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A Family of Poisson Structures on Hermitian Symmetric Spaces

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Abstract. We investigate the compatibility of symplectic Kirillov-Kostant-Souriau structure and Poisson-Lie structure on coadjoint orbits of semisimple Lie group. We prove that they are compatible for an orbit compact Lie group iff the orbit is hermitian symmetric space. We prove also the compatibility statement for non-compact hermitian symmetric space. As an example we describe a structure of symplectic leaves on $\mathbf{CP}^n$ for this family. These leaves may be considered as a perturbation of Schubert cells. Possible applications to infinite-dimensional examples are discussed.

1. Introduction

It became clear recently that quantum groups play a fundamental role in modern field-theoretical models. An interpretation of quantum groups as new symmetries of chiral conformal field theory was discussed in [AGS] and [FG]. A quasiclassical counterpart of quantum groups, Poisson-Lie groups was identified with classical symmetries of classical chiral theory [F, F-G].

By a well-known prescription of Kirillov-Kostant-Souriau coadjoint orbits $\mathcal{O}$ of a compact semisimple Lie group G have a natural symplectic structure and hence are Poisson manifolds. The developments of the Poisson-Lie group theory [ST, W] make it possible to introduce another Poisson structure on $\mathcal{O}$ which is induced from the Poisson-Lie structure on the compact semisimple Lie group G. We denote these structures by π_{Kir} and $\pi_{\mathrm{P\text{-}L}}$, respectively. The main result of this paper is to show that π_{Kir} and $\pi_{\mathrm{P\text{-}L}}$ are compatible on the orbit $\mathcal{O}$ iff $\mathcal{O}$ is a hermitian compact symmetric space (h.c.s.s.) occurring in the Cartan list.

On the other side, using the duality between compact and non-compact symmetric spaces, we can show that $\pi_{\mathrm{P\text{-}L}}$ and an imaginary part of the canonical hermitian form are compatible for non-compact hermitian symmetric spaces. The consideration of symplectic leaves of the family $\pi_{\mathrm{Kir}}+\pi_{\mathrm{P\text{-}L}}$ in CP^n provides an interesting example of the deformation of the Schubert cells (see Sect. 7).

The infinite-dimensional analogues of π_{Kir} and $\pi_{\text{P-L}}$ are so-called first and second Gelfand-Dikii Hamiltonian structures on the affine space of differential operators $\mathscr{L}_n = \{L = \partial^n + u_{n-1}\partial^{n-1} + \ldots + u_0\}$. We briefly discuss the Manin triple, connected with the Lie algebra of symbols of pseudodifferential operators. It endows Lie algebras of differential operators (DOP) and symbols of integral Volterra operators (IO) with Lie bialgebra structures. Action of DOP on $\mathscr{L}_n$ with second Gelfand-Dikii structure is the action of the Lie bialgebra on Poisson manifold [R]. Lie bialgebra structure on DOP is supposed to be coboundary after some extension. On the other hand, we show that the Lie bialgebra structure for IO is not coboundary one. The analogous result holds for Borel subgroup in the finite-dimensional case.

We feel it would be interesting to find more deep connections between hermitian symmetric spaces and their infinite-dimensional counterparts in context of differential operators. Another way to treat the second Gelfand-Dikii structure in terms of Poisson-Lie group theory are [STS] and [Z]. Preliminary version of [Z] was quite suggestive for us in doing the present work.

2. Poisson-Lie Groups. Generalities

2.1. We start with a description of some basic facts about Poisson-Lie groups. Most of this material may be found in [D, ST, LW]. We follow notations of [LW].

Let π be a *Poisson structure* on a manifold M. This means that π is a bivector field, $\pi \in \Lambda^2 TM$ such that corresponding *Poisson bracket*

$$\{f, g\} = \langle \pi, df \wedge dg \rangle \quad \text{for all} \quad f, g \in C^\infty(M)$$

endows $\text{Fun}(M)$ with Lie algebra structure. The pair (M, π) is called a *Poisson manifold.*

2.2. Lie group G is called *Poisson-Lie group* if it is a Poisson manifold such that the multiplication $m: G \times G \to G$ is a morphism of Poisson manifolds [the Poisson structure on $G \times G$ is the product of Poisson structures on factors and a map $f: M \to N$ between two Poisson manifolds is *Poisson morphism* if $\{f^*g, f^*h\}_M = f^*\{g, h\}_N$, for arbitrary functions $g, h \in \mathbf{C}^\infty(N)$].

2.3. Let $\mathbf{g}$ be Lie algebra, $\mathbf{g}^*$ be dual vector space to $\mathbf{g}$. We say that $\mathbf{g}$ is a *Lie bialgebra* if there is a Lie algebra structure $[\,,\,]_*$ on $\mathbf{g}^*$ such that the map $\delta: \mathbf{g} \to \Lambda^2\mathbf{g}$ *(called the co-bracket)*, dual to the bracket $[\,,\,]_*: \Lambda^2\mathbf{g}^* \to \mathbf{g}^*$ is a 1-cocycle with respect to the adjoint action of $\mathbf{g}$ on $\Lambda^2\mathbf{g}$.

2.4. Let G be connected and a simply connected Lie group, and let $\mathbf{g}$ be its Lie algebra. Then [D] there is one-to-one correspondence between Poisson-Lie group structures on G and Lie bialgebra structures on $\mathbf{g}$.

As V. Drinfeld showed every Poisson-Lie structure on a semisimple connected G has the following form:

$$\pi(g) = l_{g*}(\mathbf{r}) - r_{g*}(\mathbf{r}), \tag{1}$$

where l_{g*} and r_{g*} denote tangent maps of left and right translations by $g \in G$. The element $\mathbf{r} \in \Lambda^2\mathbf{g}$ satisfies the following condition:

$$[[\mathbf{r}, \mathbf{r}]] := [\mathbf{r}^{12}, \mathbf{r}^{13}] + [\mathbf{r}^{12}, \mathbf{r}^{23}] + [\mathbf{r}^{12}, \mathbf{r}^{23}] \in \Lambda^3\mathbf{g}, \tag{2}$$

where the r.h.s. is invariant under the adjoint action of $\mathbf{g}$.

