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Tetrahedral Zamolodchikov Algebras Corresponding to Baxter's L-Operators

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Abstract. Tetrahedral Zamolodchikov algebras are structures that occupy an intermediate place between the solutions of the Yang-Baxter equation and its generalization onto 3-dimensional mathematical physics – the tetrahedron equation. These algebras produce solutions to the tetrahedron equation and, besides, specific "two-layer" solutions to the Yang-Baxter equation. Here the tetrahedral Zamolodchikov algebras are studied that arise from L-operators of the free-fermion case of Baxter's eight-vertex model.

Introduction

The tetrahedron equation is a generalization of the Yang-Baxter equation, which is fundamental in studying the exactly solved models in $1+1$-dimensional mathematical physics, onto the $2+1$-dimensional case. Nontrivial solutions of the tetrahedron equation do exist. They were found by Zamolodchikov [1, 2] for the tetrahedron equation "with variables on the faces" and by this author [3, 4] – for the equation "with variables on the links". In the latter case, the solution consists of the commutation relation matrices of the so-called tetrahedral Zamolodchikov algebra – a structure designed to span the gap between the Yang-Baxter and tetrahedron equations.

The existence of a large family of the tetrahedral Zamolodchikov algebras was shown in the papers [3, 4]. However, there are some difficulties here: firstly, the explicit calculation of the commutation relation matrices and, secondly, the verification of whether those matrices really satisfy the tetrahedron equation. These difficulties have been overcome in the mentioned works only in one particular "trigonometrical" case. In addition to the solutions of the tetrahedron equation, the tetrahedral Zamolodchikov algebras produce by themselves the "two-layer" solutions to the Yang-Baxter equation, and here again only the trigonometrical case has been studied [5].

In the present paper, the commutation relation matrices S are calculated in a more general case, with the trigonometrical functions replaced by elliptic ones. The key role is played by so-called "vacuum covectors" (or vacuum row vectors) – the covectors of a specific tensor product form that retain this form under the action of S. Somewhat unexpectedly, the result is that the commutation relation matrices, in a sense, do not depend upon the modulus k of elliptic functions. This result has two sides: we do not obtain new solutions of the triangle equation but, due to this very fact, we obtain a lot of new "two-layer" solutions of the Yang-Baxter equation.

Now some words about the contents of the following sections. In Sect. 1, the commutation relation matrix S of the tetrahedral Zamolodchikov algebra is shown to have, even in the most general situation considered in [4], a rather large family of vacuum covectors transformed by S into themselves, so that this 8×8-matrix has 6 eigencovectors with an eigenvalue 1. The definition of the tetrahedral Zamolodchikov algebra is recalled in Theorem 1.2. In Sect. 2, we show that some symmetry imposed on the algebra makes S have one more family of vacuum covectors, in many respects very similar to the former one. This enables us to find 2 more eigencovectors of S, with an eigenvalue -1. The matrix elements of S may now be found rather easily. Moreover, the needed calculation turns out to have been already done in papers [3, 4]. The "two-layer" solutions of the Yang-Baxter equation are constructed in Sect. 4. Some discussion is given in Sect. 5.

1. General Properties of the Tetrahedral Zamolodchikov Algebras Arising from the Felderhof L-Operators

We will need some 2-dimensional complex linear spaces. They will be denoted by the letter V with subscripts. Let us fix the bases in these spaces and regard them as consisting of column vectors. The tensor products of such spaces will also consists of column vectors, so that, in the usual way,

$$\begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} xz \\ xt \\ yz \\ yt \end{pmatrix} .$$

The linear operators we will deal with are thus identified with square matrices.

Definition. The Felderhof L-operator [7] is a linear operator acting in a tensor spaces, given by a matrix of the form

$$\begin{pmatrix} a_+ & 0 & 0 & d \\ 0 & b_- & c & 0 \\ 0 & c & b_+ & 0 \\ d & 0 & 0 & a_- \end{pmatrix}$$

with the condition

$$a_+a_- + b_+b_- = c^2 + d^2 . \tag{1.2}$$

The first of the two spaces in whose tensor product a Felderhof L-operator acts will be from now on the same for all of them. We denote it as V_0. The second space may be different for different operators, and is denoted as $V_1, V_2, \dots$.

The Felderhof L-operators are solutions to the Yang-Baxter equation. To be exact, let $L = L_{01}$ and $M = M_{02}$ be Felderhof L-operators acting in $V_0 \otimes V_1$ and $V_0 \otimes V_2$ respectively. Each of them may be considered as acting in $V_0 \otimes V_1 \otimes V_2$ if multiplied by an identity operator in the lacking space. Generally, let us identify a linear operator with its product by an identity operator. The Yang-Baxter equation is

$$R_{12}L_{01}M_{02} = M_{02}L_{01}R_{12}\,, \tag{1.3}$$

the linear operator R_{12} acting, of course, in $V_1 \otimes V_2$.

