

Asymptotic Differentiable Structure on Cantor Set

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Abstract. We study hyperbolic maps depending on a parameter ε . Each of them has an invariant Cantor set. As ε tends to zero, the map approaches the boundary of hyperbolicity. We analyze the asymptotics of scaling function of the invariant Cantor set as ε goes to zero. We show that there is a limiting scaling function of the limiting map and this scaling function has dense jump discontinuities because the limiting map is not expanding. Removing these discontinuities by continuous extension, we show that we obtain the scaling function of the limiting map with respect to a Ulam-von Neumann type metric.

0. Introduction

A differentiable structure on the topological Cantor set determined by the intrinsic scaling function of a geometric Cantor set on the line has been introduced by Sullivan [4]. Consider a family of geometric Cantor sets on the line. What will happen to such differentiable structures if all gaps of geometric Cantor sets finally close to form an interval (a one dimensional differentiable manifold)? It is the question we will discuss in this paper. The main results (Theorems 1 and 2) say that for a family of geometric Cantor sets on the line generated by a family of nonlinear folding maps, the scaling functions tend to the scaling function of a sequence of partitions on the interval [2] as all gaps finally close. And moreover, the scaling function of the sequence of partitions on the interval has countably many jump discontinuities, and is uniformly continuous at the rest of the points on the dual Cantor set. If we replace all the discontinuities by the continuous extension of the values on the rest of the points, we get a Hölder continuous function on the dual Cantor set. This function is the scaling function of a sequence of partitions with respect to a Ulam-von Neumann type metric [6] (we call this metric an orbifold metric (with respect to the Lebesgue metric)). In the other words, if we consider the interval with this Ulam-von Neumann type metric as a Riemannian manifold (an orbifold, compare to [5]), the limit of differentiable structures on the topological Cantor set is the differentiable structure of this Riemannian manifold.

This paper is organized as follows: in Sect. 1, we recall a short definition of a scaling function for both the geometric Cantor set on the line and a sequence of partitions on the line. Following the definitions we state the main results (Theorem 1 and 2). In Sect. 2, we show two uniform estimates of the nonlinearities of long compositions of any map in a family (Lemma 2 and 3) and the proofs of the main results.

1. Statements of Main Results

Suppose I is the interval $[-1, 1]$ and f is a C^1 folding map from I to the real line with a unique critical point 0. We say 0 is a *power law critical point* if there is a number $\gamma > 1$ such that the limits of $f'(x)/|x|^{\gamma-1}$ exist and equal nonzero numbers as x tends to zero from above and below. The number γ is called an *exponent* of f . Henceforth, we will assume that f is increasing on $[-1, 0]$ and decreasing on $[0, 1]$, and $f(0) \geq 1, f(-1) = f(1) = -1$, and that 0 is the power law critical point of f .

Let g_0 from I to $[-1, 0]$ and g_1 from I to $[0, 1]$ be the left and right inverse branches of f . For a sequence $w = i_0 \dots i_n$ of symbols $\{0, 1\}$, let $g_w = g_{i_0} \circ \dots \circ g_{i_n}$ be the composition of g_{i_0} to g_{i_n} and $I_2 = g_w(I)$ be the image of I under g_w . We use η_n to denote the collection of intervals I_w for all w of length $n + 1$ and $\lambda_n = \max_{I_w \in \eta_n} |I_w|$ to denote the maximum of the lengths of the intervals in η_n .

Definition 1. We say f is in \mathcal{H} , the space of hyperbolic maps, if (a) $f(0) > 1$, (b) f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$, and (c) there are constants $K > 0$ and $0 < \mu < 1$ such that $\lambda_n \leq K\mu^n$ for all integers $n > 0$.