This condition is called a *modified Yang-Baxter equation* and the bracket $[[\,,\,]]: \Lambda^2\mathbf{g}\otimes\Lambda^2\mathbf{g}\to\Lambda^3\mathbf{g}$ is a Schouten-Nijenhuis bracket. Here $\mathbf{r}^{12}$, e.g., denotes an element $\mathbf{r}^{12}=\mathbf{r}\otimes\mathbf{1}_3\in(\mathbf{g}\otimes k)^{\otimes 3}$ ($k=\mathbf{R},\mathbf{C}$); $\mathbf{r}$ is called a *classical* $\mathbf{r}$*-matrix.*

The condition (2) ensures that the bracket $[\,,\,]^*$ on $\mathbf{g}^*$, satisfies the Jacobi identity. The corresponding Lie bialgebra structure is calculated in the obvious way: the co-bracket δ is given by

$$\delta(x)=d_e\pi(x)=L_{\bar{x}}\pi(e)=\left.\frac{d}{dt}\right|_{t=0} r_{(e^{-tx})_*}\pi(e^{tx})=\mathrm{ad}_x(\mathbf{r}),$$

where $d_e\pi$ is the intrinsic derivative of a polyvector field on G with $\pi(e)=0$, $\bar{x}$ is any vector field on G with $\bar{x}(e)=x$ and $L_{\bar{x}}$ denotes the Lie derivative [LW].

The Poisson structures of the form (1) are called *coboundary* or $\mathbf{r}$-matrix structures. Since for a connected semisimple or a compact Lie group G every 1-cocycle is a coboundary, one has

Proposition 2.1. *The Poisson-Lie structures on a connected semisimple or a compact Lie group G are in one-to-one correspondence with the solutions* $\mathbf{r}\in\Lambda^2\mathbf{g}$ *of the modified Yang-Baxter equation.*

2.5. Let $\mathbf{g}$ be a Lie bialgebra. There is a unique Lie algebra structure on the vector space $\mathbf{g}\oplus\mathbf{g}^*$ such that
1) $\mathbf{g}$ and $\mathbf{g}^*$ are Lie subalgebras;
2) the symmetric bilinear form on $\mathbf{g}\oplus\mathbf{g}^*$ given by the relation $\langle X+\xi, Y+h\rangle =\langle X,\eta\rangle+\langle Y,\eta\rangle$ for all $X,Y\in\mathbf{g}$, $\xi,\eta\in\mathbf{g}^*$ is invariant.

This structure is given by $[X,\xi]=-\mathrm{ad}^*_X(\xi)+\mathrm{ad}^*_\xi(X)$ for $X\in\mathbf{g}$, $\xi\in\mathbf{g}^*$, where ad^* is the coadjoint action. This Lie algebra is denoted by $\mathbf{g}\bowtie\mathbf{g}^*$ and $(\mathbf{g}\bowtie\mathbf{g}^*,\mathbf{g},\mathbf{g}^*)$ is an example of a *Manin triple.* In general, a Manin triple is a decomposition of a Lie algebra $\mathbf{g}$ with a non-degenerate invariant scalar product $\langle\,,\,\rangle$ into direct sum of isotropic with respect to $\langle\,,\,\rangle$ vector spaces, $\mathbf{g}=\mathbf{g}_+\oplus\mathbf{g}_-$ such that $\mathbf{g}_\pm$ are Lie subalgebras of $\mathbf{g}$. It is well-known that there is one-to-one correspondence between Lie bialgebras and Manin triples.

2.6. Let G be a connected simply connected Poisson-Lie group, $\mathbf{g}$ its Lie algebra and $(\mathbf{g}\bowtie\mathbf{g}^*,\mathbf{g},\mathbf{g}^*)$ the Manin triple. By duality, $(\mathbf{g}^*\bowtie\mathbf{g},\mathbf{g}^*,\mathbf{g})$ is also a Manin triple. Then $\mathbf{g}^*$ is a Lie bialgebra. This enables us to consider a connected and simply connected Lie group G^* with a Poisson-Lie structure π^* and with the *tangent Lie bialgebra* $\mathbf{g}^*$. The Poisson-Lie group (G^*,π^*) is called *dual* to (G,π).

There are natural left and right actions of dual Poisson-Lie group G^* on G. These actions are called left (right) *dressing transformations.* The dressing transformations are Poisson actions as Semenov-Tyan-Shansky [ST] proved.

3. Poisson-Lie Subgroups and Cosets

Now we pass to Poisson-Lie subgroups. We should like to observe briefly well-known facts about Poisson-Lie cosets. The main sources of references are [ST] and [LW].

3.1. A Lie subgroup H of a Poisson-Lie group G is called a *Poisson-Lie subgroup* if it has its own Poisson-Lie structure and the inclusion $i: H\subset G$ is a Poisson map.

Proposition 3.1 [ST]. *The following conditions are equivalent:*
1) *H is a Poisson-Lie subgroup;*

2) *(assuming H is connected)* $\mathbf{h}^\perp \subset \mathbf{g}^*$ *is an ideal (where* $\mathbf{h}$ *is the Lie algebra of H);*
3) *H is invariant under the right (or the left) action of* G^* *on G.*

3.2. A coset space G/H with a Poisson structure is called a *Poisson coset space of a Poisson-Lie group G*, if the natural map $G \to G/H$ is a Poisson map.

Clearly, the only possible Poisson structure on G/H making it a Poisson coset space can be defined as follows. Consider a space $\mathscr{F}(G/H)$ of functions on G/H as H-right invariant functions on G. Then take their Poisson bracket. If the result is H-invariant then we get the desired Poisson bracket on G/H. Hence we have the following

Proposition 3.2. *The following conditions are equivalent:*
1) G/H *is a Poisson coset space;*
2) $\pi_l(h) = 0$ *in* $\Lambda^2(\mathbf{g}/\mathbf{h})$, $h \in H$;
3) $\pi_r(h) = 0$ *in* $\Lambda^2(\mathbf{g}/\mathbf{h})$, $h \in H$;
4) *(assuming H is connected)* $\mathbf{h}^\perp$ *is a subalgebra in* $\mathbf{g}^*$.
Here $\pi_{l,r}: G \to \Lambda^2\mathbf{h}$, $\pi_l(g) = l_{g_*^{-1}}\pi(g)$, $\pi_r(g) = r_{g_*^{-1}}\pi(g)$ *for* $g \in G$.

Let $H^\perp$ be a connected subgroup in G^* with the Lie algebra $\mathbf{h}^\perp$. Then the following properties may be extracted easily from [LW]:
1) A natural action $G \times G/H \to G/H$ is a Poisson map.
2) (Assuming G^* acts on G) symplectic leaves of the Poisson structure on G/H are the orbits of $H^\perp$-action.

Clearly, condition 4) of Proposition 3.2 is weaker than condition 2) of Proposition 3.1. Hence for a Poisson-Lie subgroup H the space G/H is a Poisson coset space, but to get a Poisson coset space G/H the subgroup H need not necessarily be a Poisson-Lie subgroup. In other words the set of Poisson-Lie subgroups is "smaller" than the set of subgroup-stabilizers for a given Poisson-Lie group.

4. Iwasawa Decomposition, Manin Triples, and Poisson-Lie Structures for Real Lie Groups

In this section we apply the generalities of Sects. 2 and 3 to real forms of a semisimple complex Lie group. We recall briefly the Iwasawa decomposition of complex semisimple Lie group [He].

