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Generalized Drinfel'd-Sokolov Hierarchies

II. The Hamiltonian Structures

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Abstract. In this paper we examine the bi-Hamiltonian structure of the generalized KdV-hierarchies. We verify that both Hamiltonian structures take the form of Kirillov brackets on the Kac-Moody algebra, and that they define a coordinated system. Classical extended conformal algebras are obtained from the second Poisson bracket. In particular, we construct the $W_n^{(l)}$ algebras, first discussed for the case $n = 3$ and $l = 2$ by Polyakov and Bershadsky.

1. Introduction

This paper is a continuation of [1], where we generalized the Drinfel'd-Sokolov construction of integrable hierarchies of partial differential equations from Kac-Moody algebras, see [2]. The work of Drinfel'd and Sokolov, itself, constituted a generalization of the original Korteweg-de Vries (KdV) hierarchy, the archetypical integrable system. The main omission from our previous paper was a discussion of the Hamiltonian formalism of these integrable hierarchies, which is the subject of this present paper. A Hamiltonian analysis of these integrable systems allows a much deeper insight into their structure, in particular important algebraic structures are encountered such as the Gel'fand-Dikii algebras [3], or classical W-algebras, which arise as the second Hamiltonian structure of the A_n-hierarchies of Drinfel'd and Sokolov. Though we shall say more about this later, the exemplar of this connexion is found in the original KdV hierarchy whose second Hamiltonian structure is the Virasoro algebra. The new hierarchies of [1] lead amongst other things to the $W_N^{(l)}$-algebras for $1 \leq l \leq N-1$, introduced in [4].

A feature often encountered in the Hamiltonian analysis of integrable hierarchies, is the presence of two *coordinated* Poisson structures which we designate $\{\phi, \psi\}_1$ and $\{\phi, \psi\}_2$. The property of *coordination* implies that the one-parameter family of brackets

$$\{\phi, \psi\} = \{\phi, \psi\}_1 + \mu\{\phi, \psi\}_2 ,$$

μ arbitrary, is also a Poisson structure, which is a non-trivial statement as regards the Jacobi identity. We say a system has a *bi-Hamiltonian structure* if the brackets are coordinated and if the Hamiltonian flow can be written in two equivalent ways

$$\dot{\phi} = \{H_2, \phi\}_1 = \{H_1, \phi\}_2 .$$

Under various general assumptions, the existence of a bi-Hamiltonian structure implies the existence of an infinite hierarchy of flows, that is, an infinite set of Hamiltonians $\{H_i\}$, such that

$$\partial_{t_i} \phi = \{H_{i+1}, \phi\}_1 = \{H_i, \phi\}_2 ,$$

where the Hamiltonians are in involution with respect to both Poisson brackets, whence the flows ∂_{t_i} commute. In the example of the KdV hierarchy, which has as its first non-trivial flow the original KdV equation,

$$\frac{\partial u}{\partial t_1} = -\frac{1}{4} u''' + \frac{3}{2} u u' ,$$

where prime indicates differentiation with respect to x, the two Poisson structures are

$$\begin{aligned} \{u(x), u(y)\}_1 &= 2\delta'(x-y) , \\ \{u(x), u(y)\}_2 &= \tfrac{1}{2}\,\delta'''(x-y) - 2u(x)\,\delta'(x-y) - u'(x)\,\delta(x-y) . \end{aligned} \tag{1.1}$$

One notices that the second structure is nothing but the Virasoro algebra, as was already mentioned. There are hierarchies that do not admit a bi-Hamiltonian structure, for example the *modified* KdV hierarchy (mKdV) (and its generalizations [2, 5]) which, as is well known, is related to the KdV hierarchy by the *Miura Map* [2, 6]. The Miura map takes a solution $\nu(x)$ of the mKdV hierarchy into a solution of the KdV hierarchy by

$$u(x) = -\,\nu'(x) - \nu(x)^2 .$$

This non-invertible mapping is in fact a Hamiltonian map from the single Hamiltonian structure of the mKdV hierarchy to the second Hamiltonian structure of the KdV hierarchy. The existence of a *modified* hierarchy associated to a KdV hierarchy is a feature also encountered in the generalizations of [1, 2]. In fact, the situation is richer than this, since there exists a tower of *partially modified* hierarchies (pmKdV), at the top of which is the KdV hierarchy and at the bottom its associated modified hierarchy [1]. Each of these hierarchies has a Miura transform connecting it with the hierarchies above. We shall show that the KdV hierarchies of [1] admit a bi-Hamiltonian structure, whereas for the partially modified hierarchies we only obtain a single Hamiltonian structure. The Miura map is proved to be Hamiltonian, connecting the pmKdV Hamiltonian structure to the second Hamiltonian structure of the KdV hierarchy.

In order to make this paper reasonably self-contained, we briefly review in Sect. 2 relevant details of [1], highlighting those aspects which are important for the construction of the Hamiltonian structures. Section 3 is the main body of the paper, in which the two Poisson brackets are proposed and skew-symmetry and the Jacobi identity are checked. In fact, the stronger statement of coordination is proved. Section 4 discusses the way in which the hierarchies lead to extended conformal algebras. Section 5 discusses the partially modified KdV hierarchies and their associated Miura mappings, proving in particular that the Miura map is a Hamiltonian mapping. Section 6 is devoted to applying the preceding formalism to a number of examples. In particular, we consider the Drinfel'd-Sokolov KdV hierarchies for the untwisted Kac-Moody algebras, the fractional KdV hierarchy of [7], and various other cases.

2. Review

In this section we summarize certain salient aspects of [1], to which one should refer for further details.

The central object in the construction of the hierarchies is a Kac-Moody algebra $\hat{g}$, realized as the loop algebra $\hat{g} = g \otimes \mathbf{C}[z, z^{-1}] \oplus \mathbf{C}d$, where g is a finite Lie algebra. The derivation d is chosen to induce the *homogeneous* gradation, so that $[d, a \otimes z^n] = na \otimes z^n \;\forall a \in g$. One can define other gradations as follows [8]:

Definition 2.1. A *gradation of type* $\mathbf{s}$, is defined via the derivation $d_{\mathbf{s}}$ which satisfies

$$[d_{\mathbf{s}}, e_i \otimes z^n] = (nN + s_i) e_i \otimes z^n ,$$

where e_i, $i = 1, \ldots, \mathrm{rank}(g)$, are the raising operators associated to the simple roots of g, in some Cartan-Weyl basis of g, $N = \sum_{i=0}^{\mathrm{rank}(g)} k_i s_i$, where k_i are the Kac-Labels of g, and $\mathbf{s} = (s_0, s_1, \ldots, s_{\mathrm{rank}(g)})$ is a vector of $\mathrm{rank}(g) + 1$ non-negative integers.

Each derivation can be expressed in the following way:

$$d_s = N(d + \delta_{\mathbf{s}} \cdot H) , \qquad \delta_{\mathbf{s}} = \frac{1}{N} \sum_{k=1}^{\mathrm{rank}(g)} \left(\frac{2}{\alpha_k^2}\right) s_k \omega_k , \tag{2.1}$$

where α_i are the simple roots of g, H is the Cartan subalgebra of g and the ω_i are the fundamental weights ($\alpha_i \cdot \omega_j = (\alpha_i^2/2)\delta_{ij}$). Observe that the difference $d_s - Nd$ is an element of the Cartan subalgebra of g.

Under a gradation of type $\mathbf{s}$, $\hat{g}$ is a $\mathbf{Z}$-graded algebra:

$$\hat{g} = \bigoplus_{i \in \mathbf{Z}} \hat{g}_i(\mathbf{s}) .$$

The homogeneous gradation corresponds to $\mathbf{s}_{\mathrm{hom}} \equiv (1, 0, \ldots, 0)$.

An important rôle is played by the *Heisenberg subalgebras* of $\hat{g}$, which are maximal nilpotent subalgebras of $\hat{g}$, see [9] for a definition. It is known that, up to conjugation, these are in one-to-one correspondence with the conjugacy classes of the Weyl group of g [9]. We denote these subalgebras as $\mathscr{H}[w]$, where $[w]$ indicates the conjugacy class of the Weyl group of g.

Remark. For an element $\Lambda \in \mathscr{H}[w]$, the Kac-Moody algebra has the decomposition $\hat{g} = \mathrm{Ker}(\mathrm{ad}\,\Lambda) \oplus \mathrm{Im}(\mathrm{ad}\,\Lambda)$. In [1], a distinction was made between hierarchies of type I and type II. This referred to whether the element Λ was *regular*, or not – regularity implying that $\mathrm{Ker}(\mathrm{ad}\,\Lambda) = \mathscr{H}[w]$. In what follows we shall restrict ourselves to the former case.

Remark. Associated to each Heisenberg subalgebra there is a distinguished gradation of type $\mathbf{s}$, which we denote $\mathbf{s}[w]$, with the property that $\mathscr{H}[w]$ is an invariant subspace under $\mathrm{ad}(d_{\mathbf{s}[w]})$ [1].

One can introduce the notion of a partial ordering on the set of gradations of type s. We say $\mathbf{s} \succeq \mathbf{s}'$ if $s_i \neq 0$ whenever $s'_i \neq 0$.

Lemma 2.1. [1]. *An important property of this partial ordering is that if* $\mathbf{s} \succeq \mathbf{s}'$ *then the following is true*

(i) $\hat{g}_0(\mathbf{s}) \subseteq \hat{g}_0(\mathbf{s}')$,
(ii) $\hat{g}_j(\mathbf{s}) \subset \hat{g}_{\geq 0}(\mathbf{s}')$ *or* $\hat{g}_{\leq 0}(\mathbf{s}')$, *depending on whether* $j > 0$ *or* $j < 0$, *respectively,*
(iii) $\hat{g}_j(\mathbf{s}') \subset \hat{g}_{>0}(\mathbf{s})$ *or* $\hat{g}_{<0}(\mathbf{s})$, *depending on whether* $j > 0$ *or* $j < 0$, *respectively.*

In the above, we have used the notation $\hat{g}_{>a}(\mathbf{s}) = \bigoplus_{i>a} \hat{g}_i(\mathbf{s})$ and so on, to indicate subspaces of $\hat{g}$.

The construction of the hierarchies relies on the matrix Lax equation. First of all, associated to the data $(\Lambda, \mathbf{s}, [w])$ one defines the object

$$L = \partial_x + q + \Lambda, \tag{2.2}$$

where Λ is a constant element of $\mathcal{H}[w]$ with well defined positive $\mathbf{s}[w]$-grade i. By constant we mean $\partial_x \Lambda = 0$. The fact that it is possible to choose Λ to have a well defined $\mathbf{s}[w]$-grade follows from the second remark above. The *potential* q is defined to be an element of $C^\infty(\mathbf{R}/\mathbf{Z}, Q)$, where Q is the following subspace of $\hat{g}$:

$$Q = \hat{g}_{\geq 0}(\mathbf{s}) \bigcap \hat{g}_{<i}(\mathbf{s}[w]), \tag{2.3}$$

where $\mathbf{s}$ is any other gradation such that $\mathbf{s} \preceq \mathbf{s}[w]$. The potentials are taken to be periodic functions, so as to avoid technical complications [2].

In this paper our interest is principally in the KdV-type hierarchies, for which the gradation s is the homogeneous gradation. For these systems the analysis of Drinfel'd and Sokolov in [2] generalizes, leading to a bi-Hamiltonian structure. Thus, for brevity we introduce the following notation – superscripts will denote $\mathbf{s}[w]$-grades, so that $\hat{g}^j \equiv \hat{g}_j(\mathbf{s}[w])$, and subscripts will indicate homogeneous grade.

