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Random Walk Representations and the Mayer Expansion in the N-Vector Model

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Abstract. We consider a random walk representation of non-abelian statistical models, and apply it to represent the free energy in terms of the correlation functions of random walks. This enables us to find an analytic region of the free energy with respect to the inverse temperature. This method can be applied to the block spin transformations.

1. Introduction

Quark confinements and non-existence of phase transitions in two dimensional non-Abelian statistical models have been long standing problems in modern physics. The author recently considered non-Abelian models by means of the renormalization group method, and investigated a cluster expansion which uses the random walk representations [4, 7].

We start with the expression of the model, say ν dimensional Heisenberg model with $O(N)$ symmetry (N-vector model):

$$\langle\cdot\rangle \equiv \frac{1}{Z_\Lambda(J)}\int(\cdot)\exp\left[-\frac{J}{2}\sum_{|x-y|=1}(\phi(x)-\phi(y))^2\right]\prod d\phi(x)\,, \tag{1}$$

where $\phi(x)\in S^{N-1}$ for all lattice points $x\in\Lambda(\subset Z^\nu)$, $J>0$ is the coupling constant (= inverse temperature) and the integration $d\phi(x)$ is over the $N-1$ dimensional sphere S^{N-1}. Finally $Z_\Lambda(J)$ is the normalization constant chosen so that $\langle 1\rangle = 1$. There are several studies on the thermodynamic properties of this system. Among them is

Theorem (Brydge-Froehlich-Spencer-Sokal [4], see also Simon-Lieb [9]). *If $J < N/2\nu$, there exist strictly positive constants m and C such that*

$$\langle\phi(0)\,\phi(x)\rangle \le C\exp[-m|x|]\,.$$

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It is a long standing problem whether or not this holds for any (real) J if $N > 2$ and $\nu = 2$. We want to represent the free energy $\lim |\Lambda|^{-1} \log(Z_\Lambda) \equiv \alpha(J)$ as a function of J and obtain the region of J in which α is analytic. Our main results are summarized in the following theorems:

Theorem A. (1) *The free energy $\alpha(J)$ is represented in terms of the truncated correlation (Ursell) functions of random walks.*
(2) *The free energy $\alpha(J)$ is analytic in J in the region*

$$\Omega(N, \nu) = \left\{ J; |J| < J_c \equiv (1 - \varepsilon) \frac{N}{2\nu} \right\},$$

where ν stands for the dimensions of the lattice space and $\lim_{\nu \to \infty} \varepsilon = 0$.

Theorem B. *The correlation functions decay exponentially fast for $J \in \Omega$.*

For smaller $\nu (\nu = 2, 3)$, we have numerical bounds on J_c which are less than $N/2\nu$ but are close to $N/2\nu$. In the final section of this paper, we discuss how we can apply the present method to the block spin transformation.

Remark 1. A similar investigation was done by Kupiainen [8] by using another type of random walk representation. But for $\nu = 2$, his result is $J_c \leq \text{const} \log N$.

2. Random Walk Representation of the Free Enery

Introducing the Fourier-Laplace transformation and integrating over the spin variables [4] which is now standard, we start with the following random walk representation of the partition function:

$$\begin{aligned} Z_\Lambda(J) &= \int \exp\left[-\frac{J}{2} \sum (\phi(x) - \phi(y))^2 \right] \prod d\phi(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2} \right)^n \sum_{\omega_1, \ldots, \omega_n} \frac{J_{\omega_1} \ldots J_{\omega_n}}{2^{|\omega_1|} \ldots 2^{|\omega_n|}} \exp[-V(\omega_1, \ldots, \omega_n)] \end{aligned} \tag{2}$$

(except for a constant coefficient) where $\omega = \{\{b_0, \ldots, b_L\}; b_j = (x_j, x_{j+1}), x_j \in Z^\nu, |x_j - x_{j+1}| = 1, x_{L+1} = x_0\}$ are closed loops made by nearest neighbor bonds in Z^d and $|\omega_i|$ is the length of the loop ω_i which equals $\sum_x v_i(x) \equiv |v_i|$, the sum of the visiting numbers. Moreover

$$J_\omega = \prod_{b \in \omega} J_b, \tag{3a}$$

$$\exp[-V(\omega_1, \ldots, \omega_n)] = \prod_{x \in \cup \omega_i} \frac{[N/2 - 1]!}{[v(x) + N/2 - 1]!}, \tag{3b}$$

where

$$v(x) = \sum_i v_i(x) \tag{3c}$$

with $v_i(x) \in \{0, 1, \ldots\}$ being the visiting number of ω_i at $x \in Z^\nu$. Then the standard Mayer expansion method implies that

$$\begin{aligned} \alpha(J) &= \lim |\Lambda|^{-1} \log Z_\Lambda(J) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2}\right)^n \sum_{\omega_1, \ldots, \omega_n} \frac{1}{|X|} \frac{J_{\omega_1} \ldots J_{\omega_n}}{2^{|\omega_1|} \ldots 2^{|\omega_n|}} [\exp[-V(\omega_1, \ldots, \omega_n)]]^T , \end{aligned} \tag{4}$$

where $0 \in X \equiv \cup \operatorname{supp} \omega_i$ and $[\ldots]^T$ means the truncated correlation functions of random walks:

$$\begin{aligned} [\exp[-V(\omega_1, \ldots, \omega_n)]]^T &= [\exp[-V(\omega_1, \ldots, \omega_n)] - \sum_{t_1 \cup t_2 = t} \exp[-V(t_1) - V(t_2)] \\ &\quad + 2 \sum_{t_1 \cup t_2 \cup t_3 = t} \exp[-\sum V(t_i)] - \ldots . \end{aligned} \tag{5}$$