4.1. Let $\mathbf{g}$ be a semisimple Lie algebra over $\mathbf{C}$, $\mathbf{g}^0$ be the same algebra considered over $\mathbf{R}$ ($\dim_{\mathbf{R}}\mathbf{g}^0 = 2\dim_{\mathbf{C}}\mathbf{g}$. Let $\mathbf{u} \subset \mathbf{g}^0$ be a compact real form of $\mathbf{g}$, $\mathbf{h}_c \subset \mathbf{u}$ be Cartan subalgebra of $\mathbf{u}$ and $\mathbf{h} = \mathbf{C}\mathbf{h}_c$ be a complexification of $\mathbf{h}_c$. If we choose the system Δ_+ of positive roots of $\mathbf{h}$ in $\mathbf{g}$, then we obtain a decomposition

$$\mathbf{g}^0 = \mathbf{u} + \mathbf{a} + \mathbf{n}_+ , \tag{3}$$

where $\mathbf{n}_+ = \sum_{\alpha \in \Delta_+} \mathbf{g}_\alpha$ and $\mathbf{a} \subset \mathbf{h}$, $\mathbf{a} = i\mathbf{h}_c$ is a non-compact part of the Cartan subalgebra $\mathbf{h}$.

Let also G^0 be a connected Lie group with the Lie algebra $\mathbf{g}^0$ (G^0 is a complex semisimple Lie group) U be its maximal compact subgroup with Lie algebra $\mathbf{u}$, A and N_+ be the connected subgroups of G^0 with Lie algebras $\mathbf{a}$ and $\mathbf{n}_+$. Then the mapping

$$U \times A \times N_+ \to G^0 . \tag{4}$$

$(u, a, n) \to uan$ is a diffeomorphism. This diffeomorphism is a specialization of general Iwasawa decomposition to a case of complex semisimple Lie groups [He].

4.2. Now we recall the results by Lu-Weinstein on Poisson-Lie structures generated by this decomposition. Let $\mathbf{b} = \mathbf{a} + \mathbf{n}_+$ and let B be the corresponding connected Lie subgroup of G^0. Then the triple $(\mathbf{g}^0, \mathbf{u}, \mathbf{b})$ is actually a Manin triple. Indeed, we need only an invariant inner product. Let us assume that $\mathbf{u}$ is a fixed point set of a standard Chevalley antiinvolution σ of $\mathbf{g}$: $\sigma(E_{\pm\alpha}) = -E_{\mp\alpha}$, $\sigma(H_\alpha) = -H_\alpha$, $\sigma(\lambda X) = \bar{\lambda}\sigma(X)$, where $\{E_\alpha, H_\alpha\}$ is a Chevalley basis of $\mathbf{g}$ with respect to a fixed Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$. If $K(X, Y)$ is a Killing form in $\mathbf{g}$, then

$$K(\sigma X, \sigma Y) = \overline{K(X, Y)} \tag{5}$$

and the imaginary part $\operatorname{Im} K(X, Y)$ of the Killing form is non-degenerated in $\mathbf{g}^0$. By the relation (5), $\operatorname{Im} K(X, Y)$ vanishes on $\mathbf{u}$ and on $\mathbf{b}$. Hence $(\mathbf{g}^0, \mathbf{u}, \mathbf{b})$ is a Manin triple and U and B are dual Poisson-Lie groups. The dressing transformations (4) are $u \mapsto u'$, where u' is solution of the equation

$$b^{-1} \cdot u = u' \cdot b', \qquad u, u' \in U, \qquad b, b' \in B,$$

for fixed u and b.

4.3. We can generalize this construction to other real forms of a complex semisimple Lie group. Again we start from the complex semisimple Lie algebra $\mathbf{g}$ and its compact subalgebra $\mathbf{u}$, which is defined by some Chevalley antiinvolution σ. We may describe other real forms $\mathbf{g}^\tau$ of $\mathbf{g}$ by antiinvolutions τ of $\mathbf{g}$, which commute with σ: $\mathbf{g}^\tau = \{g \in \mathbf{g}^0,\ \tau(g) = g\}$.

Let $\theta = \sigma\tau$ be Cartan involution and $\mathbf{h}$ be θ-stable Cartan subalgebra of $\mathbf{g}$, generated over $\mathbf{C}$ by the Cartan subalgebra $\mathbf{h}_\tau$ of $\mathbf{g}^\tau$. If we choose the system Δ_+ of positive roots of $\mathbf{h}$ in $\mathbf{g}$, then the imaginary part of the Killing form $\operatorname{Im} K(X, Y)$ vanishes in $\mathbf{g}^\tau$ and in $\mathbf{b} = \mathbf{n}_+ + \mathbf{a}$, where $\mathbf{n}_+ = \sum_{\alpha \in \Delta_+} \mathbf{g}_\alpha$, $\mathbf{a} = i \cdot \mathbf{h}_\tau$. Indeed, $K(\theta X, \theta Y) = K(X, Y)$ as θ is an automorphism of $\mathbf{g}$ and

$$K(X, Y) = K(\sigma(\tau X), \sigma(\tau Y)) = \overline{K(\tau X, \tau Y)}.$$

The following proposition describes the Poisson-Lie structures for simple real Lie groups with a compact Cartan subgroup.

Proposition 4.1. *Let $\mathbf{h}_\tau$ be a compact Cartan subalgebra of $\mathbf{g}^\tau$. Then the triple*

$$(\mathbf{g}^0, \mathbf{g}^\tau, \mathbf{b}) \tag{6}$$

is a Manin triple with respect to a pairing $\operatorname{Im} K(X, Y)$.

Proof. We have to prove only that $\mathbf{g}^\tau \cap \mathbf{b} = 0$. Due to the assumption, $\theta(h) = h$ for any $h \in \mathbf{h}$, and θ leaves invariant the root spaces of $\mathbf{h}$ in $\mathbf{g}$, $\theta(E_\alpha) = c(\alpha) \cdot E_\alpha$ for some $c(\alpha) \in \mathbf{C}$. So, $\tau(E_\alpha) = -c(\alpha) E_{-\alpha}$ and $\mathbf{g}^\tau \cap \mathbf{b} = 0$.

Remark. More subtle considerations show that the statement of this proposition is valid for the Lie algebras $sl(n, \mathbf{R})$ and $so(2p+1, 2q+1)$ which do not contain compact Cartan subalgebras, if we start, following [Ga], from the maximally compact Cartan subalgebra. The only simple Lie algebras, for which our arguments do not work, are $A2 = su^*(2n)$, $E1$ and $E4$ in terms of Cartan classification [He].

