{2l-1}(\phi) \,,$$

$$\phi W_{2k+1}(\phi) = 2 W_{2k}(\phi) \,. \tag{6.18}$$

This ring also has the automorphism

$$\rho \to \frac{1}{2}\rho\phi \ , \quad \phi \to \phi \ , \tag{6.19}$$

though now it corresponds to the geometric symmetry corresponding to a θ rotation by π, which is not a group element. No connection to the D-series is evident.

7. CP^2/D_4

Our techniques allow us to compute higher dimensional orbifolds as well. In this section, we consider the orbifold CP^2/D_4, with the group generators acting by the matrices

$$r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{7.1}$$

The reason for considering this orbifold is that as we let the area of the CP^1 go to infinity, we can obtain a nonabelian conformal orbifold theory. Nonabelian orbifolds have not been heavily studied (though see [14]) and little is known about their twist fields. To see how this limit arises, consider the point $p = (1, 0, 0)$. This point is fixed under the entire group D_4. Thus, in the conformal limit $\beta \to 0$, the space around p becomes C^2 and the action of the group is given by the differential action near p, which is the linear action defined by the bottom two entries of the matrices in (7.1).

There are no subtleties in the computation of the ring for this theory. The ring contains three observables for each conjugacy class, fifteen total. Some group elements have fixed spheres, leaving us with a twisted volume form V_g as an observable. This is equal to $X \cdot 1_g$.

The ring of observables is generated by three elements

$$\chi \equiv \beta^{-1/3} X \ , \quad \mu \equiv \frac{1}{2}\beta^{-1/6} \mathbf{r} \ , \quad \alpha \equiv \frac{1}{2}\beta^{-1/3} \mathbf{A}_\theta \ , \tag{7.2}$$

where $\mathbf{r}$ represents the composite operator associated to the nontrivial 0-form on the fixed sphere of r, and $\mathbf{A}_\theta$ represents the operator corresponding to the fixed point $(1, 0, 0)$. The defining relations are

$$\alpha^7 = \alpha \ , \quad \mu^2 = \alpha^4 \ , \quad \mu\alpha^6 = \mu \ , \quad \chi^3 = 1 \ , \quad \mu\chi = \mu\alpha^4 \ , \quad \alpha\chi = \alpha^5 \ . \tag{7.3}$$

The group automorphism takes the form

$$\alpha \to \alpha \ , \quad \mu \to \mu\alpha^3 \ , \quad \chi \to \chi \tag{7.4}$$

in this presentation of the ring.

We leave to further study the consideration of orbifolds by other groups and higher dimensional spaces, though we expect orbifolds of CP^1 by the exeptional discrete subgroups have a soliton spectrum described by the associated extended Dynkin diagrams [9, 28, 35].

8. Methods for Computing Twist Field Correlations

Our observables are nothing but twist fields – they create twisted chiral-primary states in the full non-topological sigma model. With our knowledge of the ring, we have the β-dependence of the theory (which means scale dependence since $\beta = e^{-A}$). There is another theory we could have gotten from the original sigma model which is the complex conjugate theory, obtained by performing the twist of the $N = 2$ theory so that *anti*-holomorphic maps were the instantons. The ring of this theory is obtained by complex conjugation. Now we can use recent non-perturbative results [6] for computing the metric

$$g_{i\bar{j}} = \langle \bar{j} | i \rangle \tag{8.1}$$

as a function on coupling constant space. This is the metric of the full non-topological sigma model, restricted to the chiral states, and is closely related to Zamolodchikov's metric [34] (see [6] for a discussion). In reference [6], the authors derived differential equations for (8.1). We will consider here the scale-dependence of this metric. The non-trivial input is that as the area of the $\mathbf{CP}^1$ goes to infinity the curvature goes to zero, so there is no curvature anomaly and we have a conformal field theory. So we expect good behavior of $g_{i\bar{j}}$ as $\beta \to 0$. As was discussed in [8] and [7], demanding finiteness in this limit can be enough to specify the exact form of solution to these equations.

