Commun. Math. Phys. 154, 63–84 (1993)

Communications in Mathematical Physics

A Concept of the Mass Center of a System of Material Points in the Constant Curvature Spaces

G. A. Galperin

Department of Mathematics, Northwestern University, Evanston, IL 60208, USA

Received June 15, 1990; in revised form January 15, 1993

Abstract. This article demonstrates that in the Lobatchevsky space and on a sphere of arbitrary dimensions, the concept of the mass center of a system of mass points can be correctly defined. Presented are: a uniform geometric construction for defining the mass center; hyperbolic and spheric "lever rules"; the theorem of uniqueness for determining the mass center in these spaces. Among the compact manifolds, only the sphere possesses this property.

1. Preliminary. Statement of the Main Results

The classical definition of the centroid (A, m) of a system of material points can be stated as follows: a point A with mass m is called *the centroid* of a system of material points $A_1, \ldots, A_k$ with masses $m_1, \ldots, m_k$ in the Euclidean space $\mathbf{R}^n$ if

$$m \cdot \overrightarrow{OA} = \sum_{i=1}^{k} m_i \cdot \overrightarrow{OA_i}\,, \quad \text{and} \quad m = \sum_{i=1}^{k} m_i\,,$$

where $O \in \mathbf{R}^n$ is an arbitrary point (Fig. 1). Then the mass $m = \sum_{i=1}^{k} m_i$ is located in the point A.

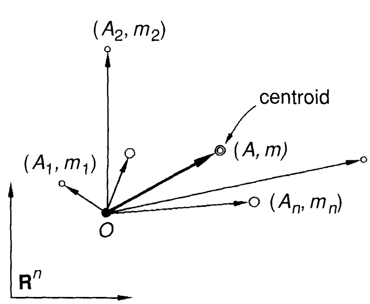

Fig. 1

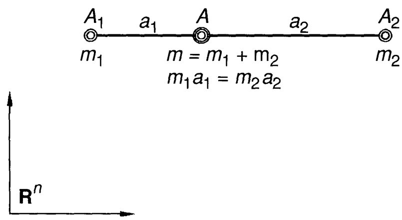

Fig. 2

This definition is based on the existence of a *linear structure* in $\mathbf{R}^n$. The centroid satisfies the following properties a) if we decompose the system of points $A_1, \ldots, A_k$ into subsystems and replace each subsystem by its centroid, we obtain a new system with the same centroid; and b) the definition of a centroid is invariant under the group of isometrics of $\mathbf{R}^n$.

There is another definition of a centroid, an inductive one which is based on the existence of the *Euclidean structure* in $\mathbf{R}^n$. First we define a centroid for a system of two material points as a material point (A, m) for which "the Euclidean rule of the lever" is fulfilled:

$$m_1 a_1 = m_2 a_2 \,, \tag{1}$$

where $a_1 = |A_1 A|$, $a_2 = |A_2 A|$ (A belongs to the segment $A_1 A_2$); and

$$m = m_1 + m_2 \quad \text{(Fig. 2)}\,. \tag{2}$$

To determine the centroid of $k > 2$ material points is necessary to replace any $k-1$ of them $A_1, \ldots, A_{k-1}$ by their centroid and then find the centroid of two remaining points $(A, m) \cup (A_k, m_k)$ as above.

This definition is correct as well, although it is not simple to check property a). The following question arises: *is it possible to give a consistent definition of a centroid for a system of material points situated in a space of constant non-zero curvature – for instance, in the spherical space S^n and in the hyperbolic space Λ^n (Lobatchevsky space)?*

Neither S^n nor Λ^n have a linear structure, so the above vectorial definition does not work. Also the Euclidean lever rule does not work because Axiom a) fails. To see this it is sufficient to consider an isosceles triangle ABC (AB is its base) with the equal masses at its vertices. It follows from the Euclidean lever rule that the centroid of this point system is the intersection point of the medians of ABC and that the medians are divided by this point in ratio 1:2. But this does not hold for arbitrary triangles in S^n or Λ^n (Fig. 3).

Nevertheless a definition of centroid in spherical and hyperbolic spaces does exist and it is a unique one. There are special *"rules of the lever"* for the spaces S^n and Λ^n; the mass of the centroid in the case of S^n is less than the sum of masses and in the case Λ^n is more than this sum.

The main results of the article are:

1) A definition of centroid in the space Λ^n is given based on Special Relativity Theory (the so-called *"relativistic centroid"*);

2) A uniform definition of centroid in the spaces $\mathbf{R}^n$, S^n, and Λ^n is given (the so-called *"model centroid"*). It is shown that *relativistic* and *model* centroids of given points system in Λ^n coincide as well as *classical* and *model* centroids of given points system in $\mathbf{R}^n$;

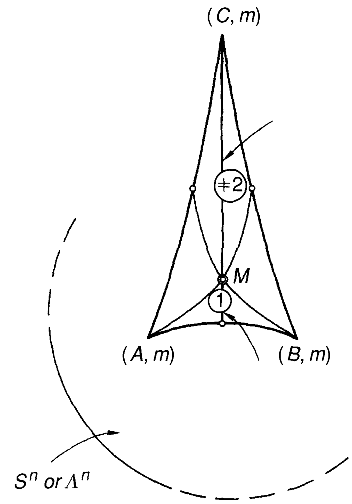

Fig. 3

3) A natural system of axioms of centroid is formulated (the so-called "*axiomatic centroid*". It is shown that the *model* centroids satisfy this system of axioms in the spaces S^n, Λ^n, and $\mathbf{R}^n$.
4) A uniqueness theorem is proven for the spaces S^n, Λ^n, and $\mathbf{R}^n$ with $n \geq 2$. It asserts that the *axiomatic* centroid coincides with the *model* one.
5) Manifolds on which the notion of centroid can be defined are clarified.

2. "Relativistic" Centroid in the Lobatchevsky Space Λ^n

The definition of centroid in case of Λ^n arises from relativistic dynamics. Let A be some inertial frame of reference in $\mathbf{R}^n$. Consider the space $\mathscr{V}_A$ of velocities of relativistic particles in $\mathbf{R}^n$ viewed in this frame of reference. Velocities $\{v\}$ are normalized by $c = 1$, where c is the light velocity (so for every scalar v, $0 < v < 1$). It is known [1] that $\mathscr{V}_A$ is Lobatchevsky space realized as a unit ball with the centre denoted by the same letter A. Let B be a free particle in $\mathbf{R}^n$. Denote by $v_{A|B}$ its (vector) velocity in the frame of reference A. Consider such a point $B \in \mathscr{V}_A$ (that is denoted by the same letter as the particle) that $\overrightarrow{AB} = v_{A|B}$. Thus there is an infinite number of particles in $\mathbf{R}^n$ corresponding to every given point B of $\mathscr{V}_A$: all these particles have the same velocity $v_{A|B}$ which is equal to $\overrightarrow{AB}$. Each point F of the boundary sphere S_A of the ball $\mathscr{V}_A$ corresponds to the velocity of photon $F \in \mathbf{R}^n$ moving in direction $\overrightarrow{AF}$ in R^n. So, S_A is the *absolute* of the Lobatchevsky space $\mathscr{V}_A$ (Fig. 4).

Consider the space $\mathscr{V}_B$ corresponding to another frame of reference B. A natural map L_{AB} from $\mathscr{V}_A$ to $\mathscr{V}_B$ arises: and this map is a projective transformation. When speaking about spaces of velocities we will sometimes omit the name of the frame of reference. Metrics $\|\cdot\|$ in the space of velocities $\mathscr{V}$ is given by the formula

$$\|XY\| = 1/2 \ln(1 + v_{X|Y})/(1 - v_{X|Y}) \tag{3a}$$

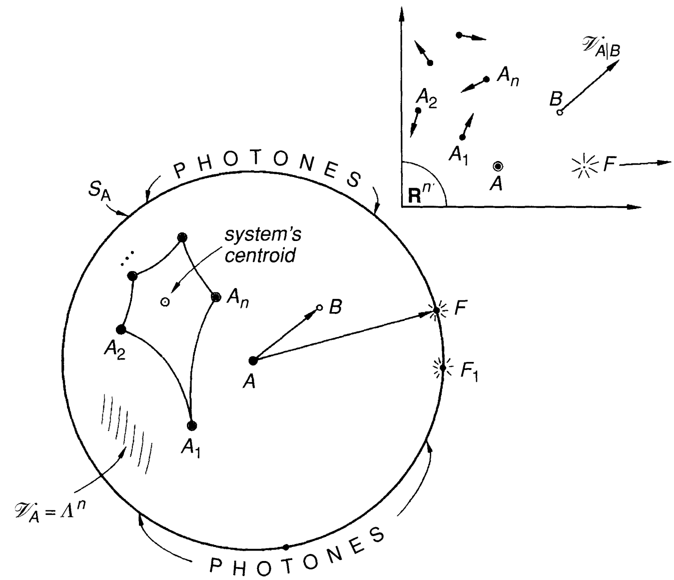

Fig. 4

A, B, A_1, A_2, ..., A_n are particles;
F, F_1 are photones;
$\mathscr{V}_{A|B} = \tanh \|AB\|$;
$\mathscr{V}_{A|F} = 1$

or equivalently

$$v_{X|Y} = \tanh \|XY\|\,, \tag{3b}$$

where X and Y are two points in the space $\mathscr{V}$. The model of Lobatchevsky space constructed above is, in fact, the well-known Cayley-Klein Model.