On the group level we have the pairs of dual Poisson-Lie groups G^τ, B. Contrary to the compact case, the dressing transformations for non-compact Lie groups may be defined only locally, because there is no global analog of Iwasawa decomposition of G^0 in this case.

4.4. S. Majid noticed that in the compact case Lie bialgebra structure for $\mathbf{u}$ is a real form of the standard Lie bialgebra structure for $\mathbf{g}=\mathbf{C}\mathbf{u}$, described by the Drinfeld-Jimbo **r**-*matrix* [M, D].

This structure is given by the following Manin triple $(\mathbf{u_C}\bowtie\mathbf{b_C}, \mathbf{u_C}, \mathbf{b_C}) = (\mathbf{g}\oplus\mathbf{g}, \mathbf{g}_1, \mathbf{g}_2)$, where $\mathbf{g}_1=\{(X,Y)\in\mathbf{g}\oplus\mathbf{g}\,|\,X=Y\}\simeq\mathbf{g}$, $\mathbf{g}_2=\mathbf{h_C}\oplus\mathbf{n_C}$ and $\mathbf{h_C}=\{(X,-X)\,|\,X\in\mathbf{h}\}$, $\mathbf{n_C}=\{(X,Y)\,|\,X\in\mathbf{n}_-,\ Y\in\mathbf{n}_+\}$. The inner product of (X_1,Y_1) and (X_2,Y_2) is $\frac{1}{2}(K(X_1,X_2)-K(Y_1,Y_2))$ and identifies $\mathbf{g}_2=\mathbf{h_C}\oplus\mathbf{n_C}$ with $\mathbf{g}_1^*=\mathbf{u}_\mathbf{C}^*$.

In terms of the Drinfeld-Jimbo r-matrix

$$\mathbf{r}=\frac{i}{2}\sum_{\alpha\in\Delta^+} E_\alpha\wedge E_{-\alpha}\,.$$

Majid's statement may be explained as a formula for the cobracket

$$\delta(X)=[X\otimes 1+1\otimes X,\mathbf{r}_0]\,,\qquad X\in\mathbf{u}\,,\tag{7}$$

where $\mathbf{r}_0\in\Lambda^2\mathbf{u}$ is a real form of $i\mathbf{r}$:

$$\mathbf{r}_0=\tfrac{1}{4}\sum_{\alpha\in\Delta^+} V_\alpha\wedge W_\alpha\tag{8}$$

and $V_\alpha, W_\alpha\in\mathbf{u}$, $V_\alpha=E_\alpha-E_{-\alpha}$, $W_\alpha=i(E_\alpha+E_{-\alpha})$.

An analogous statement holds for non-compact real Lie algebras with a compact Cartan subalgebra. The cobrackets for the Lie bialgebra structure (6) may be described up to the constant coefficients by the formulas (7) and (8), where $V_\alpha=E_\alpha+\tau(E_\alpha)$, $W_\alpha=i(E_\alpha-\tau(E_\alpha))$.

4.5. A non-compact real Lie group may be equipped with different Poisson-Lie structures. The first reason is the existence of different conjugacy classes of Cartan subgroups. For example, we may consider $sl(n,\mathbf{R})$ as a split real form of $sl(n,\mathbf{C})$. We see that the standard Manin triple for $sl(n,\mathbf{C})$ from 4.4 is defined over $\mathbf{R}$ and thus defines the bialgebra structure for $sl(n,\mathbf{R})$. This bialgebra structure cannot be isomorphic to that from Proposition 4.1, because the kernels of the cobrackets in these two cases are non-conjugated Cartan subalgebras.

On the other hand, let us consider the Manin triple (6) for the case $\mathbf{g}^\tau=su(p,q)$. It looks like

$$(sl(n,\mathbf{C}), su(p,q), \mathbf{b})\,,\qquad n=p+q\,,$$

where

$$\mathbf{b}=\begin{pmatrix}\lambda_1 & & b_{i,j}\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

and $\lambda_1,\lambda_2,\ldots,\lambda_n\in\mathbf{R}_+$, $b_{i,j}\in\mathbf{C}$, $i<j$.

The algebra $\mathbf{b}$ does not depend on p and q, but for different p and q we obtain non-isomorphic cobrackets for $\mathbf{b}$, that dualize the brackets from $su(p,q)$.

A classification of Poisson-Lie structures, associated with the Drinfeld-Jimbo r-matrix in the case of simple real Lie groups was given in [CGO].

4.6. Let P be a parabolic subgroup in G^0. Every coadjoint orbit $\mathcal{O}$ of a compact simple group U for a fixed point l is naturally isomorphic to a coset U/P_u, where $P_u=U\cap P$.

Proposition 4.2 (Lu-Weinstein). *$U \cap P$ is a Poisson-Lie subgroup of the Poisson-Lie group U.*

Hence every coadjoint orbit $\mathcal{O}$ of the compact semisimple Poisson-Lie group U has the natural Poisson structure π converting $\mathcal{O}$ into a Poisson-Lie coset space. The symplectic leaves of this induced Poisson structure on $\mathcal{O}$ are precisely the B-orbits on $G^0/P \simeq U/U \cap P$ or Bruhat cells. We prefer to call this Poisson structure a *Poisson-Lie structure* to indicate its Poisson-Lie group origins.

Analogously, let G^τ be a real form of G^0 with a compact Cartan subgroup and let the parabolic subalgebra $\mathbf{p}$ contains compact Cartan subalgebra $\mathbf{h}_c$. Then it is not difficult to see that the Poisson-Lie structure is well-defined on a coset space $G^\tau/P \cap G^\tau$.

4.7. We should like to point out that this situation is specific for compact groups and may be false for other Poisson-Lie groups.

Let us consider the dual Poisson-Lie group $U^* = B$ for instance. Consider $U = SU(n)$ and the element $\lambda = \sum (e_{i+1,i} - e_{i,i+1})$,

$$\lambda = \begin{bmatrix} 0 & -1 & & & 0 \\ 1 & 0 & -1 & & \\ & 1 & & \ddots & \\ & & \ddots & & -1 \\ 0 & & & 1 & 0 \end{bmatrix} \in \mathbf{u} \simeq su(n).$$

We compute the stabilizer of λ in $\mathbf{b}$

$$\mathrm{Stab}(\lambda) = \begin{bmatrix} 0 & \mu_1 & & & c_{ij} \\ & 0 & \mu_2 & & \\ & & & \ddots & \\ & & & & \mu_{n-1} \\ 0 & & & & 0 \end{bmatrix}, \quad \mu_i \in \mathbf{R}, \quad c_{ij} \in \mathbf{C}.$$

Clearly,

$$(\mathrm{Stab}(\lambda))^\perp = \begin{bmatrix} i\lambda_1 & \nu_1 & & & 0 \\ -\nu_1 & i\lambda_2 & & & \\ & & \ddots & & \\ & & \ddots & & \nu_{n-1} \\ 0 & & & -\nu_{n-1} & i\lambda_n \end{bmatrix}, \quad \lambda_i, \nu_i \in \mathbf{R}, \quad \sum_{i=1}^{n} \lambda_i = 0,$$

and $(\mathrm{Stab}((\lambda))^\perp$ *is not a subalgebra* in $SU(n)$. Hence $\mathcal{O}_\lambda$ is not a Poisson-Lie coset space.