Theorem 1.1. *The existence of a non-zero operator R_{12} that satisfies Eq.* (1.3) *is equivalent to the fact that each of the values*

$$\Gamma = \frac{2cd}{a_+b_- + a_-b_+}\,, \qquad h = \frac{a_-^2 + b_+^2 - a_+^2 - b_-^2}{2(a_+b_- + a_-b_+)}$$

is the same for L_{01} and M_{02}. With this, R has the same form (1.1, 1.2).

Proof is given in papers [6–8].

Now let us recall some results from Sects. 5 and 6 of [4]. If L-operators L and M satisfy Eq. (1.3), then there exists, in addition to R_{12}, an operator R^1_{12} such that

$$(R^1_{12})^T L_{01}M_{02} = M_{02}L_{01}R^1_{12}\,,$$

and

$$(R^1_{12})^T \neq R^1_{12}$$

(the superscript T denotes matrix transposing). As to the operator R_{12}, let us rename it as R^0_{12}.

Let one more Felderhof L-operator $N = N_{03}$ be given with the same Γ and h as L and M (the subscripts denote, as before, the numbers of the spaces in which an operator acts). Consider symmetrical operators R^0_{13} and R^0_{23} and non-symmetrical ones $\tilde{R}^1_{23}$ and $\tilde{\tilde{R}}^1_{23}$ that satisfy equations

$$(\tilde{R}^a_{13})^T L_{01}N_{03} = N_{03}L_{01}\tilde{R}^a_{13}\,,$$
$$(\tilde{\tilde{R}}^a_{23})^T M_{02}N_{03} = N_{03}M_{02}\tilde{\tilde{R}}^a_{23}\,,$$

with $a = 0, 1$. The tildes are here due to the fact that each pair of L-operators has, of course, its own R-operators. However, to avoid bulky notations, we allow ourselves to omit these tildes and distinguish the R-operators by their indices.

Theorem 1.2. *The constructed R-operators are generators of a tetrahedral Zamolodchikov algebra,* i.e. *the equalities*

$$R^a_{12}R^b_{13}R^c_{23} = \sum_{d,e,f=0}^{1} S^{abc}_{def} R^f_{23}R^e_{13}R^d_{12}\,, \tag{1.4}$$

$a, b, c = 0, 1$, hold. These equalities, with general L, M, N, determine the matrix $S = (S^{abc}_{def})$ uniquely.

Proof can be found in [4]. It is based upon the fact that both the products $R^a_{12}R^b_{13}R^c_{23}$ and $R^f_{23}R^e_{13}R^d_{12}$ transform the so-called vacuum vectors of the operator LMN into those of the operator NML. As is shown in Sect. 5 of paper [4], linear operators that perform such a transformation of the vacuum vectors form a 8-dimensional linear space. So, for the two mentioned types of R-operator products, their linear spans are bound to coincide (each being 8-dimensional, too). This leads to the linear dependences (1.4).

Note that the "trigonometrical" case of the paper [5] is degenerate – there the linear spans are 6-dimensional. However, they coincide as well.

Theorem 1.3. *The symmetrical R-operators introduced above satisfy by themselves the Yang-Baxter equation:*

$$R_{12}^0 R_{13}^0 R_{23}^0 = R_{23}^0 R_{13}^0 R_{12}^0 \,.$$

Proof can be drawn immediately from the parametrization of the Felderhof L-operators given in paper [8] (see formulae (3) therein).

Now introduce the Felderhof L-operators

$$L_\alpha = \left(\mathbb{1} \otimes \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) L \left(\mathbb{1} \otimes \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right), \tag{1.5}$$

$$M_\beta = \left(\mathbb{1} \otimes \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\right) M \left(\mathbb{1} \otimes \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\right), \tag{1.6}$$

$$N_\gamma = \left(\mathbb{1} \otimes \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\right) N \left(\mathbb{1} \otimes \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\right),$$

α, β, γ being complex numbers. This means, for example, that if L is given by (1.1), then

$$L_\alpha = \begin{pmatrix} \alpha^2 a_+ & 0 & 0 & \alpha d \\ 0 & b_- & \alpha c & 0 \\ 0 & \alpha c & \alpha^2 b_+ & 0 \\ \alpha d & 0 & 0 & a_- \end{pmatrix}.$$

The newly constructed operators have the same values Γ as L, M, N do. The value h, for the matrix L_α, is replaced by

$$h_\alpha = \frac{a_-^2 - b_-^2 + \alpha^4(b_+^2 - a_+^2)}{2\alpha^2(a_+ b_- + a_- b_+)},$$

and analogous values h_β, h_γ (with their own $a_\pm$, $b_\pm$) correspond to the matrices M_β, N_γ.

