

Singularity, Complexity, and Quasi-Integrability of Rational Mappings

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Abstract. We investigate global properties of the mappings entering the description of symmetries of integrable spin and vertex models, by exploiting their nature of birational transformations of projective spaces. We give an algorithmic analysis of the structure of invariants of such mappings. We discuss some characteristic conditions for their (quasi)-integrability, and in particular its links with their singularities (in the 2-plane). Finally, we describe some of their properties *qua* dynamical systems, making contact with Arnol'd's notion of complexity, and exemplify remarkable behaviours.

1. Introduction

We want to analyze in detail some realizations of Coxeter groups [1, 2] by birational transformations of projective spaces which have been shown to appear in the description of the symmetries of quantum integrable systems [3–5, 6–10].

The first motivation to look at these realizations resides of course in their relations with the star-triangle and the Yang-Baxter equations or their higher dimensional generalizations such as the tetrahedron equations. A characteristic feature of the orbits of the known solutions of the Yang-Baxter equations under these groups is that they are confined to subvarieties of high codimension of the parameter space (actually curves), *signaling the existence of an unexpectedly large number of algebraically independent invariants*. The discovery and the analysis of the possible invariants is a decisive step in the study of the Yang-Baxter (tetrahedron, ...) equations, in particular for what concerns the so-called baxterization problem [11].

Another motivation is to use these realizations to construct discrete time evolution maps, as it is usual in the study *à la* Poincaré of dynamical systems, by iterating some element of the group. One of the main questions in this setting is again to bring to light the possible presence of invariants and invariant tori [12–14, 15–19, 6–10].

If a realization admits algebraic invariants, we will say it has a property of quasi-integrability.

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In [6, 7] it was shown, among other things how the direct graphical investigation of the group action by means of numerical calculations (“drawing the picture”) leads to a nice representation of the orbits. We want to elaborate on the problem and discuss the deeper structure of the non-linear realizations by birational transformations of projective spaces.

We first recall which kind of infinite Coxeter groups arise in the theory of solvable models of statistical mechanics and collect some facts about birational maps which we shall use in the sequel.

We then analyze the cohomological structure of the possible invariants. We give the general (albeit formal) solution to the problem in terms of an algorithmic research of linear systems on \mathbb{P}_n satisfying a covariance property with respect to the realizations we handle. We then specialize to the case $n = 2$, analyze the singular locus of the group, and discuss some illustrative examples.

In the final section we make contact with Arnol’d’s notion of complexity [20, 21], which measures the growth of the topological non-triviality of the intersection of a fixed subvariety and the image of another one under an iteration map. We will argue how our notion of quasi-integrability is related to a *polynomial* growth of the complexity while the generic one is exponential.

2. Coxeter Groups and Birational Realizations

One of the outcomes of [6–10] is the construction of a number of groups generated by involutions and of various rational realizations on projective spaces.

Consider the Coxeter group G generated by ν involutions I_1, I_2, \dots, I_k , ($k = 1 \dots \nu$), verifying no relations other than the involution property. The group G is infinite and there are two essentially different situations.

If $\nu = 2$, the group is the infinite dihedral group $\mathbb{Z}_2 \times \mathbb{Z}$, and all elements may uniquely be written $I_1^\alpha (I_1 I_2)^q$, with $\alpha = 0, 1$ and $q \in \mathbb{Z}$. The number of elements of given length l is 2.

If $\nu \geq 3$, the number of elements of length l grows exponentially with l , and the group is in a sense bigger (still countable).

As an example for the groups described in [6–10], the number ν of generators depends on the dimension d of the lattice: it is just 2^{d-1} so that if $d = 2$, G is generated by two involutions and if $d \geq 3$, G is generated by more than three involutions.

One may then construct various realizations Γ of G by explicit transformations of some projective space. They are obtained by specifying the realization of the generators. Since it is precisely the realizations that we want to study here, and especially the problem of the existence of invariants, we will mainly talk about Γ and not G , and use the same notation I_k for the generators of G and their representatives in Γ .

The realizations Γ we consider are essentially obtained from operations on matrices, especially matrix inversions, and transpositions of their entries, the matrices being originally matrices of Boltzmann weights of statistical mechanical spin and vertex models on the lattice or R -matrices of 2-dimensional field theories. The projective space we consider is just the space of entries of the matrices up to a common multiplicative factor.

Let us describe here typical examples of such realizations.

Suppose m is a $q \times q$ matrix. The ordinary matrix inverse I defines an involutive rational transformation of \mathbb{P}_{q^2-1} , which reads in homogeneous coordinates:

$$I: m_{ij} \rightarrow \text{cofactor of } m_{ij} . \tag{1}$$

We may also consider the element by element inverse (so-called Hadamard inverse):

$$J: m_{ij} \rightarrow 1/m_{ij} . \tag{2}$$

These two inverses appear in the study of spin models. Notice that $I^2 = J^2 = 1$ and there is no other relation between I and J . In particular I and J do not commute and $\varphi = IJ$ is of infinite order.