The function $q(x)$ plays the rôle of the phase space coordinate in this system. However, there exist symmetries in the system corresponding to the gauge transformation

$$L \to SLS^{-1}, \tag{2.4}$$

with S being generated by x dependent functions on the subalgebra $P \subset \hat{g}$, where

$$P = \hat{g}_0(\mathbf{s}) \bigcap \hat{g}_{<0}(\mathbf{s}[w]). \tag{2.5}$$

The *phase space* of the system $\mathcal{M}$ is the set of gauge equivalence classes of operators of the form $L = \partial_x + q + \Lambda$. The space of functions $\mathcal{F}$ on $\mathcal{M}$ is the set of gauge invariant functionals of q of the form

$$\varphi[q] = \int_{\mathbf{R}/\mathbf{Z}} dx\, f(x, q(x), q'(x), \ldots, q^{(n)}(x), \ldots).$$

It is straightforward to find a basis for $\mathcal{F}$, the gauge invariant functionals. One simply performs a non-singular gauge transformation to take q to some canonical form q^{can}. The components of q^{can} and their derivatives then provide the desired basis. For instance, for the generalized A_n-KdV hierarchies of Drinfel'd and Sokolov, q consists of lower triangular $n+1$ by $n+1$ dimensional matrices, while the gauge group is generated by strictly lower triangular matrices. A good gauge slice, and the choice made in [2], consists of matrices of the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ u_1 & u_2 & \dots & u_n & 0 \end{pmatrix} . \tag{2.6}$$

The u_i's and their derivatives provide a basis for $\mathscr{F}$.

The outcome of applying the procedure of Drinfel'd and Sokolov to (2.2) is that there exists an infinite number of commuting flows on the gauge equivalence classes of L. These flows have the following form. For each element of the Centre of Ker(ad Λ) with positive $\mathbf{s}[w]$-grade, which we denote by K, there are two gauge equivalent ways of writing the flows:

$$\frac{\partial L}{\partial t_b} = [A(b)^{\geq 0}, L] , \qquad \frac{\partial L}{\partial t'_b} = [A(b)_{\geq 0}, L] ,$$

where the superscript and subscript ≥ 0 refer to projections onto non-negative components in $\mathbf{s}[w]$-grade and $\mathbf{s}$-grade respectively. In the case where Λ is regular, $K = \mathscr{H}[w]^{>0}$ and so we can construct a flow for each element of the Heisenberg algebra with positive grade (the type II hierarchies require a somewhat different treatment). The generator $A(b)$ is constructed from the Heisenberg algebra via the transformation $A(b) = \Phi^{-1} b \Phi$, where $b \in K$, and $\Phi = 1 + \sum_{j<0} \Phi^j$, $\Phi^j \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathrm{Im}(\mathrm{ad}\,\Lambda) \cap \hat{g}^j)$, is the unique transformation which takes L to

$$\mathscr{L} = \Phi L \Phi^{-1} = \partial_x + \Lambda + \sum_{j<i} h^j , \tag{2.7}$$

where $h^j \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathscr{H}[w])$ with $\mathbf{s}[w]$-grade j. The equations of motion take the following form in the coordinates q^{can}:

$$\frac{\partial L^{\mathrm{can}}}{\partial t_b} = [A(b)_{\geq 0} + \theta_b, L^{\mathrm{can}}] ,$$

where $L^{\mathrm{can}} = L(q^{\mathrm{can}})$, and $\theta_b \in C^\infty(\mathbf{R}/\mathbf{Z}, P)$ is the generator of an infinitesimal gauge transformation which compensates for the fact that a flow will generically take q out of the gauge slice.

The quantities h^j are the conserved densities for the flows, that is, there exist quantities a^j such that

$$\partial_t h^j + \partial_x a^j = 0 .$$

These conserved densities are, in fact, the Hamiltonian densities for the hierarchies. In [1] it was shown that $a^j =$ constant for $j \geq 0 (j < i)$, and therefore the quantities h^j for $i > j \geq 0$ are constant under all flows in the hierarchy. This is an important observation to which we return in Sect. 3.6.

3. The Hamiltonian Structures

In this section we explicitly construct the two coordinated Hamiltonian structures of the KdV-type hierarchies (defined by the requirement that $\mathbf{s}$ is the homogeneous gradation). The first Hamiltonian structure is a direct generalization of the first Hamiltonian structure of the KdV hierarchy, while the second involves a classical r-matrix. Our approach follows that of Drinfel'd and Sokolov [2].

3.1. Preliminaries

For each $b \in \mathscr{H}[w]^{>0}$, there are four ways to write the flow:

$$\frac{\partial L}{\partial t_b} = [A(b)^{\geq 0}, L] = -[A(b)^{<0}, L], \tag{3.1}$$

$$\frac{\partial L}{\partial t'_b} = [A(b)_{\geq 0}, L] = -[A(b)_{<0}, L], \tag{3.2}$$

where, as before, $A(b) = \Phi^{-1} b \Phi$. The flows defined by (3.1) and (3.2) only differ by a gauge transformation. Indeed, by applying Lemma 2.1 we have

$$A(b)^{<0} = A(b)_{<0} + A(b)_0^{<0},$$
$$A(b)_{\geq 0} = A(b)^{\geq 0} + A(b)_0^{<0},$$

and so the flows are related by the infinitesimal gauge transformation generated by $A(b)_0^{<0} \in C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}_0 \cap \hat{g}^{<0})$:

$$\frac{\partial L}{\partial t_b} = \frac{\partial L}{\partial t'_b} - [A(b)_0^{<0}, L].$$

The flows along t_b and t'_b are, of course, identical on the phase space $\mathscr{M}$.

There is a natural inner product on the functions $C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g})$, defined as follows:

$$(A, B) = \int_{\mathbf{R}/\mathbf{Z}} dx \langle A(x), B(x) \rangle_{\hat{g}},$$

where $\langle\, , \rangle_{\hat{g}}$ is the Killing form of $\hat{g}$. Explicitly

$$\langle a \otimes z^n, b \otimes z^m \rangle_{\hat{g}} = \langle a, b \rangle_g \delta_{n+m,0},$$

where $\langle\, , \rangle_g$ is the Killing form of g. With respect to an arbitrary gradation we can express the inner product in terms of the (suitably normalized) Killing form of the finite Lie algebra $\hat{g}_0(\mathbf{s})$:

$$(A, B) = \sum_{k \in \mathbf{Z}} \int dx \langle A_k(x), B_{-k}(x) \rangle_{\hat{g}_0(\mathbf{s})},$$

where A_k and B_k are the components of A and B of grade k in the $\mathbf{s}$-gradation, and $\langle\, , \rangle_{\hat{g}_0(\mathbf{s})}$ is the Killing form of the finite Lie algebra $\hat{g}_0(\mathbf{s})$. The inner product does not depend on the particular gradation chosen, as long as the Killing forms of the finite algebras are suitably normalized.

The first stage of the programme is to rewrite (3.1) and (3.2) in Hamiltonian form. In order to accomplish this, components of $A(b)$ have to be related to the Hamiltonians of the flows, which are in turn constructed from the conserved densities h^j.

Definition 3.1. For a constant element $b \in \mathscr{H}[w]^{>0}$, we define the following functional of q:

$$H_b[q] = (b, h(q)),$$

where $h(q) = \sum_{j<i} h^j$ is the sum of the conserved densities of (2.7).

Next we introduce the functional derivatives of functionals of q.

Definition 3.2. For a functional φ of q we define its functional derivative $d_q\varphi \in C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}_{\leq 0})$ via

$$\frac{d}{d\varepsilon}\varphi[q+\varepsilon r]\bigg|_{\varepsilon=0} \equiv (d_q\varphi, r),$$

for all $r \in C^\infty(\mathbf{R}/\mathbf{Z}, Q)$.

Observe that the functional derivative $d_q\varphi$ is valued in the subalgebra $\hat{g}_{\leq 0}$. This is connected to the choice of the space $Q = \hat{g}_{\geq 0} \cap \hat{g}^{<i}$, and is explained by a group theoretic formulation of the generalized KdV hierarchy, which will be discussed in another publication.

Since $r \in C^\infty(\mathbf{R}/\mathbf{Z}, Q)$, there is an ambiguity in the definition of the functional derivative $d_q\varphi$ corresponding to the fact that terms in the annihilator of Q are not fixed by the definition. Thus $d_q\varphi$ is defined up to terms in $\hat{g}^{\leq -i}$, the annihilator of Q in $\hat{g}_{\leq 0}$. In fact we can interpret the functional derivative as taking values in the quotient algebra $\hat{g}_{\leq 0}/\hat{g}^{\leq -i}$. Part of the analysis of the Poisson structure in later sections involves proving that the Poisson brackets are well defined given that

$$d_q\varphi \in C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}_{\leq 0}/\hat{g}^{\leq -i}),$$

i.e. that terms in $\hat{g}^{\leq -i}$ do not contribute. The fact that the second Poisson structure is well defined is linked to gauge invariance.

The definition of the functional derivative, Definition 3.2, is related to the familiar notion of functional derivative in the following way. If we introduce some basis $\{e_\alpha\}$ for $\hat{g}$, with dual basis $\{e_\alpha^*\} \in \hat{g}$ under the inner product, then if $q = \sum q_\alpha e_\alpha$ the derivative

$$d_q\varphi = \sum_\alpha \frac{\delta\varphi}{\delta q_\alpha} e_\alpha^* \bmod C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}^{\leq -i}),$$

where $\delta\varphi/\delta q_\alpha$ is the conventional definition of a functional derivative.

Now we present two central theorems.

Theorem 3.1. *The functional derivative of $H_b[q]$ is:*

$$d_q H_b = A(b)_{\leq 0} \bmod C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}^{\leq -i}).$$

Proof. Consider Definition 3.2 of the functional derivative:

$$\frac{d}{d\varepsilon} H_b[q+\varepsilon r] = \frac{d}{d\varepsilon}(b, h(q+\varepsilon r)) = \left(b, \frac{d}{d\varepsilon}\mathscr{L}(\varepsilon)\right), \tag{3.3}$$

where $\mathscr{L}(\varepsilon) = \mathscr{L}(q+\varepsilon r)$, using (2.7). Now we use the relation $\mathscr{L}(\varepsilon) = \Phi(\varepsilon)L(\varepsilon)\Phi^{-1}(\varepsilon)$ to evaluate

$$\frac{d}{d\varepsilon}\mathscr{L}(\varepsilon) = \Phi(\varepsilon)r\Phi^{-1}(\varepsilon) + \left[\frac{d\Phi(\varepsilon)}{d\varepsilon}\Phi^{-1}(\varepsilon), \mathscr{L}(\varepsilon)\right]. \tag{3.4}$$

Substituting this into (3.3), and using the identity

$$(A,[B,C]) = -(B,[A,C]),$$

along with the fact that $[\mathscr{L},b]=0$, we have

$$\frac{d}{d\varepsilon}H_b[q+\varepsilon r]\bigg|_{\varepsilon=0} = (b,\Phi r\Phi^{-1}) = (\Phi^{-1}b\Phi,r),$$

So finally

$$d_qH_b[q] = (\Phi^{-1}b\Phi)_{\leq 0} \bmod C^\infty(\mathbf{R}/\mathbf{Z},\hat{g}^{\leq -i}),$$

as claimed.

Lemma 3.1. *The quantities $A(b)$ occurring in the time evolution equations* (3.1), (3.2) *possess the following symmetry:*

$$A(zb)^{j+N}_{k+1} = zA^j_k(b).$$

Proof. $A(zb)=\Phi^{-1}(zb)\Phi = zA(b)$, and since z carries homogeneous grade 1 and $\mathbf{s}[w]$-grade N, the result follows trivially.

Remark. A special case is the relation $A(zb)_{\leq 0} = zA_{<0}(b)$. The fact that this relation only holds for the homogeneous gradation is ultimately the reason why the KdV hierarchies, for which $\mathbf{s}=\mathbf{s}_{\text{hom}}$, admit two Hamiltonian structures, whereas the partially modified KdV hierarchies only exhibit a single Hamiltonian structure.