Let us see how this works. Consider a Z_n orbifold of $\mathbf{CP}^1$, as in [8]. In order to consider the β behavior of the theory, we must find the operator corresponding to a perturbation in β. Because we constructed the action from the Kähler form, X is the operator corresponding to β variation. Actually, $-\ln\beta = A$ multiplies the X term, so the operator corresponding to β is properly $C_\beta = -\frac{1}{\beta}X$. The differential equation for the metric g is [6]

$$\partial_{\bar{\beta}}(g\partial_\beta g^{-1}) = [C_\beta, gC_\beta^\dagger g^{-1}] . \tag{8.2}$$

The metric $g_{i\bar{j}}$ represents a fusion of topological and anti-topological (in which the anti-holomorphic maps are instantons) theories. The states in these two theories are related by the real structure matrix:

$$\langle \bar{j} | = \langle i | M_j^i . \tag{8.3}$$

The topological metric is $\eta_{ij} = \langle \phi_i \phi_j \rangle$. From (8.3) and the definition (8.1), we see

$$M = \eta^{-1} g . \tag{8.4}$$

The CPT conjugate of $|i\rangle$ is $|\bar{i}\rangle$. Acting twice by CPT is the identity, so we see

$$MM^* = (\eta^{-1}g)(\eta^{-1}g)^* = 1 . \tag{8.5}$$

For our $\mathbf{CP}^1/Z_n$ example, we have two observables in each sector, corresponding to the two fixed points (north and south poles). The metric g is block diagonal in each sector, while the metric η relates h- and h^{-1}-twisted sectors (since it involves no "out" states). There is a symmetry ($h \to h^{-1}$ or $z \to z^{-1}$) equating the north pole in the h-sector to the h^{-1}-twisted south pole (analogous to the symmetry giving rise to the automorphism (7.4)). Consider $h \in Z_n$. On the h and h^{-1} subspace, with basis $\{1_{h,a}, 1_{h,b}, 1_{h^{-1},a}, 1_{h^{-1},b}\}$ we have

$$g = \begin{pmatrix} a & c & 0 & 0 \\ c^* & b & 0 & 0 \\ 0 & 0 & b & c^* \\ 0 & 0 & c & a \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{8.6}$$

where we have used hermiticity and the aforementioned symmetry (note that a and b are real). Applying (8.5), we find $ab = 1, c = 0$. Note that $h = h^{-1} \Rightarrow a = b = 1$. Now g depends only on $|\beta|$ [8], so we can define

$$x = 4|\beta|^{1/2}, \quad u(x) = 2\log(a|\beta|^{(n-2l)/2n}) . \tag{8.7}$$

We find from (8.2) that u obeys a special form of the Painlevé III equation:

$$u'' + \frac{1}{x} u' = 4\sinh u . \tag{8.8}$$

Now we must require that

$$u \to r\log x + s, \quad r = 2\left(\frac{n - 2l}{n}\right) \tag{8.9}$$

in order for a to be finite at $x = 0$. It turns out [19] that restricting the coefficient on the logarithm in (8.9) determines s by the equation

$$e^{s/2} = \frac{1}{2^r} \frac{\Gamma\left(\dfrac{1}{2} - \dfrac{r}{4}\right)}{\Gamma\left(\dfrac{1}{2} + \dfrac{r}{4}\right)} . \tag{8.10}$$

Resolving the morass, we find

$$a(0) = \frac{\Gamma\left(\dfrac{l}{n}\right)}{\Gamma\left(1 - \dfrac{l}{n}\right)} , \tag{8.11}$$

which we use to derive the proper normalization of the twist fields.