Here $t = \{1, \ldots, n\}$ and $t = \bigcup_1^k t_i$ means the partitions of t into non-overlapping subsets t_i and for $t_i = \{a_1, \ldots, a_j\}$,

$$\exp[-V(t_i)] = \exp[-V(\omega_{a_1}, \ldots, \omega_{a_j})] .$$

In order to represent the truncated correlation functions of the random walks, we introduce truncated potentials $V^T(\omega_1, \ldots, \omega_n)$ defined by the following formulas:

$$\begin{aligned} V(\omega_1) &= V^T(\omega_1) , \\ V(\omega_1, \omega_2) &= \sum_{t \subset \{1,2\}} V^T(t) , \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ V(\omega_1, \ldots, \omega_n) &= \sum_{t \subset \{1, \ldots, n\}} V^T(t) . \end{aligned} \tag{6a}$$

Therefore we conversely have

$$V^T(\omega_1, \ldots, \omega_k) = \sum_{t \subset \{1, \ldots, k\}} (-1)^{k-|t|} V(t) , \tag{6b}$$

where t runs over all possible subsets of $\{1, 2, \ldots, k\}$. Given a subset t, we define the composite random walk denoted by $\omega_{[t]}$ or simply by $[t]$. This is the random walk that is equal to $\bigcup_{i \in t} \omega_i$ together with the visiting numbers. We similarly define many body potentials $V^T([t_1], \ldots, [t_n])$ for composite random walks:

$$\begin{aligned} V^T([t_1]) &= V([t_1]) , \\ V^T([t_1], [t_2]) &= V([t_1], [t_2]) - V([t_1]) - V([t_2]) , \end{aligned}$$

and etc. Using Eq. (6b), we easily find that

$$V^T([t_1], \ldots, [t_n]) = \sum_{t'_1, \ldots, t'_n} V^T(t'_1, \ldots, t'_n) , \tag{7}$$

where t'_i are subsets of t_i different from $\emptyset$ and the truncated functions on the right-hand side are *the truncated functions with respect to all walks contained in* $\cup t'_i$.

The Ursell functions have complicated expressions when we have manybody interaction terms in the potentials. We first explain our notation following [2] and [3].

For $t = \{1, 2, \ldots, n\}$, let $t_1 = \{1\}$, and let $\{2, 3, \ldots, n\} = \bigcup_{i=2}^{k} t_i$ be any partition of $\{2, \ldots, n\}$. Given k, let η_k be a map $\{1, 2, \ldots, k-1\} \to \{1, 2, \ldots, k-1\}$ such that $\eta_k(i) \leq i$. Moreover let $X_1 = t_1 = \{1\}$, $X_i = X_{i-1} \cup t_i$, $i = 2, \ldots, k$. Then for any subset $b \subset t$, define functions s_i so that

$$s_i(b) = \begin{cases} s_i & \text{if } X_i \cap b \neq \emptyset, \text{ and } X_i^c \cap b \neq \emptyset \\ 1 & \text{otherwise.} \end{cases} \tag{8}$$

Namely $s_i(b) = 1$ whenever $b \subset X_i$ or $b \subset X_i^c$ and $s_i(b) = s_i$ otherwise. Let

$$\begin{aligned} W(\{t_i\}_i^k; \{s_i\}_1^{k-1}) &\equiv \sum_{b \subset t} s_1(b) \ldots s_{k-1}(b) V^T(b) \\ &= \sum_{1 \leq u < j \leq k} s_i \ldots s_{j-1} \mathscr{W}(i,j) + \sum V([t_i]), \end{aligned} \tag{9a}$$

where

$$\mathscr{W}(i,j) \equiv V^T([i],[j]) + V^T\left([i], \left[\bigcup_{i+1}^{j-1} t_k\right], [j]\right), \tag{9b}$$

and this form follows by the definition of the many body potential (6a) and we identified the subscript i with the set of loops t_i. (Note that $V([t]) = V(t) = V(\omega_1, \ldots, \omega_n)$.) This is a convex linear combination of $V(t_i, t_{i+1}, \ldots, t_j)$, $1 \leq i \leq j \leq k$ and thus satisfies the bound (in fact we show $\mathscr{W} \geq 0$, see Lemma 2)

$$\begin{aligned} \exp[-W(\{t_i\}_1^k; \{s_i\}_1^{k-1})] &\leq \exp[-\sum V(t_i)] \\ &= \prod_i \prod_{x \in \bigcup_{j \in t_i} \omega_j} \frac{\left[\frac{N}{2} - 1\right]!}{\left[\frac{N}{2} + \sum_j v_j(x) - 1\right]!}, \end{aligned} \tag{10}$$

where $\sum_{j \in t_i} v_j(x)$ is the sum of the visiting numbers of $\{\omega_j; j \in t_i\}$ at x.