Obviously, there exists a one-parameter family of triples (α, β, γ) such that

$$h_\alpha = h_\beta = h_\gamma \,. \tag{1.7}$$

We will sometimes denote such triples by one letter $\zeta = (\alpha, \beta, \gamma)$. Under conditions (1.7), one can construct the R-operators for $L_\alpha, M_\alpha, N_\gamma$ in the same way as it was done for L, M, N. Consider symmetrical R-operators (but omit the superscript 0)

$$R_{\alpha\beta} = (R_{\alpha\beta})_{12}\,, \qquad R_{\alpha\gamma} = (R_{\alpha\gamma})_{13}\,, \qquad R_{\beta\gamma} = (R_{\beta\gamma})_{23}\,,$$

so that, e.g.,

$$R_{\alpha\beta} L_\alpha M_\beta = M_\beta L_\alpha R_{\alpha\beta} \,. \tag{1.8}$$

From (1.5), (1.6) and (1.8) follows

$$R_{12}^T(\zeta) LM = MLR_{12}(\zeta)\,, \tag{1.9}$$

wherein

$$R_{12}(\zeta) = \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\right) R_{\alpha\beta} \left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \beta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right). \tag{1.10}$$

Further, from Theorem 1.3 with L, M, N replaced by $L_\alpha, M_\beta, N_\gamma$ follows

$$R_{\alpha\beta}R_{\alpha\gamma}R_{\beta\gamma} = R_{\beta\gamma}R_{\alpha\gamma}R_{\alpha\beta}\,,$$

and this leads to the equation

$$R_{12}(\zeta)R_{13}(\zeta)R_{23}(\zeta) = R_{23}(\zeta)R_{13}(\zeta)R_{12}(\zeta)\,, \tag{1.11}$$

with $R_{13}(\zeta)$ and $R_{23}(\zeta)$ defined by the formulae analogous to (1.10).

Theorem 1.4. *The commutation relation matrix S of the tetrahedral Zamolodchikov algebra possesses a one-parameter family of "vacuum covectors" mapped into themselves:*

$$\sum_{a,b,c=0}^{1} X_a(\zeta)Y_b(\zeta)Z_c(\zeta)S^{abc}_{def} = X_d(\zeta)Y_e(\zeta)Z_f(\zeta)\,,$$

or simply

$$(X(\zeta)\otimes Y(\zeta)\otimes Z(\zeta))S = X(\zeta)\otimes Y(\zeta)\otimes Z(\zeta)\,. \tag{1.12}$$

The linear space generated by the covectors $X(\zeta)\otimes Y(\zeta)\otimes Z(\zeta)$ is, in the general position, 6-dimensional.

Proof. As follows from Sect. 5 of paper [4], the operators $R_{12}(\zeta)$ that satisfy Eq. (1.9) lie in a 2-dimensional linear space, i.e.

$$R_{12}(\zeta) = \sum_{a=0}^{1} X_a(\zeta)R^a_{12}\,. \tag{1.13}$$

In this way the covector $X(\zeta)$ arises, and $Y(\zeta)$ and $Z(\zeta)$ are constructed analogously from $R_{13}(\zeta)$ and $R_{23}(\zeta)$. The equality (1.12) follows then from (1.11) and (1.4).

As to the space generated by $X(\zeta)\otimes Y(\zeta)\otimes Z(\zeta)$ being 6-dimensional, this fact is examined in detail in the following section for the operators L, M, N possessing some special symmetry. In the general case, nothing essentially new arises, so we will allow ourselves not to write down the corresponding formulae.

2. The Case of Baxter's L-Operators

2.1. Vacuum Covectors Mapped into Themselves

Consider now the Felderhof L-operators (1.1, 1.2) with $a_+ = a_-$, $b_+ = b_-$. Such Felderhof L-operators are, at the same time, a particular case of the Baxter 8-vertex model L-operators. We will use the following parametrization:

$$a_\pm = cn\lambda\,, \quad b_\pm = sn\lambda dn\lambda\,,$$
$$c = dn\lambda\,, \quad d = ksn\lambda cn\lambda\,.$$

Herein k is the modulus of the elliptic functions, λ is the so-called spectral parameter. Evidently, Γ and h from Theorem 1.1 take now the values

$$\Gamma = k\,, \qquad h = 0\,.$$

Taking this into account, let us fix k and denote the L-operators as $L_{0i}(\lambda)$, 0, i being the numbers of spaces.

Let us choose the L-operators from the previous section in the form

$$L = L_{01}(\lambda_1), \qquad M = L_{02}(\lambda_2), \qquad N = L_{03}(\lambda_3),$$

$\lambda_1, \lambda_2, \lambda_3$ being complex numbers.