It is of course possible to define all kinds of block inverses, the size of the blocks ranging from the full matrix size (for I) to 1 (for J).

One may also mix I with linear transformations. This happens already when I and J defined by (1) and (2) are collineated, i.e. when there exists a linear transformation C such that $I = C^{-1}JC$ (see [6]). We may also mix I with permutations of the entries.

Let us describe here the transpositions of entries which appear in the study of vertex models on a d -dimensional lattice [9, 10].

Suppose M is a multiindex matrix of size $q^d \times q^d$ written in the form $M_{j_1 j_2 \dots j_d}^{i_1 i_2 \dots i_d}$. There exist d different partial transpositions t_1, t_2, \dots, t_d with the evident definition:

$$(t_k M)_{j_1 \dots j_k \dots j_d}^{i_1 \dots i_k \dots i_d} = M_{j_1 \dots j_k \dots j_d}^{i_1 \dots j_k \dots i_d} . \tag{3}$$

We clearly have a product and an inverse I for these multiindex matrices. We may define $2^{(d-1)}$ new inversions by:

$$I_\kappa = t_{\alpha_1} \dots t_{\alpha_s} I t_{\alpha_{s+1}} \dots t_{\alpha_d} = t_{\alpha_{s+1}} \dots t_{\alpha_d} I t_{\alpha_1} \dots t_{\alpha_s} ,$$

where $\kappa = (\{\alpha_1, \dots, \alpha_s\}, \{\alpha_{s+1}, \dots, \alpha_d\})$ is a partition of $\{1, \dots, d\}$.

These various inverses yield involutive (bi)rational mappings of $\mathbb{P}_{q^{2d}-1}$. They are related by collineations, easy to write from the representation of the partial transpositions. Note that the product of all t_k 's is the full transposition t and commutes with all the inverses. Such realizations appear in the study of vertex models.

We may further enrich the representations by imposing constraints on the entries of the matrices, provided the transformations are compatible with these constraints (see [6] for the notion of admissible patterns). This yields realizations on projective spaces of lower dimensions.

Needless to say that along these ideas, one may construct a variety of involutions acting on various projective spaces.

We stress that, at the level of the realization, there may exist additional relations between the generators, possibly making it finite [as in Example (6.4)].

3. Some Facts about Rational Mappings

In this section we collect some results about rational and birational mappings between algebraic varieties (see for example [22–25]).

Definition 3.1. A correspondence $\varphi_Z: X \rightarrow Y$ between algebraic varieties X and Y is an algebraic subset $Z \subset X \times Y$. φ_Z is a rational map if there is a Zariski open set $U \subset X$ on which the correspondence is one to one.

If $Z \subset X \times Y$ is a rational map, the inverse correspondence is defined by the graph $Z^{-1} := \{(y, x) \in Y \times X \mid (x, y) \in Z\}$. If the correspondences Z and Z^{-1} are both rational mappings, then Z (or φ_Z) is called a birational transformation. A birational map is a biholomorphism except on subvarieties of codimension at least two.

A linear system D on X is a non-empty linear subspace of the space of global sections of some line bundle \mathcal{L} over X . Its base locus is the set of common zeroes of all sections in D . A remarkable result is that [23] there exists a one to one correspondence between linear systems on X of dimension d with base locus of codimension not less than 2 and rational maps $X \xrightarrow{\phi} \mathbb{P}_{d-1}$ up to projective automorphisms of \mathbb{P}_{d-1} .

For what we are concerned with, the paradigm of the rational map is the so-called σ -process or blow up of a point.

We refer to [24, 25] for the general definitions. We shall call Hadamard inversion and generically denote by J the prototypical birational mapping in \mathbb{P}_n . Let x_0, x_1, \dots, x_n and y_0, y_1, \dots, y_n be coordinates in two different copies of \mathbb{P}_n and let us consider the algebraic set $Z \subset \mathbb{P}_n \times \mathbb{P}_n$ given by the n equations $x_0 y_0 = x_1 y_1 = \dots = x_n y_n$. By definition, the graph of J is Z .

It is valuable to specialize to $n = 2$: outside the triangle $x_0 x_1 x_2 = 0$, Z is the graph of the map $[x_0, x_1, x_2] \rightsquigarrow [1/x_0, 1/x_1, 1/x_2]$. The generic point on the line $x_i = 0$ is sent into the point p_i whose only non-vanishing coordinate is y_i , while to the points p_i corresponds the entire line $y_i = 0$. One can say that J blows up p_i to the line $x_i = 0$ and blows down the line $x_i = 0$ to the point p_i .