Suppose f is a map in \mathcal{H} . Then $C_f = \bigcap_{n=0}^{+\infty} f^{-n}(I)$, the maximal invariant set of f , is a geometric Cantor set on the line (the proof is to use the naive distortion lemma). The scaling function of this geometric Cantor set has been defined in [4] as follows: suppose $w_n = i_n \dots i_1 i_0$ is a finite sequence of 0 and 1. Take $v_{n-1} = i_n \dots i_1$ and define $s(w_n) = |I_{w_n}|/|I_{v_{n-1}}|$, where $I_{w_n} = g_{w_n}(I), I_{v_{n-1}} = g_{v_{n-1}}(I)$ and $|\cdot|$ is the Lebesgue measure. Suppose

$$C^* = \{a^* = \dots i_n \dots i_1 i_0 \mid i_n \in \{0, 1\}\}$$

is the dual Cantor set. If $\lim_{n \rightarrow +\infty} s(w_n)$ exists for every $a^* = \dots w_n$ in C^* , we define a function $s(a^*) = \lim_{n \rightarrow +\infty} s(w_n)$ on C^* . This function is the scaling function of C_f . It is a C^1 -invariant (in other words, it is the same for both f and $h \circ f \circ h^{-1}$ whenever h is a C^1 -diffeomorphism of I).

Theorem A (Sullivan [4]). *Suppose f is a map in \mathcal{H} . The scaling function s of C_f exists and is Hölder (see [1] for the definition of a Hölder continuous function on a symbolic space).*

Now we consider a map f with $f(0) = 1$, which maps I into itself. In this case, the interval I is the maximal invariant set of f and we define an *orbifold metric* on I (associated with f) as

$$dy = \frac{dx}{(1 - x^2)^{\frac{\gamma-1}{\gamma}}}$$

Remark 1. The orbifold metric dy can be thought of as a singular metric with respect to the Lebesgue metric dx and the corresponding change of coordinate is $h_\eta(x) = -1 + b \int_{-1}^x dt / (1 - t^2)^{(\gamma-1)/\gamma}$, where b is determined by $h(1) = 1$.

Lemma 1. Suppose $\tilde{f} = h_\gamma \circ f \circ h_\gamma^{-1}$ (it is $f : (I, dy) \mapsto (I, dy)$). Then \tilde{f} is continuous and $\tilde{f}|[-1, 0]$ and $\tilde{f}|[0, 1]$ are C^1 -embeddings.

Proof. It is clearly that \tilde{f} is continuous and $\tilde{f}'(x) \neq 0$ for x in $(0, 1)$. By the definition of the power law critical point in this paper, it is easy to check that the one-sided derivatives of $\tilde{f}(x)$ at $0, -1$ and 1 are not zero either, and that $\tilde{f}'(x)$ is continuous on $[-1, 0]$ and $[0, 1]$.

Definition 2. We say f is on \mathcal{BH} , the boundary of hyperbolicity, if (1) $f(0) = 1$, (2) $\tilde{f}|[-1, 0]$ and $\tilde{f}|[0, 1]$ are $C^{1+\alpha}$ embeddings for some $0 < \alpha \leq 1$, and (3) there are constants $K > 0$ and $0 < \mu < 1$ such that $\lambda_n \leq K\mu^n$ for all integers $n > 0$.

Remark 2. Let $r_-(x)$ and $r_+(x)$ be $f'(x)/|x|^{\gamma-1}$ on $[-1, 0]$ and $[0, 1]$, respectively. Then (2) is equivalent to say that f, r_- and r_+ are $C^{1+\alpha'}$ for some $0 < \alpha' \leq 1$. Some examples of maps on \mathcal{BH} can be found in [3].

Although the maximal invariant set of f on \mathcal{BH} is the interval I , we can still define the scaling function of the sequence of partitions of I similarly: suppose $\eta = \{\eta_n\}_{n=0}^{+\infty}$ is the sequence of partitions of I determined by f . For every finite sequence $w_n = i_n \dots i_1 i_0$ of 0 and 1 , take $v_{n-1} = i_n \dots i_1$ and define $s(w_n) = |I_{w_n}|/|I_{v_{n-1}}|$, where $I_{w_n} = g_{w_n}(I)$, $I_{v_{n-1}} = g_{v_{n-1}}(I)$ and $|\cdot|$ is the Lebesgue measure. If $\lim_{n \rightarrow +\infty} s(w_n)$ exists for every $a^* = \dots w_n$ in C^* , we define a function $s(a^*) = \lim_{n \rightarrow +\infty} s(w_n)$ on C^* . This function is the scaling function of η . It is again a C^1 -invariant.