Using the relative formulae for momentum $p = mv(1 - v^2)^{-1/2}$ and energy $E = m(1 - v^2)^{-1/2}$ of a relativistic particle X with the rest-mass m in *inertial* frame of a reference A, we find the values p and E through the distance $a = \|XY\|$ for which $\tanh(a) = v$:

$$p = m \sinh(a)\,, \qquad E = m \cosh(a)\,, \qquad E^2 - p^2 = m^2\,. \tag{4}$$

Identify Λ^n with some $\mathscr{V}_0$ and consider a number of points $A_1, \ldots, A_k$ with the masses $m_1, \ldots, m_k$. We now define their centroid. Choose some corresponding relativistic particles $A_1, \ldots, A_k$ in $\mathbf{R}^n$ and consider their centroid A. The corresponding point A in $\mathscr{V}_0$ is the desired centroid of the initial system $A_1, \ldots, A_k$.

The velocity v, and hence the position, of point $A \in \mathscr{V}_0$ ($v = \overrightarrow{OA}$) can be defined according to the conservation of momentum low. According to the conservation of energy law the centroid's energy can be defined as the sum of the component particle's energies. Then, according to the formula $E = m\cosh(a)$, the necessary value of mass m is found which must be located at the point $A \in \mathscr{V}_0$. The couple (A, m) is called the "*relativistic*" centroid of the system $(A_1, m_1) + \ldots + (A_k, m_k)$. The correctness of this definition will be established in the next section; but now we will obtain some explicit formulae for the relativistic lever rule.

Calculate the centroid (A, m) of two-points system $(A_1, m_1) + (A_2, m_2)$ in Λ^n. We set $\|A_1 A\| = a_1$, $\|AA_2\| = a_2$ and consider the ball $\mathscr{V}_A$. Suppose that corresponding relativistic particles A_1, and A_2 in $\mathbf{R}^n$ are moving along the same straight line with the velocities $v_1 = \tanh(a_1)$ and $v_2 = \tanh(a_2)$ with respect to their centroid $A \in \mathbf{R}^n$. Next, suppose at some moment they bind in this centroid together (particles A_1 and A_2 position in $\mathbf{R}^n$ can be varied arbitrarily: it does not depend on the points A_1, A_2 position in the ball $\mathscr{V}_0$). Point A is the center of the ball $\mathscr{V}_A$ and represents the zero-velocity while bound particles momentum of pulse have equal values but differ only by the sign:

$$m_1 \sinh(a_1) = m_2 \sinh(a_2)\,. \tag{5}$$

The total energy $E = m_1 \cosh(a_1) + m_2 \cosh(a_2)$ equals $m \cosh(0) = m$ whence the mass situated in the centroid $A \in \mathscr{V}_A$ of points $A_1, A_2 \in \mathscr{V}_A$ equals

$$m = m_1 \cosh(a_1) + m_2 \cosh(a_2)\,. \tag{6}$$

Formulae (5), (6) give "*relativistic rule of lever*". It is quite clear how to define the centroid of several points inductively (through adding the new points to arbitrary chosen of points one by one).

3. The "Model" Centroids in the Space of Constant Curvature

We now give the uniform definition of centroid in the n-dimensional space $\mathbf{X}$ of constant curvature ($\mathbf{X} = \mathbf{R}^n, \Lambda^n, S^n$) for a system of material points.

Let a certain model of space $\mathbf{X}$, i.e. a hypersurface $M_\mathbf{X}$ exist in the Euclidean space $\mathbf{R}^{n+1}$ with Cartesian coordinate system $x_0, x_1, \dots, x_n$; $A_1, A_2, \dots, A_k$ are the points on $M_\mathbf{X}$ with their masses $m_1, m_2, \dots, m_k$. Consider a ray starting at the origin of $\mathbf{R}^{n+1}$ with direction $\sum_{i=1}^{k} m_i \overrightarrow{OA_i}$. The point of intersection of this ray with $M_\mathbf{X}$ shall be called the *centroid* of the points $A_1, A_2, \dots, A_k$, and locate a mass m determined by the formula:

$$m \cdot \overrightarrow{OA} = \sum_{i=1}^{k} m_i \cdot \overrightarrow{OA_i} \quad \text{(Fig. 5a)}\,. \tag{7}$$

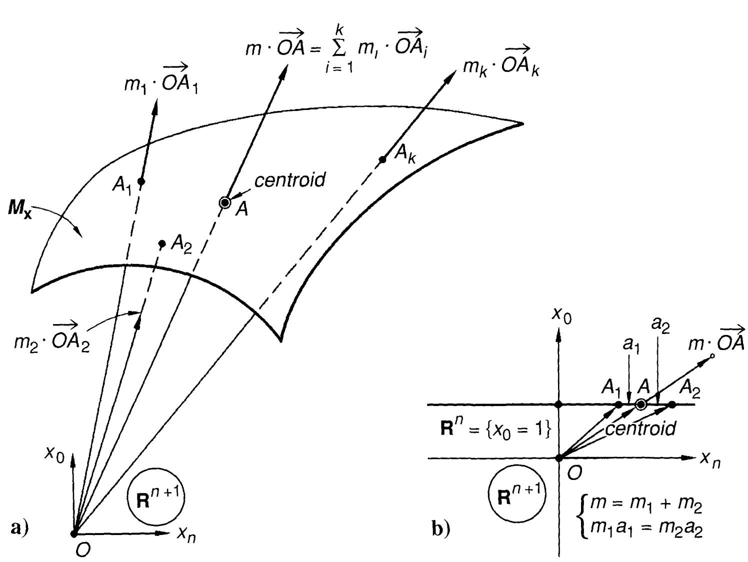

Fig. 5

The couple (A, m) we shall call a *"model" centroid* of a system $(A_1, m_1) \cup (A_2, m_2) \cup (A_3, m_3) \cup \ldots \cup (A_k, m_k)$. The correctness of given definitions (i.e., the results independence of new points joining order) is followed by the commutativity of a vector's addition.

We shall render concrete this definition for each space from $\mathbf{R}^n, \Lambda^n, S^n$ calling obtained centroids *"Euclidean," "hyperbolic,"* and *"spherical"* correspondingly.

3.1. Euclidean Centroid. Imagine that $\mathbf{R}^n$ is a n-dimensional plane $\{x_0 = 1\}$ in the space $\mathbf{R}^{n+1}$. The x_0-coordinates of all the vectors $\overrightarrow{OA}$, $i = 1, \ldots, k$, are equal to 1 and hence the x_0-coordinate of vector in right-hand side of (7) is equal to $\sum_{i=1}^{k} m_i$. If $A \in \{x_0 = 1\}$ is the model centroid of the system $\{A_i\}$, then the first coordinate of vector $\overrightarrow{OA}$ is equal to 1; therefore, (7) at once implies that

$$m = \sum_{i=1}^{k} m_i \quad \text{(Fig. 5b)}. \tag{8}$$

It is simple to obtain that centroid of two points which belongs to the plane $\mathbf{R}^n$ satisfies the *Euclidean rule of lever* (1).

Thus the fact that the *classical* centroid of the system of material points coincides with the *model* one in $\mathbf{R}^n$ is proven.

3.2. Hyperbolic Centroid. The model of Lobatchevsky space Λ^n of the curvature 1 in the space $\mathbf{R}^{n+1}$ with the origin O is the upper sheet of two-sheeted hyperboloid $\{\mathbf{x} : \|\mathbf{x}\| = [\mathbf{x}, \mathbf{x}] = 1\}$ situated in the half-space $[\mathbf{x}, \mathbf{x}] > 0$, where $\mathbf{x} \in \mathbf{R}^{n+1}$ and $[\mathbf{x}, \mathbf{y}]$ is pseudoscalar product in $\mathbf{R}^{n+1}$ which is defined by the quadratic form:

$$[\mathbf{x}, \mathbf{y}] = x_0 y_0 - x_1 y_1 - \ldots - x_n y_n \,. \tag{9}$$

If A and B are two points belonging to this hyperboloid, and $\overrightarrow{OA} = \mathbf{x}$, $\overrightarrow{OB} = \mathbf{y}$, then the distance r induced by the pseudoeuclidean metric between them is given by the formula:

$$\cosh r = [\mathbf{x}, \mathbf{y}] \quad \text{(Fig. 6a)}. \tag{10}$$

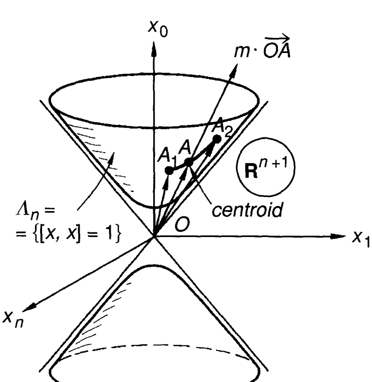

Fig. 6 a)

The *hyperbolic rotations* of space $\mathbf{R}^{n+1}$ are the linear transformations of $\mathbf{R}^{n+1}$ with determinant equal to 1, preserving the form $[\mathbf{x}, \mathbf{y}]$ and translating each sheet of cone $[\mathbf{x}, \mathbf{x}] = 0$ to itself. These rotations give the motions of space Λ_n in constructed model. The cone $[\mathbf{x}, \mathbf{x}] = 0$ is an absolute of space Λ^n in this model.