Finally, we recall the following properties, which will be used in what follows. From the description of a birational mapping as an algebraic set in the product $X \times Y$ it is apparent that φ blows up p to a divisor D if and only if φ^{-1} blows down D to p , and it is also evident that different points p_1 and p_2 cannot be blown up to the same divisor D (otherwise the inverse map would not be rational).

Also, it is a standard result that blowing up a point adds a free factor \mathbb{Z} in the Picard group of X . It follows that if φ is a birational map in \mathbb{P}_2 which blows up n points, then it must blow down exactly n exceptional divisors, as explained above.

4. Invariants and Quasi-Integrability

Let Γ be a group of birational transformations in \mathbb{P}_n . A meromorphic function $\Delta: \mathbb{P}_n \rightarrow \mathbb{P}_1$ deserves to be called a Γ -invariant if it satisfies

$$\Delta(g(x)) = \Delta(x) \quad \forall g \in \Gamma. \tag{4}$$

Since a meromorphic function on an algebraic variety can be thought of as the ratio of two sections of a suitable line bundle, we are naturally lead to the following scheme.

A Γ one-cocycle is a collection of sections $a(g, x)$ of some line bundles over \mathbb{P}_n satisfying the cocycle condition:

$$a(g_1 g_2, x) = a(g_1, g_2 x) \cdot a(g_2, x). \tag{5}$$

A section σ of a line bundle will be called *a-covariant* (for some cocycle a) if the equation

$$\sigma(gx) = a(g, x) \cdot \sigma(x) \tag{6}$$

holds. This equation may be reformulated in group cohomology terms [26] as $a = \delta\sigma$ meaning that a is actually a coboundary.

Finding a Γ -invariant is equivalent to finding two sections σ_1 and σ_2 verifying Eq. (6) for the *same* a . This means that we are interested in the coboundaries of the 0-cochains rather than the cohomology groups. The strategy is to find 1-cocycles admitting a sufficient number of primitives.

As a side remark, we notice that these equations, which will play a prominent role in the sequel, are well defined at all points of \mathbb{P}_n . In fact for any birational transformation g , the singular locus is a subvariety of codimension greater than or equal to two in \mathbb{P}_n , and the equations above admit a unique holomorphic extension to the whole of \mathbb{P}_n , even if they have meaning in the point set sense only on the nonsingular locus of g .

We are interested in the case where Γ is a Coxeter group, i.e. is generated by ν involutions I_k of degree d_k (the degree is a natural notion in terms of the homogeneous coordinates $[x_0, \dots, x_n]$). A Γ one-cocycle will be completely specified by the assignment of ν sections $a(I_k, x)$. Remarkably, the possible values of $a(I_k, x)$ may be found explicitly.

Each involution I_k defines a characteristic polynomial ϕ_k of degree $d_k^2 - 1$ in the following manner. The I_k being involutions, I_k^2 appears as the multiplication by a degree $d_k^2 - 1$ polynomial $\phi_k(x_0, \dots, x_n)$. We then have the following

Lemma 4.1. *If $a(g, x)$ is a trivial Γ cocycle, the sections $a(I_k, x)$ divide a suitable power of ϕ_i for $i = 1, \dots, \nu$.*

Proof. Suppose $a = \delta\sigma$. Then, from the coboundary equation (6) for $g = I_k^2$ we get that $a(I_k^2, x) = \phi(x)^m$, with $m = \deg(\sigma)$. The assertion follows from the cocycle condition (5).

Definition 4.1. We shall say that Γ is collineated when its generators are all conjugated by means of elements of $PGL(n + 1, \mathbb{C})$ to a standard one, K .

This is the case for a number of models among which are the Baxter model ($n = 3$) and the examples of Sect. 6.

If the characteristic polynomial ϕ_K factorizes into $\phi_K = \prod_{l=0}^j p_l^{d_l}$, then it is clear that for every i there exists a set of global homogeneous coordinates $[X_0^{(i)}, \dots, X_n^{(i)}]$ in which $a(I_i, x)$ will be the product of the same polynomials p_l , possibly weighted with different exponents d'_l .

Remark. When K is the Hadamard inversion J in \mathbb{P}_n a straightforward computation shows that $\phi_J = \prod_{l=0}^n x_l^{(n-1)}$ so that we get

Proposition 4.1. *A coboundary for a collineated Coxeter group of birational transformations with generators conjugated to the Hadamard inversion J in \mathbb{P}_n is a monomial in some suitable homogeneous coordinates, with coefficient ± 1 .*

It is apparent that our cohomological setting gives an algorithmic prescription for the search for invariants: find first the characteristic polynomials ϕ_k , which is straightforward, then the possible coboundaries, which is a factorization problem, and check how many primitives they have, which amounts to solving a linear problem.

The realization Γ in \mathbb{P}_n admits p invariants $\Delta_1, \dots, \Delta_p$ if there exists an a -covariant linear system L_p^d of projective dimension p and degree d for some cocycle a . It is not guaranteed that the orbits of a realization admitting p invariants lie on subvarieties of dimension $n - p$. Indeed the question of the algebraic independence of the invariants has to be examined further [27].