Then $R^a_{ij} = R^a_{ij}(\lambda_i, \lambda_j)$ $(a = 0, 1; 1 \le i < j \le 3)$ can be chosen in the form

$$R^0_{ij}(\lambda_i, \lambda_j) = f_0(\lambda_i, \lambda_j) \begin{pmatrix} a_0 & & & d_0 \\ & b_0 & c_0 & \\ & c_0 & b_0 & \\ d_0 & & & a_0 \end{pmatrix},$$

$$R^1_{ij}(\lambda_i, \lambda_j) = f_1(\lambda_i, \lambda_j) \begin{pmatrix} -a_1 & & & -d_1 \\ & -b_1 & -c_1 & \\ & c_1 & b_1 & \\ d_1 & & & a_1 \end{pmatrix}.$$

Herein

$$\begin{aligned} a_0 &= cn(\lambda_i - \lambda_j), & b_0 &= sn(\lambda_i - \lambda_j) dn(\lambda_i - \lambda_j), \\ c_0 &= dn(\lambda_i - \lambda_j), & d_0 &= k sn(\lambda_i - \lambda_j) cn(\lambda_i - \lambda_j), \\ a_1 &= cn(\lambda_i + \lambda_j), & b_1 &= sn(\lambda_i + \lambda_j) dn(\lambda_i + \lambda_j), \\ c_1 &= dn(\lambda_i + \lambda_j), & d_1 &= k sn(\lambda_i + \lambda_j) cn(\lambda_i + \lambda_j), \end{aligned}$$

$f_0(\lambda_i, \lambda_j)$ and $f_1(\lambda_i, \lambda_j)$ are arbitrary multipliers that will be written also as simply f_0 and f_1.

Theorem 2.1. *In our "Baxterian" case the operator S has two invariant subspaces – the "even" subspace and the "odd" subspace, generated by the products $R^a_{12} R^b_{13} R^c_{23}$ with the even and the odd sum $a + b + c$, respectively.*

Proof. Let us introduce the matrix $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the products of R-operators that stand in the left-hand side and right-hand side of the definition (1.4) of S. They commute with $\sigma \otimes \sigma \otimes \sigma$ if the sum $a + b + c$ is even, and anti-commute if the sum is odd. Therefrom the invariance of the corresponding subspaces is easily drawn. Q.E.D.

Remark. As concerns the action of S upon the covectors (i.e. "from the right"), here, of course, the "even" and "odd" invariant subspaces also arise. As we will show in the end of this subsection, the 6-dimensional subspace generated by the vacuum covectors, from Theorem 1.4 contains the whole "even" subspace.

Let us examine how the covector $X(\zeta) = (X_0 \ X_1)$ and the numbers α and β from the previous section are connected in our "Baxterian" case. Consider the matrix elements from the secondary diagonal of the operator $R_{12}(\zeta)$, on the one hand, as expressed by the formula (1.13), and on the other hand, as obtained from the elements of the symmetrical operator $R_{\alpha\beta}$ according to formula (1.10). This symmetry leads to two conditions:

$$x = \frac{f_0 d_0}{f_1 d_1} \cdot \frac{1 - \alpha^2\beta^2}{1 + \alpha^2\beta^2}, \tag{2.2}$$

$$x = \frac{f_0 c_0}{f_1 c_1} \cdot \frac{\beta^2 - \alpha^2}{\beta^2 + \alpha^2}, \tag{2.3}$$

where

$$x = x(\zeta) = \frac{X_1}{X_0}\,. \tag{2.4}$$

Certainly, by eliminating x from (2.2) and (2.3) one would get the old condition $h_\alpha = h_\beta$ (1.7).

Now let us choose

$$\frac{f_0}{f_1} = \sqrt{\frac{c_1 d_1}{c_0 d_0}} \tag{2.5}$$

and express α and β through x according to (2.2, 2.3). The result is as follows:

$$\frac{1-\alpha^4}{1+\alpha^4} = (\varrho_{12} + \varrho_{12}^{-1})\frac{x}{x^2+1}\,, \tag{2.6}$$

$$\frac{1-\beta^4}{1+\beta^4} = (\varrho_{12} - \varrho_{12}^{-1})\frac{x}{x^2-1}\,, \tag{2.7}$$

where

$$\varrho_{12} = \sqrt{\frac{c_1 d_0}{c_0 d_1}}\,, \tag{2.8}$$

the ratio under the square root composed, of course, of matrix elements of the operators R_{12}^a.

Let us do the same for the operators R_{12}^b and R_{23}^c. Denote

$$y = \frac{Y_1}{Y_0}\,, \qquad z = \frac{Z_1}{Z_0}\,,$$

as in (2.4). The result is

$$\frac{1-\alpha^4}{1+\alpha^4} = (\varrho_{13} + \varrho_{13}^{-1})\frac{y}{y^2+1}\,, \tag{2.9}$$

$$\frac{1-\gamma^4}{1+\gamma^4} = (\varrho_{13} - \varrho_{13}^{-1})\frac{y}{y^2-1}\,, \tag{2.10}$$

$$\frac{1-\beta^4}{1+\beta^4} = (\varrho_{23} + \varrho_{23}^{-1})\frac{z}{z^2+1}\,, \tag{2.11}$$

$$\frac{1-\gamma^4}{1+\gamma^4} = (\varrho_{23} - \varrho_{23}^{-1})\frac{z}{z^2-1}\,, \tag{2.12}$$