Let $(A_1, m_1) \cup (A_2, m_2) \cup \ldots \cup (A_k, m_k)$ be a k-points material particles system belonging to the upper sheet of the hyperboloid $\{||\mathbf{x}|| = 1\}$. According to the general definition (7) we shall find a ray with source O and directed by the vector $\sum m_i \overrightarrow{OA_i}$ and this ray's point of intersection A with the hyperboloid. This is the obtained material point (A, m) which we shall call the *"hyperbolic centroid"*.

Let us calculate a two-points system's $(A_1, m_1) + (A_2, m_2)$ centroid on the hyperboloid $\{||\mathbf{x}|| = 1\}$. Denote $\mathbf{z}_1 = \overrightarrow{OA_1}$, $\mathbf{z}_2 = \overrightarrow{OA_2}$; $||\mathbf{z}_1|| = ||\mathbf{z}_2|| = 1$. We shall find the length of vector $m_1\mathbf{z}_1 + m_2\mathbf{z}_2$ in pseudoeuclidean metric:

$$||m_1\mathbf{z}_1 + m_2\mathbf{z}_2||^2 = m_1^2||\mathbf{z}_1||^2 + 2m_1m_2[\mathbf{z}_1, \mathbf{z}_2] + m_2^2||\mathbf{z}_2||^2 = m_1^2 + m_2^2 + 2m_1m_2\cosh r\,,$$

where r is the distance between points A_1 and A_2 on the hyperboloid. Notice that for $\mathbf{z} = \overrightarrow{OA}$, $||\mathbf{z}|| = 1$ and that vector $\mathbf{z}$ can be obtained from vector $m_1\mathbf{z}_1 + m_2\mathbf{z}_2$ by its decreasing $||m_1\mathbf{z}_1 + m_2\mathbf{z}_2||$ times; on the other hand, this value must be equal to the mass m which is at the point A. Therefore,

$$m = \sqrt{m_1^2 + m_2^2 + 2m_1m_2\cosh r}\,. \tag{11}$$

Thus, the mass m which is at the point A is equal to the length of vector $m_1\mathbf{z}_1 + m_2\mathbf{z}_2$ in the pseudoeuclidean metric. Hence,

$$\overrightarrow{OA} = \mathbf{z} = (m_1\mathbf{z}_1 + m_2\mathbf{z}_2)/m = (m_1\mathbf{z}_1 + m_2\mathbf{z}_2)/\sqrt{m_1^2 + m_2^2 + 2m_1m_2\cosh r}\,. \tag{12}$$

The formulae (11), (12) show us that the motion of *hyperbolic* centroid is invariant with respect to the space's Λ^n motion group. We now introduce the *hyperbolic rule of the lever.*

If a_1 is the distance in the Lobatchevsky metric between A_1 and A, then

$$\cosh a_1 = [\mathbf{z}_1, \mathbf{z}] = (m_1[\mathbf{z}_1, \mathbf{z}_1] + m_2[\mathbf{z}_1, \mathbf{z}_2])/m = (m_1 + m_2\cosh r)/m\,,$$

and similarly

$$\cosh a_2 = (m_2 + m_1\cosh r)/m\,.$$

Hence,

$$\begin{aligned} m &= m^2/m = (m_1^2 + m_2^2 + 2m_1m_2\cosh r)/m \\ &= m_1(m_1 + m_2\cosh r)/m + m_2(m_2 + m_1\cosh r)/m \\ &= m_1\cosh a_1 + m_2\cosh a_2\,. \end{aligned} \tag{13}$$

From

$$\sinh a_1 = \sqrt{\cosh^2 a_1 - 1} = \sqrt{\frac{(m_1 + m_2\cosh r)^2}{m_1^2 + m_2^2 + 2m_1m_2\cosh r} - 1} = \frac{m_2\sinh r}{m}$$

and

$$\sinh a_2 = \frac{m_1\sinh r}{m}$$

we obtain

$$m_1\sinh a_1 = m_2\sinh a_2 \quad \text{(Fig. 6b)}\,. \tag{14}$$

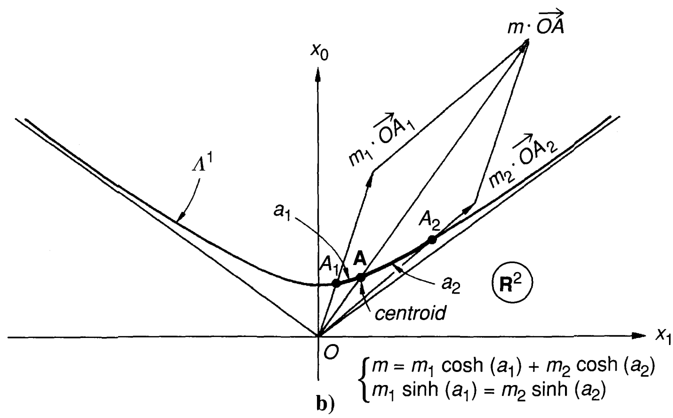

Fig. 6

Formulae (13) and (14) – "*hyperbolic rule of lever*" – completely coincide with formulae (6) and (5) respectively. Thus, the fact that the points systems *relative* centroid coincides with the *hyperbolic* one in Λ^n is shown. Therefore both are defined.

Note that the mass which is situated at the centroid according to the formulae (11) and (13) exceeds the sum of the particle masses. But in the case when the distance between points are small, the mass of centroid is roughly equal to the sum of masses of these points; however, the greater the distance between points the greater the mass in centroid. The mass at the centroid rises exponentially as $r \to \infty$.

3.3. Spherical Centroid. The model of n-dimensional spherical space of constant positive curvature 1 is a sphere, S^n, located in the space $\mathbf{R}^{n+1}$ with the center in the origin O, with equation $x_0^2 + \ldots + x_n^2 = 1$.

The distance between two points is induced by euclidean metric in $\mathbf{R}^{n+1}$ and equals the length of arc on the great circle which connects these points. If the distance between the sphere's S^n points A and B is equal to r, $0 \le r \le \pi$, and $\overrightarrow{OA} = \mathbf{x}$, $\overrightarrow{OB} = \mathbf{y}$, then

$$\cos r = (\mathbf{x}, \mathbf{y}), \tag{15}$$

where $(\mathbf{x}, \mathbf{y}) = x_0 y_0 + x_1 y_1 + \ldots + x_n y_n$ is vector dot product in $\mathbf{R}^{n+1}$. We will denote the length of vector $\mathbf{x} \in \mathbf{R}^{n+1}$ by $|\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})}$.

The motions of sphere S^n are induced by rotations of space $\mathbf{R}^{n+1}$ around the center O with respect to which the sphere is invariant.

Let $(A_1, m_1) \cup \ldots \cup (A_k, m_k)$ be a k-material points system on sphere S^n. According to general definition (7) the point (A, m) is defined on the sphere. We will call the point (A, m) "*spherical centroid*" of this system (Fig. 7a). Let us now calculate the spherical centroid of two points $A_1, A_2 \in S^n$ having masses m_1, m_2. Denote $\overrightarrow{OA_1} = \mathbf{z}_1$, $\overrightarrow{OA_2} = \mathbf{z}_2$, $|\mathbf{z}_1| = |\mathbf{z}_2| = 1$. The distance between points A_1 and A_2 is determined by the formula

$$\cos r = (\mathbf{z}_1, \mathbf{z}_2).$$

Calculations similar to those made in the hyperbolic case show that the mass which is located at the point $A \in S^n$ is equal to the length of $m_1 \mathbf{z}_1 + m_2 \mathbf{z}_2$:

$$m = \sqrt{m_1^2 + m_2^2 + 2 m_1 m_2 \cos r}. \tag{16}$$

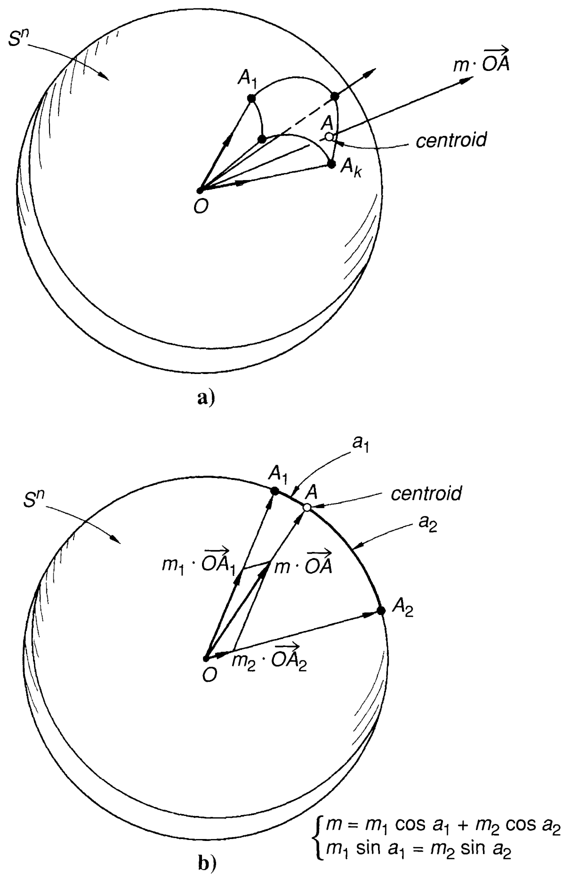

Fig. 7

Let a_1 be the spherical distance between A and A_1, a the spherical distance between A and A_2. Then

$$\cos a_1 = (z_1, (m_1 z_1 + m_2 z_2)/m) = (m_1 + m_2 \cos r)/m\,,$$
$$\cos a_2 = (m_2 + m_1 \cos r)/m\,,$$

and hence

$$\left\{\begin{aligned} m &= m_1(m_1 + m_2 \cos r)/m + m_2(m_2 + m_1 \cos r)/m \\ &= m_1 \cos a_1 + m_2 \cos a_2 \\ m_1 \sin a_1 &= m_2 \sin a_2 \quad \text{(Fig. 7b)}\,. \end{aligned}\right. \tag{17}$$