One has to realize how exceptional is the existence of any invariant of the birational realization, as the following argument shows.

Let us consider covariance with respect to one generator K of Γ . Suppose K is of degree d_K , and suppose we are looking for an invariant of degree m . Clearly from Eq. (6) the degree q of the cocycle is related to d_K and m by $m(d_K - 1) = q$. The dimension of the space $\mathcal{P}(m, n)$ of homogeneous polynomials of degree m in n variables is $\binom{n+m-1}{m-1}$. The requirement of K -covariance selects a linear system

L^K in $\mathcal{P}(m, n)$, of generic dimension at most $\frac{1}{2} \binom{n+m-1}{m-1}$. Imposing the same condition for another generator I leads to look for the intersection of hyperplanes of at most complementary dimensions in $\mathcal{P}(m, n)$. This intersection is generically empty. As a consequence we have

Proposition 4.2. *The action of a generic Coxeter group of birational transformations in \mathbb{P}_n does not admit any non-trivial invariant.*

In the particular case of collineated realizations, this leads to the further property, which we prove for $\nu = 2$ for simplicity.

Proposition 4.3. *The set of collineated groups admitting a non-trivial invariant has the structure of a quasi-projective variety.*

Proof. The group $PGL(n+1, \mathbb{C})$ admits a natural structure of quasi-projective variety, since it is identified with the complement of the degree $(n+1)$ hypersurface $\det A = 0$ in $\mathbb{P}^{(n+1)^2-1}$. Choosing K as prototypical generator of Γ , I is specified (in the collineated case) by the choice of an element $C \in PGL(n+1, \mathbb{C})$ by $I = C^{-1}KC$. If for any polynomial P we define $P_C := P(C^{-1}x)$ and set $y = Cx$, then the covariance equation with respect to I reads

$$P_C(Ky) = a_I(C^{-1}y)P_C(y).$$

For each possible cocycle a , this is an algebraic equation in the parameters t_α of the matrix C . The coordinates of any basis $\{e_\alpha\}$ in L^I with respect to a basis in $\mathcal{P}(m, n)$ completed from a basis of L^K may be expressed as algebraic functions of the t_α 's. The condition of nontriviality of the intersection is a condition on the rank of the matrix

$$B := \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ e_\alpha^2(t_1, \dots, t_{(n+1)^2-1}) & \dots \end{bmatrix}$$

which in turn is an algebraic condition on the $e_\alpha^i(t_1, \dots, t_{(n+1)^2-1})$, concluding the proof.

5. Realizations in the 2-Plane

Let us discuss mappings in \mathbb{P}_2 . Here our analysis is made quite complete by the fact that all rational maps from an algebraic surface can be described in terms of σ processes and also that singularities of birational maps can occur only at points. Moreover a theorem of Noether [25] assures that the group of birational maps is generated by the inversion J together with the projective group $PGL(3, C)$. We will restrict ourselves here to $\nu = 2$, i.e. Γ is generated by two involutions I and K .

Definition 5.1. The singular locus of Γ , $S(\Gamma)$ is defined as

$$S(\Gamma) = \{x \in \mathbb{P}_2 \mid \exists g \in \Gamma \text{ s.t. } g \text{ blows } x \text{ up}\}.$$

Since Γ is generated by I and K , $S(\Gamma)$ is obtained by the action of Γ on the singular points of I and K .

Definition 5.2. We will say that Γ is properly singular if $S(\Gamma)$ contains at least four points in general position, i.e. such that no three of them are aligned.

Let $\Pi(\Gamma)$ be the set of singular divisors:

$$\Pi(\Gamma) = \{\pi \in \text{Div}(\mathbb{P}_2) \mid \exists g \in \Gamma \text{ s.t. } g \text{ blows } \pi \text{ down to a point}\}.$$

Let us suppose that Γ admits a rational invariant Δ and let us look at what happens at points in $S(\Gamma)$. The equation $\Delta(g(x)) = \Delta(x)$ is clearly meaningless at the indetermination points of Δ . Noticing that $\forall x \in S(\Gamma)$, $\exists \pi_x \in \Pi(\Gamma)$ such that Π_x is blown down to x by some element $g \in \Gamma$, we can conclude that $x \in S(\Gamma)$ is either an indetermination point of Δ or Δ is constant along the corresponding divisor π_x , i.e. π_x belongs to the pencil of curves $\Delta = \text{const}$. Then we can state the

Proposition 5.1. *If $S(\Gamma)$ is infinite, and Γ is properly singular, then Γ does not admit any invariant.*