This example helps one to prove

Proposition 4.3. *The Poisson-Lie structure on B is not a coboundary structure.*

If it were a coboundary structure then it would be $\pi_l(s) = \mathbf{r} - \mathrm{Ad}_{s^{-1}}(\mathbf{r})$ for some $s \in S$, where S is the stabilizer of λ in $\mathbf{b}$. However, this stabilizer acts trivially on $\mathbf{b}/\mathrm{Stab}(\lambda)$ and hence $\pi_l(s)$ would be zero in $\Lambda^2(\mathbf{b}/\mathrm{Stab}(\lambda))$ and due to Proposition 3.2

we conclude that $(\mathrm{Stab}(\lambda))^{\perp}$ would be a subalgebra in $\mathbf{b}^* = \mathbf{u}$ but it is not. This provides the desired contradiction.

We shall indicate an analogue of this example in the infinite-dimensional situation in Sect. 8.

5. Poisson Structures on Hermitian Symmetric Spaces

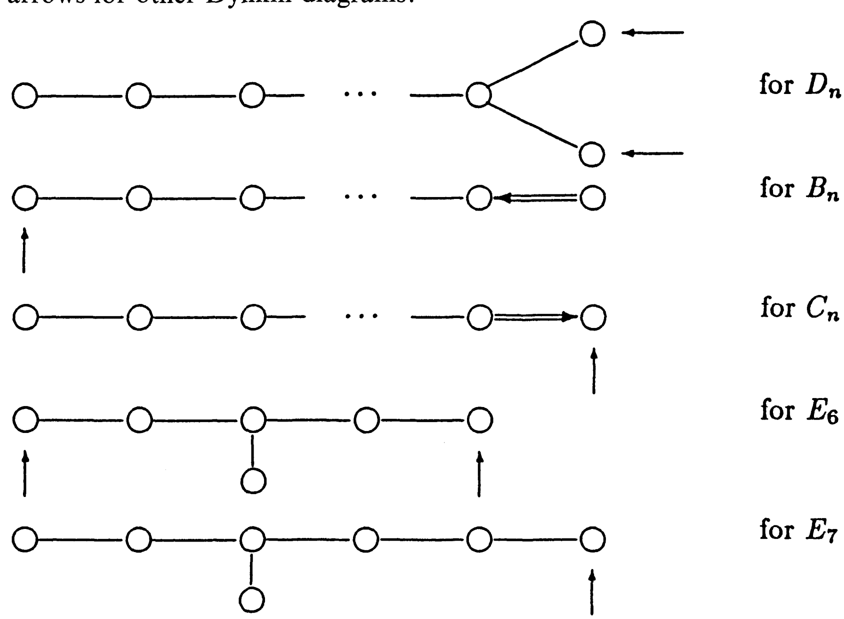
We want to discuss whether the pair of Poisson $(\pi_{\mathrm{Kir}}, \pi_{\mathrm{P\text{-}L}})$ structures for a coadjoint orbit $\mathcal{O}$ are compatible. Here π_{Kir} is symplectic Kirillov structure and $\pi_{\mathrm{P\text{-}L}}$ is the Poisson-Lie structure induced by the coboundary Poisson-Lie structure on U. We investigate also the compatibility of the pair of Poisson structures $(\pi_{\mathrm{Her}}, \pi_{\mathrm{P\text{-}L}})$ for non-compact hermitian symmetric spaces, where π_{Her} is an imaginary part of the canonical invariant hermitian form.

5.1. Recall that two Poisson structures (π_1, π_2) are called *compatible* if every linear combination $\lambda_1\pi_1 + \lambda_2\pi_2$ is again a Poisson structure. To verify the compatibility of π_1 and π_2 we need to verify that Schouten-Nijenhuis bracket $[[\pi_1, \pi_2]]$ is equal to zero whence $[[\pi_1, \pi_2]] = [[\pi_2, \pi_2]] = 0$.

5.2. Let U be a compact real form of a simple complex Lie group G. Let P be a parabolic subgroup in G, $P_U = P \cap U$, $\mathbf{p}$ and $\mathbf{p}_{\mathbf{u}}$ be Lie algebras of P and P_U. Denote by Δ_P the following subset of positive roots $\Delta_P = \{\alpha \in \Delta_+, E_{-\alpha} \notin \mathbf{p}\}$. Let $\mathcal{O}_l \simeq U/P_U$ be a coadjoint orbit for U with a fixed point l.

Theorem 5.1. *The Kirillov structure π_{Kir} on $\mathcal{O}_l$ is compatible with the Poisson-Lie structure $\pi_{\mathrm{P\text{-}L}}$ if and only if this orbit $\mathcal{O}_l$ is a compact hermitian symmetric space.*

5.3. **Remarks.** 1) It is evident that U/P_U is a compact complex manifold (a flag manifold). One needs to verify the condition that U/P_U is a symmetric space.
2) If U/P_U is a symmetric space then P is the maximal parabolic subgroup (that is only one simple root α belongs to Δ_P). The root α is arbitrary for A_n and is marked by arrows for other Dynkin diagrams:



5.4. Proof of Theorem 5.1. We need to calculate the Schouten-Nijenhuis bracket $[[\pi_{\mathrm{Kir}}, \pi_{\mathrm{P\text{-}L}}]]$ on U/P_U. Consider functions on U/P_U as P_U invariant functions on U and compute the Schouten-Nijenhuis bracket for some bivector fields π^0_{Kir} and $\pi^0_{\mathrm{P\text{-}L}}$ on U (π^0_{Kir} and $\pi^0_{\mathrm{P\text{-}L}}$ coincide with π_{Kir} and $\pi_{\mathrm{P\text{-}L}}$ on P_U-invariant functions). Then we apply $[[\pi^0_{\mathrm{Kir}}, \pi^0_{\mathrm{P\text{-}L}}]]$ to P_U-invariant functions and get the bracket $[[\pi_{\mathrm{Kir}}, \pi_{\mathrm{P\text{-}L}}]]$ on U/P_U.