Formulae (17) give a *spherical rule of lever*; together with formula (16) they prove that the definition of *spherical centroid* is invariant with respect to the action of the group of spherical motions. The mass which is located at the centroid is always less than the sum of the composing points masses and the greater the distance between points A_1 and A_2 the less it is: the value of m is minimum when $r = \pi$ and equals $|m_1 - m_2|$. When the points are close the mass m becomes roughly equal to the sum of the points masses (spherical geometry approaches the Euclidean one). It should be noted that unlike the Euclidean and Lobatchevsky spaces, in the spherical space the points centroid is not always be defined. Namely, it can not be defined for two points if they have equal masses and are antipodal: for more than two points their centroid can not be defined if $m_1\overrightarrow{OA_1} + \ldots + m_k\overrightarrow{OA_k} = 0$. It is more suitable however to consider that (in these cases) centroid is defined but a zero mass is located there; therefore, to point the centroid's geometric position is impossible. To be in analogy with A^n we shall consider S^n as velocities space of some abstract "mechanical" points system in some abstract space **K**: since $A_1, A_2 \in S^n$ are the points with distance a between them, we shall introduce a formal notation for "relative" particles A_1, A_2 velocities in the "space" **K** according to the formula $v = v_{A_1|A_2} = \tan a$. Then the particle's A_2 momentum of pulse with respect to "frame of reference" connected with the "particle" A_1 can be inscribed in a form $p = m \sin a$, and in the centroid's "frame" A is fulfilled an equality $m_1 \sin a_1 = m_2 \sin a_2$, where a_1 and a_2 are the distances from A to A_1 and to A_2 respectively. The particle's A "energy" in the frame of reference connected with the particle A can be inscribed in a form $E = m \cos a$. The formulae for "momentum of pulse" and "energy" by means of the velocity can be inscribed in a form:

$$p = m \sin a = mv/\sqrt{1+v^2}\,, \qquad E = m \cos a = m/\sqrt{1+v^2}\,.$$

When $a \to \pi/2$, then $v \to \infty$; therefore the "velocities" in this " mechanics" are not limited. Is there any physical reality to this model? That is a problem for physicists.

4. The "Axiomatic" Centroid

The notion of centroid can be axiomatized in a natural way. Before giving the necessary axioms we shall introduce some general definitions. The couple $\mathbf{a} = (A, m)$ where $A \in \mathbf{X}$ is geometric point, m is a mass concentrated in the point A shall be called a *material point* in space $\mathbf{X}$. The couple $(A, 0)$ is considered as empty set according to definition. Thus, the set of material points in space X is a *cone*, i.e. the right product $\mathbf{X} \times \mathbf{R}_+$, in which $\mathbf{X} \times \{0\}$ is contracted to one point: $\forall A, B \in \mathbf{X} \Rightarrow (A, 0) = (B, 0)$. We shall denote the material points by small Latin boldface letters.

A set of material points we shall call a *material system* which we shall denote by some Gothic letter: $\mathfrak{A} = \{\mathbf{a}_i\} = \{(A_i, m_i)\}$. In particular, one material point (A, m) is a (one-point) system.

We shall define now the multiplication operation of material points system by the real non-negative numbers: if $\lambda \in \mathbf{R}$ is a real non-negative number, let be according definition:

$$\lambda \cdot \mathbf{a} = \lambda \cdot (A, m) \stackrel{\text{def}}{=} (A, \lambda m)\,, \quad \text{and} \quad \lambda \cdot \mathfrak{A} = \{\lambda \mathbf{a}_i\} = \{(A_i, \lambda m_i)\}\,.$$

Thus when multiplied by λ every material points mass increases λ times although its geometric position does not change.

Let us define the *union* of two material systems. Let $\mathfrak{A} = \{\mathbf{a}_i\} = \{(A_i, m_i)\}$ and $\mathfrak{B} = \{\mathbf{b}_j\} = \{(B_j, m_j)\}$ be two arbitrary systems; generally speaking, the points sets $\{A_i\}$ and $\{B_j\}$ may have common elements and the common elements masses should not coincide necessarily. We set by definition:

$$\mathfrak{A} \cup \mathfrak{B} = \{\mathbf{a}_i\} \cup \{\mathbf{b}_j\} = \{\mathbf{c}_k\} = \{(C_k, m_k)\}\,,$$

where

$$\{C_k\} = \{A_i\} \cup \{B_j\},$$

$$m_k = \begin{cases} m_i, & \text{if point } C_k \text{ coincides with the point } A_i \text{ and } C_k \notin \{B_j\}; \\ m_j, & \text{if point } C_k \text{ coincides with the point } B_j \text{ and } C_k \notin \{A_i\}; \\ m_i + m_j, & \text{if point } C_k \text{ coincides with the coinciding points } A_i, B_j. \end{cases}$$

In other words, the common material points masses in the system $\mathfrak{A}$ and $\mathfrak{B}$ are added. On the set of systems $\{\mathfrak{A}\}$ a natural topology induced by the right product [excepting points in form $(A, 0)$] can be introduced: two material points (A_1, m_1) and (A_2, m_2) are considered as close ones if points $A_1, A_2 \in \mathbf{X}$ are close in the topology of space $\mathbf{X}$ and the numbers m_1 and m_2 close ones in straight lines $\mathbf{R}$ topology (i.e., $|m_1 - m_2|$ is small); all the points of form $(A, 0)$ naturally are considered as close ones. Two systems $\mathfrak{A} = \{\mathbf{a}_i\}$ and $\mathfrak{B} = \{\mathbf{b}_i\}$ with the *same number* of material points in them are considered as close ones if material points $\mathbf{a}_i$ and $\mathbf{b}_i$ they consist of are close. Closeness of systems $\mathfrak{A}$ and $\mathfrak{B}$ in the indicated topology we shall denote by the mark $\sim$.

The *"axiomatic"* centroid of the material points system $\mathfrak{A}$ is a material point $\mathbf{a}$ posed by the special map $\mathbb{U}: \{\mathfrak{A}\} \to \{a\}$ (from all material points systems set to all material points set). The mapping $\mathbb{U}$ satisfies the natural axiom's system as follows:

Axiom 1 (immovability axiom). $\mathbb{U}\{(A, m)\} = (A, m)$: the centroid of a one material point system coincides with this point.

Axiom 2 (induction axiom). $\mathbb{U}(\mathfrak{A} \cup \mathfrak{B}) = \mathbb{U}(\mathbb{U}(\mathfrak{A}) \cup \mathbb{U}(\mathfrak{B}))$: the centroid of sum of two material points systems $\mathfrak{A}$ and $\mathfrak{B}$ coincides with two points system's centroid, where the first point is the systems $\mathfrak{A}$ centroid, and the second is the systems $\mathfrak{B}$ centroid.

Axiom 3 (multiplication axiom). $\mathbb{U}(\lambda \cdot \mathfrak{A}) = \lambda \cdot \mathbb{U}(\mathfrak{A})$: the geometric position of any material points systems $\mathfrak{A}$ centroid coincides with material points systems $\lambda \cdot \mathfrak{A}$ one, every mass of which is increased with respect to the systems $\mathfrak{A}$ points masses the same number λ of times and the mass of which is being located in systems $\mathfrak{A}$ centroid.

Axiom 4 (invariance of centroid with respect to the space's motions). Let $G: \mathbf{X} \to \mathbf{X}$ be the motion group (group of isometrics) of space $\mathbf{X}$. Then for all elements $g \in G$,

$$\mathbb{U} \circ g = g \circ \mathbb{U}:$$

the centroid of a system of material points $\mathfrak{A}$ goes, by the motion $\mathbf{g}$, to the point of $\mathbf{X}$ such that the centroid of $g(\mathfrak{A})$ is situated, i.e. $\mathbb{U}(g(\mathfrak{A})) = g(\mathbb{U}(\mathfrak{A}))$.

Axiom 5 (continuity axiom). If $\mathfrak{A} \sim \mathfrak{B}$, then $\mathbb{U}(\mathfrak{A}) \sim \mathbb{U}(\mathfrak{B})$: close systems have close centroids.

Theorem 1. *In a given space* $\mathbf{X}$ *with constant curvature* ($\mathbf{R}^n, \Lambda^n$ *or* S^n) *a unique mapping* $\mathbb{U}$ *exists, which satisfies Axioms 1–5.*

The proof of existence is, in a fact, given above. Actually, it is evident that the *model* centroid of points system in the space $\mathbf{X}$ satisfies all these axioms. Thus, the nontrivial part of the theorem is the assertion of uniqueness of the mapping $\mathbb{U}$. In other words, an assertion coinciding the system's $\mathfrak{A}$ *axiomatic* centroid with the *model* one. To prove this assertion the next item serves.