Proof. With the above observation in mind, the only case we must rule out is that Γ admits an infinite number of singular points and only a finite number of singular divisors, since by Bezout's theorem, any rational degree d invariant admits at most d^2 indetermination points. Let us suppose that $\#S(\Gamma) = \infty$ and $\#\Pi(\Gamma) = N$ and let us consider the set $\mathcal{P} = \{H_\alpha\}$ ($\alpha = 1 \dots 2^N$) of parts of $\Pi(\Gamma)$. We can make a partition of Γ into 2^N disjoint subset $\Gamma = \bigcup_{\alpha=1}^N \Gamma_\alpha$, where

$$\Gamma_\alpha = \{g \in \Gamma \mid g \text{ blows down exactly all the divisors in } H_\alpha\}.$$

At least one of the Γ_α , say Γ_0 is infinite. If we consider any pair of elements (h_0, l_0) in Γ_0 , the product $h_0 \cdot l_0^{-1}$ is a birational map without singularities, and hence an element $D \in PGL(3, \mathbb{C})$. Since Γ_0 is infinite, we can arrange things so that D is a non-trivial word of even length. Since words of odd length in Γ are involutions one has:

$$DKD = K, \quad DID = I. \tag{7}$$

It follows that D permutes the points of $S(\Gamma)$, and some power D^k must be the identity. As a consequence, there is some non-trivial product $(IK)^l = \mathbf{1}$, meaning that actually Γ is finite, and contradicting the infiniteness of $S(\Gamma)$.

Remark. The above proposition proves that a necessary condition for existence of an invariant is the finiteness of the singular locus. It is tempting to conjecture that this is also a sufficient condition for any properly singular realization. This is unfortunately not the case, as the example of Sect. (6.6) will show.

The next step is to try to relate the singular locus of the group with the singular locus of the would-be invariant. Let the Δ be a rational invariant for Γ and let $S(\Delta)$ be the set of its indetermination points. If $y \in S(\Delta)$ and $y \notin S(\Gamma)$, the (finite) orbit Γ_y of y is made out of indetermination points of Δ , on which every element $g \in \Gamma$ is regular. Since Γ_y is of even order, say $2l$, y is a fixed point for $(IK)^{n \cdot l}$, $n \in \mathbb{Z}$. Let $\tilde{\mathbb{P}}_2^y \xrightarrow{pr} \mathbb{P}_2$ be the 2-plane \mathbb{P}_2 blown up at y . Since we are working with \mathbb{P}_2 , Δ

extends to a function $\tilde{\Delta}$ on $\tilde{\mathbb{P}}_y^2$ with no indetermination points in a neighbourhood of the exceptional divisor $E = pr^{-1}(y)$. The restriction $\phi = \tilde{\Delta}|_E$ is a meromorphic map on \mathbb{P}_1 whose value is determined by the limits of $\Delta(x)$ along lines in \mathbb{P}_2 passing through y .

We will say that Γ is non-degenerate if for every $g \in \Gamma$ the tangent map to g is non-nilpotent at every isolated fixed point of g . Then we can state the

Proposition 5.2. *Let Γ be a non-degenerate Coxeter group of birational transformations of \mathbb{P}_2 . If Γ admits an invariant Δ , then $S(\Delta) \subset S(\Gamma)$.*

Proof. Suppose $y \in S(\Delta)$ and $y \notin S(\Gamma)$; the non-degeneracy of Γ ensures that there is at least one line in $PT_{\mathbb{P}_2}(y)$ whose orbit under the tangents to $(IK)^{n \cdot l}$, $l \in \mathbb{Z}$ is infinite [recall that $(IK)^l$ is a minimal length element in Γ which fixes y]. Hence ϕ as defined above assumes the same value on an infinite number of points, i.e. it is constant over the whole of E_y . But this in turn implies that Δ is well defined at y which contradicts $y \in S(\Delta)$ and ends the proof.

Notice that the converse inclusion $S(\Gamma) \subset S(\Delta)$ does not hold in general. If the inclusion is strict, we have seen that for any $x \in S(\Gamma)$ and not in $S(\Delta)$, the invariant Δ is constant along some divisor π_x . This gives a useful information about the invariant (see Example 6.3).

As for what the singularities of the generic curve Σ_λ of the pencil are related to the singular orbit $S(\Gamma)$ the following considerations hold. By Bertini's theorem, generic curves are smooth outside the base locus $S(\Delta)$ whence the chain of inclusions

$$S(\Sigma_\lambda) \subset S(\Delta) \subset S(\Gamma). \tag{8}$$

Moreover, exploiting the genus formula for singular curves one can give relations¹ between the degree of d of the invariant and $S(\Gamma)$. If Δ is a degree d curve in \mathbb{P}_2 with singular locus $\text{Sing}(\Sigma)$, then [22]

$$g(\Sigma) = \frac{(d-1) \cdot (d-2)}{2} - \sum_{p \in \text{Sing}(\Sigma)} \delta_p$$

with δ_p depending on the type of the singularity at p . Since Σ_λ is irreducible and admits Γ as an infinite group of automorphisms, thanks to the inclusion (8) and to the fact that $\delta_p = 0$ if p is not in $\text{Sing}(\Sigma_\lambda)$, one has

$$\frac{(d-1) \cdot (d-2)}{2} - \sum_{p \in S(\Gamma)} \delta_p = \begin{cases} 0 \\ 1 \end{cases}.$$

This relation shows that there is a balance between the degree of the invariants and the number and nature of the singular points of the generic curve, and consequently of Γ .