ϱ_{13} and ϱ_{23} constructed from matrix elements of the corresponding R-operators according to the same formula (2.8). Eliminating α, β and γ, one gets

$$(\varrho_{12} + \varrho_{12}^{-1})\frac{x}{x^2+1} = (\varrho_{13} + \varrho_{13}^{-1})\frac{y}{y^2+1}\,, \tag{2.13}$$

$$(\varrho_{12} - \varrho_{12}^{-1})\frac{x}{x^2-1} = (\varrho_{23} + \varrho_{23}^{-1})\frac{z}{z^2+1}\,, \tag{2.14}$$

$$(\varrho_{13} - \varrho_{13}^{-1})\frac{y}{y^2-1} = (\varrho_{23} - \varrho_{23}^{-1})\frac{z}{z^2-1}\,. \tag{2.15}$$

Equation (2.13) determines an elliptic curve in the space of the variables x, y. The two other equations, given a point (x, y) in this curve, determine uniquely the corresponding value of z (without contradicting each other – see Subsect. 2.3 for details). Each of the functions x, y, z has 2 poles in the elliptic curve. Consequently, the components of the covector $(1\ x) \otimes (1\ y) \otimes (1\ z)$, which is proportional to $X(\zeta) \otimes Y(\zeta) \otimes Z(\zeta)$, have altogether 6 poles. Therefrom the last statement of the Theorem 1.4 follows in our "Baxterian" case.

Among the triples (x, y, z) that satisfy (2.13–2.15), there are $(0,0,0)$, $(0,\infty,\infty)$, $(\infty,0,\infty)$, $(\infty,\infty,0)$. The corresponding products $X(\zeta) \otimes Y(\zeta) \otimes Z(\zeta)$ (wherein if, say, $x = \infty$, then $X(\zeta) = (0\ 1)$) form a basis in the "even" subspace of covectors. This proves the remark following Theorem 2.1.

As to the eigencovectors in the "odd" subspace with an eigenvalue 1, they can be obtained in explicit form by projecting the other solutions of the system (2.13–2.15) onto the "odd" subspace, so as to get $(0, z, y, 0, x, 0, 0, xyz)$. Such solutions are, e.g.,

$$x = \varrho_{12}\,, \quad y = \varrho_{13}\,, \quad z = \varrho_{23}\,;$$
$$x = \varrho_{12}\,, \quad y = \varrho_{13}^{-1}\,, \quad z = \varrho_{23}^{-1}$$

and so on.

2.2. *One more Family of Vacuum Covectors*

On more one-parameter family of vacuum covectors has been found out due to the symmetry $a_+ = a_-$, $b_+ = b_-$. Let us construct from the L-operators L, M, N given by (2.1) the new ones,

$$L' = (\mathbb{1} \otimes \sigma)\, L(\sigma \otimes \mathbb{1})\,,$$
$$M' = (\mathbb{1} \otimes \sigma)\, M(\sigma \otimes \mathbb{1})\,,$$
$$N' = (\mathbb{1} \otimes \sigma)\, N(\sigma \otimes \mathbb{1})\,,$$

wherein, as before, $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The following way of writing down these operators will also be used:

$$L' = \sigma_1 L \sigma_0\,, \qquad M' = \sigma_2 M \sigma_0\,, \qquad N' = \sigma_3 N \sigma_0\,,$$

the subscript of σ meaning the number of the space in which it acts.

The adding of the prime to L-operators is equivalent to replacements

$$a_+ = a_- \leftrightarrow c\,, \qquad b_+ = b_- \leftrightarrow d\,.$$

Recalling a well-known transformation of the elliptic functions, one can say also that the L-operators remain the operators of the same form, while their spectral parameters and the modulus of the elliptic functions take new "primed" values

$$\lambda_i' = k\lambda_i\,, \qquad k' = k^{-1}\,.$$

Hence, one can perform all the constructions already done for L, M, N, also for L', M', N', with adding primes to all the letters. In particular, the equation analogous to (1.11) holds

$$R_{12}'(\zeta')\, R_{13}'(\zeta')\, R_{23}'(\zeta') = R_{23}'(\zeta')\, R_{13}'(\zeta')\, R_{12}'(\zeta')\,, \tag{2.17}$$

with which the covectors $X'(\zeta')$, $Y'(\zeta')$, $Z'(\zeta')$ are associated.