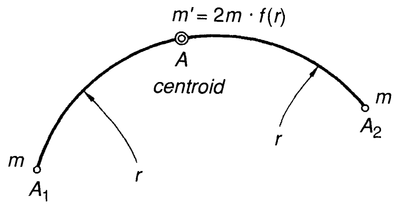

Fig. 8

5. Proof of the Theorem (the Uniqueness of Mapping $\mathbb{U}$)

The proof of the theorem will be fulfilled by a few steps. When proving, unless otherwise stipulated, we consider the "axiomatic" definition of the centroid.

Step 1: Two-point System Equal Masses. We shall consider the system $\mathfrak{A}$ of two points $(A_1, m), (A_2, m)$ with equal masses m, situated in the ends of the segment A_1A_2 with its length $2r$. Let this system's centroid be $\mathbb{U}(\mathfrak{A}) = (A, m')$.

Lemma 1. a) *Point A is the midpoint of the segment A_1A_2.*
b) *The function $f = f(r)$ exists dependent on the distance r only, such that*

$$m' = 2m \cdot f(r) \quad \text{(Fig. 8)}. \tag{19}$$

Proof. a) Denote the midpoint of segment A_1A_2 as B and let us prove that $A = B$. For proving it we shall consider the space $\mathbf{X}$ rotation about the point B such that the points A_1 and A_2 exchange places with each other. In every space $\mathbf{R}^n, \Lambda^n, S^n$ such a rotation exists, and moreover the point B remains fixed: we shall denote such a rotation by $R_B^{180^\circ}$. Then $R_B^{180^\circ}\mathfrak{A} = \mathfrak{A}$. According to Axiom 4,

$$\mathbb{U}(R_B^{180^\circ}\mathfrak{A}) = R_B^{180^\circ}(\mathbb{U}(\mathfrak{A})),$$

hence

$$\mathbb{U}(\mathfrak{A}) = R_B^{180^\circ}(\mathbb{U}(\mathfrak{A})) \Rightarrow A = R_B^{180^\circ}A;$$

therefore, the points A and B coincides and assertion a) is proven.
b) Let us prove the second part of the lemma. Generally speaking, the function $f = m'/m$ depends not only on the distance r but on the segment's A_1A_2 position in the space as well as the mass m. However, Axiom 4 (centroid invariance with respect to $\mathbf{X}$ motions) implies that m' does not depend on the segment's A_1A_2 position and consequently the function f does not either.

We shall consider now two systems: $\mathfrak{A}_1 = (A_1, m_1) \cup (A_2, m_1)$ and $\mathfrak{A}_2 = (A_1, m_2) \cup (A_2, m_2)$ and prove that $f(m_1, r) = f(m_2, r)$.

Actually, $\mathfrak{A}_2 = m_2/m_1\mathfrak{A}_1$, and from multiplication Axiom 3 we have

$$\mathbb{U}(\mathfrak{A}_2) = m_2/m_1 \cdot \mathbb{U}(\mathfrak{A}_1);$$

then $m_2' = (m_2/m_1)m_1'$. Therefore,

$$2m_2 f(m_2, r) = m_2/m_1 \cdot 2m_1 f(m_1, r) \Rightarrow f(m_2, r) = f(m_1, r).$$

Lemma 1 is proved. □

Later on it will be natural to consider the function f as an *even* one: $f(-r) = f(r)$.

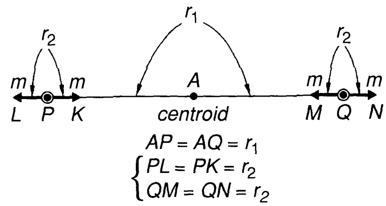

Fig. 9

Step 2. The System of Equations for Function f

Lemma 2. *The following equality holds:*

$$f(r_1)f(r_2) = 1/2(f(r_1 - r_2) + f(r_1 + r_2)) . \tag{20}$$

Proof. Let us consider $r_2 \leq r_1$. Let PQ be a segment with its length $2r_1$ and A its midpoint. Lay aside on the straight line PQ from the both points P and Q the segments PL, PK and QM, QN with their length r_2 (points K and M lie inside the segment PQ). We shall put to the points L, K, M, N the same masses equal to m and obtain the four point system $\mathfrak{A}$. Let us find the centroid $\mathbb{U}(\mathfrak{A})$ two ways (Fig. 9).

We shall use the induction Axiom 2, Lemma 1 and the immovability Axiom 1:

$$\begin{aligned}\mathbb{U}(\mathfrak{A}) &= \mathbb{U}\{\mathbb{U}\{(L,m) \cup (K,m)\} \cup \mathbb{U}\{(M,m) \cup (N,m)\}\} \\ &= \mathbb{U}\{(P, 2mf(r_2)) \cup (Q, 2mf(r_2))\} = (A, 4mf(r_2)f(r_1)) .\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{U}(\mathfrak{A}) &= \mathbb{U}\{\mathbb{U}\{(L,m) \cup (N,m)\} \cup \mathbb{U}\{(K,m) \cup (M,m)\}\} \\ &= \mathbb{U}\{(A, 2mf(r_1 + r_2)) \cup (A, 2mf(r_1 - r_2))\} \\ &= (A, 2m(f(r_1 + r_2) + f(r_1 - r_2))) .\end{aligned}$$

Comparing the right parts of the obtained equalities we get formula (20). Lemma 2 is proved. □

Lemma 3. *If r_1, r_2 are the legs of a right triangle and l is its hypotenuse in the space* **X** *then*

$$f(l) = f(r_1)f(r_2) . \tag{21}$$

Proof. Consider two-dimensional plane π containing a right triangle with its legs r_1, r_2 and hypotenuse l. We shall mark on an arbitrary segment PQ in the plane π with its length $2r_1$ its midpoint A. Then we shall draw in the plane π the straight lines through points P and Q which are perpendicular to the line PQ. Put off the segments with their length r_2 on these lines at both sides from the points P and Q: $PK = PL = r_2$, $QM = QN = r_2$. In the obtained rectangle $\Pi = KLMN$, the diagonals KM and LN intersect at point A and are divided by it in halves (this follows from rectangle Π central symmetry with respect to the point A). The half of the length of each diagonal we shall denote by l. All the right triangles AKP, APL, AMQ, and AQN have the same legs r_1 and r_2, and hypotenuses l (Fig. 10).

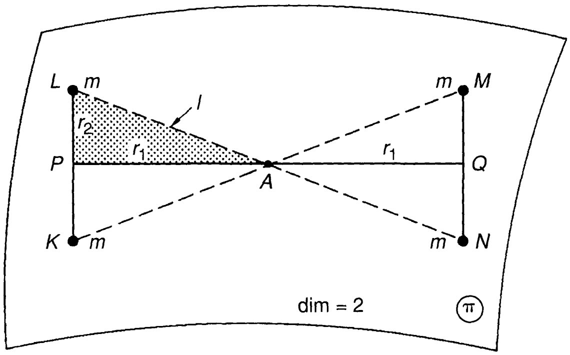

Fig. 10

Let us locate the same masses m at all vertices K, L, M, N of rectangle Π and find the centroid of obtained four point system $\mathfrak{A}$.

Axiom 2 and Lemma 1 imply:

$$\mathbb{U}\{(K,m)\cup(L,m)\}=(P,2mf(r_2)), \qquad \mathbb{U}\{(M,m)\cup(N,m)\}=(Q,2mf(r_2)).$$

Now we can find the system's $\mathfrak{A}$ centroid from Axiom 2:

$$\mathbb{U}(\mathfrak{A})=\mathbb{U}\{P,2mf(r_2))\cup(Q,2mf(r_2))\}=(A,4mf(r_2)f(r_1)).$$

Let us do it another way:

$$\begin{aligned}\mathbb{U}(\mathfrak{A})&=\mathbb{U}\{\mathbb{U}\{(L,m)\cup(N,m)\}\cup\mathbb{U}\{(M,m)\cup(K,m)\}\}\\&=\mathbb{U}\{(A,2mf(l))\cup(A,2mf(l))\}=(A,4mf(l)).\end{aligned}$$

Comparing right parts of the obtained equalities we obtain the assertion of Lemma 3. Lemma 3 is proved. □

Thus, function f satisfies the system of Eq. (20) and (21).

Step 3. Possible Forms of the Function f. Let us find the all possible solutions of Eqs. (20) and (21). As far as we are concerned, the masses will be non-negative, and hence, we have $f(r)\geq 0$ for any $r\geq 0$. Axiom 1 implies $f(0)=1$. Then the continuity Axiom 5 implies that f is *continuous*: a slight displacement of one system points brings a slight centroid change, i.e., the slight change of the mass located in the centroid. Let $r_1=r_2=r$ in formula (20). Then

$$f^2(r)=1/2(f(0)+f(2r))=1/2(1+f(2r)). \tag{22}$$

Continuity of the function f implies the existence of a neighborhood of the point $r=0$ in which $f(r)>0$ for any r. Two cases are possible:

Case 1. There is a point x_0 such that $0<f(x_0)<1$. Then there is $\alpha\in[0,\pi/2]$ such that $f(x_0)=\cos\alpha$. Substituting $r=x_0/2$ in formula (22):

$$f^2(x_0/2)=(1+\cos\alpha)/2=\cos^2\alpha/2.$$

Hence, $f(x_0/2)=\cos\alpha/2$ [as far as $f(x_0/2)>0$]. Similarly

$$f(x_0/4)=\cos\alpha/4, \qquad f(x_0/8)=\cos\alpha/8,\ldots,f(x_0/2^k)=\cos\alpha/2^k,\ldots.$$

Setting $c=\alpha/x_0$, we have that for any $x=x_0/2^k$, $k=1,2,3,\ldots,f(x)=\cos cx$.