Let $I^* = \{a^* = \dots i_n \dots i_0 \mid \in C^*, i_{n_k} = 1 \text{ for infinitely many } k\}$ and O^* be the complement of I^* in C^* . The set O^* is countable.

Theorem 1. Suppose f is a map on \mathcal{BH} and $\eta = \{\eta_n\}_{n=0}^{+\infty}$ is the sequence of partitions of I determined by f . The scaling function s of η exists, and is continuous at every point a^* in I^* and discontinuous at every point a^* in O^* . Moreover, every a^* in O^* is a jump discontinuity of s and the restriction $s|I^*$ is uniformly continuous (see Fig. 1).

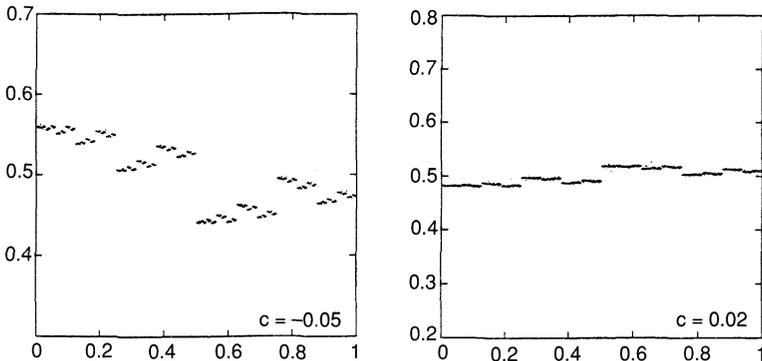


Fig. 1. The graphs of the scaling functions for $f_c(x) = -x^2 + 2 + cx^2(4 - x^2)$

Suppose f is on \mathcal{BH} and $\eta = \{\eta_n\}_{n=0}^{+\infty}$ is the sequence of partitions of I determined by f . Let $\tilde{s}(w_n) = \|I_{w_n}\|/\|I_{v_{n-1}}\|$, where $w_n = v_{n-1}i_0$ and $\|\cdot\|$ is the measure on I induced by the orbifold metric dy . If $\tilde{s}(a^*) = \lim_{n \rightarrow +\infty} \tilde{s}(w_n)$ exists for every $a^* = \dots w_n$ in C^* , then \tilde{s} is defined as the scaling function of η under dy .

Corollary 1. *Suppose f is on \mathcal{BH} . The scaling function \tilde{s} of η under the orbifold metric dy exists and is Hölder, and moreover, it is the continuous extension of $s|I^*$ to C^* (see Fig. 1).*

A family $\{f_\varepsilon\}_{0 \leq \varepsilon \leq 1}$ in $\mathcal{H} \cup \mathcal{BH}$ with $f_\varepsilon(0) = 1 + \varepsilon$ is said to be regular if

- (i) $F(x, \varepsilon) = f_\varepsilon(x)$ is a C^1 function of $[-1, 1] \times [0, 1]$ and $f'_\varepsilon(x)$ are uniformly α -Hölder continuous functions of x ,
- (ii) f_ε has the same exponent γ at 0 for every ε in $[0, 1]$ and $r_-(x, \varepsilon) = f'_\varepsilon(x)/|x|^{\gamma-1}$ on $[-1, 0] \times [0, 1]$ and $r_+(x, \varepsilon) = f'_\varepsilon(x)/|x|^{\gamma-1}$ on $[0, 1] \times [0, 1]$ are continuous and $r_+(x, \varepsilon)$ and $r_-(x, \varepsilon)$ are uniformly α -Hölder continuous functions of x , and
- (iii) there are two constants $K > 0$ and $0 < \mu < 1$ such that $\lambda_{n,\varepsilon} \leq K\mu^n$ for all $n > 0$ and $0 \leq \varepsilon \leq 1$.