6. Examples

We describe here some specific examples in \mathbb{P}_2 with two generators I and J . The physical origin of the models we will be dealing with is to be found notably among

¹ We thank M. Talon for useful remarks on this point

two-dimensional spin models with interaction along the edges [3], and this explains the terminology we use.

We fix J to be the Hadamard inversion $[x_i] \rightsquigarrow [1/x_i]$, and I to be $I = C^{-1}JC$ with the collineation matrix $C \in PGL(3, \mathbb{C})$. We will concentrate on the parametrization by C and examine some algebraic families in $PGL(3, \mathbb{C})$. For some of them Γ admits a non-trivial algebraic invariant.

We can associate to Γ a diagram D_Γ whose vertices are the points in $S(\Gamma)$ and where two vertices p_1 and $p_2 \in S_\Gamma$ are joined by an edge if either $p_1 = Jp_2$ or $p_1 = Ip_2$. The edges are oriented if I (resp. J) can be applied in one direction only. The diagram does not characterize the examples, but is a useful tool. In particular the families we give here have been obtained by deformations of given collineations, demanding the stability of the topology of the diagram.

The singular points of J are $P_1 = [1, 0, 0]$, $P_2 = [0, 1, 0]$, $P_3 = [0, 0, 1]$, and the one of I are $Q_i = C^{-1}(P_i)$, ($i = 1, 2, 3$).

One should notice that a number of the families we produce in this way fall into the general form found in [28] (i.e. verifying $C([1, 0, 0]) = [1, 1, 1]$ and $C^2([1, 0, 0] = [1, 0, 0])$, but with *non-integer entries*. This general form depends on four parameters:

$$\begin{bmatrix} 2 & \alpha & \beta \\ 2 & -1 + \gamma & -1 - \gamma \\ 2 & -1 - \delta & -1 + \delta \end{bmatrix} \tag{9}$$

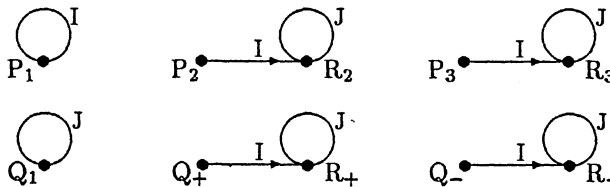
6.1. The Z_5 Family. The general Z_5 (five-state chiral Potts) model is described by a 5×5 cyclic matrix. We may consider its 2-parameter reduction obtained by imposing that the matrix is symmetric. It falls into a family of quasi-integrable models parameterized by a complex number q , which (whenever this makes sense) is identifiable with the square root of the number of states [29]. The collineation matrix is

$$C_{Z_5}(q) = \begin{bmatrix} 2 & q^2 - 1 & q^2 - 1 \\ 2 & -1 + q & -1 - q \\ 2 & -1 - q & -1 + q \end{bmatrix}. \tag{10}$$

Its singular set is made of *ten* points: P_1, P_2, P_3 which are the usual singular points of J , $Q_1 = [1, 1, 1]$ and $Q_\pm = [-4, 1 \pm q, 1 \mp q]$, the singular points of I , and $R_2 = I(P_2)$, $R_3 = I(P_3)$, $R_\pm = J(Q_\pm)$. All these points are indetermination points for the invariant

$$\Delta_{Z_q} = \frac{(x - z)(y - z)((q - 1)(x^2 + y^2) + 2(q + 1)xy)}{(2 + (q - 2)(zx + zy) + 2xy)(x - y)^2}.$$

Notice that here $S(\Delta) = S(\Gamma)$. The diagram D_Γ is the following:



6.2. *The BMV Family.* The matrix of (Boltzmann) weights of the BMV model [6] is

$$\begin{bmatrix} x & y & z & y & z & z \\ z & x & y & z & y & z \\ y & z & x & z & z & y \\ y & z & z & x & z & y \\ z & y & z & y & x & z \\ z & z & y & z & y & x \end{bmatrix}.$$