Lemma 4. *For any* $x = x_0/2^k$, $k = 1, 2, \ldots$, *the equality*

$$f(nx) = \cos(c \cdot nx)$$

holds for any positive integer n.

Proof. Induction by n. When $n = 1$ the formula is correct. Let it be correct for given positive integer n. We shall set $r_1 = nx$, $r_2 = x$ in formula (20):

$$f(nx)f(x) = 1/2(f((n-1)x) + f((n+1)x)),$$

then

$$f((n+1)x) = 2\cos(c \cdot nx)\cos(cx) - \cos(c \cdot (n-1)x) = \cos(c \cdot (n+1)x). \quad \text{Q.E.D.}$$

Thus, the fact that for any x in the form $x = x_0 \cdot m/2^k$, where m and k are arbitrary positive integers $f(x) = \cos(cx)$, where $c = \alpha/x_0$. If $f(x_0) = 1 = \cos 0$, then $f(x) = 1$ for any value of argument in the form $x = x_0 \cdot m/2^k$. Hence the continuity of function f implies

Assertion 1. In Case 1 for any x, $f(x) = \cos cx$ where $c = \alpha/x_0$. When $\alpha = 0$, $f(x) \equiv 1$.

Case 2. There is a point x_0 such that $f(x_0) > 1$.

Assertion 2. In the Case 2 for any x, $f(x) = \cosh(cx)$ where $c = \alpha/x_0$.
The proof is similar (with substitution of cosh for cos) to the proof of Assertion 1.

Step 4. The connection between the Space **X** *and the Function* f. We shall render concrete the space **X** now and use Lemma 3.

I. **X = R**^{n}. For Euclidean space $l = \sqrt{r_1^2 + r_2^2}$ (Fig. 11a). Lemma 3 implies the equality

$$f(\sqrt{r_1^2 + r_2^2}) = f(r_1)f(r_2). \tag{23}$$

We shall prove that *the solution of Eq.* (23) *is* $f(r) \equiv 1$ *only.*
Indeed, let $f(r) = g(r^2)$, then from (23),

$$g(r_1^2 + r_2^2) = g(r_1^2)g(r_2^2).$$

Setting $x = t_1^2$, $y = r_2^2$ we obtain that for any non-negative x, y,

$$g(x+y) = g(x)g(y). \tag{24}$$

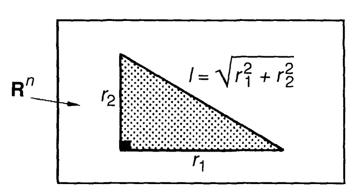

Fig. 11 **a)**

All the solutions of Eq. (24) in the class $C(\mathbf{R})$ have the form $g(x) = e^{\lambda x}$, $\lambda \in \mathbf{R}$, hence

$$f(r) = e^{\lambda r^2} . \tag{25}$$

On the other hand the Assertions 1 and 2 imply that$f(r)$ can have either the form

$$f(r) = \cos(c \cdot r) \tag{26}$$

or the form

$$f(r) = \cosh(c \cdot r) . \tag{27}$$

Comparing function (25) growth rate and one of any of functions (26), (27) it is simple to get $\lambda = c = 0$, hence $f(r) \equiv 1$. Q.E.D.

Thus the fact that *for space* **R** *the centroid of two points with equal masses at the midpoint of the segment connecting them and the mass which is located at the centroid is equal to the points' masses sum* is proved.

II. $\underline{\mathbf{X} = S^n}$. For a right triangle with legs r_1 and r_2 and hypotenuse l in the space S^n the next equality is fulfilled

$$\cos l = \cos r_1 \cdot \cos r_2 \quad \text{(Fig. 11b)} . \tag{28}$$

On the other hand, the formula (21) holds: $f(l) = f(r_1)f(r_2)$. We shall substitute $f(r) = \cos(cr)$ into it:

$$\cos(c \cdot l) = \cos(c \cdot r_1) \cos(c \cdot r_2) . \tag{29}$$

Given the values of l, r_1, and r_2 close to zero we shall expend $\cos(c \cdot l)$, $\cos(c \cdot r)$, and $\cos(c \cdot r_2)$ into Taylor series until the fourth order term,

$$\cos(c \cdot l) = 1/2(cl)^2 - 1/24(cl)^4 + O((cl)^6), \dots$$

(second degree members are not sufficient for proving!) and then substitute these series into (29). With regard to (28), after some calculations, we obtain:

$$c^2 - c^4 = 0 ;$$

hence either $c = 0$ or $c = 1$. If $c = 0$ then $f(r) \equiv 1$; if $c = 1$ then $f(r) = \cos r$ and Eq. (29) is converted into Eq. (28).

In case $f(r) = \cosh(c{\cdot}r)$, substituting this function into (21) we obtain the equation

$$\cosh(c \cdot l) = \cosh(c \cdot r_1) \cos(c \cdot r_2) . \tag{30}$$

After expanding the functions $\cosh(c \cdot l)$, $\cosh(c \cdot r_1)$, $\cosh(c \cdot r_2)$ via Taylor series until the fourth order term, we get the following equation for c:

$$c^2 + c^4 = 0 ;$$

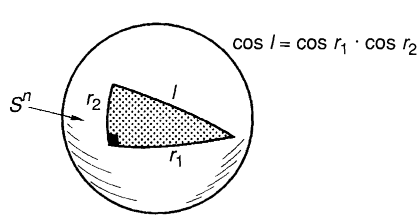

Fig. 11 **b)**

hence, $c = 0$: in this case $f(r) \equiv 1$. We shall prove that the case $f(r) \equiv 1$ appearing twice must be excluded.

The formula (19) for $f \equiv 1$ implies that the mass being located in centroid of two-point system is equal to their sum. Consider, on the sphere, the isosceles triangle ABC ($AB = BC$) with a small base AC and the material system: $\mathfrak{A} = \{(A, m) \cup (C, m) \cup (B, m) \cup (B, m)\}$. Let D be the middle of segment AC, K – the middle of AB, L – the middle of BC, E – the middle of BD, M – the middle of KL. Then with regard to $f \equiv 1$,

$$\mathbb{U}(\mathfrak{A}) = \mathbb{U}\{(D, 2m) \cup (B, 2m)\} = (E, 4m)$$

and, on the other hand,

$$\begin{aligned}\mathbb{U}(\mathfrak{A}) &= \mathbb{U}\{\mathbb{U}\{(A, m) \cup (B, m)\} \cup \mathbb{U}\{(C, m) \cup (B, m)\}\} \\ &= \mathbb{U}\{(K, m) \cup (L, m)\} = (M, 4m)\,.\end{aligned}$$

This implies the coincidence of points E and M, which is false for the *spherical* triangle ABC. Therefore the case $f \equiv 1$ must be excluded. We get that *for spherical space S^n the mass which is located at $\mathbb{U}\{A_1, m), (A_2, m)\}$ is equal to $m' = 2m \cos r$, where r is a half of segment length A_1A_2.*

III. $\underline{\mathbf{X} = \Lambda^n}$. For a right triangle with its legs r_1 and r_2 and hypotenuse l in the Lobatchevsky space Λ^n the next equality is valid:

$$\cosh(l) = \cosh(r_1) \cdot \cosh(r_2) \quad \text{(Fig. 11c)}\,. \tag{31}$$

Again, from formula (21): $f(l) = f(r_1) \cdot f(r_2)$ with regard to the fact that either $f(r) = \cos(c \cdot r)$ or $f(r) = \cosh(c \cdot r)$ or $f(r) \equiv 1$. By the reasoning that has been done in the previous item, we obtain that the solution of Eq. (21) may be only

$$f(r) = \cosh(r)\,.$$

Thus the fact that *in the case of space Λ^n the mass being located at the points (A_1, m) and (A_2, m) centroid is equal to $m' = 2m \cosh(r)$, where r is a half of distance between these points*, is proved.

Step 5. General Case of Two Different Masses. Let us consider, in the space $\mathbf{X}$, the material points (A, m_1) and (B, m_2) forming the system $\mathfrak{A}$. Hence by the formulae of item 3 we can define this system's *model* centroid – the material point $\mathbb{U}_{\text{mod}}(\mathfrak{A}) = (C, m)$ the position C of which is determined by the formulae (1), (2), (13), (14), (17) depending on the space's $\mathbf{X}$ form. We shall prove that by the same formulae the *axiomatic* centroid can be calculated. We shall denote the axiomatic centroid as $\mathbb{U}_{\text{ax}}(\mathfrak{A})$.

Let a_1 and a_2 be the lengths of the segments which the segment AB by the point C is divided to.

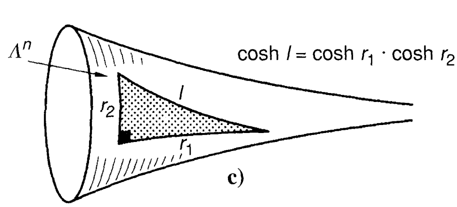

Fig. 11

Assertion 3. Suppose that the segments a_1 and a_2 are dyadically dependent, i.e. $l \cdot a_1 = k \cdot a_2 = lk \cdot a$ (k and l are positive integer), and $k + l = 2^q$. Then

$$\mathbb{U}_{\mathrm{ax}}(\mathfrak{A}) = (C, m).$$

Proof. We shall prove the assertion by induction on the exponent q. For $q = 1$, $a_1 = a_2$ and $m_1 = m_2$, it is established in Step 4 that for such a system $\mathbb{U}_{\mathrm{ax}}(\mathfrak{A}) = (C, m)$.