Let us consider the standard decomposition

$$\mathbf{u}=\mathbf{h}+\sum_{\alpha\in\Delta_+}\mathbf{R}V_\alpha\oplus\mathbf{R}W_\alpha\,, \tag{9}$$

where $V_\alpha=E_\alpha-E_{-\alpha}$, $W_\alpha=i(E_\alpha+E_{-\alpha})$. We assume that $\mathbf{h}\subset\mathrm{Stab}(l)$ for a fixed point l of U/P_U and identify l with some $h\in\mathbf{h}$ using the Killing form K. Then $\alpha\in\Delta_P$ iff $h(\alpha)=0$ and one can take

$$\mathbf{r}_P=\frac{1}{2}\sum_{\alpha\in\Delta_P}\frac{1}{h(\alpha)}V_\alpha\wedge W_\alpha$$

and

$$\pi^0_{\mathrm{Kir}}(g)=l_{g*}(\mathbf{r}_P)\,.$$

Furthermore, $\pi^0_{\mathrm{P\text{-}L}}(g)=l_{g*}(\mathbf{r}_0)-r_{g*}(\mathbf{r}_0)$, where $g\in U$ and

$$\mathbf{r}_0=\tfrac{1}{4}\sum_{\alpha\in\Delta_+}V_\alpha\wedge W_\alpha\,.$$

Lemma 5.1.

$$[[\pi^0_{\mathrm{Kir}}(g), r_{g*}(\mathbf{r}_0)]]=0\,.$$

Proof of Lemma 5.1. Indeed, $\pi^0_{\mathrm{Kir}}(g)=l_{g*}(\mathbf{r}_P)$ is composed from infinitesimal right shifts and $r_{g*}(\mathbf{r}_0)$ is a right-invariant bivector. On the other hand, $r_{g*}(\mathbf{r}_0)$ is composed from infinitesimal left shifts and $l_{g*}(\mathbf{r}_P)$ is left-invariant.

Lemma 5.2.

$$l_{g^{-1}_*}[[\pi^0_{\mathrm{Kir}}(g), l_{g*}(\mathbf{r}_0)]]=2[\mathbf{r}_P^{13},\mathbf{r}_0^{12}]+2[\mathbf{r}_P^{12},\mathbf{r}_0^{13}]+\mathrm{cycl}(1,2,3)\,, \tag{10}$$

Proof of Lemma 5.2. A direct computation.

Now we get a purely algebraic problem: does the r.h.s. of (10) equal to zero in $\Lambda^3(\mathbf{u}/\mathbf{p_u})$? Instead of $\mathbf{u}$ one can consider $\mathbf{g}=\mathbf{u}\otimes\mathbf{C}$ and $\mathbf{p}=\mathbf{p}_u\otimes\mathbf{C}$. Then $V_\alpha\wedge W_\alpha=2iE_\alpha\wedge E_{-\alpha}$. We assume that E_α are normalized $[K(E_\alpha, E_{-\beta})=\delta_{\alpha,\beta}]$ with respect to the Killing form K and we denote by $N_{\alpha,\beta}$ the corresponding structure constants:

$$[E_\alpha, E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta}\,. \tag{11}$$

We need also well known identities for the structure constants $N_{\alpha,\beta}$ (see [He]), which are direct consequences of an invariance of a Killing form

$$N_{\alpha,\beta}=N_{-\alpha-\beta,\alpha}=N_{\beta,-\alpha-\beta}\,.$$

Lemma 5.3.

$$\begin{aligned}&[E^1_\alpha\wedge E^3_{-\alpha}, E^1_\beta\wedge E^2_{-\beta}]\\ &=\tfrac{1}{4}[N_{\alpha,\beta}E_{\alpha+\beta}\otimes E_{-\beta}\otimes E_{-\alpha}-N_{-\alpha,\beta}E_{\beta-\alpha}\otimes E_{-\beta}\otimes E_\alpha\\ &-N_{\alpha,-\beta}E_{\alpha-\beta}\otimes E_\beta\otimes E_{-\alpha}+N_{-\alpha,-\beta}E_{-\alpha-\beta}\otimes E_\beta\otimes E_\alpha]\,.\end{aligned}$$

Proof. Straightforward calculation.

Lemma 5.4. *Consider two bivectors* $\mathbf{r}_c = \sum_{\alpha\in\Delta_+} c(\alpha)E_\alpha \wedge E_{-\alpha}$ *and* $\mathbf{r}_d = \sum_{\beta\in\Delta_+} d(\beta) \times E_\beta \wedge E_{-\beta}$. *Then* $E_{\alpha+\beta}\otimes E_{-\alpha}\otimes E_{-\beta}$ *enters in* $([\mathbf{r}_c^{13}, \mathbf{r}_d^{12}] + [\mathbf{r}_d^{13}, \mathbf{r}_c^{12}] + \mathbf{r}_{123})$ *with a coefficient*

$$N_{\alpha,\beta}[c(\alpha)d(\beta) - c(a+\beta)d(\alpha) - c(\beta)d(\alpha+\beta)] + N_{\alpha\beta}[d(\alpha)c(\beta) - d(a+\beta)c(\alpha) - d(\beta)c(\alpha+\beta)].$$

Proof of Lemma 5.4. One needs to collect carefully similar terms. The non-zero coefficient before $E_{\alpha+\beta}\otimes E_{-\alpha}\otimes E_{-\beta}$ occurs three times only:
(i) in $[E_\alpha^{(1)} \wedge E_{-\alpha}^{(3)}, E_\beta^{(1)} \wedge E_{-\beta}^{(2)}]$;
(ii) in $[E_{\alpha+\beta}^{(1)} \wedge E_{-\alpha-\beta}^{(3)}, E_\alpha^{(2)} \wedge E_{-\alpha}^{(3)}]$;
(iii) in $[E_{\alpha+\beta}^{(1)} \wedge E_{-\alpha-\beta}^{(3)}, E_\beta^{(3)} \wedge E_{-\beta}^{(2)}]$.

To complete the proof of the theorem we consider

$$c_h(\alpha) = \begin{cases} \dfrac{1}{h(\alpha)}, & \text{if } h(\alpha) \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

and $d_0(\alpha) \equiv 1$. Then (11) gives that the coefficient before $E_{\alpha+\beta}\otimes E_{-\alpha}\otimes E_{-\beta}$ in $l_{g^{*}_{1}}[\pi^0_{\text{Kir}}, \pi^0_{\text{D-S}}]$ is

$$-2N_{\alpha,\beta}c_h(\alpha+\beta). \tag{12}$$

If $c_h(\alpha) \neq 0$ and $c_h(\beta) \neq 0$ then using equation

$$(c_h(\alpha) + c_h(\beta))c_h(\alpha+\beta) = c_h(\alpha)c_h(\beta)$$

we conclude that $c_h(\alpha+\beta) \neq 0$.