For a function η_k, we have the tree graph $T = T(\eta_k)$ over t regarding $(i+1, \eta_k(i))$ as the bond connecting two vertices $i+1$ and $\eta_k(i) \in \{1, 2, \ldots, i\}$. Set $\eta_k(i) = j$ for simplicity. We define the reduced two-body potential $\mathscr{V}(j, i+1)$, $(j < i)$ by

$$\mathscr{V}(j, i+1) \equiv \sum_{t'_j \subset t_j} \sum_{t'_{j+1} \subset t_{j+1}} \ldots \sum_{t'_i \subset t_i} V^T(t'_j, \ldots, t'_i, t_{i+1}), \tag{11}$$

where $t'_{j-1}, \ldots, t'_i$ may be empty but t'_j is not empty. If some $t'_i, \ldots, t'_{j-1}$ are empty, we just neglect those in V^T. Thus we have

$$\mathscr{V}(j, i+1) = V^T([j], i+1) + V^T([j], [t], i+1), \tag{12}$$

where $[t]$ means the composite random walk made by single random walks in t_l, $(l = j+1, \ldots, i)$.

Then we have the following theorem whose proof is essentially in [3]:

Theorem 1.

$$[\exp[-V(\omega_1, \ldots, \omega_n)]]_T = \sum_{k=2}^{n} \sum_{t_1, \ldots, t_k} \sum_{\eta_k} \int_0^1 ds_1 \ldots \int_0^1 ds_{k-1} f(\eta_k, s) \times \prod_{i} \mathscr{V}(\eta_k(i), i+1) \exp[-W(\{t_i\}; \{s_i\})], \quad (13)$$

where $t_1 = 1$ and $t_2 \cup \ldots \cup t_k$ changes over all partitions of $\{2, \ldots, n\}$ into $k-1$ subsets, and

$$f(\eta_k, s) = \prod_{i=1}^{k-1} s_{i+1} \ldots s_{\eta(i)} \,. \quad (14)$$

3. Truncated Correlation Functions

We here estimate $\mathscr{V}(i, j+1)$ and $\mathscr{W}(i, j+1)$ which are expressed in terms of V^T. To do so, we start with the estimate of more general $V^T(t) = V^T(\omega_1, \ldots, \omega_n)$. Let $x \in Z^d$ be a point in $\cap \operatorname{supp} \omega_i$. Then if $n > i$,

$$V^T(t_k) = \sum f_N(x)\,, \quad (15a)$$

$$f_N(x) = \sum_{t'} (-1)^{n-|t'|} \log \left[\left(\frac{N}{2} + v(t') - 1 \right)! \right], \quad (15b)$$

$$v(t') = \sum_{i \in t'} v_i(x)\,, \quad (15c)$$

where $t' \subset t$ including the case of $t' = \emptyset$. For example, we have

$$f_N(x) = \log \frac{\left[\frac{N}{2} + v_1(x) - 1\right]!}{\left[\frac{N}{2} - 1\right]!}\,,$$

for $n = 1$, and for $n = 2$ we have

$$f_N(x) = \log \frac{\left[\frac{N}{2} + v_1(x) + v_2(x) - 1\right]! \left[\frac{N}{2} - 1\right]!}{\left[\frac{N}{2} + v_1(x) - 1\right]! \left[\frac{N}{2} + v_2(x) - 1\right]!}\,.$$

Now we see

$$\begin{aligned} f_{N+2} - f_N &= \sum_{t'} (-1)^{n-|t'|} \log \left(\frac{N}{2} + v(t') \right) \\ &= \int_0^1 \ldots \int_0^1 ds_1 \ldots ds_n \left(\prod \frac{\partial}{\partial s_i} \right) \log \left(\frac{N}{2} + \sum v_i s_i \right) \\ &= (-1)^{n-1} (n-1)! \int_0^1 \ldots \int_0^1 \prod ds_i \frac{\prod v_i}{\left[\frac{N}{2} + \sum v_i s_i\right]^n}\,. \end{aligned}$$

This recursion formula is easily solved and yields the following representation for $f_N(x)$:

$$f_N(x) = \sum_{k=0}^{\infty}(-1)^n\,(n-1)!\int_0^1 ds_1 \ldots \int_0^1 ds_n\,\frac{\prod v_i}{\left[\frac{N}{2}+\sum v_i s_i + k\right]^n}$$

$$= \sum_{k=0}^{\infty}(-1)^n\,(n-1)!\int_0^{2v_1/N} dy_1 \ldots \int_0^{2v_n/N} dy_n\,\frac{1}{\left[1+\sum y_i + \frac{2k}{N}\right]^n}\,. \tag{16a}$$

The series converges absolutely for $n > 1$ and is equal to

$$(-1)^n\,(n-1)!\,\frac{N}{2(n-1)}\left(1+O\left(\frac{n}{N}\right)\right)$$
$$\times \int_0^{2v_1/N} \cdots \int_0^{2v_n/N} \prod dy_i\,\frac{1}{[1+\sum y_i]^{n-1}}\,, \tag{16b}$$

where $O(n/N) < 2(n-1)/N$ is the correction term caused by replacing the summation with the integral. See Remark 2 below. We let $t = \bigcup_2^i t_i$ and let $[t]$ stand for the composite random walk made by single random walks contained in t. Assume $x \in t_1 \cap t \cap t_{i+1}$, and assume that t_1 visits $x\,v_1$ times, t visits $x\,v_2$ times and t_{i+1} visits $x\,v_3$ times, and finally each loop $\omega_1, \ldots, \omega_n$ in t_{i+1} visits $x\,v_i'$ times. Then $v_3 = \sum v_i'$ and the following two lemmas and a corollary are immediate:

Lemma 2. $\mathscr{V}(1, i+1) = \sum_x \mathscr{V}(1, i+1)(x)$, *where*

$$\mathscr{V}(1,i+1)(x) = \sum_{k=0}^{\infty}(-1)^{n+1}\,(n+1)!\int_0^{2v_1/N} dy_1 \int_{2v_2/N}^{\infty} dy_2$$
$$\times \int_0^{2v_1'/N} \cdots \int_0^{2v_n'/N} \prod dy_i'\,\frac{1}{\left[1+y_1+y_2+\sum y_i' + \frac{2k}{N}\right]^{n+2}}\,. \tag{17}$$

Corollary 3. *Let $n_j \geq 1$ be the number of loops contained in the composite loop $\omega_{[t_j]}$ and let $v_j(x) \in N$ be the visiting number of the loop ω_j at x. Then*

$$|\mathscr{V}(i,j)(x)| \leq \left(1+O\left(\frac{n_j}{N}\right)\right)\left(\frac{2}{N}\right)^{n_j}(n_j-1)!\,v_{[t_i]}(x)\prod_{s=1}^{n_j} v_{j_s}(x)\,. \tag{18}$$

Lemma 4. $\mathscr{W}(1, i+1) = \sum_x \mathscr{W}(1, i+1)(x)$, *where*

$$\mathscr{W}(1,i+1)(x) = \sum_{k=0}^{\infty} 2\int_0^{2v_1/N}\int_{2v_2/N}^{\infty}\int_0^{2v_3/N}\frac{\prod dy_i}{\left[1+y_1+y_2+y_3+\frac{2k}{N}\right]^3}\,. \tag{19}$$

Remark 2. It is easy to see that the correction term $O(n/N)$ is bounded by $2(n-1)/N$. This may be large, but on the other hand, we have the factor $(2/N)^n(1+2/N)^{-1}\dots(1+2(n-1)/N)^{-1}$ which comes from the n-ple self-crossing points in $\omega_{[t]}$. Therefore these correction terms are cancelled by these factors in $\exp[-V(\omega_{[t]})]$.

4. Convergence of the Mayer Expansion

Let $t=\{1,2,\dots,n\}$ and let $t=t_1\cup\dots\cup t_k$ be partitions of t into sets of loops ($t_1=1$). Let $0\in X\equiv\cup\operatorname{supp}\omega_i$. The n-loop contribution K_n to α is given by

$$K_n(J)=\frac{N^n}{n!2^n}\sum_{\omega_1,\dots,\omega_n}\frac{1}{|X|}\prod\frac{J_{\omega_i}}{2^{|\omega_i|}}\sum_k\sum_{t_1,\dots,t_k}\times\left(\prod_{i=1}^{k-1}\int_0^1 ds_i\right)\mathscr{U}(s)e^{-\mathscr{W}(s)-\sum V(t_i)}, \tag{20}$$

where we set

$$\mathscr{U}(s)=\prod_{i=2}^{k}U(i),\qquad \mathscr{W}(s)=\sum_{i=1}^{k-1}s_iW(i+1)$$

together with the following notation:

$$\begin{aligned}
&U(2)=\mathscr{V}(1,2),\\
&U(3)=\mathscr{V}(2,3)+s_i\mathscr{V}(1,3),\\
&\dots\ \dots\ \dots\dots\dots\dots\\
&U(k)=\mathscr{V}(k-1),k)+s_{k-2}\mathscr{V}(k-2,k)+\dots+s_1s_2\dots s_{k-2}\mathscr{V}(1,k),\\
&W(2)=\mathscr{W}(1,2),\\
&W(3)=\mathscr{W}(2,3)+s_1\mathscr{W}(1,3),\\
&\dots\ \dots\ \dots\dots\dots\dots\\
&W(k)=\mathscr{W}(k-1,k)+s_{k-2}\mathscr{W}(k-2,k)+\dots+s_1s_2\dots s_{k-2}\mathscr{W}(1,k).
\end{aligned}$$

Here $\operatorname{sgn}(U(i))=(-1)^{n_i}$ and then

$$\operatorname{sgn}(K_n)=(-1)^{n-1}. \tag{21}$$

For later purposes, we define the following quantities:

$$|v_{[i]}|\equiv|v_{[t_i]}|=\sum_x v_{t_{[i]}}(x)=\sum_{j\in t_i}\sum_x v_j(x), \tag{22}$$

$$I_k\equiv\int_0^1\dots\int_0^1\prod_1^{k-1}ds_i|v_{[1]}|\,(|v_{[2]}|+s_1|v_{[1]}|)\dots(|v_{[k-1]}|+s_{k-2}|v_{[k-2]}|+\dots+s_{k-2}s_{k-3}\dots s_2s_1|v_{[1]}|). \tag{23}$$

(The integrand does not depend on s_{k-1}, but so does $\mathscr{W}(s)$.) We will need the following lemma which goes back to [1–3].