Let this assertion be correct for the exponent $q - 1 \geq 1$. We shall prove it for the exponent q. Considering $k < l$ we have $a_1 < a_2$ and $m_1 > m_2$.

Let us represent the system $\mathfrak{A} = (A, m_1) \cup (B, m_2)$ in the form $\mathfrak{A} = (A, m_1 - m_2) \cup (A, m_2) \cup (B, m_2)$. Then according to Axioms 2 and 1 we have

$$\mathbb{U}_{\mathrm{ax}}(\mathfrak{A}) = \mathbb{U}_{\mathrm{ax}}\{(A, m_1 - m_2) \cup \mathbb{U}_{\mathrm{ax}}\{(A, m_2) \cup (B, m_2)\}\}.$$

The result obtained in Step 1 implies

$$\mathbb{U}_{\mathrm{ax}}\{(A, m_2) \cup (B, m_2)\} = (D, 2m_2 \cdot f((a_1 + a_2)/2)),$$

where D is the middle of segment AB, f is either the function cosh, or cos or identically 1 (for the spaces Λ^n, S^n, and $\mathbf{R}^n$ correspondingly). Therefore,

$$\mathbb{U}_{\mathrm{ax}}(\mathfrak{A}) = \mathbb{U}_{\mathrm{ax}}\{(A, m_1 - m_2) \cup (D, m')\},$$

where $m' = 2m_2 \cdot f((a_1 + a_2)/2)$. It should be noted that the points' $(A, m_1 - m_2)$ and (D, m') *model* centroid coincides with the initial points (A, m_1) and (B, m_2), i.e., is equal to (C, m). However, the segment AD is divided by the point C to the segments a_1 and $a_2' = a_2 - 1/2(a_1 + a_2) = 1/2(a_2 - a_1)$ for which $a_1 = ka$, $a_2' = l'a$, where

$$k + l' = k + 1/2(l - k) = 1/2(l + k) = 2^{q-1}.$$

Hence the induction's supposition implies:

$$\mathbb{U}_{\mathrm{ax}}\{(A, m_1 - m_2) \cup (D, m')\} = \mathbb{U}_{\mathrm{mod}}\{(A, m_1 - m_2) \cup (D, m')\} = (C, m),$$

hence, $\mathbb{U}_{\mathrm{ax}}(\mathfrak{A}) = (C, m)$, Q.E.D. □

For finishing the proof we use Axiom 5 only which asserts the continuity of mapping $\mathbb{U}_{\mathrm{ax}}$. Thus we obtain that for any value of the relation a_1/a_2, rational or irrational,

$$\mathbb{U}_{\mathrm{ax}}(\mathfrak{A}) = \mathbb{U}_{\mathrm{mod}}(\mathfrak{A}).$$

The Theorem is completely proved. □

6. The Spaces in which Centroid can be Defined

Let us pose a QUESTION: *what are the spaces* **X** *(smooth manifolds) where a centroid satisfying Axioms 1–5 can be defined?*

From this moment on we shall use "centroid" without the "axiomatic" because only such ones will be considered below.

Let **X** be a smooth real manifold of dimension n, G its motion group assumed to be a Lie group (of course, another supposition can be done, in particular we can suppose the group G to be missing; see below). We do not consider **X** to be a metric space as we did it in the previous items. The "motion group" of **X** we consider as a group G operating transitively on $\mathbf{X}$. Among all the manifolds $\mathbf{X}$ we select these in which the centroid of any material points system can be defined. We shall give a description of such manifolds.

Toward this goal we shall extend the definition of a material point introduced in item 4. Besides the points with non-negative masses $m \geq 0$, we shall consider points with negative masses. All the old points and new ones introduced we shall call "*material*" as before. We remember that all the material points with zero-mass are identified with the point $(A, 0)$, where A is an arbitrary point of the manifold $\mathbf{X}$. The operation of scalar multiplication can be extended as well: now the product operation introduced in item 4 can be fulfilled by any real number $\lambda \in \mathbf{R}$ – positive or negative. All the definitions introduced in item 4 remain the same and can be extended for imaginary material points in a natural way.

Now we shall do a simple but important methodological step – in the set of all material points we introduce the *operation of addition* "+": we shall call the two point centroid (which exists and belongs to manifold $\mathbf{X}$ according to our supposition) their *sum*, i.e.,

$$(A_1, m_1) + (A_2, m_2) = (A, m) \stackrel{\text{def}}{=} \mathbb{U}\{(A_1, m_1) \cup (A_2, m_2)\}.$$

Hence the set $\mathbf{E}$ of all material points is converted into linear space over the field of real numbers $\mathbf{R}$: $\mathbf{E}$'s zero is the arbitrary material point with mass $m = 0$ (in other words, the set $\{(A, 0)\}$, where A is any point from manifold $\mathbf{X}$); the negative vector of $\mathbf{a} = (A, m)$ is the vector $-\mathbf{a} = (A, -m)$ because

$$\begin{aligned} \mathbf{a} + (-\mathbf{a}) &= (A, m) + (A, -m) = \mathbb{U}\{(A, m) \cup (A, -m)\} \\ &= \mathbb{U}(A, 0) = (A, 0) = 0 \in E. \end{aligned}$$

All the axioms of linear space can be verified.

More strictly, consider the set $\mathbf{E}$ of all possible differences $(A, m) - (B, m')$, where $m, m' > 0$. On this set we introduce the equivalence relation $\sim$: we shall say that $\mathbf{a} - \mathbf{b} \sim \mathbf{c} - \mathbf{d}$ if $\mathbf{a} + \mathbf{d} = \mathbf{c} + \mathbf{b}$, where the "+" operation is finding the centroid of points with *positive* masses. It should be noted that all differences $(A, 0) - (B, 0)$ can be identified with the points $(C, 0)$ and all differences $(A, m) - (B, 0)$ – with the point (A, m). The addition operation on the equivalence classes is defined as follows:

$$\begin{aligned} &[(A_1, m_1) - (A_2, m_2)] \oplus [(B_1, m_1') - (B_2, m_2')] \\ &\quad = [(A_1, m_1) + (B_1, m_1')] - [(A_2, m_2) + (B_2, m_2')]. \end{aligned}$$

This operation $\oplus$ converts $\mathbf{E}$ into space. As far as manifold $\mathbf{X}$ is n-dimensional, the following assertion holds.

Assertion 4. Dimension of space $\mathbf{E}$ is equal to $n + 1$.

Proof. Let us consider a set $\mathbf{K} = \{(\mathbf{X}, m)\} = \mathbf{X} \times \mathbf{R}/\sim (A, 0)$. This is a cone which has dimension $n+1$, and is a convex cone because for all $\mathbf{a} = (A, m)$ and $\mathbf{b} = (B, m)$ the segment $\lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$ belongs to $\mathbf{K}\,\forall\lambda \in [0, 1]$. Besides, the space $\mathbf{E}$ is generated by all differences $\mathbf{a} - \mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbf{K}$, i.e., $\mathbf{E}$ is generated by the cone $\mathbf{K}$. It implies that as a basis for $\mathbf{E}$ we can choose vectors $\mathbf{e}_1, \ldots, \mathbf{e}_s \in \mathbf{K}$. Hence, the convex hull of these vectors, $\mathbf{Y} = \left\{ \sum\limits_{i=1}^{s} \lambda_i \mathbf{e}_i | \forall \lambda_i \geq 0 \right\}$, belongs to $\mathbf{K}$: $\mathbf{Y} \subset \mathbf{K}$, and on the other hand, $\dim \mathbf{Y} = \dim \mathbf{E} = s > n$. Thus $n + 1 = \dim \mathbf{K} \geq \dim \mathbf{Y} = \dim \mathbf{E} = s > n$. Therefore, $s = n + 1$, i.e., $\dim \mathbf{E} = n + 1$, Q.E.D. □

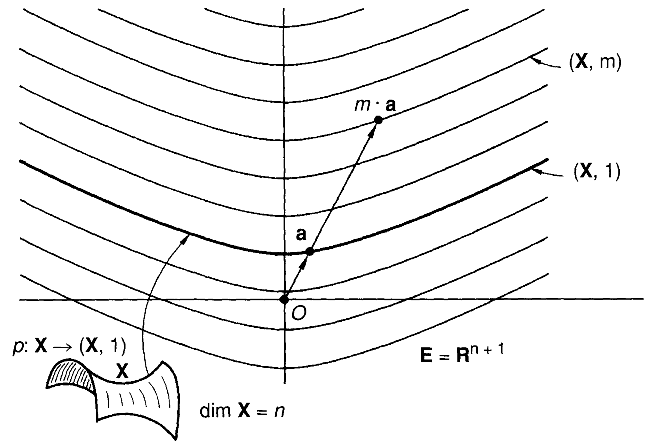

Fig. 12

The set of material points with mass 1 forms in the space $\mathbf{E} = \mathbf{R}^{n+1}$ a smooth n-dimensional manifold identified with the manifold $\mathbf{X}$ under the natural enclosure $p: \mathbf{X} \to \mathbf{E}$ which maps the point $A \in \mathbf{X}$ to the vector $\mathbf{a} = (A, 1) \in \mathbf{E}$. Thus $\mathbf{X}$ is an n-dimensional manifold which is enclosed into the linear space, and we shall denote for obvious reasons $(\mathbf{X}, 1)$. It should be noted that for fixed number $m \in \mathbf{R}$ the manifold $\{(A, m)\}$, where A passes through all points of manifold $\mathbf{X}$, comes out from manifold $(\mathbf{X}, 1)$ by its homothety with the center in origin of $\mathbf{E} = \mathbf{R}^{n+1}$ and the coefficient of homothety equal to m.