If π_{Kir} and $\pi_{\text{P-L}}$ are compatible then from (12) we get the following condition:

$$\alpha, \beta \in \Delta_P \quad \text{implies} \quad N_{\alpha,\beta} = 0 \quad \text{or} \quad \alpha+\beta \notin \Delta_P. \tag{13}$$

Evidently, the condition (13) is equivalent to the following: $\mathbf{u}$ admits the decomposition $\mathbf{u} = \mathbf{p}_u \otimes \mathbf{p}_1$, where $\mathbf{p}_1 = \sum_{\alpha\in\Delta_P} \mathbf{R}V_\alpha \otimes \mathbf{R}W_\alpha$; and, in fact, $\mathbf{u}$ is the orthogonal symmetric Lie algebra [He, Chap. 5, Sect. 1]. That is, U/P_U is a symmetric space.

Conversely, let U/P_U be a symmetric space. Then (12) is verified, hence every commutator of $[E_\alpha^{(1)} \wedge E_{-\alpha}^{(3)}, E_\beta^{(1)} \wedge E_{-\beta}^{(2)}]$ type is zero in $\Lambda^3(\mathbf{g}/\mathbf{p}_u \otimes \mathbf{C})$. Hence the result.

5.5. Corollary. *Consider a family* π_λ *of Poisson structures on the orbit* $\mathcal{O}_l = U/P_U$ *which is a compact hermitian symmetric space,* $\pi_\lambda = \pi_{\text{P-L}} + \lambda\pi_{\text{Kir}}$, $\lambda \in \mathbf{R}$. *Then the Poisson-Lie group U acts on this Poisson manifold* $(\mathcal{O}_l, \pi)$ *in a Poisson way.*

5.6. Let us recall that for a compact symmetric space which looks like U/P_U we can realize the *dual* non-compact symmetric space $\tilde{U}/P_{\tilde{U}}$ in such a way that the Lie algebra $\tilde{\mathbf{u}} = \mathbf{g}^\tau \subset \mathbf{g}$ is presented as

$$\tilde{\mathbf{u}} = \mathbf{h}_c \oplus \sum_{\alpha\in\Delta_+} \mathbf{R}\tilde{V}_\alpha \oplus \mathbf{R}\tilde{W}_\alpha,$$

where $\tilde{V}_\alpha = E_\alpha + \tau(E_\alpha)$, $\tilde{W}_\alpha = i(E_\alpha - \tau(E_\alpha))$, and, more concretely, $\tilde{V}_\alpha = V_\alpha$, $\tilde{W}_\alpha = W_\alpha$ if $\alpha \notin \Delta_P$ and $\tilde{V}_\alpha = iV_\alpha$, $\tilde{W}_\alpha = iW_\alpha$ if $\alpha \in \Delta_P$. Here, as usually, $V_\alpha = E_\alpha - E_{-\alpha}$, $W_\alpha = i(E_\alpha + E_{-\alpha})$. The real form $\tilde{\mathbf{u}}$ has the same compact Cartan subalgebra $\mathbf{h}_c$ and

$\tilde{U}/P_{\tilde{U}}$ is a Poisson-Lie coset space (see Sect. 4.4) with respect to Poisson-Lie structure

$$\mathbf{r}_0=\tfrac{1}{4}\sum_{\alpha\in\Delta_+}\tilde{V}_\alpha\wedge\tilde{W}_\alpha .$$

The homogeneous space $\tilde{U}/P_{\tilde{U}}$ can be also realized as an orbit of $\tilde{U}$ in $\tilde{\mathbf{u}}^*$ and the Kirillov bivector field π_{Kir} coincides in this case with bivector field π_{Herm} corresponding to imaginary part of invariant hermitian form

$$\pi_{\mathrm{Kir}}=\pi_{\mathrm{Herm}}=l_{g*}(\mathbf{r}_P),$$

where

$$\mathbf{r}_P=\tfrac{1}{4}\sum_{\alpha\in\Delta_P}\frac{1}{h(\alpha)}\tilde{V}_\alpha\wedge\tilde{W}_\alpha .$$

The computations from the previous sections prove the following

Proposition 5.1. *Poisson structures π_{Herm} and $\pi_{\text{P-L}}$ are compatible for non-compact hermitian symmetric spaces.*

Hence we have

Theorem 5.2. *For any hermitian symmetric space Poisson structures π_{Kir} and $\pi_{\text{P-L}}$ are compatible.*

6. Examples

The most popular examples of compact hermitian symmetric spaces are grassmannians G_n^k of k-dimensional subspaces in n-dimensional complex vector space. We give here the explicit expressions of $\pi_{\text{P-L}}$ and π_{Kir} for the projective spaces $\mathbf{C}P^n=G_{n+1}^1$ and discuss the structure of symplectic leaves of $\pi_{\text{P-L}}+\lambda\pi_{\mathrm{Kir}}$.

The direct calculations show that in affine part $\mathbf{C}^n\subset\mathbf{C}P^n$ we have the following expressions for $\pi_{\text{P-L}}$ and for π_{Kir}:

$$\begin{aligned}\pi_{\text{P-L}}=&-i\sum_i(1+|z_i|^2)\Big(\sum_k|z_k|^2-\sum_{k>i}|z_k|^2\Big)\partial_{z_i}\wedge\partial_{\bar z_i}\\&-\frac{i}{2}\sum_{i<j}(z_iz_j\partial_{z_i}\wedge\partial_{z_j}-\bar z_i\bar z_j\partial_{\bar z_i}\wedge\partial_{\bar z_j})\\&+\sum_{i\neq j}z_i\bar z_j\Big(1+2\sum_k|z_k|^2\Big)\partial_{z_i}\wedge\partial_{\bar z_j}\end{aligned}$$

and

$$\pi_{\mathrm{Kir}}=-\frac{i}{2}\Big(1+\sum_k|z_k|^2\Big)\Big(\sum_{i\neq j}(z_i\bar z_j\partial_{z_i}\wedge\partial_{\bar z_j})+\sum_i(1+|z_i|^2)\partial_{z_i}\wedge\partial_{\bar z_i}\Big).$$

Here $\partial_{z_i}=\partial/\partial_{z_i}$, $\partial_{\bar z_i}=\partial/\partial_{\bar z_i}$.

In the simplest case of $\mathbf{C}P^1$,

$$\pi_{\text{P-L}}+\lambda\pi_{\mathrm{Kir}}=-\frac{i}{2}(1+|z|^2)(\lambda+(\lambda+2)|z|^2)\partial_z\wedge\partial_{\bar z}.$$

This bivector defines a symplectic structure if λ does not belong to a segment $[-2,0]$. If $\lambda=0$ or $\lambda=-2$ then there is one point as degenerated symplectic leaf ($z=0$ for $\lambda=0$ and $z=\infty$ for $\lambda=-2$); and for $\lambda\in(-2,0)$ the collection of symplectic points on a circle $|z|^2 = -\frac{\lambda}{\lambda+2}$ may be considered as a deformation of a Schubert cell $z=0$.