Theorem 2. *Suppose $\{f_\varepsilon\}_{0 \leq \varepsilon \leq 1}$ in $\mathcal{H} \cup \mathcal{BH}$ is regular and $\{s_\varepsilon\}_{0 \leq \varepsilon \leq 1}$ is the family of corresponding scaling functions. Then*

- (I) for $\varepsilon_0 \in (0, 1]$, s_ε converges to s_{ε_0} uniformly on C^* as ε tends to ε_0 ,
- (II) for every $a^* \in C^*$, $s_\varepsilon(a^*)$ tends to $s_0(a^*)$ as ε goes to zero, and
- (III) $s_\varepsilon|I^*$ converges to $s_0|I^*$ as ε goes to zero.

2. The Proofs of the Main Results

Suppose $\{f_\varepsilon\}_{0 \leq \varepsilon \leq 1}$ in $\mathcal{H} \cup \mathcal{BH}$ with $f_\varepsilon(0) = 1 + \varepsilon$ is a regular family. Let U_ε be the interval bounded by $f_\varepsilon^{-1}(0)$ and V_ε be the closure of the complement of U_ε in I for every ε in $[0, 1]$ (see Fig. 2).

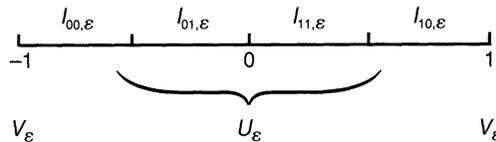


Fig. 2

Lemma 2. *There is a constant $K > 0$ such that for every ε in $[0, 1]$, if $f_\varepsilon^{oi}(x)$ and $f_\varepsilon^{oi}(y)$ are in the same connected component of V_ε for $i = 0, 1, \dots, n - 1$, then*

$$\frac{|(f_\varepsilon^{on})'(x)|}{|(f_\varepsilon^{on})'(y)|} \leq \exp(K|f_\varepsilon^{on}(x) - f_\varepsilon^{on}(y)|^\alpha).$$

Proof. It is the naive distortion lemma since $f'_\varepsilon(x) \neq 0$ for x in V_ε .

Lemma 3. *There is a constant $K > 0$ such that for every ε in $[0, 1]$, every pair x and y in an interval of $\eta_{m,\varepsilon}$ and every $0 < n \leq m$, if $f_\varepsilon^{on}(x)$ and $f_\varepsilon^{on}(y)$ are in U_ε , then*

$$\frac{|(f_\varepsilon^{on})'(x)|}{|(f_\varepsilon^{on})'(y)|} \leq \exp(K|f_\varepsilon^{on}(x) - f_\varepsilon^{on}(y)|^\alpha).$$

Proof. Let x_i and y_i be the images of x and y under f_ε^{oi} . Then x_i and y_i lie in a same interval of $\eta_{(m-i),\varepsilon}$. For every positive integer $n \leq m$,

$$\frac{(f_\varepsilon^{on})'(x)}{(f_\varepsilon^{on})'(y)} = \prod_{i=0}^{n-1} \frac{f'_\varepsilon(y_i)}{f'_\varepsilon(x_i)}.$$

This product can be factored into two products,

$$\prod_{x_i, y_i \in V_\varepsilon} \frac{f'_\varepsilon(y_i)}{f'_\varepsilon(x_i)} \quad \text{and} \quad \prod_{x_i, y_i \in V_\varepsilon} \frac{f'_\varepsilon(y_i)}{f'_\varepsilon(x_i)}.$$

Since (i) and (iii) in the definition of a regular family, there is a constant $K_1 > 0$ such that

$$\prod_{x_i, y_i \in V_\varepsilon} \frac{|f'_\varepsilon(y_i)|}{|f'_\varepsilon(x_i)|} \leq K_1.$$

To estimate the second product, let

$$h_\varepsilon(x) = -1 + b_\varepsilon \int_{-1}^x \frac{dt}{((1 + \varepsilon) - t^2)^\frac{\gamma-1}{\gamma}}$$

be the family of changes of coordinates on I (where h_0 is the orbifold metric we mentioned in Sect. 1). Let

$$\tilde{f}_\varepsilon(x) = h_\varepsilon^{-1} \circ f_\varepsilon \circ h_\varepsilon(x).$$