On the other hand, the "primed" R-operators are connected with the "unprimed" ones by formulae analogous to (2.16). To be exact,

$$(R')^0_{ij} = \sigma_j R^0_{ij} \sigma_i = \sigma_i R^0_{ij} \sigma_j \,, \tag{2.18}$$

$$(R')^1_{ij} = \sigma_j R^1_{ij} \sigma_i = -\sigma_i R^1_{ij} \sigma_j \,. \tag{2.19}$$

Now rewrite (2.17) in the form

$$\begin{aligned} &\sigma_2 R'_{12}(\zeta')\sigma_1 \cdot \sigma_1 R'_{13}(\zeta')\sigma_3 \cdot \sigma_3 R'_{23}(\zeta')\sigma_2 \cdot \\ &\quad = \sigma_2 R'_{23}(\zeta')\sigma_3 \cdot \sigma_3 R'_{13}(\zeta')\sigma_1 \cdot \sigma_1 R'_{12}(\zeta')\sigma_2 \,. \end{aligned} \tag{2.20}$$

According to (2.18–2.19), each of the factors separated by dots is a linear combination of "unprimed" R-operators. For example,

$$\sigma_2 R'_{12}(\zeta')\sigma_1 = \sum_{a=0}^{1} X'_a(\zeta') R^a_{12} \,,$$

$$\sigma_1 R'_{12}(\zeta')\sigma_2 = \sum_{a=0}^{1} (X'_-)_a(\zeta') R^a_{12} \,,$$

wherein if $X' = (x_0 \;\; x_1)$ then $X'_- = (x_0 \;\; -x_1)$. Thus, the relation (2.20) provides one more family of the vacuum covectors of the (unprimed) operator S, in addition to that constructed in Theorem 1.4:

$$(X'(\zeta') \otimes Y'_-(\zeta') \otimes Z'(\zeta'))S = X'_-(\zeta') \otimes Y'(\zeta') \otimes Z'_-(\zeta') \,. \tag{2.21}$$

Let us introduce the ratios x', y', z' of the first coordinates of the covectors $X'(\zeta'), Y'(\zeta'), Z'(\zeta')$ to their zero coordinates, as in formula (2.4). What is the connection among x', y' and z'? The answer is given, of course, by the old formulae (2.13–2.15), with the primes added to x, y, z, and ϱ_{ij} [see (2.8)] replaced by

$$\varrho'_{ij} = \sqrt{\frac{a_1 b_0}{a_0 b_1}} \,. \tag{2.22}$$

Here in the right-hand side there are, of course, the matrix elements of R^a_{ij}. Note that the interchange $a \leftrightarrow c$, $b \leftrightarrow d$ of the matrix elements of the R-operator that corresponds to the adding of the prime, does not change the ratio f_0/f_1 (2.5).

The next theorem summarizes these considerations.

Theorem 2.2. *If the operator S is constructed from the Baxter's L, M, N, then it has, in addition to the vacuum covectors of Theorem 1.4, one more one-parameter family of vacuum covectors that has been described in this subsection. S is determined uniquely by its action upon the vacuum covectors (formulae* (1.12) *and* (2.21)*). S is an involution: $S^2 = \mathbb{1}$, the eigensubspace with an eigenvalue -1 being 2-dimensional and lying in the "odd" subspace.*

The last statement in the theorem, of course, applies to the action of S upon the vectors as well as upon the covectors.

Proof. It follows from (2.21) that if one decomposes

$$X'(\zeta') \otimes Y'_-(\zeta') \otimes Z'(\zeta')$$

into the sum of its "even" and "odd" parts according to the decomposition of the covector space into the direct sum, then under the action of S the "even" part remains unchanged, while the "odd" part changes its sign. These "odd" parts sweep a 2-dimensional subspace, while the "even" parts sweep the whole "even" subspace as well as covectors of Theorem 1.4 do. Thus, the two families of vacuum covectors sweep two 2-dimensional subspaces in the "odd" subspace, the corresponding eigenvalues of the operator S being $+1$ and -1. Q.E.D.

The explicit expressions for the eigencovectors with the eigenvalue -1 can be obtained in the same manner as those for the eigencovectors with the eigenvalue $+1$ were obtained in the end of Subsect. 2.1. With the eigenvalues being $+1$ and eigencovectors known, it is not very hard to calculate the matrix elements of S. However, the author prefers to profit by calculations already done in papers [3, 4]. The possibility of this is shown in the next subsection.

2.3. *Further Properties of S, and its Matrix Elements in Explicit Form*

We have seen that S is determined by its vacuum covectors, while the covectors are determined by the values ϱ_{ij} (2.8) and ϱ'_{ij} (2.22), $1 \le i < j \le 3$. Let us reconsider the system of Eqs. (2.13–2.15). It has, of course, an infinite set of solutions (x, y, z), each solution corresponding to a vacuum covector.

Lemma. *The system* (2.13–2.15) *has an infinite set of solutions provided one of the two following equalities holds:*

$$\varrho_{12}^2 - \varrho_{13}^2 + \varrho_{23}^2 - \varrho_{12}^2\varrho_{13}^2\varrho_{23}^2 = 0\,, \tag{2.23}$$

$$\varrho_{12}^{-2} - \varrho_{13}^{-2} + \varrho_{23}^{-2} - \varrho_{12}^{-2}\varrho_{13}^{-2}\varrho_{23}^{-2} = 0\,. \tag{2.24}$$

Proof. After being raised to the power -2, each equation of the system (2.13–2.15) becomes linear with respect to

$$x^2 + x^{-2}\,, \qquad y^2 + y^{-2}\,, \qquad z^2 + z^{-2}\,. \tag{2.25}$$

This system always has a solution, e.g. $x = \varrho_{12}$, $y = \varrho_{13}$, $z = \varrho_{23}$. Thus, the system has an infinite set of solutions if and only if its determinant, made up of the coefficients at the unknowns, vanishes. The determinant, in its turn, vanishes if (2.23) or (2.24) holds. So, the lemma is proven.