3.2. *The First Hamiltonian Structure*

The First Hamiltonian Structure is derived by considering the equation for the flow in the form

$$\frac{\partial L}{\partial t_b} = -[A(b)_{<0},L].$$

Recall that q has $\mathbf{s}[w]$-grade in the range $-N+1$ to $i-1$. Since the maximum $\mathbf{s}[w]$-grade of L is i, it is easy to see that the terms that are needed to express the flow are only those components of $A(b)_{<0}$ with $\mathbf{s}[w]$-grade from $-N+1-i$ to -1. From Theorem 3.1 we have the relation $d_qH_{zb}=A(zb)_{\leq 0}$, and so using Lemma 3.1 we may re-express these quantities in terms of $A(b)_{<0}$:

$$z\sum_{k=-N+1-i}^{-1} A(b)^k_{<0} = d_qH_{zb} \bmod C^\infty(\mathbf{R}/\mathbf{Z},\hat{g}^{\leq -i}).$$

Therefore, the flow can be written as

$$\frac{\partial L}{\partial t_b} = -\left[\frac{1}{z}\,d_qH_{zb},L\right]_{\geq 0}, \tag{3.5}$$

where the restriction to homogeneous grade ≥ 0 is crucial and ensures that the right-hand side is contained in $C^\infty(\mathbf{R}/\mathbf{Z},Q)$, as required. Notice that the contribution from terms in the functional derivative of grade less than $1-i$ cannot contribute to (3.5) because of the projection.

So for a functional φ of q:

$$\frac{\partial \varphi}{\partial t_b} = \left(d_q\varphi, \frac{\partial q}{\partial t_b}\right) = -(d_q\varphi, z^{-1}[d_q H_{zb}, L]_{>0}).$$

Written in this form, the restriction to positive homogeneous grade is redundant, being automatically ensured because $d_q\varphi$ has strictly non-positive homogeneous grade.

The candidate Poisson bracket for the First Hamiltonian structure is thus

$$\{\varphi, \psi\}_1 = (d_q\varphi, z^{-1}[d_q\psi, L]), \tag{3.6}$$

for two functionals of q. Of course, we must check that (3.6) is a well defined Poisson bracket, and we must also consider the rôle of gauge invariance. This we shall do in Sects. 3.4 and 3.5.

3.3. The Second Hamiltonian Structure

The Second Hamiltonian Structure results from considering the flow written in the form

$$\frac{\partial L}{\partial t_b} = [A(b)_{\geq 0}, L].$$

Firstly, we split $A(b)_{\geq 0}$ into the terms of zero and positive homogeneous grade:

$$A(b)_{\geq 0} = A(b)_0 + A(b)_{>0}. \tag{3.7}$$

The terms of zero grade, $A(b)_0$, can have $\mathbf{s}[w]$-grade between $-N+1$ and $N-1$. The functional derivative $(d_q H_b)_0$ gives the component of $A(b)_0$ with $\mathbf{s}[w]$-grade between the greater of $-i+1$ or $-N+1$, and $N-1$, i.e.

$$A(b)_0 = (d_q H_b)_0 + \Psi,$$

where Ψ represents the sum of terms of homogeneous grade zero and $\mathbf{s}[w]$-grade less than $1-i$, which is zero if $i \geq N$. The terms of positive homogeneous grade in (3.7), can be re-expressed using Lemma 3.1:

$$A(b)_{>0} = zA(z^{-1}b)_{\geq 0}.$$

Collecting these results, we have

$$\frac{\partial L}{\partial t_b} = [(d_q H_b)_0, L] + z[A(z^{-1}b)_{\geq 0}, L] + [\Psi, L]. \tag{3.8}$$

We can ignore the term involving Ψ since this is just a gauge transformation. Then we notice that the second term in (3.8) is equal to $z\partial L/\partial t_{z^{-1}b}$, which we can express in terms of functional derivatives of H_b using the first Hamiltonian structure (3.5). So (3.8) becomes

$$\frac{\partial L}{\partial t_b} = [(d_q H_b)_0, L] - z[z^{-1}d_q H_b, L]_{\geq 0}. \tag{3.9}$$

This can be written slightly differently by using $z[z^{-1}a, b]_{\geq 0} = [a, b]_{>0}$.

The candidate Poisson bracket on functionals of q is thus

$$\{\varphi, \psi\}_2 = (d_q\varphi, [d_q\psi_0, L] - [d_q\psi, L]_{>0}). \tag{3.10}$$

The above expression can be rewritten in a form that is more suitable for our later discussions:

$$\{\varphi, \psi\}_2 = (d_q\varphi_0, [d_q\psi_0, L]) - (d_q\varphi_{<0}, [d_q\psi_{<0}, L])\,, \tag{3.11}$$

where we have used the fact that $d_q\varphi = d_q\varphi_0 + d_q\varphi_{<0}$, and that the inner product matches terms of opposite grade. In the following sections we discuss the rôle of gauge symmetry, and whether these brackets define a symplectic structure.

3.4. Gauge Invariance

It has already been mentioned in Sect. 2 that the hierarchies exhibit a gauge symmetry. More specifically the form of L is preserved under the transformation

$$L \mapsto SLS^{-1}\,, \tag{3.12}$$

or equivalently

$$q \mapsto \tilde{q} = S(q + \Lambda)S^{-1} - \Lambda + S\partial_x S^{-1}\,, \tag{3.13}$$

where S is an x-dependent element of the group generated by the subalgebra $P = \hat{g}_0 \cap \hat{g}^{<0}$. For the KdV-type hierarchies considered in this section, P is of maximum dimension for a given conjugacy class $[w]$. As discussed in [1], the flow equations of the hierarchy should be understood as equations on the gauge equivalence classes of L under (3.12). For the generalized KdV hierarchy, we have proposed two Hamiltonian structures. The fact that the hierarchies define dynamics on gauge equivalence classes implies that these Hamiltonian structures should respect the gauge symmetry, i.e. that the Poisson structure is well defined on gauge invariant functionals. However, before proceeding with the discussion of gauge invariance, it is necessary to prove that the brackets are actually well defined as functionals of q, i.e. that the ambiguity of the functional derivatives $d_q\varphi$ and $d_q\psi$, consisting of terms in $C^\infty(\mathbf{R}/\mathbf{Z}, g^{\leq -i})$, do not contribute to the bracket. This is obtained as a corollary of the following lemma.

Lemma 3.2. *For* $\Psi \in C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}^{\leq -i})$*:*
(i) $(\Psi, z^{-1}[d_q\varphi, L]) = 0\,,$
(ii) $(\Psi_{<0}, [d_q\varphi_{<0}, L]) = 0\,,$
(iii) $(\Psi_0, [d_q\varphi_0, L]) = 0\,.$

Proof. The maximum $\mathbf{s}[w]$-grade of L is i, and that of $d_q\varphi$ is $N-1$, therefore the maximum $\mathbf{s}[w]$-grade of $z^{-1}[d_q\varphi, L]$ is $i-1$. Since Ψ only has $\mathbf{s}[w]$-grade less than or equal to $-i$, the first part of the lemma follows. To prove the second part of the lemma we notice that the maximum $\mathbf{s}[w]$-grade of $d_q\varphi_{<0}$ is -1, by Lemma 2.1. Therefore the maximum grade of the second term in the right side of the inner product is $i-1$, showing that the expression vanishes. To show that the third expression vanishes is more subtle and relies on the fact that the brackets are properly defined on gauge invariant functionals, a point that we shall come to shortly. Notice, first of all, that the projection Ψ_0 is the generator of an infinitesimal gauge transformation. The variation of q under this transformation is $\delta_\varepsilon q = \varepsilon[\Psi_0, L]$. Since $\varphi[q]$ is gauge invariant we have

$$0 = \frac{d}{d\varepsilon}\,\varphi[q + \delta_\varepsilon q]\bigg|_{\varepsilon=0} = (d_q\varphi, [\Psi_0, L])\,. \tag{3.14}$$

But Ψ_0 has $\mathbf{s}[w]$-grade $\leq -i$, therefore $[\Psi_0, L]$ has $\mathbf{s}[w]$-grade less than or equal to zero, so the only term which can contribute to the right-hand side of (3.14) is

$$(d_q\varphi_0, [\Psi_0, L]) = -(\Psi_0, [d_q\varphi_0, L]),$$

which follows from the invariance of the Killing form. But this is zero by (3.14), and so the lemma is proved.

We now establish how the functional derivatives transform under gauge transformations.

Lemma 3.3. *Under the gauge transformation* (3.3), *the functional derivative of a gauge invariant functional* φ *transforms as:*

$$d_q\varphi \mapsto Sd_q\varphi S^{-1}.$$

Proof. Consider the definition of the functional derivative

$$\frac{d}{d\varepsilon}\varphi[q+\varepsilon r]\bigg|_{\varepsilon=0} = (d_q\varphi, r),$$

for $r \in Q$. Since $\varphi[q]$ is a gauge invariant functional, we perform the gauge transformation $\tilde{q}+\varepsilon r \mapsto q + \varepsilon S^{-1}rS$ and obtain

$$\begin{aligned}\frac{d}{d\varepsilon}\varphi[\tilde{q}+\varepsilon r]\bigg|_{\varepsilon=0} \equiv \frac{d}{d\varepsilon}\varphi[q+\varepsilon S^{-1}rS]\bigg|_{\varepsilon=0} &= (d_q\varphi, S^{-1}rS) \\ &= (Sd_q\varphi S^{-1}, r),\end{aligned}$$

using the ad-invariance of the inner product. Therefore, from the definition of the functional derivative we have

$$d_{\tilde{q}}\varphi = Sd_q\varphi S^{-1}.$$

Remember that the functional derivatives are only defined modulo terms of $\mathbf{s}[w]$-grade less than $1-i$, and it is in this sense that the equality holds.

Proposition 3.12. *The Poisson brackets* (3.6) *and* (3.11) *of two gauge invariant functionals of q are gauge invariant functionals of q.*

Proof. For the first Poisson bracket, (3.6), the transformed bracket is

$$(d_{\tilde{q}}\varphi, z^{-1}[d_{\tilde{q}}\psi, \tilde{L}]) = (Sd_q\varphi S^{-1}, z^{-1}[Sd_q\psi S^{-1}, SLS^{-1}]) = (d_q\varphi, z^{-1}[d_q\psi, L]),$$

where the last manipulation follows from the ad-invariance of the inner product. The proof for the second Poisson bracket proceeds in the same spirit, although in this case it also depends on the fact that S has zero homogeneous grade.

3.5. The Jacobi Identity

In this section we verify that both (3.6) and (3.11) define Poisson brackets. This entails checking that the brackets are skew symmetric and that the Jacobi identity is satisfied. In fact, we shall prove the stronger statement that they are coordinated.

In order to demonstrate that the Jacobi identity is satisfied, we first make the following digression. Consider a Lie algebra g, with a Lie bracket denoted $[\,,]$. Suppose we have an endomorphism $R \in \operatorname{End} g$, then we can define a new bracket operation

$$[x,y]_R = [Rx,y] + [x,Ry]\,, \tag{3.15}$$

$\forall x, y \in g$, see [10]. If the Jacobi identity is satisfied in $[\,,]_R$ then there exists a new Lie algebra structure on the underlying vector space of g, denoted g_R. The Jacobi identity translates into the following condition on R:

$$[Rx,Ry] - R([Rx,y] + [x,Ry]) = \lambda[x,y]\,, \tag{3.16}$$

for some proportionality constant λ. This equation is known as the *modified Yang-Baxter Equation* (mYBE) [10].

The simplest example of this procedure is when g has the vector space decomposition $g = a + b$, where a and b are subalgebras of g. If P_a and P_b are the projectors onto these subalgebras, then we can define the new Lie algebra g_R via $R = (P_a - P_b)/2$. In this case

$$[x,y]_R = [P_a x, P_a y] - [P_b x, P_b y]\,,$$

which implies $g_R \cong a \oplus b$, the mYBE then being satisfied with $\lambda = -\frac{1}{4}$.