Then we can factor the second product into three factors,

$$\prod_{x_i, y_i \in U_\varepsilon} \frac{f'_\varepsilon(y_i)}{f'_\varepsilon(x_i)} = \prod_{x_i, y_i \in U_\varepsilon} \frac{h'_\varepsilon(y_i)}{h'_\varepsilon(x_i)} \cdot \prod_{x_i, y_i \in U_\varepsilon} \frac{\tilde{f}'_\varepsilon(h_\varepsilon(y_i))}{\tilde{f}'_\varepsilon(h_\varepsilon(x_i))} \cdot \prod_{x_i, y_i \in U_\varepsilon} \frac{h'_\varepsilon(f(x_i))}{h'_\varepsilon(f(y_i))}.$$

Further, the last one can be factored into

$$\prod_{x_i, y_i \in U_\varepsilon} \frac{(1 + f_\varepsilon(y_i))^\frac{\gamma-1}{\gamma}}{(1 + f_\varepsilon(x_i))^\frac{\gamma-1}{\gamma}} \quad \text{and} \quad \prod_{x_i, y_i \in U_\varepsilon} \frac{(1 - f_\varepsilon(y_i))^\frac{\gamma-1}{\gamma}}{(1 - f_\varepsilon(x_i))^\frac{\gamma-1}{\gamma}}.$$

Since $h'_\varepsilon(x)$ and $\tilde{f}'_\varepsilon(x)$ are uniformly α' -Hölder continuous functions on each of $I_{01,\varepsilon}$ and $I_{11,\varepsilon}$ ($U_\varepsilon = I_{01,\varepsilon} \cup I_{11,\varepsilon}$), there is a constant $K_2 > 0$ so that

$$\prod_{x_i, y_i \in U_\varepsilon} \frac{|h'_\varepsilon(y_i)|}{|h'_\varepsilon(x_i)|} \leq K_2 \quad \text{and} \quad \prod_{x_i, y_i \in U_\varepsilon} \frac{|\tilde{f}'_\varepsilon(h_\varepsilon(y_i))|}{|\tilde{f}'_\varepsilon(h_\varepsilon(x_i))|} \leq K_2.$$

It is clearly that there is a constant $K_2 > 0$ such that

$$\prod_{x_i, y_i \in U_\varepsilon} \frac{(1 + f_\varepsilon(y_i))^\frac{\gamma-1}{\gamma}}{(1 + f_\varepsilon(x_i))^\frac{\gamma-1}{\gamma}} \leq K_3.$$

To finish the proof we write

$$\frac{1 - f_\varepsilon(y_i)}{1 - f_\varepsilon(x_i)} = 1 + \frac{f_\varepsilon(x_i) - f_\varepsilon(y_i)}{1 - f_\varepsilon(x_i)}.$$

From the facts that $f_\varepsilon(1) = f_\varepsilon(-1) = -1$ and $f^{oj}(x_i)$ is in U_ε for some $i < j \leq n$ (which is assured by the assumption of the lemma), we can find a constant $K_4 > 0$ such that

$$\prod_{x_i, y_i \in U_\varepsilon} \frac{(1 - f_\varepsilon(y_i))^{\frac{\gamma-1}{\gamma}}}{(1 - f_\varepsilon(x_i))^{\frac{\gamma-1}{\gamma}}} \leq K_4.$$

Then $K = K_1 K_2^2 K_3 K_4$ is a constant which satisfies the lemma.

A straightforward calculation from Lemma 2 and 3 to the cases that $a^* = \dots w_n$ is in O^* and is in I^* , respectively, implies that there are constants $K = K(a^*) > 0$ and $0 < \alpha \leq 1$ and a subsequence $\{n_i\}$ of the integers such that for all $m \geq k \geq n_i$, and ε and ε' in $[0, 1]$,

$$|s_\varepsilon(w_k) - s_{\varepsilon'}(w_m)| \leq K(|I_{w_{n_i}, \varepsilon}|^\alpha + |I_{w_{n_i}, \varepsilon'}|^\alpha) + |s_\varepsilon(w_{n_i}) - s_{\varepsilon'}(w_{n_i})| \quad (*)$$

Proof of Theorem 1. The existence of the scaling function s of the sequence of partition η follows from the inequality (*) as $\varepsilon = \varepsilon' = 0$.