The inversion I is just the matrix inversion. This model pertains to the one-parameter family of quasi-integrable models whose collineation matrix is

$$C_{\text{BMV}}(w) = \begin{bmatrix} 1 & w - 1 & w \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \tag{11}$$

and is reached for the value $w = 3$. The singular locus is made out of P_1, P_2, P_3 and

$$Q_1 = [1, 1, 1], Q_2 = [w - 1, -(w + 1), w - 1], Q_3 = [1, 1, -1],$$

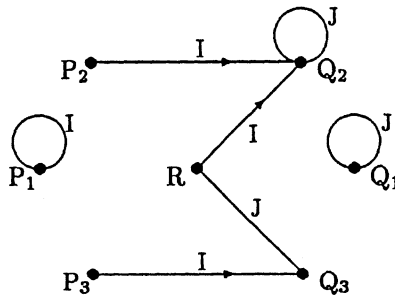
together with an extra point $R = J(Q_2) = [w^2 - 1, (w - 1)^2, w^2 - 1]$. The case $w = 1$ is singular. The family admits the invariant

$$\Delta_{\text{BMV}}(w) = \frac{P_w^2(x, y, z)Q_w(x, y, z)}{(y + z)^4(x - z)^2(x - y)},$$

where

$$\begin{aligned} P_w(x, y, z) &= (1 - w)(z^2 - xy) + (w - 3)z(x - y), \\ Q_w(x, y, z) &= (1 - w^2)(y^3 - xz^2) + (w^2 - 4w - 1)y^2(x - z) \\ &\quad + 2(w - 1)^2yz(x - y). \end{aligned}$$

Here again $S(\Delta) = S(\Gamma)$ and the singular graph is as follows:



6.3. *The Symmetric Ashkin-Teller Model.* By symmetric Ashkin-Teller model we understand the 4-state spin model with the cyclic and symmetric matrix of Boltzmann weights

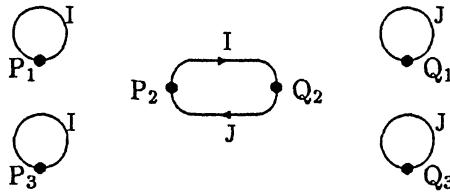
$$\begin{bmatrix} x_0 & x_1 & x_2 & x_1 \\ x_1 & x_0 & x_1 & x_2 \\ x_2 & x_1 & x_0 & x_1 \\ x_1 & x_2 & x_1 & x_0 \end{bmatrix}$$

for which the matrix inversion is collineated to the Hadamard inversion by means of the matrix

$$C_{AT} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}. \tag{12}$$

The singular points of the matrix inversion are $Q_1 = [1, 1, 1]$, $Q_2 = [1, 0, -1]$, $Q_3 = [1, -1, +1]$.

The singular diagram is drawn below



It contains a loop connecting the two points P_2 and Q_2 . This means that the points $P_2 = [0, 1, 0]$ and $Q_2 = C^{-1}(P_2)$ are not singular for IJ and hence it is no surprise that they are non-singular for the invariant

$$\Delta_{AT} = \frac{y^2 - xz}{y(x - z)}.$$

There is a strict inclusion of $S(\Delta)$ in $S(\Gamma)$, since the points P_2 and Q_2 are not in $S(\Delta)$. The corresponding divisors $\prod_{P_2} = \{\text{the line } y = 0\}$ (resp. $\prod_{Q_2} = \{\text{the line } x = z\}$), are indeed curves in the pencil $\Delta = \text{const}$.

6.4. A Finite Realization. It is instructive to consider the group described by the collineation matrix

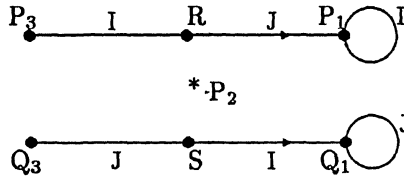
$$C_F(q) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & q & -(1 + q) \\ 1 & 0 & -1 \end{bmatrix} \tag{13}$$

which is invertible for $q \neq 0$. It is apparent that I and J share $P_2 = [0, 1, 0]$ as a common singular point. Apart from P_2 , I admits as singular points $Q_1 = [1, 1, 1]$ and $Q_3 = [q, -(2 + q), -q]$. The singular graph D_Γ depicted below contains two more points $R = J(Q_3)$ and $S = I(Q_2)$. This model admits at least two algebraically independent invariants, which can be taken to be

$$\Delta_F^{(1)} = \frac{z^4 + x^4}{z^2 x^2},$$

$$\Delta_F^{(2)} = \frac{q^2 z x (y - z)^2 + q [(z^2 - xy)^2 - z(z - 2x)(x^2 + y^2)] + 2(z^2 - xy)^2}{(z - y)^2 (z^2 + x^2)}.$$

This, together with Bezout's theorem tells us that the orbit under Γ of any point $p \in \mathbb{P}_2$ is finite and of order not greater than 16, i.e. Γ is finite. A direct inspection shows that $(IJ)^4 = \mathbf{1}$ and there are additional invariants, of course not algebraically independent from the two previous ones.