One can verify that the values ϱ_{ij} defined as in (2.8) satisfy Eq. (2.23). Further, the system (2.13–2.15) has the following evident property: it does not change when the triple $(\varrho_{12}, \varrho_{13}, \varrho_{23})$ is replaced by any solution (x_0, y_0, z_0) of the system. Thus, the solutions of the system (2.13–2.15) possess by themselves one of the properties of the type (2.23) or (2.24), namely (2.23) because of the possibility to continuously deform (x, y, z) into $(\varrho_{12}, \varrho_{13}, \varrho_{23})$:

$$x^2 - y^2 + z^2 - x^2y^2z^2 = 0\,. \tag{2.26}$$

The left-hand side of Eq. (2.26) may be viewed as a scalar square of the covector $(0, z, y, 0, x, 0, 0, xyz)$, if we define a scalar product between two "odd" covectors by the formula

$$\begin{aligned}&\langle (0, u_1, u_2, 0, u_3, 0, 0, u_4), (0, v_1, v_2, 0, v_3, 0, 0, v_4)\rangle \\ &\quad = u_1v_1 - u_2v_2 + u_3v_3 - u_4v_4\,.\end{aligned} \tag{2.27}$$

Then Eq. (2.26) means that the "odd" parts of the covectors $X(\zeta)\otimes Y(\zeta)\otimes Z(\zeta)$ lie in a 2-dimensional isotropic (i.e. that coincides with its orthogonal complement with respect to the scalar product) subspace. It is easy to count that 2-dimensional isotropic subspaces in a 4-dimensional space form a 1-dimensional submanifold in the manifold of all the 2-dimensional subspaces. From all that follows that the eigensubspace of the operator S with the eigenvalue 1 (in the "odd" covector space) is determined by merely one complex number. Of course, the same is true for the eigensubspace with the eigenvalue -1. Thus, S actually depends on two parameters instead of the four: $\lambda_1, \lambda_2, \lambda_3$ and k. The following calculations are in full agreement with this conclusion.

According to Eqs. (2.8) and (2.22),

$$\varrho_{ij}^2 = \frac{\dfrac{sn\cdot cn}{dn}(\lambda_i-\lambda_j)}{\dfrac{sn\cdot cn}{dn}(\lambda_i+\lambda_j)}, \tag{2.28}$$

$$(\varrho'_{ij})^2 = \frac{\dfrac{sn\cdot dn}{cn}(\lambda_i-\lambda_j)}{\dfrac{sn\cdot dn}{cn}(\lambda_i+\lambda_j)}. \tag{2.29}$$

Define the quantities

$$s_{ij} = \frac{sn(\lambda_i-\lambda_j)}{sn(\lambda_i+\lambda_j)},$$

$$c_{ij} = \frac{\dfrac{cn}{dn}(\lambda_i-\lambda_j)}{\dfrac{cn}{dn}(\lambda_i+\lambda_j)}.$$

One can verify by standard means that

$$\frac{1-s_{ij}c_{ij}}{c_{ij}-s_{ij}} = \frac{cn}{dn}2\lambda_i, \tag{2.30}$$

$$\frac{1+s_{ij}c_{ij}}{c_{ij}+s_{ij}} = \frac{cn}{dn}2\lambda_j. \tag{2.31}$$

For the given pair (i,j), the equalities (2.30, 2.31) enable one, as well, to express s_{ij} and c_{ij} through $\frac{cn}{dn}2\lambda_i$ and $\frac{cn}{dn}2\lambda_j$. There is no need to write down the explicit formulae; note only that they contain a square root, but this little non-uniqueness doesn't influence our conclusions. Finally, the ratios (2.28, 2.29) are expressed through s_{ij} and c_{ij}.

Thus, one can change the modulus k of elliptic functions without changing the operator S. For this, it is sufficient that the values

$$\frac{cn}{dn}2\lambda_1, \quad \frac{cn}{dn}2\lambda_2, \quad \frac{cn}{dn}2\lambda_3$$

remain invariable. This permits one to take $k\to 0$, the case in which S has been already found in the previous author's works [3, 4] (by direct calculation, according to Definition (1.4) of S).