Applying this formalism to our situation, we consider the vector-space decomposition $\hat{g}_{\leq 0} = \hat{g}_0 + \hat{g}_{<0}$, into the subalgebras $\hat{g}_0$ and $\hat{g}_{<0}$ of the Lie algebra $\hat{g}_{\leq 0}$. The importance of this decomposition is that the second Hamiltonian structure may be succinctly rewritten as

$$\{\varphi,\psi\}_2 = (q + \Lambda, [d_q\varphi, d_q\psi]_R) - (d_q\varphi, (d_q\psi)')\,,$$

where $R = (P_0 - P_{<0})/2$, half the difference of the projector onto the subspace of zero homogeneous grade and the projector onto the subspace of strictly negative homogeneous grade. Notice that only the terms of zero homogeneous grade, $d_q\varphi_0$ and $d_q\psi_0$ contribute to the last term.

In fact, we may combine the first and second Poisson brackets into one elegant expression using the following lemma.

Lemma 3.4. *The one-parameter family of endomorphisms of the Lie algebra $\hat{g}_{\leq 0}$ defined by*

$$R_\mu = R - \mu \cdot \frac{1}{z}\,,$$

where $R = (P_0 - P_{<0})/2$ and $\mu \in \mathbf{C}$, satisfies the modified Yang-Baxter Equation.

Proof. It is useful to define $\sigma = -\mu/z$, with the property that $\sigma : \hat{g}_{\leq 0} \to \hat{g}_{<0}$. In order to prove the lemma we must demonstrate that R_μ satisfies the mYBE. The left-hand side of (3.16) is equal to

$$\begin{aligned}&[(R+\sigma)x,(R+\sigma)y] - (R+\sigma)([(R+\sigma)x,y] + [x,(R+\sigma)y])\\ &\quad= [Rx,Ry] - R([Rx,y] + [x,Ry]) - \sigma^2([x,y]) - 2R\sigma([x,y])\,,\end{aligned}$$

where we used the fact that $[\sigma(x), y] = \sigma[(x,y])$. Now, $-2R$ acts as the identity on $\sigma([x,y])$, and hence the above expression is equal to

$$-\left(\frac{1}{2} - \frac{\mu}{z}\right)^2 [x,y]\,,$$

verifying that the mYBE is satisfied by R_μ.

Since the endomorphism R_μ satisfies the mYBE, there exists a Poisson bracket on the dual space given by the Kirillov bracket construction [11]. Up to a term involving a derivative, which can be interpreted as a central extension of $\hat{g}_0$ and causes no problem in the proof of the Jacobi identity, this is the previously constructed bi-Hamiltonian structure of the space of gauge invariant functions on Q, Eqs. (3.6), (3.11). We summarize the Hamiltonian structure in the form of a theorem.

Theorem 3.2. *There is a one parameter family of Hamiltonian structures on the gauge equivalence classes of the generalized* KdV *hierarchy given by*

$$\{\varphi,\psi\}_\mu = (q+\Lambda, [d_q\varphi, d_q\psi]_{R_\mu}) - (d_q\varphi, (d_q\psi)'), \tag{3.17}$$

where $[\,,\,]_{R_\mu}$ *is the Lie algebra commutator constructed from* $R_\mu = (P_0 - P_{<0})/2 - \mu/z$. *Expanding in powers of* μ, $\{\,,\,\}_\mu = \mu\{\,,\,\}_1 + \{\,,\,\}_2$, *we obtain the two coordinated Hamiltonian structures on* $\mathscr{M}$

$$\{\varphi,\psi\}_1 = -(d_q\varphi, z^{-1}[d_q\psi, L]),$$
$$\{\varphi,\psi\}_2 = (q+\Lambda, [d_q\varphi, d_q\psi]_R) - (d_q\varphi, (d_q\psi)').$$

where $R = (P_0 - P_{<0})/2$. *Under time evolution in the coordinate* t_b, *the following recursion relation holds:*

$$\frac{\partial\varphi}{\partial t_b} = \{\varphi, H_{zb}\}_1 = \{\varphi, H_b\}_2. \tag{3.18}$$

Recall that our analysis has concentrated on the KdV-type hierarchies defined by the two gradations $(\mathbf{s}_{\text{hom}}, \mathbf{s}[w])$. In obtaining the Poisson brackets from the dynamical equations, Sects. 3.1 and 3.2, we have employed special properties of the homogeneous gradation. This dependence on the homogeneous gradation can be observed in the formulae for the Poisson brackets, the first Poisson structure involving a factor of z^{-1} while the second is expressed in terms of the R-operator $R = (P_0 - P_{<0})/2$. However, gauge invariance removes explicit dependence of the second Poisson bracket on the homogeneous gradation. More explicitly, it is possible to express the second Poisson bracket in terms of an arbitrary gradation $\mathbf{s}$, satisfying the inequalities $\mathbf{s}_{\text{hom}} \preceq \mathbf{s} \preceq \mathbf{s}[w]$. This is accomplished through the use of the following lemma.

Lemma 3.5. *Consider a Lie algebra* g *with the subalgebras* $A, C, A+B, B+C$. *Then if* $R_A = (P_A - P_{B+C})/2$, $R_C = (P_{A+B} - P_C)/2$, *the Lie algebra commutators satisfy:*

$$[X,Y]_{R_C} = [X,Y]_{R_A} + [P_BX, Y] + [X, P_BY], \qquad \forall X, Y \in g.$$

The proof of this lemma is just a question of writing out the Lie brackets, and so is omitted. In actual fact, we are interested in a centrally extended version of this lemma applied to the centrally extended Lie algebras $([\,,\,]_R, \omega_{\text{hom}})$, $([\,,\,]_{R[\mathbf{s}]}, \omega_{\mathbf{s}})$, where $\omega_{\mathbf{s}}$ is the central extension of $\hat{g}_0(\mathbf{s})$, $\omega_{\mathbf{s}}(X,Y) = (X', P_{0[\mathbf{s}]}Y)$. With this modification, the lemma still holds, the additional terms involving the central extension $\omega(X,Y) = (X', Y)$. We have the following proposition.

Proposition 3.2. *The second Poisson bracket between gauge invariant functions can be expressed in the form*

$$\{\varphi,\psi\}_2 = (q+\Lambda, [d_q\varphi, d_q\psi]_{R[\mathbf{s}]}) - (P_{0[\mathbf{s}]}d_q\varphi, (d_q\psi)'),$$

where $R[\mathbf{s}] = (P_{\geq 0[\mathbf{s}]} - P_{<0[\mathbf{s}]})/2$, *with the arbitrary gradation* $\mathbf{s}$ *satisfying* $\mathbf{s}_{\text{hom}} \preceq \mathbf{s} \preceq \mathbf{s}[w]$.

Proof. The proof follows from Lemma 3.5 with $A = \hat{g}_0 \cap \hat{g}_{\geq 0}(\mathbf{s})$, $B = \hat{g}_0 \cap \hat{g}_{<0}(\mathbf{s}) \subset \hat{g}_0 \cap \hat{g}^{<0}$, $C = \hat{g}_{<0}$, along with the fact that the additional terms vanish owing to the gauge invariance of functionals.

The importance of this proposition will become apparent in our later analysis of the partially modified KdV hierarchies, [1], in Sect. 5, for which the gradation $\mathbf{s}$ is chosen to be more general than the homogeneous gradation hitherto considered.

3.6. Centres

In this section we point out that the Poisson brackets defined in Theorem 3.2 sometimes admit non-trivial centres. The existence of these centres is directly related to the Hamiltonian densities h^j with $i > j \geq 0$. As we have already remarked these densities are constant under all the flows of the hierarchy, and so not all the functionals on $\mathscr{M}$ are dynamical. We shall show below that the densities are centres of the Poisson bracket algebra.

Before we proceed to the proposition we first establish a useful lemma.

Lemma 3.6. *The functional* $\Theta_f = (f, h(q))$, *where* $h(q)$ *was defined in Definition 3.1, for*

$$f \in C^\infty(\mathbf{R}/\mathbf{Z}, \oplus_{j=1-i}^{0} \mathscr{H}^j[w]),$$

satisfies

$$d_q\Theta_f = \Phi^{-1} f \Phi \bmod C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}^{\leq -1}).$$

Proof. One follows the steps of Theorem 3.1 up to Eq. (3.4),

$$\frac{d}{d\varepsilon}\Theta_f[q+\varepsilon r]\bigg|_{\varepsilon=0} = \left(f, \Phi(\varepsilon) r \Phi^{-1}(\varepsilon) + \left[\frac{d\Phi(\varepsilon)}{d\varepsilon}\Phi^{-1}(\varepsilon), \mathscr{L}(\varepsilon)\right]\right)\bigg|_{\varepsilon=0}. \tag{3.19}$$

However, in this case f is not a constant and so (3.19) equals

$$(\Phi^{-1} f \Phi, r) + \left(f', \frac{d\Phi(\varepsilon)}{d\varepsilon}\Phi^{-1}(\varepsilon)\right)\bigg|_{\varepsilon=0}.$$

The second term cannot contribute because $(d\Phi(\varepsilon)/d\varepsilon)\Phi^{-1}(\varepsilon)$ has $\mathbf{s}[w]$-grade < 0 and f' has $\mathbf{s}[w]$-grade ≤ 0, and so the lemma is proved.

Proposition 3.3. *The functionals of the form* $\Theta_f = (f, h)$ *for*

$$f \in C^\infty(\mathbf{R}/\mathbf{Z}, \oplus_{j=1-i}^{k} \mathscr{H}^j[w]),$$

are centres of the first Poisson bracket algebra, for $k = 0$, *and centres of the second Poisson bracket algebra, for* $k = -1$.

Proof. Lemma 3.6 implies that $d_q\Theta_f = \Phi^{-1} f \Phi$, modulo terms of $\mathbf{s}[w]$-grade less than $1 - i$, which will not contribute to the Poisson brackets owing to Lemma 3.2. We have

$$\{\varphi, \Theta_f\}_1 = -(d_q\varphi, z^{-1}[\Phi^{-1} f \Phi, L]).$$

This is zero because $[\Phi^{-1}f\Phi, L] = -\Phi^{-1}f'\Phi$, using the definition of Φ in Sect. 2, Eq. (2.7), and the fact that $z^{-1}\Phi^{-1}f'\Phi$ has homogeneous grade < 0. This proves that Θ_f is a centre of the first Hamiltonian structure. For the second structure

$$\begin{aligned}\{\varphi, \Theta_f\}_2 &= (d_q\varphi_0, [(\Phi^{-1}f\Phi)_0, L]) - (d_q\varphi_{<0}, [(\Phi^{-1}f\Phi)_{<0}, L]) \\ &= (d_q\varphi, [(\Phi^{-1}f\Phi)_0, L]) + (d_q\varphi_{<0}, \Phi^{-1}f'\Phi)\,, \end{aligned} \tag{3.20}$$

which follows because $(\Phi^{-1}f\Phi)_{<0} = \Phi^{-1}f\Phi - (\Phi^{-1}f\Phi)_0$. The second term in (3.20) is zero owing to mismatched homogeneous grade. If the $\mathbf{s}[w]$-grade of f is less than zero, the first term above induces a gauge transformation of φ, which is zero because φ is a gauge invariant functional, and so the proposition is proved. Notice that the proof for the second structure does not cover the case when f has zero $\mathbf{s}[w]$-grade.

It is an obvious corollary of the proposition that the densities h^j, for $0 \le j < i$ are non-dynamical, as was proved in [1] directly. It is interesting to notice that h^0 is *not* a centre of the second Poisson bracket algebra, even though it is non-dynamical, a point which will be apparent in the examples considered in Sect. 6.