6.5. *The Symmetric Z_7 Model.* The symmetric Z_7 model [6] may be defined by the collineation matrix

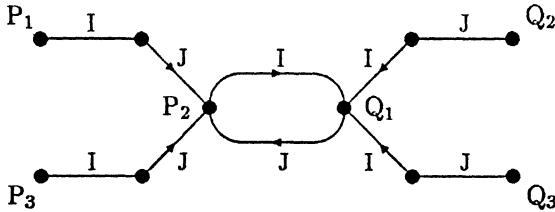
$$C_{Z_7} = \begin{bmatrix} 2 & 6 & 6 \\ 2 & -1 - i\sqrt{7} & -1 + i\sqrt{7} \\ 2 & -1 + i\sqrt{7} & -1 - i\sqrt{7} \end{bmatrix}. \tag{14}$$

From the point of view of dynamical systems the model shows chaotic properties in \mathbb{P}_2 . It has an *infinite* singular orbit $S(\Gamma)$. According to Proposition 5.1, it does not admit any invariant, as it is confirmed by a direct inspection of the orbits. This model behaves actually like a generic element of the family (9).

6.6. *A Finite Diagram Model (FDM).* Here Γ is generated by the Hadamard inverse J and $I = D^{-1}JD$ with

$$D = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Notice that D is not of the form (9). The singular diagram is finite but Γ does not have any invariant. This provides a remarkable example for what we said in Sect. (5), i.e. that the finiteness of the singular orbit is only a necessary condition for the existence of a non-trivial invariant



7. About Complexity

In this section we will discuss the notion of complexity of the groups we have been considering.

As we said earlier, we may consider the group G as generated by involutions and relations. If the number ν of generators is bigger than 2, G is “exponentially big” [30]. We will not comment on this here, and limit ourselves to $\nu = 2$, as one would concentrate on 1-dimensional subgroups of differentiable groups, and examine the realizations Γ .

What we want to point out is that there is a very diverse behaviour of these realizations, *even in the case $\nu = 2$* , manifesting itself in the complexity of the

representing transformations. The notion of complexity we appeal to is a simple form of the one introduced in [20].

Suppose F is a diffeomorphism of a compact smooth n -manifold M . Let S_k and R_l submanifolds of M of dimension respectively k and l . Let S_k^m be the m^{th} iterate of S_k by F and $T_{S,R}(m)$ the intersection

$$T_{S,R}(m) = R_l \cap S_k^m .$$

Arnol'd [20] defines the *complexity* $C_{S,R}(m)$ as

$$C_{S,R}(m) = \sum b_p(T_{S,R}(m)) ,$$

the b_p 's being the Betti numbers. He also proves that for a sufficiently generic choice of S_k and R_l the complexity grows at most exponentially with m .

The analysis of complexity of plane mappings has already been achieved for the case of *polynomial and polynomially invertible* transformations [31,21]. We are interested in a wider class of transformations, as the one described in Sect. (5), where Γ is generated by two *rational* transformations.

Although these transformations are not diffeomorphisms there is a natural measure of the complexity by the degree of iterates.

Generically, if the two generating involutions are of degree u and v respectively, the degree of $\varphi = IK$ is $w = uv$, so that $\text{deg } \varphi^{(k)} = w^k$. If Σ_1 and Σ_2 are two linear subspaces of \mathbb{P}_2 then $\varphi^{(k)}(\Sigma_1)$ should be for generic Γ a curve of degree w^k , and the complexity $C_{\Sigma_1, \Sigma_2}(k)$ would then be exactly w^k .

However, due to the fact that we are working with projective space, there is a simple mechanism for the lowering of the degree of the iterates $\varphi^{(k)}$, for one has to factorize out common factors from the expressions of the homogeneous coordinates of $\varphi^{(k)}$. This provides a variety of behaviours for the degree d as a function of the order of iteration k , lying between exponential growth and periodicity, with the particular case of polynomial (or polynomially bounded) growth.

The outcome of our analysis is that there is a connection between the existence of an invariant and a polynomial (as opposed to exponential) rate of growth, as shown in Table 1, inferred from the results of the direct calculation of the first few iterations on the examples of Sect. (6) where the degree of φ is 4.

Table 1. Behaviour of degrees

Model	Growth	Rank of invariants
Z_5	$d(k + 1) + d(k - 1) - 2d(k)$ periodic of period 3	One
BMV	$d(k + 1) + d(k - 1) - 2d(k)$ bounded	One
Finite	$d(k)$ periodic of period 4	Two
$A - T$	$d(k) = 4k$	One
FDM	$d(k) = 4^k$ (generic)	None
Z_7	$d(k + 1) - d(k) = f_{2k}\{f_k\}$ (a Fibonacci sequence)	None

A more detailed analysis of these properties, together with the study of other properties of realizations of Coxeter groups *qua* dynamical systems such as finite (periodic) orbits is the matter of further investigations (see [32,33]).

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