Lemma 5. *Let* $M = \sum_1^n |v_i|$. *Then* $M \geq 2n$ *and*

$$I_k < \left(\frac{M}{k}\right)^k e^k .$$

Proof. We set

$$\mathbf{v}(s, i-1) = |v_{[i-1]}| + s_{i-2}|v_{[i-2]}| + \ldots + s_{i-2}s_{i-3}\ldots s_2 s_1 |v_{[1]}| .$$

Multiplying $\exp[\alpha \mathbf{v}(s, k-1) s_{k-1}] \geq 1$ with $\alpha > 0$, we have the estimate

$$\begin{aligned}\int_0^1 & ds_{k-1} \mathbf{v}(s, k-1) \exp[\alpha \mathbf{v}(s, k-1) s_{k-1}] \\ &= \alpha^{-1}(\exp[\alpha |v_{[k-1]}| + \alpha \mathbf{v}(s, k-2) s_{k-2}] - 1) \\ &\quad < \alpha^{-1} \exp[\alpha |v_{[k-1]}|] \exp[\alpha \mathbf{v}(s, k-2) s_{k-2}] .\end{aligned}$$

Repeating these, we have $I_k < (\alpha)^{-k} \exp\left[\sum_1^{k-1} \alpha |v_{[i]}|\right]$ which holds for any $\alpha > 0$. Then we set $\alpha = k/\sum |v_{[i]}|$ to get the result. Q.E.D.

Suppose loops $\{\omega_1, \ldots, \omega_n\}$ and partitions $t_1 = \{1\}, t_2, \ldots, t_k$ of $\{1, 2, \ldots, n\}$ are given. Let η be tree graphs over $t_1, \ldots, t_k$, $C_n = \frac{1}{n!}\left(\frac{N}{2}\right)^n$ and consider

$$\begin{aligned} C_n \sum_{\text{transl.}} \frac{1}{|X|} \prod \frac{J_{\omega_i}}{2^{|\omega_i|}} \sum_\eta \int \cdots \int \prod ds_i f(\eta, s) \\ \times \prod \mathscr{V}(\eta(i), i+1)\, e^{-W(s) - \sum V(t_i)} , \end{aligned} \tag{24}$$

where $\sum_{\text{transl.}}$ means the sum over all possible translations of ω_i, $i > 1$. For the above quantity to be different from zero, all loops $\omega_{i'}$ contained in ω_{t_i} must cross each other at a same point, and all composite loops ω_{t_i} must cross each other. From Eq. (18) in Corollary 3, we have

$$\sum_{\text{transl.}} |\mathscr{V}(i, j)| \leq \left(1 + O\left(\frac{n_j}{N}\right)\right) \left(\frac{2}{N}\right)^{n_j} (n_j - 1)!\, |v_{[i]}| \prod_{s=1}^{n_j} |v_{j_s}| ,$$

where we considered the translations of ω_{j_s}, $j_s \in t_j$ only. Thus we finally have

$$\begin{aligned} |\text{Eq. (24)}| &\leq \frac{N}{2n!} \prod (n_i - 1)! \left\{ \prod \frac{J_{\omega_i}}{2^{|\omega_i|}} |v_i| \exp[-V(\omega_i)] \right\} I_k \\ &\leq \frac{N}{2n!} \prod \left\{ (n_i - 1)! \frac{J_{\omega_i}}{2^{|\omega_i|}} |v_i| \exp[-V(\omega_i)] \right\} \left(\frac{\sum_{i=1}^n \sum_x v_i(x)}{k} \right)^k e^k , \end{aligned} \tag{25}$$

where we used Lemma 5 and we omitted for simplicity the correction term $O(n/N)$ cancelled by the marginal term in $\exp[-V(\omega)]$, see Remark 2. Suppose $n_2, \ldots, n_k$

with $n_i \geq 1$ and $\sum_2^k n_i = n - 1$ are given. Then the number of the partitions of $\{2, \ldots, n\}$ into these groups is

$$\frac{(n-1)!}{n_2! \ldots n_k!},$$

and the number of the solutions of the equation $\sum_2^k n_k = n - 1$ with $n_i \geq 1$ is $\binom{n-2}{k-2}$. Therefore $|K_n|$ is bounded by

$$\sum_{0 \in \omega_i} \frac{N}{n} \prod \frac{J_{\omega_i} |v_i| \exp[-V(\omega_i)]}{2^{|\omega_i|}} \left\{ \sum_{k=2}^{n} H_k \left(\frac{\sum |v_i|}{k} \right)^k e^k \right\} \tag{26a}$$

$$\leq \sum_{0 \in \omega_i} \frac{N}{n} \prod \frac{J_{\omega_i} |v_i| \exp[-V(\omega_i)]}{2^{|\omega_i|}} \left\{ \sum_{k=2}^{n} \binom{n-2}{k-2} \left(\frac{\sum |v_i|}{k} \right)^k e^k \right\}, \tag{26b}$$

where H_k is defined by

$$H_k \equiv \sum_{n_2 + \ldots + n_k = n-1} \frac{1}{n_2 \ldots n_k} \leq \binom{n-2}{k-2}. \tag{27}$$