4. Conformal Symmetry

It is proved in [2] that the KdV hierarchies exhibit a scale invariance, i.e. under the transformation $x \mapsto \lambda x$, for constant λ, each quantity in the equations can be assigned a scaling dimension such that the equations are invariant. The original KdV equation provides a typical example; the appropriate transformations are $x \mapsto \lambda x$, $u \mapsto \lambda^{-2}u$ and $t \mapsto \lambda^3 t$. In this section, we prove that this scaling invariance generalizes to the hierarchies defined in [1], and are, in fact, symmetries of the second symplectic structure. By generalizing this result to arbitrary conformal (analytic) transformations, $x \mapsto y(x)$, we surmise that the second Poisson bracket algebra contains (as a subalgebra) the algebra of conformal transformations, i.e. a Virasoro algebra. This would imply that the second Poisson bracket algebra is an extended chiral conformal algebra, generalizing the occurrence of the W-algebras as the second Poisson bracket algebra of the hierarchies of Drinfel'd and Sokolov.

Consider the transformation $x \to \lambda x$ on the Lax operator $L = \partial_x + \Lambda + q(x)$. In order that this rescaling can be lifted to a symmetry of the equations of motion, it is necessary that the form of L is preserved. To this end we consider the transformation $z \to \tilde{z} = \lambda^{-N/i}z$, with a simultaneous adjoint action by:

$$U = \lambda^{-(N/i)\delta_{\mathbf{s}[w]}\cdot H} \equiv \exp\left(-\frac{N}{i}(\log\lambda)\delta_{\mathbf{s}[w]}\cdot H\right), \tag{4.1}$$

where $\delta_{\mathbf{s}[w]}$ is defined in (2.1). By means of this transformation we are lead to the following proposition.

Proposition 4.1. *The dynamical equation of the hierarchy generated by the Hamiltonian H_{bj} (with respect to the second Hamiltonian structure), is invariant under the transformations*

$$x \mapsto \lambda x\,, \qquad t_{bj} \mapsto \lambda^{j/i}t_{bj}\,, \qquad q^k \mapsto \lambda^{k/i-1}q^k\,,$$

where i is the $\mathbf{s}[w]$-grade of Λ, and q^k is the component of q with $\mathbf{s}[w]$-grade k.

Proof. We consider the following transformation of L

$$L(x, q; z) = \lambda U L(y, \tilde{q}; z\lambda^{-N/i})U^{-1}\,,$$

where $y = \lambda x$. First of all notice that under the transformation

$$\Lambda(z) = \lambda U \Lambda(\tilde{z}) U^{-1},$$

which ensures that L preserves its form. Secondly, under this transformation the coefficient of q with $\mathbf{s}[w]$-grade k transforms as $\tilde{q}^k = \lambda^{k/i-1} q^k$. In order to derive the rescaling of the time parameter t_{bj}, consider the change in the transformation $L = \Phi^{-1} \mathscr{L} \Phi$ in Eq. (2.7). If $\tilde{\Phi}$ is the transformation which conjugates $L(y, \tilde{q}; \tilde{z})$ into the Heisenberg subalgebra, then the relationship between Φ and $\tilde{\Phi}$ is

$$\Phi(x, q; z) = U \tilde{\Phi}(y, \tilde{q}; \tilde{z}) U^{-1},$$

because adjoint action by U preserves the decomposition $\hat{g} = \mathrm{Im}(\mathrm{ad}\,\Lambda) \oplus \mathrm{Ker}(\mathrm{ad}\,\Lambda)$, and the eigenspaces $\hat{g}^j$. Therefore $(\Phi^{-1} b^j \Phi) = \lambda^{j/i} U(\tilde{\Phi}^{-1} \tilde{b}^j \tilde{\Phi}) U^{-1}$, where $\tilde{b}^j$ is equal to b^j with z replaced by $\tilde{z}$. Since adjoint action by U commutes with the projections $P_{\geq 0}$, $P_{<0}$, we deduce that the time evolution equations, (3.1), (3.2), are invariant if $\tilde{t}_{bj} = \lambda^{j/i} t_{bj}$.

Notice that the cumulative effect of the transformation on z and the conjugation by U is equivalent to a global rescaling of the gradation $\mathbf{s}[w]$, i.e. action by $\exp(-\log\lambda\, \mathrm{ad}(d_{\mathbf{s}[w]})/i)$.

We have shown that the equations of the hierarchy are quasi-homogeneous under the lift of the transformation $x \mapsto \lambda x$. However one can lift the more general conformal transformations $x \mapsto y(x)$ onto the phase space in a similar manner, by altering the global $\mathbf{s}[w]$-rescaling, to a local rescaling. Thus we transform $z \mapsto \tilde{z} = (y')^{-N/i} z$, and alter (4.1) to:

$$U[y] = (y')^{-(N/i)\delta_{\mathbf{s}[w]} \cdot H},$$

where $y' = dy/dx$. The transformation of the Lax operator is now

$$L(x, q; z) = y' U[y] L(y, \tilde{q}; \tilde{z}) U[y]^{-1},$$

corresponding to the following transformation of the potential q

$$q(x; z) = y' U[y] \tilde{q}(y; \tilde{z}) U[y]^{-1} + \frac{N}{i} \left(\frac{y''}{y'} \right) \delta_{\mathbf{s}[w]} \cdot H. \tag{4.2}$$

This conformal transformation induces a corresponding conformal transformation on the space of gauge equivalence classes, i.e. on $\mathscr{M}$. This follows because the gauge group is generated by $\hat{g}_0 \cap \hat{g}^{<0}$, and under a $\mathbf{s}[w]$-rescaling this subalgebra is invariant. Thus if $L = S^{-1} L' S$, with S in the gauge group, then under a conformal transformation the corresponding Lax operators are related by the gauge transformation defined by $U[y]^{-1} S U[y]$. We observe that under a conformal transformation, the non-dynamical functionals on $\mathscr{M}$ are also subject to transformation.

With the conformal transformation $x \to y(x)$, the manipulations in the proof of Proposition 4.1 can now be reproduced, with λ replaced by y', however, one now finds the flow variables transform in a more complicated way:

$$t_{bj} \mapsto \tilde{t}_{bj} = (y')^{j/i} t_{bj} + \dots,$$

where the dots represent terms which depend on the variables t_{bk}, with $k < j$, which vanish for the scale transformation.

Example. Let us consider the case of the original KdV hierarchy. We want to determine how the gauge invariant function u, defined in Sect. 2, transforms under the transformation (4.2). The form of L in this case is

$$L = \partial_x + \begin{pmatrix} a & 0 \\ b & -a \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix},$$

and the gauge invariant function is $u = a^2 + b - a'$. Under the transformation (4.2) one finds

$$a = y'\tilde{a} + \frac{y''}{2y'}, \qquad b = (y')^2\tilde{b},$$

and so

$$\begin{aligned} u &= (y')^2(\tilde{a}^2 + \tilde{b} - \partial_y \tilde{a}) - \frac{y'''}{2y'} + \frac{3}{4}\left(\frac{y''}{y'}\right)^2 \\ &= (y')^2\tilde{u} - \tfrac{1}{2}\mathscr{S}(y), \end{aligned}$$

where $\mathscr{S}(y) = y'''/y' - \frac{3}{2}(y''/y')^2$ is the Schwartzian derivative of y with respect to x. Thus $u(x)$ transforms like a projective connexion or Virasoro generator. This gives a hint of a hidden conformal symmetry in the system.

Below, we show that the second Poisson bracket algebra, for a general hierarchy, is invariant under an arbitrary conformal transformation.

We now wish to determine how the functional derivatives of the gauge invariant functionals transform under the conformal transformation (4.2). Recall that the functional derivative $d_q\varphi$ of a functional $\varphi \in \mathscr{F}$ is a gauge invariant element of $C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}_{\leq 0}/g^{\leq -\imath})$. We use the notation $d_q\varphi(x;z)$ for the functional derivative, explicitly indicating the integration variable, $x \in \mathbf{R}/\mathbf{Z}$ and loop variable z in Definition 3.2, i.e.

$$\frac{d}{d\varepsilon}\varphi[q + \varepsilon r]\bigg| = \int dx \langle d_q\varphi(x;z), r(x;z)\rangle, \tag{4.3}$$

for arbitrary $r \in C^\infty(\mathbf{R}/\mathbf{Z}, Q)$. Our notation mirrors that of the finite dimensional case: the functional derivative $d_q\varphi$ taking values in the "cotangent space" at $q \in \mathscr{M}$. In the calculation of the functional derivative $d_q\tilde{\varphi}$, it is necessary to take proper account of the variables x and z in the functional derivative; the relationship between $d_q\tilde{\varphi}$ and $d_{\tilde{q}}\varphi$ involving a transformation of these variables similar to that occurring in (4.2).

Lemma 4.1. *The transformation* (4.2) *induces the following transformation on the functional derivatives*

$$d_q\tilde{\varphi}(x;z) = U[y]\, d_{\tilde{q}}\varphi(y(x);\tilde{z})U[y]^{-1},$$

where $\tilde{\varphi}[q] = \varphi[\tilde{q}]$ *is the pull-back of the gauge invariant functional* φ.

Proof. Consider the definition of $d_q\varphi(x;z)$ in (4.3). Making the dependence on all the variables explicit, under (4.2) we have $\tilde{\varphi}[q + \varepsilon r] = \varphi[\tilde{q} + \varepsilon\tilde{r}]$, with $\tilde{r}(y(x);\tilde{z}) = (y')^{-1}U[y]^{-1}r(x;z)U[y]$, and so from (4.3) we deduce

$$\begin{aligned} \int dx \langle d_q\tilde{\varphi}(x;z), r(x;z)\rangle &= \int dy (y')^{-1}\langle d_{\tilde{q}}\varphi(y;\tilde{z}), U^{-1}r(x;z)U\rangle \\ &= \int dx \langle U d_{\tilde{q}}\varphi(y;\tilde{z})U^{-1}, r(x;z)\rangle, \end{aligned}$$

hence the lemma is proved.

Proposition 4.2. *The transformation* (4.2) *is a Poisson mapping of the second Poisson structure.*

Proof. We have to show that $\{\tilde{\varphi},\tilde{\psi}\}_2[q] = \{\varphi,\psi\}_2[\tilde{q}]$. First of all, $U[y]$ carries zero grade, therefore from Lemma 4.1 $d_q\tilde{\varphi}(x;z)_0 = Ud_{\tilde{q}}\varphi(y;\tilde{z})_0U^{-1}$; similarly for $d_q\tilde{\varphi}(x;z)_{<0}$. This means, using the expression (3.10) and Lemma 4.1,

$$\begin{aligned}\{\tilde{\varphi},\tilde{\psi}\}_2[q] &= \int dx\langle Ud_{\tilde{q}}\varphi(y;\tilde{z})_0U^{-1},[Ud_{\tilde{q}}\psi(y;\tilde{z})_0U^{-1},y'UL(\tilde{q})U^{-1}]\rangle\\ &\quad-\int dx\langle Ud_{\tilde{q}}\varphi(y;\tilde{z})_{<0}U^{-1},[Ud_{\tilde{q}}\psi(y;\tilde{z})_{<0}U^{-1},y'UL(y,\tilde{q};\tilde{z})U^{-1}]\rangle\\ &= \{\varphi,\psi\}_2[\tilde{q}]\,,\end{aligned}$$

owing to the ad-invariance of the Killing form, and the fact that the factor of y' transforms the measure in just the right way, $dx \mapsto dy$. It is important that the inner product on the Kac-Moody algebra pairs terms of opposite grade so that it is invariant under the transformation $z \mapsto \tilde{z}$, i.e.

$$\langle A(z),B(z)\rangle = \langle A(\tilde{z}),B(\tilde{z})\rangle\,.$$

The fact that the first symplectic structure does not respect the conformal symmetry is due to the presence of the z^{-1} term in (3.6).