The explicit expressions for the matrix elements of S, with our choice (2.5) of the ratio f_0/f_1, are

$$S_{000}^{000} = S_{011}^{011} = S_{101}^{101} = S_{110}^{110} = 1\,,$$
$$S_{010}^{001} = S_{001}^{010} = -S_{111}^{100} = -S_{100}^{111} = \sqrt{\operatorname{cth}(\varphi_1-\varphi_3)}\,\sqrt{\operatorname{th}(\varphi_2-\varphi_3)}\,,$$
$$S_{100}^{001} = S_{111}^{010} = -S_{001}^{100} = -S_{010}^{111} = \sqrt{\operatorname{th}(\varphi_1-\varphi_2)}\,\sqrt{\operatorname{th}(\varphi_2-\varphi_3)}\,,$$
$$S_{111}^{001} = S_{100}^{010} = S_{010}^{100} = S_{001}^{111} = \sqrt{\operatorname{th}(\varphi_1-\varphi_2)}\,\sqrt{\operatorname{cth}(\varphi_1-\varphi_3)}\,,$$

the other matrix element equal to 0. Here the values φ_1, φ_2, φ_3 are given by

$$\operatorname{th}\varphi_i = \frac{1-\dfrac{cn}{dn}2\lambda_i}{1+\dfrac{cn}{dn}2\lambda_i}\,. \tag{2.32}$$

Thus, we can conclude this section by formulating its results as the following theorem.

Theorem 2.3. *The commutation relation matrix S of the tetrahedral Zamolodchikov algebra corresponding to Baxter's L-operators $L_{0i}(\lambda_i)$ depends actually on only 2 differences of the values φ_i (2.32) (while a priori S depends on 4 arguments k, λ_1, λ_2, λ_3). The explicit form of the matrix elements is as given in the preceding paragraph.*

3. The "Two-Layer" Solutions to the Yang-Baxter Equation

Solutions to the Yang-Baxter equation of a specific "2-layer" form can be constructed out of the R-operators described in Subsect. 2.1, as well as it was done in paper [5] for the trigonometrical case.

Such an R-operator depends on three variables: $R_{ij}^a = R_{ij}^a(\lambda_i, \lambda_j, k)$. Consider the 4-dimensional spaces

$$W_1 = V_1 \otimes V_1\,, \qquad W_2 = V_2 \otimes V_2\,, \qquad W_3 = V_3 \otimes V_3$$

and the operators

$$R_{ij}(\lambda_i, \lambda_j, k_1, \mu_i, \mu_j, k_2) = \sum_{a=0}^{1} R_{ij}^a(\lambda_i, \lambda_j, k_1) \otimes R_{ij}^a(\mu_i, \mu_j, k_2) \tag{3.1}$$

acting in $W_i \otimes W_j$. What must be the arguments λ_i, μ_i, k_1, k_2 for the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{3.2}$$

to hold (for each R_{ij}, the corresponding arguments are implied)?

Using (1.4), one can verify that Eq. (3.2) holds if

$$S(\lambda_1, \lambda_2, \lambda_3, k_1)\, S^T(\mu_1, \mu_2, \mu_3, k_2) = \mathbb{1}\,.$$

Then, from the expressions for the matrix elements of S (Subsect. 2.3) one sees that S transforms into S^T under the change

$$\varphi_i \to \text{const} - \varphi_i\,,$$

with $\sqrt{\operatorname{th}(\varphi_1-\varphi_2)}$ and $\sqrt{\operatorname{th}(\varphi_2-\varphi_3)}$ multiplied by $\sqrt{-1}$, and $\sqrt{\operatorname{cth}(\varphi_1-\varphi_3)}$ multiplied by $(-\sqrt{-1})$. Taking into account that, besides, $S = S^{-1}$, we get the following theorem.

Theorem 3.1. *The "two-layer" operators* R_{ij} *given by* (3.1) *satisfy the Yang-baxter equation* (3.2) *if the sum*

$$\varphi_i(\lambda_i, k_1) + \varphi_i(\mu_i, k_2),$$

with φ_i *determined by* (2.32), *does not depend on* $i = 1, 2, 3$.

It is seen from Theorem 3.1 that if one fixes 5 independent arguments of the operator R_{12}, then the operators R_{13} and R_{23} satisfying Eq. (3.2) form a one-parameter family.

4. Discussion

In this work, the commutation relation matrix S of the tetrahedral Zamolodchikov algebra is calculated in a more general case than in papers [3, 4]: the tetrahedral algebras now depend upon the modulus k of the elliptic functions, while in the mentioned works it was equal to zero. The main result is that, actually, no new matrices are obtained in this way, the "old" matrices thus demonstrating a sort of universality. This reminds us of the uniqueness of the "static limit" solutions to the tetrahedron equation with variables on the faces ([10], Sect. 6).

It is worth mentioning that if one imposes upon the Felderhof L-operators the condition $d = 0$, but not the conditions $a_+ = a_-$, $b_+ = b_-$ [see formulae (1.1, 1.2)], then the operators S obtained starting from such L-operators will again coincide with the already known ones. This may be concluded from a simple consideration that will be presented elsewhere. Thus, only the case of the most general L-operators still remains unstudied.

As concerns the already constructed S-operators, it would be very interesting to know whether there exists any 2-dimensional "exactly solved" spin models connected with them, like the 1-dimensional spin chains are connected with the solutions of the Yang-Baxter equation.

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