Moreover setting $M = \sum |v_i| (\geq 2n)$, we have

$$\sum_{k=2}^{n} \binom{n-2}{k-2} \left(\frac{M}{k} \right)^k e^k$$
$$= \left\{ \sum_{k=2}^{n} \binom{n-2}{k-2} \left(\frac{n}{eM} \right)^{n-k} \left(\frac{n}{k} \right)^k \right\} \exp\left[n \log \frac{eM}{n} \right].$$

Here

$$\sum_{k=2}^{n} \binom{n-2}{k-2} \left(\frac{n}{eM} \right)^{n-k} \left(\frac{n}{k} \right)^k \leq (1 + C_1)^n, \tag{28a}$$

$$\exp\left[n \log \frac{M}{n} \right] \leq \exp[C_2 \sum |v_i|], \tag{28b}$$

with $C_1 \leq 1/2$, $C_1 \to 0$ as $M/n \to \infty$, and $C_2 \leq 1/e$, $C_2 \to 0$ as $M/n \to \infty$, since $M = \sum |v_i| \geq 2n$. In order to improve C_i numerically, we may take the maximum of the left-hand side of Eq. (28a) with respect to k. Then we see that the maximum is attained by $k = (\sqrt{3} - 1)n + O(1)$, which implies that $(1 + C_1)^n$ can be replaced by $\text{const}\, n^\alpha (1 + C_1)^n$, $C_1 < 0.4274$. To be more accurate, we may refine the bound (27) for H_k. Let $m \in \{k - 1, \ldots, 1, 0\}$ be the number of i such that $n_i = 1$. Then we have

$$H_k \leq \sum_{m=0}^{k-1} \binom{k-1}{m} \binom{n-k-1}{k-m-2} \frac{1}{2^{k-1-m}}.$$

We again use the Stirling formula and take the maximum to find that

$$\sum_{k=2}^{n} H_k \left(\frac{n}{eM}\right)^{n-k} \left(\frac{n}{k}\right)^k \leq Cn^\alpha (1 - C_1)^n ,$$

where $C_1 < 0.2581$, and C and α are some positive constants.

Lemma 6.

$$|K_n| < CNn^\alpha \prod \left[\sum_{0\in\omega_i} \frac{J_{\omega_i}|v_i| \exp[-V(\omega_i)]}{2^{|\omega_i|}} (1 + C_1) \exp[1 + C_2|v_i|] \right] , \qquad (29)$$

where $C = O(1)$, $C_1 < 0.2581$, $C_2 < 1/e$ uniformly in n and α is a positive constant. Moreover C_1 and C_2 converge to 0 as $\sum_i |v_i|/n \to \infty$.

Remark 3. There are several methods to improve these bounds [1, 2]. Since $\exp[C_2|v_i|] = \exp[C_2|\omega_i|]$, the factor $\exp[C_2|v_i|]$ is absorbed by replacing J_b with $\exp[C_2] J_b = \exp[1/e] J_b = 1.4446 J_b$ (i.e. scaling) in the definition of J_{ω_i}.

5. Numerical Evaluation of the Convergent Radius

Before proving next theorem, we estimate the number of loop diagrams of length $2m$. Let p_i and q_i be the numbers of unit walks $x \to x + e_i$ and $x - e_i \to x$ respectively, contained in ω. The necessary and sufficient condition for a walk ω of length $2m$ to form a loop is that $p_i = q_i = n_i$ and $\sum n_i = m$. Then the number of loops of length $2m$ is given by

$$l_m \equiv \sum_{n_1+\ldots+n_\nu=m} \frac{(2m)!}{(n_1! \ldots n_\nu!)^2} = O(\nu^m) . \qquad (30)$$

For small m, they are:

$$\begin{aligned} l_1 &= 2\nu , \\ l_2 &= 12\nu^2 - 6\nu , \\ l_3 &= 120\nu^3 - 180\nu^2 + 80\nu , \\ l_4 &= 1680\nu^4 - 5040\nu^3 + 6740\nu^2 - 2326\nu , \end{aligned}$$

and it is easy to see

$$l_m \leq \frac{1}{\nu} (2\nu)^{2m} \quad \text{for} \quad m \leq m(\nu)$$

for some constant $m(\nu)$ ($m(\nu) \to \infty$). By the Stirling formula, we have

$$l_m = O(1) 2^{2m} \sum_{n_1+\ldots+n_\nu=m} \left(\frac{m!}{n_1! \ldots n_\nu!}\right)^2$$

and then $O(1) 2^{2m} \nu^m < l_m$. One can also show that $l_m = O(m^{-\alpha}) (2\nu)^{2m}$. The number of random walks of length $2m$ which are not necessarily loops is

$$\sum_{p_1+q_1+\ldots+p_\nu+q_\nu=2m} \frac{(2m)!}{p_1! q_1! \ldots p_\nu! q_\nu!} = (2\nu)^{2m} .$$