Since the second Poisson bracket is preserved by the conformal transformations one expects the symmetry to be generated by some functional on phase space, say $T^{\text{vir}}(x)$, which would satisfy the (classical) Virasoro algebra:

$$\{T^{\text{vir}}(x),T^{\text{vir}}(y)\}_2 = (c/2)\delta'''(x-y) - 2T^{\text{vir}}(x)\delta'(x-y) - T^{\text{vir}\prime}(x)\delta(x-y)\,.$$

The component $L_{-1} = \int dx\, T^{\text{vir}}(x)$ would generate translations in x, i.e.

$$\frac{\partial\phi(x)}{\partial x} = \left\{\phi(x),\int dy\,T^{\text{vir}}(y)\right\}_2\,.$$

When the centres are set to zero x becomes identified with t_Λ, and so such transformations are generated by the flow associated with the Hamiltonian H_Λ. Therefore, up to a possible total derivative, the Virasoro generator should be equal to the Hamiltonian density h^{-i}, which one can readily confirm has scaling dimension two, as required.

This leads us to the conclusion that the second Poisson bracket algebra of a generalized KdV hierarchy contains as a subalgebra the Virasoro algebra, and is thus a (classical) chiral extended conformal algebra. We shall explicitly construct the Virasoro generator for the examples that are considered in Sect. 6.

5. Modified Hierarchies and Miura Map

In this section we consider how our formalism extends to the various *partially modified* hierarchies that are associated to a given KdV hierarchy. These modified hierarchies are constructed by considering a Lax operator of the form (2.2), where $q \in C^\infty(\mathbf{R}/\mathbf{Z},Q_{\mathbf{s}})$, $Q_{\mathbf{s}} = \hat{g}_{\geq 0}(\mathbf{s}) \cap \hat{g}^{<i}$. The space $Q_{\mathbf{s}}$ is a certain subspace of Q labelled by another gradation of $\hat{g}$, $\mathbf{s}_{\text{hom}} \prec \mathbf{s} \preceq \mathbf{s}[w]$. The fact that $Q_{\mathbf{s}} \subset Q$ is a consequence of Lemma 2.1 which implies that $Q = Q_{\mathbf{s}} \cup Y_{\mathbf{s}}$, where $Y_{\mathbf{s}} = \hat{g}_0 \cap \hat{g}_{<0}(\mathbf{s})$. The modified hierarchies have a gauge invariance of the form (2.4) where P is replaced by $P_s = \hat{g}_0(s) \cap \hat{g}^{<0}$. Thus the phase space, denoted $\mathscr{M}_s$, of a partially modified hierarchy consists of the equivalence classes of operators of the form (2.2), with $q \in C^\infty(\mathbf{R}/\mathbf{Z}, Q_s)$, modulo the gauge symmetry (2.4) generated by P_s. Correspondingly, the space of gauge invariant functionals on Q_s is denoted $\mathscr{F}_s$. The unique hierarchy for which $P_s = \emptyset$, i.e. $\mathbf{s} \equiv \mathbf{s}[w]$ is the *modified* KdV hierarchy, whereas the hierarchies for which $\mathbf{s} \prec \mathbf{s}[w]$ are known as *partially modified* KdV hierarchies. It was shown in [1] that all the (partially) modified hierarchies associated to a KdV hierarchy can be obtained as reductions of that KdV hierarchy. Here we shall prove this in a slightly different way through an analysis of the Hamiltonian structure.

In attempting to extend the analysis of the previous sections to the partially modified KdV hierarchies, we must separate the results that explicitly require the first gradation s to be the homogeneous gradation. An essential result in this direction is Proposition 3.2, which demonstrates that the second Poisson bracket can be defined without resorting to the homogeneous gradation. From this result alone, we can predict that the partially modified KdV hierarchies are Hamiltonian with respect to the Poisson bracket

$$\{\varphi, \psi\}_s = (q_s + \Lambda, [d_{q_s}\varphi, d_{q_s}\psi]_{R_s}) - (d_{q_s}\varphi, (d_{q_s}\psi)'), \tag{5.1}$$

for φ and $\psi \in \mathscr{F}_s$, $R_s = (P_{0[s]} - P_{<0[s]})/2$, the Hamiltonians being defined identically to that of Definition 3.1, with the modification that $q \in Q_s$. Observe that the functional derivatives are valued in $\hat{g}_{\leq 0}(\mathbf{s})/\hat{g}^{\leq -i}$. This Hamiltonian property of the pmKdV hierarchies can be verified directly, the dynamical equations of the partially modified KdV hierarchies (3.1), (3.2) being reproduced, where subscripts now denote the gradation s. However, we derive this result through the Hamiltonian mapping properties of the Miura map.

Recall that the phase space $\mathscr{M}_s$ of a partially modified hierarchy consists of the equivalence classes of operators of the form (2.2), with $q \in C^\infty(\mathbf{R}/\mathbf{Z}, Q_s)$, modulo the gauge symmetry (2.4) generated by P_s. Thus, if we define $\mathrm{Im}_s(q)$ to denote the P_s-gauge orbit of q in Q_s, the point of the phase space $\mathscr{M}_s$ corresponding to $q \in C^\infty(\mathbf{R}/\mathbf{Z}, Q_s)$ is represented by the gauge orbit $\mathrm{Im}_s(q)$. Given the inclusion $Q_s \subset Q$, there is a corresponding inclusion of equivalence classes $\mathscr{M}_s \subset \mathscr{M}$, where the equivalence class represented by $\mathrm{Im}_s(q)$ is mapped into the equivalence class represented by $\mathrm{Im}_{\mathrm{hom}}(q) \subset Q$. This inclusion of equivalence classes is the *Miura map* $\mu : \mathscr{M}_s \subset \mathscr{M}$, [2]. Observe that the gauge group P does not preserve the subspace Q_s, i.e. the image $\mathrm{Im}(\mu) \not\subset Q_s$, where $\mathrm{Im}(\mu) = \bigcup_{q \in Q_s} \mathrm{Im}_{\mathrm{hom}}(q)$, the P-gauge orbit of Q_s in Q. There is an induced mapping $\mu^* : \mathscr{F} \to \mathscr{F}_s$ given by restriction of the functionals to the submanifold $\mathscr{M}_s$. In particular, we have the following proposition relating the Hamiltonians.

Proposition 5.1. *The Hamiltonians of the* $(\Lambda, \mathbf{s}, [w])$*-partially modified hierarchy are the restriction to* $\mathscr{M}_s$ *of the Hamiltonians of the associated* $(\Lambda, \mathbf{s}_{\mathrm{hom}}, [w])$-KdV *hierarchy.*

Proof. This follows from the observation that by restricting to the submanifold $\mathscr{M}_s$, we can perform a gauge transformation such that $q \in C^\infty(\mathbf{R}/\mathbf{Z}, Q_s)$ in (2.7). Then $\Phi^j \equiv \Phi^j_s \in C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}_{\leq 0}(\mathbf{s}) \cap \hat{g}^j)$, i.e. Φ is identical to the unique transformation employed in the $(\Lambda, \mathbf{s}, [w])$-hierarchy. Thus the restricted Hamiltonians in Definition 3.1 are identical to the Hamiltonians of the partially modified hierarchy.

Since the Hamiltonians of the partially modified hierarchy are reproduced on restriction, the dynamical equations of the partially modified hierarchy will be reproduced if the Hamiltonian structure restricts to $\mathscr{M}_{\mathbf{s}}$, i.e. if we define $I_{\mathscr{M}_{\mathbf{s}}}$ as the functionals that vanish on $\mathscr{M}_{\mathbf{s}} \subset \mathscr{M}$, then a Hamiltonian structure is induced on $\mathscr{F}_{\mathbf{s}} \cong \mathscr{F}/I_{\mathscr{M}_{\mathbf{s}}}$ if $I_{\mathscr{M}_{\mathbf{s}}}$ is an ideal of the Poisson bracket on $\mathscr{M}$. This is proved in the following proposition:

Proposition 5.2. *The second Hamiltonian structure of the generalized* KdV *hierarchy induces the following Hamiltonian structure on* $\mathscr{M}_{\mathbf{s}}$*:*

$$\{\varphi, \psi\}_{\mathbf{s}} = (q_{\mathbf{s}} + \Lambda, [d_{q_{\mathbf{s}}}\varphi, d_{q_{\mathbf{s}}}\psi]_{R_{\mathbf{s}}}) - d_{q_{\mathbf{s}}}\varphi, (d_{q_{\mathbf{s}}}\psi)'), \tag{5.2}$$

for φ *and* $\psi \in \mathscr{F}_{\mathbf{s}}$, $d_q\varphi \in C^\infty(\mathbf{R}/\mathbf{Z}, \hat{g}_{\leq 0}(\mathbf{s})/\hat{g}^{\leq -i})$, *and* $R_{\mathbf{s}} = (P_{0[\mathbf{s}]} - P_{<0[\mathbf{s}]})/2$.

Proof. The idea of the proof is to verify that if we restrict to $\mathscr{M}_{\mathbf{s}} \subset \mathscr{M}$, then the corresponding operation on the functional derivatives, i.e. taking the quotient, is well defined as a Lie algebra homomorphism. The fact that the phase space consists of gauge equivalence classes complicates this issue. Thus we proceed as follows: consider a functional $\varphi \in \mathscr{F}$. For $q \in \mathrm{Im}(\mu)$, we can perform a gauge transformation such that $q \in Q_{\mathbf{s}}$, the remaining gauge freedom being $P_{\mathbf{s}}$. With this partial gauge fixing, the derivative of φ with respect to the directions $r \in Y_{\mathbf{s}}$ are not specified, i.e. the functional derivative is defined up to $\mathrm{Ann}(Q_{\mathbf{s}}) = \hat{g}_0 \cap \hat{g}_{>0}(\mathbf{s}) \bmod \hat{g}^{\leq -\imath}$, where $\mathrm{Ann}(A) = \{l \in B^* \mid (l, a) = 0\ \forall a \in A\}$ for A a subalgebra of an algebra B. Expressing the second Poisson bracket in terms of the $\mathbf{s}$-gradation, Proposition 3.2, we observe that $\mathrm{Ann}(Q_{\mathbf{s}})$ is an ideal of the centrally extended Lie algebra $\{\hat{g}_{\leq 0}, ([\,,]_{R[\mathbf{s}]}, \omega_{\mathbf{s}})\}$. Thus the quotient is well defined, reproducing (5.2).

Observe that this proves the prediction in (5.1), made on the strength of Proposition 3.2. The first Hamiltonian structure has not been mentioned in relation to the partially modified hierarchies. This is because although the first Poisson bracket is well defined on the phase space $\mathscr{M}_{\mathbf{s}}$, it does not generate the dynamics, i.e. Eqs. (3.1), (3.2) are not reproduced. From the Hamiltonian mapping point of view, this corresponds to the fact that the space $\mathrm{Ann}(Q_{\mathbf{s}})$ is not an ideal of the Lie bracket $[\,,]$ on $\hat{g}_{\leq 0}$.

Combining these results, we obtain the following theorem.

Theorem 5.1. *The Miura map is a Hamiltonian mapping,*

$$\mu: (\mathscr{M}_{\mathbf{s}}, \{\,,\}_{\mathbf{s}}) \to (\mathscr{M}, \{\,,\}_2),$$

such that it defines a reduction of the dynamical equations of the KdV *hierarchy to those of the* pmKdV *hierarchies.*

Proof. If we consider the decomposition $q = q_{\mathbf{s}} + \check{q}$, where $q \in C^\infty(\mathbf{R}/\mathbf{Z}, Q)$, $q_{\mathbf{s}} \in C^\infty(\mathbf{R}/\mathbf{Z}, Q_{\mathbf{s}})$ and $\check{q} \in C^\infty(\mathbf{R}/\mathbf{Z}, Y_{\mathbf{s}})$, then for a gauge equivalence class in $\mathscr{M}_{\mathbf{s}}$ the Miura map is equivalent to the constraint $\check{q}_{\mathbf{s}} = 0.$, The fact that the Miura map is Hamiltonian, as follows from Proposition 5.2, implies that this constraint is preserved under all the flows. Since the Hamiltonians are reproduced on restriction to $\mathscr{M}_{\mathbf{s}}$, the time evolutions of the pmKdV hierarchies, (3.1) and (3.2), are reproduced under the Miura map.