For $\mathbf{CP}^1/D_4$, we have already solved for the ring, so we know what multiplication by X is $\left(\text{recall from (6.9) that } X = \beta^{1/2}\chi = \dfrac{\beta^{1/2}}{2}\phi_4 = \dfrac{\beta^{1/2}}{2} W_4(\phi)\right)$. For example, $\mathbf{r}X = \beta^{1/2}\mathbf{r}$ ($\mathbf{r}$ is given in (5.14)), which means that the matrix C_β has an invariant subspace of dimension one. We easily see that the right-hand side of (8.2) is zero, which, combined with the fact that the metric only depends on $|\beta|$, tells us that the normalized operator $\frac{1}{2}\mathbf{r}$ is independent of β (aside from the normalization arising from $\langle\bar{1}|1\rangle$). The same is true for $\mathbf{g}$ and θ_2. The untwisted observables were discussed in [7]. This leaves us with $\dfrac{1}{\sqrt{2}}\theta_A$ and $\dfrac{1}{\sqrt{2}}\theta_B$, where we have chosen a convenient normalization. In this subspace, the relevant matrices take the form

$$C_\beta = -\frac{1}{\beta}\begin{pmatrix} 0 & \beta^{1/4} \\ \beta^{3/4} & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} a & c^* \\ c & d \end{pmatrix}, \tag{8.12}$$

where g is a general hermitian matrix (with no components outside this subspace due to the selection rule). The reality constraint (8.5) gives us $c = 0$, $ad = 1$, so there is only one real variable, a. It is now clear that the twist operators reduce to simple Z_4 twist operators. The reason for this is that the fixed points of θ, for example, are fixed by an abelian stabilizer group. In the large limit, we are left with two copies of the Z_4 orbifold, with operators that create twisted states in both.

The situation is different for our $\mathbf{CP}^2/D_4$ orbifold. In that case, the point $p \equiv (1, 0, 0)$ was fixed under the entire nonabelian group. Now consider the theory in a neighborhood of p as we take $\beta \to 0$. As discussed in Sect. seven, this will correspond to a nonabelian orbifold of $\mathbf{C}^2$. Consider the θ-twisted sector. We have three operators. Let $\theta'/\sqrt{2}$ represent the fixed point p under θ, with $\theta_A/\sqrt{2}$ and $\theta_B/\sqrt{2}$ the operators associated to the two remaining fixed points (similarly to (5.14)). In this sub-basis we have

$$C_\beta = -\frac{1}{\beta} \begin{pmatrix} 0 & \beta^{1/4} & 0 \\ 0 & 0 & \beta^{1/2} \\ \beta^{1/4} & 0 & 0 \end{pmatrix} \tag{8.13}$$

and the topological metric

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{8.14}$$

It is clear that η is essentially $\mathrm{diag}(1, 1, -1)$, which means, from the reality condition (8.3), that the hermitian matrix g is just a unitary transformation of an element in the complexified group $SO(2, 1)$. In general, the equations resulting from (8.2) using (8.14) are quite complicated. Similar equations were studied in [22], in the context of Landau–Ginzburg models perturbed away from criticality. It is not known whether the requirement of regularity is enough to fix the values of the metric (the objects of interest to us) at the point $\beta = 0$. The equations simplify when an extra discrete symmetry requires the metric to be diagonal (e.g. a Z_3 symmetry for (8.13)). In such a case, the reality condition gives $g_{0\bar{0}} = 1$, and $g_{1\bar{1}}g_{2\bar{2}} = 1$, so we have one real parameter. Then if we define

$$x = 8|\beta|^{1/2} \quad b = 2g_{1\bar{1}} \tag{8.15}$$

we see that $b(x)$ obeys another special form of the Painlevé equation:

$$b'' = \frac{1}{b}(b')^2 - \frac{1}{x}b' + \frac{1}{x}b^2 - \frac{1}{b} . \tag{8.16}$$

This is called the Bullough–Dodd equation, and was studied in [21]. Requiring regularity of b in the limit $x \to 0$ again specifies the boundary conditions. We find $g_{1\bar{1}} = \Gamma(3/4)/\Gamma(1/4)$, so we know that a regular limit exists, though there appears to be no symmetry forcing the metric to be diagonal. We leave this question to further study.

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