By collecting terms satisfying $p_i + q_i = 2n_i$, and noticing that

$$\sum_{p+q=2n} \frac{(n!)^2}{p!q!} = \frac{2^{2n}(n!)^2}{(2n)!} \geq 1 + \sqrt{n}$$

which can be easily proved by induction, we have

$$l_m < \tau_m (2\nu)^{2m}, \qquad \tau_m = \frac{1}{1+\sqrt{m}}.$$

Here we used the inequality $\prod(1+\sqrt{n_i}) \geq 1 + \sqrt{m}$. The main contribution comes from a neighborhood of $n_1 = \ldots = n_\nu = m/\nu$. Thus if $\nu = 2$ we can improve the bound as follows (by an explicit calculation):

$$\tau_m \leq \frac{1}{1+m/2}. \tag{31}$$

Theorem 8. (1) *Let $\tilde{J}_b = \exp[C_2] J_b$. The Mayer expansion converges absolutely if*

$$\sum_{0\in\omega} \frac{\tilde{J}_\omega}{2^{|\omega|}} \exp[-V(\omega)]\, |v| < \frac{1}{e(1+C_1)}, \tag{32}$$

where ω are random loops starting from the origin.
(2) *The Mayer expansion converges absolutely in the complex region*

$$\Omega(N,\nu) \equiv \left\{ J; |J| < (1-\varepsilon)\frac{N}{2\nu} \right\}, \tag{33}$$

where ν stands for the dimension of the lattice space and $\lim_{\nu\to\infty} \varepsilon = 0$.
(3) *Let $\nu = 2$. Then the Mayer expansion converges absolutely in the complex region*

$$\Omega(N,2) = \{J; |J| < 0.102N\}. \tag{34}$$

Proof. (1) This is immediate from Lemma 6.
(2) By Eq. (26b) and Eq. (28a), it is enough to argue the convergence of the following series:

$$\sum_{n=1}^{\infty} \left\{ \sum_{M=n}^{\infty} \left\{ \sum_{m_1+m_2+\ldots+m_n=M} h(m_1)\ldots h(m_n) \right\} \exp\left[n \log \frac{2M}{n} \right] \right\}, \tag{35}$$

where

$$h(m) \equiv \sum_{\omega:|\omega|=2m} \frac{|\omega| J_\omega}{2^{|\omega|}} \exp[-V(\omega)] < 2m l_m \left(\frac{J}{N}\right)^{2m}.$$

Let $2m$ be the length of the random loop ω. At each point, the walk ω has the 2ν directions to go, and one of them is backward. If ω visits $x + e_\mu$ from x and ω goes back, ω visits x twice at least. Then

$$h(m) \leq 2m \sum_{p=0}^{\infty} \binom{2m}{p} \left(\frac{(2J-1)J}{N}\right)^{2m-p} \left(\frac{J}{N}\right)^p = 2m x^{2m},$$

where $x = (2\nu - 1)J/N + J/(N+2) < 2\nu J/N$. Thus the left-hand side of Eq. (32) is bounded by

$$\sum_{m=1}^{\infty} h(m) \le \frac{2x^2}{(1-x^2)^2}$$

which however yields a poor bound. We use the fact that the number of loop diagrams of length $2m$ is much less than $(2\nu)^{2m}$ for large ν: $l_m = O(\nu^m)$. We take ν large keeping $x = 2\nu J/N < 1$ constant. Then the following inequalities hold:

$$\sum_{m=1}^{k} h(m) < O\left(\frac{1}{\nu}\right), \qquad \sum_{m=k+1}^{k'} h(m) = O(1) \le \frac{2(k+1)x^{2(k+1)}}{(1-x^2)^2}, \tag{36}$$

where k and k' are some positive integers and $k \to \infty$, $k' \to \infty$ as $\nu \to \infty$. Then $\sum h(m) <$ const yields the bound $x < 1 - \varepsilon$ for any const < 1 as $\nu \to \infty$, where $\varepsilon \to 0$ as $k \to \infty$. Since we are interested in the upper bound, we can assume

$$h(m) = \begin{cases} O(\nu^{-1}) & \text{for } m < k \\ 2m\left(\dfrac{2\nu J}{N}\right)^{2m} & \text{for } m \ge k\,, \end{cases}$$

Or for simplicity we can assume $h(m) = (2\nu J/N)^{2m}$ for $m \ge k$ with a small change of J since m is sufficiently large. Thus using Ineq. (36), we have

$$\begin{aligned} L(n,M) &\equiv \sum_{m_1+\ldots+m_n=M} h(m_1)\ldots h(m_n) \\ &= \sum_{l=0}^{n}\sum_{s=l}^{lk}\binom{n}{l}\left\{\sum_{m_1+\ldots+m_l=s;\, m_i\le k}\prod_{1}^{l} h(m_i)\right\} \\ &\quad \times \left\{\sum_{m_{l+1}+\ldots+m_n=M-s;\, m_i>k}\prod_{l+1}^{n} h(m_i)\right\} \\ &\le \sum_{l=0}^{n}\sum_{s}\binom{n}{l}\binom{M-s-(n-l)k-1}{n-l}\left(\frac{c}{\nu}\right)^{l}\left(\frac{2\nu J}{N}\right)^{2(M-s)} \end{aligned}$$

Applying the Stirling formula to the summand, we see that $L(n,M)$ takes its maximum at $M = M_0 = (1 - O(\nu^{-1}))nk \gg n$, and decreases exponentially in $M - M_0$ and the contribution from $M < M_0/2$ is exponentially small cmpared with the one from $M \ge M_0/2$. The factor $\exp[(n/2M)\log(2M/n)]$ tends to 1 as $M \to \infty$ and thus the assertion holds.