We should emphasize that the Miura map $\mu: \mathscr{M}_{\mathbf{s}} \to \mathscr{M}$ is not invertible; it allows one to construct a solution of the KdV hierarchy in terms of a solution of the (partially) modified hierarchy, but not *vice-versa*. In addition, one can show more generally that there exists a Miura map between each partially modified hierarchy $\mu: \mathscr{M}_{\mathbf{s}_1} \to \mathscr{M}_{\mathbf{s}_2}$, whenever $\mathbf{s}_1 \succ \mathbf{s}_2$.

6. Examples

6.1. The Drinfel'd-Sokolov Generalized KdV *Hierarchies*

The Drinfel'd-Sokolov KdV hierarchies [2] are recovered from our formalism by choosing $[w]$ to be the conjugacy class containing the Coxeter element [1]. In this case $\Lambda = \sum_{i=1}^{r} e_i + ze_0$, where the e_i, for $i = 1, \ldots, r$ are the raising operators associated to the simple roots, and e_0 is the lowering operator associated to the highest root. In this case $Q = b_-$, one of the Borel subalgebras of g. The gauge freedom corresponds to n_-, where n_- is the subalgebra of g such that $b_- = n_- + h$ (as a vector space). For example in the case of A_n, choosing the defining representation, we have

$$\Lambda = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ z & & & \end{pmatrix}, \tag{6.1}$$

and Q consists of the lower triangular matrices (including the diagonal). In this example, the gauge group is generated by the strictly lower triangular matrices.

We can immediately write down the expression for the first and second Hamiltonian structures. If we define $I \equiv \sum_{i=1}^{r} e_i$ and $e \equiv e_0$, to use the notation of [2], then

$$\{\varphi, \psi\}_1 = -(d_q\varphi, [d_q\psi, e]),$$

since q has no z dependence, and

$$\{\varphi, \psi\}_2 = (d_q\varphi, [d_q\psi, \partial_x + q + I]).$$

These are exactly the two Hamiltonian structures of the generalized KdV hierarchies written down in [2] for the untwisted Kac-Moody algebras. We have not considered the case of twisted Kac-Moody algebras here, but it seems that in some cases only a single Hamiltonian structure exists, see [2].

For A_1, one finds that using the basis for $\mathscr{F}$ in (2.6) one recovers the explicit form of (1.1). The second Poisson bracket algebras are the Gel'fand-Dikii algebras [3], which are classical versions of the so-called W-algebras of [12]. For the A_n case, the scaling dimensions of the generators u_j of (2.6) are $n+2-j$, where $j = 1, \ldots, n$, and so the scaling dimensions range from 2 to $n + 1$ in integer steps. In this case $\mathscr{F}$ has a unique element of dimension 2, namely u_n, which generates the algebra of conformal transformations, or the Virasoro algebra.

6.2. The A_1 Fractional KdV*-Hierarchies*

A series of hierarchies can be associated with any choice of $[w]$, simply by choosing Λ to be any element of $\mathscr{H}[w]$ with well defined positive s$[w]$-grade. If we consider $g = A_1$ then there are two elements in the Weyl group – the identity and the reflection in the root.The identity leads to a homogeneous hierarchy which is considered in

Sect. 6.5. Choosing w to be the reflection, the Heisenberg subalgebra is spanned by (in the defining representation)

$$\Lambda^{2m+1} = z^m \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix},$$

where m is an arbitrary integer, and the superscript denotes the $\mathbf{s}[w]$-grade. When one takes Λ to be the $\mathbf{s}[w]$-grade 1 element, i.e. Λ^1, then the hierarchy which results is nothing but the usual KdV hierarchy discussed above. When one takes Λ to be an element of the Heisenberg subalgebra with grade > 1, then we have what we might call a "fractional hierarchy", to mirror the terminology of [7], because the fields q have fractional scaling dimensions.

Let us consider these hierarchies in more detail. If we take $\Lambda = \Lambda^{2m+1}$, i.e. $i = 2m+1$, then before gauge fixing, the potential is

$$q = \sum_{j=0}^{m} z^j \begin{pmatrix} a_{3j+2} & a_{3j+3} \\ a_{3j+1} & -a_{3j+2} \end{pmatrix}, \qquad a_{3m+3} = 0. \tag{6.2}$$

The gauge transformation defined in (3.13) involve the matrix

$$S = \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix},$$

and by choosing $A = a_{3m+2}$ one can generate a consistent gauge slice q^{can} of the form (6.2) with $a_{3m+2} = 0$, which generalizes the usual choice of canonical variables for the KdV hierarchy. For $m > 0$, there are $3m+1$ independent gauge invariant functionals, with m of them, corresponding to h^{2k+1} for $k = 0, \ldots, m-1$, in the center of the two Poisson brackets. Rather than present a general result for the two Hamiltonian structures, we just consider the first two cases corresponding to $i = 3$ and $i = 5$; the case $i = 1$ is, of course, just the usual KdV hierarchy whose two Poisson bracket algebras are written down in the introduction.

The Case $i = 3$. In this case there are 4 gauge invariant functionals. It is straightforward to express them in terms of the variables a_i:

$$g_1 = a_1 + 2a_2a_5 - a_5^2a_3 - a_5', \qquad g_2 = a_2 - a_3a_5,$$
$$g_3 = a_3, \qquad g_4 = a_3 + a_4 + a_5^2.$$

The g_j's have scaling dimensions $\frac{4}{3}, 1, \frac{2}{3}, \frac{2}{3}$, respectively. The non-zero brackets of the first symplectic structure are

$$\{g_2(x), g_1(y)\}_1 = (2g_3(x) - g_4(x))\,\delta(x-y),$$
$$\{g_2(x), g_3(y)\}_1 = \delta(x-y), \qquad \{g_1(x), g_1(y)\} = 2\delta'(x-y).$$

In this case the variable g_4 is in the centre as expected; indeed, if the conserved densities are constructed one finds $h^1 = g_4$. The non-zero brackets of the second structure are

$$\{g_1(x), g_3(y)\}_2 = -\delta'(x-y) + 2g_2(x)\,\delta(x-y),$$
$$\{g_2(x), g_2(y)\}_2 = -\tfrac{1}{2}\,\delta'(x-y),$$
$$\{g_2(x), g_1(y)\}_2 = g_1(x)\,\delta(x-y), \qquad \{g_2(x), g_3(y)\}_2 = -g_3(x)\,\delta(x-y).$$

Again, as expected from Sect. 3.6, g_4 is in the centre of the algebra. We recognize the algebra as the A_1 Kac-Moody algebra with non-trivial central extension. It is straightforward to write down the Virasoro generator for this algebra

$$T^{\mathrm{vir}}(x) = g_1(x)g_3(x) + g_2(x)^2 - \tfrac{1}{3}\, g_2'(x)\,.$$

The Case $i = 5$. The space of gauge invariant functions is spanned by integrals of polynomials in the seven gauge invariant functions

$$\begin{aligned}
g_1 &= a_1 + 2a_2a_8 - a_8^2a_3 - a_8'\,,\\
g_2 &= a_2 - a_8a_3\,, \qquad g_3 = a_3\,,\\
g_4 &= a_3 + a_4 + 2a_5a_8 + a_6a_7\,,\\
g_5 &= a_5 - a_6a_8\,,\\
g_6 &= a_6\,, \qquad g_7 = a_6 + a_7 + a_8^2\,.
\end{aligned}$$

The spins of the functions g_j are $\frac{6}{5}, 1, \frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{2}{5}$, respectively. The non-zero brackets of the first Poisson bracket algebra are

$$\begin{aligned}
\{g_2(x), g_1(y)\}_1 &= (2g_3(x) - g_4(x) - g_6^2(x) + g_6(x)g_7(x))\,\delta(x-y)\,,\\
\{g_2(x), g_3(y)\}_1 &= g_6(x)\,\delta(x-y)\,, \qquad \{g_2(x), g_6(y)\}_1 = \delta(x-y)\,,\\
\{g_1(x), g_1(y)\}_1 &= 2\delta'(x-y)\,, \qquad \{g_1(x), g_3(y)\}_1 = -2g_5(x)\,\delta(x-y)\,,\\
\{g_1(x), g_5(y)\}_1 &= (g_7(x) - 2g_6(x))\,\delta(x-y)\,,\\
\{g_3(x), g_5(y)\}_1 &= -\delta(x-y)\,.
\end{aligned}$$

There are two centres g_4 and g_7 as predicted by Proposition 3.3. The non-zero brackets of the second Poisson bracket algebra are

$$\begin{aligned}
\{g_2(x), g_2(y)\}_2 &= -\tfrac{1}{2}\,\delta'(x-y)\,, \qquad \{g_2(x), g_1(y)\}_2 = g_1(x)\,\delta(x-y)\,,\\
\{g_2(x), g_3(y)\}_2 &= -g_3(x)\,\delta(x-y)\,,\\
\{g_1(x), g_3(y)\}_2 &= -\delta'(x-y) + 2g_2(x)\,\delta(x-y)\,,\\
\{g_5(x), g_6(y)\}_2 &= \delta(x-y)\,.
\end{aligned}$$

Again g_4 and g_7 are centres of the algebra. The second Poisson bracket algebra is simply the direct sum of an A_1 Kac-Moody algebra, with central extension, generated by g_1, g_2 and g_3, and a "*b–c*" algebra, generated by g_5 and g_6. The Virasoro generator is, in this case, given by

$$T^{\mathrm{vir}}(x) = g_1(x)g_3(x) + g_2(x)^2 - \tfrac{1}{3}\, g_2'(x) + g_5(x)g_6'(x) - \tfrac{2}{5}\,(g_5(x)g_6(x))'\,.$$

6.3. The First Fractional A_2 KdV*-Hierarchy and* $W_3^{(2)}$

Let us consider the KdV hierarchy corresponding to the Coxeter element of the Weyl group, but in contrast to the usual Drinfel'd-Sokolov case, where Λ is given by (6.1), we now take Λ to be the element of the Heisenberg subalgebra with $i = 2$, i.e.

$$\Lambda = \begin{pmatrix} 0 & 0 & 1 \\ z & 0 & 0 \\ 0 & z & 0 \end{pmatrix}.$$

This choice corresponds to the fractional KdV hierarchy discussed in [1, 7]. Before gauge fixing the potential can be written as

$$q = \begin{pmatrix} y_1 & c & 0 \\ e & y_2 & d \\ a+bz & f & -(y_2+y_1) \end{pmatrix},$$

which, under a gauge transformation, transforms as

$$q \to \tilde{q} = \Phi\partial_x\Phi^{-1} + \Phi(q+\Lambda)\Phi^{-1} - \Lambda\,,$$

where

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & 0 \\ B & C & 1 \end{pmatrix}.$$

As shown in [1] there exists a gauge transformation given by

$$\begin{aligned} A &= \tfrac{1}{3}\,(b+c-2d)\,, \\ C &= \tfrac{1}{3}\,(2c-b-d)\,, \\ B &= y_1+y_2-dC-\frac{\beta}{\alpha}\,(cA+y_2-dC-AC)\,, \end{aligned}$$

which brings q into the canonical form

$$q^{\mathrm{can}} = \begin{pmatrix} (\alpha-\beta)U & 0 & 0 \\ G^+ & -\alpha U & 0 \\ T & G^- & \beta U \end{pmatrix} + \phi \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z & 0 & 0 \end{pmatrix},$$

where α, β are arbitrary parameters which we fix to $\alpha = 2\beta = 1$, to make the comparison with the results of [7] easier. The fields ϕ, U, $G^\pm$ and T form a basis of gauge invariant functionals of q, with spins $\frac{1}{2}, 1, \frac{3}{2}$, and 2. The field ϕ corresponds to the conserved quantity h^1 which, following Sect. 3.6, is in the center of the two Poisson brackets.