The scaling function s is continuous at every point a^* in I^* and the restriction $s|_{I^*}$ is uniformly continuous follow from the inequality (*) and the fact that $\sup_{a^* \in I^*} K(a^*) < \infty$ too.

Suppose $a^* = (\dots 00wi)$ is in O^* . Let 0_n be the finite string of zeroes of length n . Then the interval $I_{0_n w}$ is eventually in the left interval I_{00} . We use b_n and a_n to denote the lengths of $I_{0_n w}$ and $I_{0_n wi}$, respectively (where $I_{0_n wi} \subset I_{0_n w}$). Let c_n be the distance from $I_{0_n w}$ to -1 (see Fig. 3).

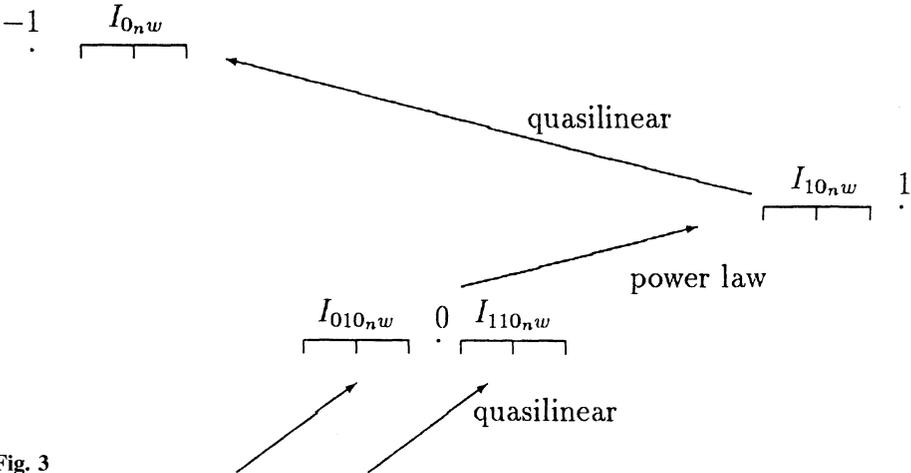


Fig. 3

Using the naive distortion lemma, we can show

$$\tau_1(a^*) = \lim_{n \rightarrow \infty} \frac{b_n}{c_n} > \tau_2(a^*) = \lim_{n \rightarrow \infty} \frac{a_n}{c_n}$$

exist. These implies that the limit of the sequence $\{s(j10_n wi)\}_{n=1}^\infty$ (where j is either 0 or 1) exists and equals

$$\frac{(1 + \tau_2(a^*))^{\frac{1}{\gamma}} - 1}{(1 + \tau_1(a^*))^{\frac{1}{\gamma}} - 1} \quad \text{or} \quad \frac{(1 + \tau_1(a^*))^{\frac{1}{\gamma}} - (1 + \tau_2(a^*))^{\frac{1}{\gamma}}}{(1 + \tau_1(a^*))^{\frac{1}{\gamma}} - 1}.$$

For a point $b^* = (\dots j10_n wi)$ in C^* , because $I_{j10_n w}$ is in U_ϵ we can use Lemma 3 to show that the error of $s(j10_n i)$ to $s(b^*)$ can be estimated by $|I_{j10_n wi}|^\alpha$, that is, there is a positive constant K_1 such that

$$|s(b^*) - s(j10_n wi)| \leq K_1 |I_{j10_n wi}|^\alpha.$$

Further,

$$\lim_{b^* \neq a^*, b^* \rightarrow a^*} s(b^*) = \lim_{n \rightarrow \infty} s(j10_n wi).$$

Since $s(a^*) = \tau_2(a^*)/\tau_1(a^*)$, we get

$$\lim_{b^* \neq a^*, b^* \rightarrow a^*} s(b^*) s(b^*) \neq s(a^*),$$

in the other words, s has jump discontinuity at a^* .

Proof of Corollary 1. It follows from the naive distortion lemma and the fact that $\tilde{s}|I^* = s|I^*$.

Proof of Theorem 2. (I) is also from the naive distortion lemma. (II) follows from the inequality (*). (III) follows from the inequality (*) and that $0 < \max_{a^* \in I^*} K(a^*) < +\infty$.

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