(3) For $\nu = 2$ we estimate Eq. (32) explicitly up to $|\omega| = 10 (m = 5)$:

$$\begin{aligned} &8\alpha^2 + 144\alpha^4 + 2400\alpha^6 + 39200\alpha^8 + 634860\alpha^{10} \\ &\quad + \sum_{m=6}^{\infty} 2m\tau_m(4\alpha)^{2m} < \frac{1}{e(1+C_1)}, \end{aligned} \tag{37}$$

where $1/e(1+C_1) > 0.2924$ and $\alpha = \tilde{J}/N = \exp[1/e]J/N$. We can numerically show that this is satisfied if $\alpha < 0.1483$, using the bound (31) for τ_m. Therefore $J/N < 0.102$. Q.E.D

Remark 4. We may be able to improve these bounds by using the fact that random walks in two dimensions visit same points many times. See [5] for a work in this direction. Self-crossing walks are, however, very weakly suppressed if N is large. Then it is open to what extent this property is responsible for the conjectured non-existence of phase transitions in two dimensions.

6. Discussions

Theorem 8 is our main conclusion. Though this bound is slightly smaller than the one obtained by Froehlich et al., singularities seem to exist in a neighborhoods of the origin whose dimaeter is very close to $N/2\nu$ (even if $\nu \leq 2$). In this sense, this result is best possible.

We would like to show that this approach can be aplied to the block spin transformations which consist of procedures to obtain effective interactions of block spins (averaged spins) defined by $(C\phi)(x) = \sum_\zeta \phi(Lx + \zeta)/L^2$, where $x \in Z^2$ and $-L/2 \leq \zeta_\mu < L/2$ (one may take $L = 2$).

We fix $(C\phi)(x)$ to be $\phi^1(x)$, and integrate over the remaining degrees of freedom. We set $\sqrt{J}\phi(x) \equiv \phi(x) \in \sqrt{J}S^{N-1}$ so that Eq. (1) is replaced by

$$\langle\cdot\rangle \equiv \frac{1}{Z}\int(\cdot)\exp\left[-\frac{1}{2}\sum(\phi(x)-\phi(y))^2\right]\prod d\phi(x)\,. \tag{38}$$

The dominant configuration of the block spins is such that the orientations of them change slowly as $x \in Z^2$ varies and $|\phi^1(x)|^2 \in [J_1 - \text{const}\,N, J_1 + \text{const}\,N]$, where $J_1 = J - \text{const}\,N$.

Then we may put

$$\phi(x) ={}^t\,(\varphi(x') + s(x), u(x)) \in R \times R^{N-1}\,, \tag{39}$$

where $\varphi(x') = |\phi^1(x')|$, $x' = ([x_1/L], [x_2/L])$ and the block averages of $\{s(x)\}$ and $\{u(x) \in R^{N-1}\}$ vanish. Since $\phi(x)$ stays on the sphere of the ball, $|u(x)| \sim (J - \varphi^2(x'))^{1/2}$.

We substitute Eq. (39) into Eq. (38) and obtain the effective interactions of the block spins by integrating over $\{s(x), u(x)\}$. Our result implies that the effective interaction obtained in this way is of short range if $\{J - \varphi^2(x')\}$ are small (but of order $O(N)$).

It is a difficult problem to what extent we can continue these steps for $\nu = 2$, because we have to show that non-local interactions not do increase (if N is large) for any numbers of the iterations. This problem remains and we will discuss this problem in the near future [7].

References

1. Battle, G.A., Federbush, P.: A Phase Cell Cluster Expansion for Euclidean Field Theories, Ann. Phys. **142**, 95–139 (1982); A Note on Cluster Expansions, Tree Graph Identities, Extra $1/N!$ Factors!!!, Lett. Math. Phys. **8**, 55–58 (1984)
2. Brydges, D., Federbush, P.: A New Form of the Mayer Expansions in Statistical Mechanics, J. Math. Phys. **19**, 2064–2067 (1978)
3. Brydges, D.: A Short Course on Cluster Expansion. In: Critical Phenomena, Random Systems, Gauge Theories, Le Houches, Session XLIII, Osterwalder, K., Stora, R. (eds.) (1986), pp. 129–183

4. Brydges, D., Froehlich, J., Spencer, T.: Random Walk Representations of Classical Spin Systems and Correlation Inequalities, Commun. Math. Phys. **83**, 123–150 (1982); Brydges, D., Froehlich, J., Sokal, A.: Random Walk Representations of Classical Spin Systems and Correlation Inequalities, Commun. Math. Phys. **91**, 117–139 (1985)
5. Hamana, Y.: On the Central Limit Theorem for the Multiple Point Range of Random Walks. Preprint, 1992, Tokyo University
6. Ito, K.R.: Renormalization group Flow of Two-Dimensional Hierarchical Heisenberg model of Dyson-Wilson Type. Commun. Math. Phys. **139**, 45–70 (1991)
7. Ito, K.R.: Random Walk Representations and the Renormalization Group Method in Non-Abelian Statistical Models. Preprint, in preparation
8. Kupiainen, A.: On the $1/n$ Expansion, Commun. Math. Phys. **73**, 273–294 (1980)
9. Simon, B.: Correlation Inequalities and the Decay of Correlations in Ferromagnets. Commun. Math. Phys. **77**, 111–126 (1980); Lieb, E.: A Refinement of Simon's Correlation Inequality, Commun. Math. Phys. **77**, 127–135 (1980)

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