Thus the space $\mathbf{E} = \mathbf{R}^{n+1}$ is stratified to manifolds $(\mathbf{X}, m)$ with their codimension 1 where m passes through all real numbers (Fig. 12). Further it would be a natural focus to confine ourselves to non-negative $m \geq 0$ only, that corresponds to the initial material points. This remark enables us to define visualized the centroid of points system on the manifold $\mathbf{X}$.

Let (A_1, m_1), (A_2, m_2) be material points on the manifold $\mathbf{X}$. We shall consider the vectors $\mathbf{d}_1 = (A_1, m_1) \in (\mathbf{X}, m_1)$, $\mathbf{d}_2 = (A_2, m_2) \in (\mathbf{X}, m_2)$ in the space $\mathbf{E}$. Their sum $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$ is the vector corresponding to the material point located in the system $\{(A_1, m_1) \cup (A_2, m_2)\}$ centroid. This vector lies on a certain ray going out from the origin $0 \in \mathbf{E}$ to the end of the vector $\mathbf{d}$ and intersects the manifold $(\mathbf{X}, 1)$ at a certain point $\mathbf{a} = (A, 1)$. Then the vector $\mathbf{d}$ comes out from the vector $\mathbf{a}$ by the multiplication by the number $m = |\mathbf{d}|/|\mathbf{a}|$ (where $|\mathbf{d}|$ and $|\mathbf{a}|$ are the length of the vectors $\mathbf{d}$ and $\mathbf{a}$ in the arbitrary metric in the space $\mathbf{E}$, euclidean for example).

As far as the point $\mathbf{d}$ lies on the manifold $(\mathbf{X}, m)$, it is the number m which is the mass to be located in the centroid of the system $(A_1, m_1) \cup (A_2, m_2)$, i.e., in the point $(A, 1)$ defined above. This construction resulting from Axioms 1–5 explains completely the correctness of the choice of the geometric definition of the model centroid for the spaces $\mathbf{R}^n$, S^n, and Λ^n. However, this construction does not necessarily prove the *uniqueness* of the definition of the centroid in this (and m.b. other) spaces because generally speaking other enclosers of manifolds $\mathbf{R}^n$, S^n, Λ^n into the linear space $\mathbf{R}^{n+1}$ and hence other solutions are a priori possible. We may take as an example the manifold $\mathbf{X} = \mathbf{R}^1$ on which for the same material points system its centroid can be defined by infinitely many ways; the centroid is defined by the enclosure of $\mathbf{X} = \mathbf{R}^1$ into the two-dimensional linear space $\mathbf{E} = \mathbf{R}^2$. For example, if $(\mathbf{X}, 1)$ is the straight line $y = 1$ we obtain the Euclidean centroid; if $(\mathbf{X}, 1)$ is enclosed

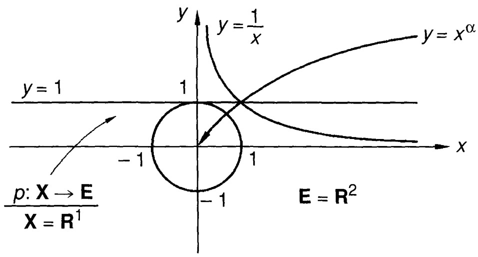

Fig. 13

in $\mathbf{R}$ as one branch of the hyperbola $y = 1/x$, the centroid of the same points system is defined by the formulae (13), (14) relating to one-dimensional Lobatchevsky space. Generally, the straight line $\mathbf{X} = \mathbf{R}^1$ may be enclosed into $\mathbf{R}^2$ as part of graph $y = x^{\alpha}$, $\alpha \in \mathbf{R}$; then the centroid of the point system is defined by the construction described above, and for every $\alpha \in \mathbf{R}$ there is its own centroid (for $\alpha = 0$ it is an Euclidean one, for $\alpha = -1$ it is a hyperbolic one, Fig. 13). However, all geometries of the straight line $\mathbf{R}$ are *isometric*; nevertheless, there are uncountable set of centroids of (one and) the same points system. Starting from dimension $n = 2$ for Euclidean space and Lobatchevsky space the centroid is defined uniquely, as it was proven in item 4.

The motion group G of manifold $\mathbf{X}$ induces the following transformation group of linear space $\mathbf{E}$ which we denote by G as well. Namely, the group G only affects the first component ($\mathbf{X}$) of the space $\mathbf{E} = \{(\mathbf{X}, m)\}$: if $A \in \mathbf{X}$, $g \in G$, $\mathbf{a} = (A, m) \in \mathbf{E}$, then

$$g\mathbf{a} = g(A, m) \stackrel{\text{def}}{=} (gA, m) .$$

It should be noted that the action of the group $G : \mathbf{E} \to \mathbf{E}$ is linear because for $\mathbf{a}_1 = (A_1, m_1)$, $\mathbf{a}_2 = (A_2, m_2)$ we have:

$$\lambda\mathbf{a}_1 + \mu\mathbf{a}_2 = \mathbf{a} = (A, m) ,$$
$$g(\lambda\mathbf{a}_1 + \mu\mathbf{a}_2) = g\mathbf{a} = g(A, m) = (gA, m) ,$$
$$g(\lambda\mathbf{a}_1) = g(A_1, \lambda m_1) = (gA_1, \lambda m_1) , \qquad g(\mu\mathbf{a}_2) = g(A_2, \mu m_2) = (gA_2, \mu m_2) ,$$

and the centroid's invariance with respect to the group G (Axiom 4) implies:

$$\begin{aligned} g(\lambda\mathbf{a}_1) + g(\mu\mathbf{a}_2) &= (gA_1, \lambda m_1) + (gA_2, \mu m) \\ &= \mathbb{U}\{(gA_1, \lambda m_1) \cup (gA_2, \mu m_2)\} = (gA, m) = g\mathbf{a} . \end{aligned}$$

Consequently, the group G is a group of *linear transformation* of the linear space $\mathbf{E}$; therefore, $\mathbf{E}$ is a *space of representations of the group* G. Then the manifold $(\mathbf{X}, 1) \subset \mathbf{E}$ [as well as all the manifolds $(\mathbf{X}, m)$] transfer into itself under the influence of the group G: $G(\mathbf{X}, 1) = (G\mathbf{X}, 1) = (\mathbf{X}, 1)$; moreover, G acts *transitively* on the manifold $(X, 1)$ (i.e., for any two points of the manifold there is a group element transferring the manifold into itself and the first point into the second one).

The manifold $(\mathbf{X}, 1)$ can be constructed in $\mathbf{E}$ from its point $\mathbf{a}$ as follows: an arbitrary vector $(A, 1) \in (\mathbf{X}, 1)$ is taken and all the elements of the group G are applied to it. Then the point $(A, 1)$ is spread in the space $\mathbf{E}$, transforming the surface $(\mathbf{X}, 1)$:

$$G(A, 1) = (\mathbf{X}, 1) .$$

Thus $(\mathbf{X}, 1)$ is an *n-manifold orbit* of the group G in the $(n + 1)$-dimensional linear space $\mathbf{E}$.

As a result we have the following

Problem. *Describe all the Lie groups for which in their certain representation theory there is an orbit with co-dimension* 1.

Another formulation:

Which of the manifolds **X** *with co-dimension* 1 *of the linear space* **E** *transfers into itself under the influence of a certain group of linear transformation?*

This problem is not completely solved yet. The solution for $n = 1$: **X** is one of the following curves on the plain **R**: either a circle or the right part of the graph $y = x^\alpha$, $x > 0$ and $\alpha \neq 0$ or the straight line $y = 1$ or the spiral $r = e^{\text{const} \cdot \varphi}$. The solution for arbitrary $n \geq 1$ is simple when the group G *is compact*. In this case, according to representation theory, G acts in the representation space as an orthogonal transformations subgroup $SO(n)$, thus its orbit is a submanifold of n-dimensional sphere S^n. However, since co-dimension of orbit is equal to 1, then the groups G orbit coincides with S^n. Therefore it is proved:

Theorem 2. *Suppose it is known that for material points system on the manifold* **X** *with the compact motion group G the notion of the centroid satisfying Axioms 1–5 can be introduced. Then* **X** *is a sphere and the centroid is defined by the unique construction described in item* 3. *The centroid can not be defined on any other compact manifold.*

Acknowledgements. I would like to thank Ya. G. Sinai for his attention to this paper and A. M. Stepin for his constructive criticism.

References

1. Dubrovskiy, V.N., Smorodinskiy, Ya.A., Surkov, E.L.: Relativistic Universe. Moscow: Nauka 1984
2. Galperin, G.A.: On the Notion of Centroid of Material Points System in Spaces of Constant Curvature. Rep. Soviet Academy of Sci. **302**, No. 5, 1039–1044 (1988)
3. Sato, M., Kimura, T.: Nagoya Math. J. **65**, 1–155 (1977)

Communicated by J. N. Mather