In terms of the combinations

$$\begin{aligned} \tilde{U} &= U + \phi^2\,, \\ \tilde{G}^\pm &= G^\pm \pm \phi' - \tfrac{3}{2}\,U\phi - \phi^3\,, \\ \tilde{T} &= T + \tfrac{3}{4}\,U^2 + (G^+ + G^-)\phi\,, \end{aligned}$$

the only non-vanishing Poisson brackets in the first Hamiltonian structure read

$$\begin{aligned} \{\tilde{U}(x), \tilde{G}^\pm(y)\}_1 &= \mp\delta(x-y)\,, \\ \{\tilde{G}^+(x), \tilde{G}^-(y)\}_1 &= -3\phi(x)\,\delta(x-y)\,, \\ \{\tilde{T}(x), \tilde{G}^\pm(y)\}_1 &= \tfrac{3}{2}\,\delta'(x-y)\,, \\ \{\tilde{T}(x), \tilde{T}(y)\}_1 &= 6\phi(x)\,\delta'(x-y) + 3\phi'(x)\,\delta(x-y)\,, \end{aligned}$$

while, in the second structure, the non-vanishing brackets are

$$\begin{aligned}
\{\tilde{U}(x), \tilde{U}(y)\}_2 &= -\tfrac{2}{3}\,\delta'(x-y)\,, \\
\{\tilde{U}(x), \tilde{G}^{\pm}(y)\}_2 &= \pm\tilde{G}^{\pm}(x)\,\delta(x-y)\,, \\
\{\tilde{G}^{+}(x), \tilde{G}^{-}(y)\}_2 &= -\delta''(x-y) + 3\tilde{U}(x)\,\delta'(x-y) \\
&\quad + \big(\tilde{T}(x) + \tfrac{3}{2}\,\tilde{U}'(x) - 3\tilde{U}^2(x)\big)\,\delta(x-y)\,, \\
\{\tilde{T}(x), \tilde{U}(y)\}_2 &= -\tilde{U}(x)\,\delta'(x-y)\,, \\
\{\tilde{T}(x), \tilde{G}^{\pm}(y)\}_2 &= -\tfrac{3}{2}\,\tilde{G}^{\pm}(x)\,\delta'(x-y) - \tfrac{1}{2}\,\tilde{G}'^{\pm}(x)\,\delta(x-y)\,, \\
\{\tilde{T}(x), \tilde{T}(y)\}_2 &= \tfrac{1}{2}\,\delta'''(x-y) - 2\tilde{T}(x)\,\delta'(x-y) - \tilde{T}'(x)\,\delta(x-y)\,.
\end{aligned}$$

As explained before, the second Hamiltonian structure is an extension of the Virasoro algebra which, in this case, corresponds to the generalized W-algebra $W_3^{(2)}$ [4], in agreement with [7], where $\tilde{T} \equiv T^{\rm vir}$ is the Virasoro generator.

For simplicity, we have only considered the case of $W_3^{(2)}$ here; however, our construction can easily be extended to more complicated cases, the complexity of the equations being the only obstacle to such an endeavor.

6.4. The Hierarchy Associated to $w = R_{\alpha_0}$ in A_2

They Weyl group of A_2 has three conjugacy classes. One contains the Coxeter element, which leads to the Drinfel'd-Sokolov KdV hierarchies and their fractional generalizations considered above. The identity element of the Weyl group leads to a *homogeneous hierarchy*, which is considered below. In this section, we consider the third possibility. We take as our representative of the conjugacy class the reflection in the root $\alpha_0 = -\alpha_1 - \alpha_2$, where α_1 and α_2 are the simple roots. For a description of how the Heisenberg subalgebra is constructed in this case, we refer to [1]. The simplest KdV hierarchy associated to this conjugacy class is obtained by taking Λ to be the element of the Heisenberg subalgebra with lowest grade, in this case $i = 2$, i.e.

$$\Lambda = \Lambda_{2,0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix}.$$

Following [1], the potential, before gauge fixing, can be written as

$$q = \begin{pmatrix} y_1 & c & 0 \\ e & y_2 & d \\ a & f & -(y_2+y_1) \end{pmatrix},$$

which under a gauge transformation changes to

$$q \to \tilde{q} = \Phi\partial_x\Phi^{-1} + \Phi(q+\Lambda)\Phi^{-1} - \Lambda\,,$$

where

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & 0 \\ B & C & 1 \end{pmatrix}.$$

In this case, there exists a gauge transformation given by

$$A = -d\,, \qquad B = y_1 + \tfrac{1}{2}(y_2 - cd)\,, \qquad C = c\,.$$

which brings q into the canonical form

$$q^{\rm can} = \begin{pmatrix} 0 & 0 & 0 \\ G^+ & 0 & 0 \\ T & G^- & 0 \end{pmatrix} + \frac{U}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, U, G^+ and T are gauge invariant functionals of q. Notice that U corresponds to the conserved quantity h^0, which, following Sect. 3.6, is only in the centre of the first Poisson bracket and not the second.

The non-vanishing Poisson brackets are the following. For the first Hamiltonian structure they are

$$\begin{aligned} \{G^+(x), G^-(y)\}_1 &= \delta(x-y)\,, \\ \{T(x), T(y)\}_1 &= -2\delta'(x-y)\,, \end{aligned}$$

while for the second structure they are

$$\begin{aligned} \{U(x), U(y)\}_2 &= -\tfrac{2}{3}\,\delta'(x-y)\,, \\ \{U(x), G^\pm(y)\}_2 &= \pm G^\pm(x)\,\delta(x-y)\,, \\ \{G^+(x), G^-(y)\}_2 &= -\delta''(x-y) + 3U(x)\,\delta'(x-y) \\ &\quad + \big(T(x) + \tfrac{3}{2}\,U'(x) - 3U^2(x)\big)\,\delta(x-y)\,, \\ \{T(x), U(y)\}_2 &= -U(x)\,\delta'(x-y)\,, \\ \{T(x), G^\pm(y)\}_2 &= -\tfrac{3}{2}\,G^\pm(x)\,\delta'(x-y) - \tfrac{1}{2}\,G'^\pm(x)\,\delta(x-y)\,, \\ \{T(x), T(y)\}_2 &= \tfrac{1}{2}\,\delta'''(x-y) - 2T(x)\,\delta'(x-y) - T'(x)\,\delta(x-y)\,. \end{aligned}$$

In this case, the extension of the Virasoro algebra described by the second Poisson bracket is again the generalized W-algebra $W_3^{(2)}$, with $T \equiv T^{\rm vir}$ being the Virasoro generator. That the same algebra should appear in this example and in that of 6.3 can clearly be seen from the definitions of the brackets. Nevertheless, even though the second Hamiltonian structures are identical, the two hierarchies of partial differential equations are completely different.

6.5. The Homogeneous Hierarchies

The homogeneous hierarchies were defined in [1]. They arise from taking $\mathbf{s}[w]$ to be the homogeneous gradation, corresponding to the identity element of the Weyl group. The simplest such hierarchy has $\Lambda = z\mu \cdot H$ and $q = f \cdot H + \sum_{\alpha \in \Phi_g} q^\alpha E_\alpha$, where $\{H, E_\alpha\, \alpha \in \Phi_g\}$ is a Cartan-Weyl basis for g, and f and q^α are the dynamical variables. In order that the hierarchy be of type I, $\mu \cdot H$ must be regular, which implies that $\mu \cdot \alpha \neq 0\ \forall \alpha \in \Phi_g$. It was observed in [1] that $h^0 = f \cdot H$, for this hierarchy, and so the variables f are constant for all the flows.

The first and second symplectic structures are easily calculated for the example at hand. In the first case one finds that the non-zero brackets are

$$\{q^\alpha(x), q^\beta(y)\}_1 = (\mu \cdot \alpha)\,\delta_{\alpha+\beta,0}\,\delta(x-y)\,,$$

and so the variables f are indeed centres as proved in Proposition 3.6. In the second case the non-zero brackets are

$$\begin{aligned}\{\nu \cdot f(x), \lambda \cdot f(y)\}_2 &= (\nu \cdot \lambda(\delta'(x-y)\,,\\ \{q^\alpha(x), \nu \cdot f(y)\}_2 &= (\alpha \cdot \nu) q^\alpha(x) \delta(x-y)\,,\\ \{q^\alpha(x), q^\beta(y)\}_2 &= \delta_{\alpha+\beta,0}(\delta'(x-y) - \alpha \cdot f(x)\delta(x-y))\\ &\quad + \varepsilon(-\alpha,-\beta) q^{\alpha+\beta}(x)\delta(x-y)\,,\end{aligned}$$

where $\varepsilon(\alpha,\beta)$ is non-zero if $\alpha+\beta$ is a root of g. So the second symplectic structure is nothing but the Kac-Moody algebra $\hat{g}$ with a central extension. Notice that f is not in the centre of the second Poisson bracket algebra, an eventuality previously encountered in Proposition 3.3. Nevertheless, f is a constant under the flows of the hierarchy because the Hamiltonians satisfy the functional equation

$$\{f(x), H\}_2 = 0\,.$$

The Virasoro generator, in this case, is constructed from the fields f and q^α via the Sugawara construction.

7. Discussion

We have presented a systematic discussion of the Hamiltonian structure of the hierarchies of integrable partial differential equations constructed in [1]. It was found that the analogues of the KdV hierarchies admit two distinct yet coordinated Hamiltonian structures, whereas the associated partially modified hierarchies only admit a single Hamiltonian structure, generalizing the results of Drinfel'd and Sokolov. In addition, we found that the Miura map between a modified hierarchy and its associated KdV hierarchy, with the second Hamiltonian structure, is a Hamiltonian map. An aspect of the analysis that we have ignored is a thorough discussion of the restriction of the phase space $\mathscr{M}$ to a symplectic leaf, i.e. the rôle of the centres. This will be considered in a later publication, where we propose a group theoretic description of these hierarchies in terms of the AKS/coadjoint formulation of integrable systems. Ultimately it would be desirable to understand the relation between the Poisson brackets on the phase space $\mathscr{M}$ presented here, and the Poisson brackets that exist on the Akhiezer-Baker functions [13]. If the analysis in [13] generalises to the continuum limit, this would lead to an interpretation of the *dressing transformation* in terms of a Hamiltonian mapping. Connected with this, we are also intrigued by the relation of our work to that of Kac and Wakimoto [14], who, following the philosophy of the Japanese school, construct a hierarchy associated to each of the basic (level 1) representations of a Kac-Moody algebra, and each conjugacy class of the Weyl group of the underlying finite Lie algebra. These hierarchies are intimately connected to the vertex operator representations of Kac-Moody algebras; see [14] for further details.

Note that only the untwisted Kac-Moody algebras have been considered in this paper; it appears that the KdV hierarchies associated to twisted Kac-Moody algebras sometimes only admit a single Hamiltonian structure, see [2].

The Second Hamiltonian structure is invariant under an arbitrary conformal transformation. These transformations include the scale transformations which reflect the quasi-homogeneity of the equations of the hierarchy, that is to say all quantities have well defined scaling dimensions such that a scale transformation leaves the equations of motion invariant. On general principles, one would expect that the second Poisson bracket algebra should contain the (chiral) algebra of conformal transformations as a subalgebra, i.e. the Virasoro algebra. This would imply that the second Poisson bracket algebra is an extended (chiral) conformal algebra, generalizing the appearance of the W_n algebras in the work of Drinfel'd and Sokolov [2] and Gel'fand and Dikii [3]. This was indeed found to be the case for the examples that were considered.

Of particular interest is the question as to whether these hierarchies have any rôle to play in the non-perturbative structure of two dimensional gravity coupled to matter systems, generalizing the known connexion of the Drinfel'd-Sokolov hierarchies. It seems that one must supplement the hierarchy with an additional equation, the so-called *string equation*, which has to be consistent with the flows of the hierarchy. Then the potentials of the hierarchy are apparently related to certain correlation functions of the field theory. Details of this will be presented elsewhere.

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While preparing this preprint we received [15], in which the Hamiltonian structures of the $W_3^{(2)}$ algebra are discussed.

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