These Poisson structures appear in the phase space with Poisson-Lie symmetries of the classical WZW model. The formulas for the corresponding singular symplectic structure [FG] coincide with them (in [FG] a parameter λ is discrete valued).

In the case of $\mathbf{C}P^n$ we can prove the very nice formula for the determinant of $\pi_{\text{P-L}} + \lambda\pi_{\text{Kir}}$:

$$\det(\pi_{\text{P-L}} + \lambda\pi_{\text{Kir}}) = \frac{-1}{4(\lambda+2)^2}\left(\prod_{i=1}^{n+1} \frac{(\lambda+2)\sum_{1\leq k\leq i}|x_k|^2 + \lambda\sum_{i<k\leq n+1}|x_k|^2}{|x_{n+1}|^2}\right)^2,$$

where $x_1 : x_2 : \ldots : x_{n+1}$ are homogeneous coordinates in $\mathbf{C}P^n$:

$$z_k = \frac{x_k}{x_{n+1}}.$$

Hence for the values of parameter λ outside the segment $[-2,0]$ we have the symplectic structure on $\mathbf{C}P^n$, for the values $\lambda=0$ or $\lambda=-2$ we obtain two opposite stratifications of $\mathbf{C}P^n$ by Schubert cells:

$$\mathbf{C}P^0 \subset \mathbf{C}P^1 \subset \ldots \subset \mathbf{C}P^n.$$

Here the stratum $\mathbf{C}P^k$ is defined by the equations

$$|x_1|^2 = |x_2|^2 = \ldots = |x_{n-k}|^2 = 0, \quad \text{if} \quad \lambda = 0$$

and

$$|x_{k+2}|^2 = |x_{k+3}|^2 = \ldots = |x_{n+1}|^2 = 0, \quad \text{if} \quad \lambda = -2.$$

If $\lambda \in (-2, 0)$ then degenerated symplectic leaves lie in the union of n complex hypersurfaces V_i:

$$V_i: \quad (\lambda+2)\sum_{1\leq k\leq i}|x_k|^2 + \lambda\sum_{i<k\leq n+1}|x_k|^2 = 0, \qquad i=1,\ldots,n,$$

which provide the deformation of the Schubert cell's stratification.

An analogous picture is valid for the non-compact case. For example, for the Poincaré unit ball $|z|<1$ which may be considered as a symmetric space

$$SU(1,1)/S(U(1)\times U(1))$$

we have

$$\pi_{\text{P-L}} + \lambda\pi_{\text{Herm}} = \frac{i}{2}(1-|z|^2)(-\lambda + (\lambda+2)|z|^2)\partial_z \wedge \partial_{\bar{z}}. \tag{14}$$

The symplectic leaves are described by the equation

$$|z|^2 = \frac{\lambda}{\lambda+2}.$$

For $\lambda<0$ this structure is symplectic, for $\lambda=0$ the origin $z=0$ is a unique degenerated symplectic leaf and for $\lambda>0$ we have a circle of degenerated symplectic points that tend to a border $|z|=1$ while λ tends to infinity.

The two brackets from the family $\pi_{\text{P-L}}+\lambda\pi_{\text{Herm}}$ have distinguished structures of symplectic leaves in $\mathbf{C}P^n$. The first one is $\pi_{\text{P-L}}=l_{g*}(\mathbf{r})-r_{g*}(\mathbf{r})$ ($\lambda=0$) and the second is $\pi_+=-l_{g*}(\mathbf{r})-r_{g*}(\mathbf{r})$ ($\lambda=-2$). Corresponding symplectic leaves are Schubert cells with respect to opposite Borel subgroups. For $-2<\lambda<0$ the leaves "interpolate" between the Schubert cells.

7. Infinite-Dimensional Remarks

In this section we discuss infinite-dimensional examples of the manifolds with a pair of compatible Poisson brackets, arising in the geometry of differential operators on a circle. In fact, these examples stimulated our finite-dimensional considerations.

On the other hand, our results are in good correspondence with Mulase's thesis [Mu] that "the space of solutions of an integrable system always looks like a factor of some general linear group over maximal parabolic subgroup." An appearance of the grassmannians or other hermitian symmetric spaces is very natural from this point of view and our results may be useful in an infinite-dimensional context as well. Infinite-dimensional analogues of symmetric spaces equipped with (pseudo) Kähler and Poisson-Lie structures seem to be useful objects for a geometric and deformation quantization of classical W-algebras.

We are grateful to Prof. A. Rosly for an enlightening discussion on the subject of this section.

7.1. We shall work within the framework of Gelfand formal variational calculus. The notations are taken from [R].

Let $\mathscr{L}_n=\{L=\partial_x^n+u_{n-1}\partial_x^{n-1}+\ldots+u_0 \mid u_i(x)\in C^\infty(S^1)\}$ be an affine infinite-dimensional space of the n^{th} order differential operators on a circle S^1. A *tangent space* to $\mathscr{L}_n$ is identified with a set of differential operators of the $(n-1)^{\text{th}}$ order.

Let $\mathscr{F}(\mathscr{L}_n)$ be a space of functions on $\mathscr{L}_n$. The element $F\in\mathscr{F}(\mathscr{L}_n)$ ("function") is a functional

$$F(u)=\int_{S^1} f(u_i^{(j)})dx\,,$$

where $u_i^{(j)}=\partial_x^j u_i$ is a polynomial with $C^\infty(S^1)$-coefficients.

To define a Poisson bracket of two functions we need, as usual, a *Hamiltonian* map $V: T^*\mathscr{L}_n\to T\mathscr{L}_n$ such that $\{f,g\}=V(df)(g)$, where $f,g\in\mathscr{F}(\mathscr{L}_n)$, $df=\delta f$ is a variation of f.

7.2. Let $CL_N(S)=\left\{a(x,\xi)=\sum_{i=-\infty}^{N} a_i(x)\xi^i \mid a_i(x)\in\mathbf{C}^\infty(S^1)\right\}$ be a space of pseudo-differential symbols of the N^{th} order. We equip the set $CL(S^1)=\bigcup_N CL_N(S^1)$ with a *symbolic multiplication*

$$a(x,\xi)\circ b(x,\xi)=\sum_{n\geqq 0}\frac{1}{n!}\partial_\xi^n a\cdot\partial_x^n b\,.$$

This converts $CL(S^1)$ into an associative algebra. Let a_+ be a "differential" part of $a\in CL(S^1)$ and $a_-=a-a_+$ be an "integral" part of a. Denote by $\mathrm{Tr}: CL(S^1)\to k$ ($k=\mathbf{R},\mathbf{C}$) the *Adler trace*

$$\mathrm{Tr}=\int_{S^1}\mathrm{res}\,,\qquad \mathrm{res}(a)=a_{-1}\,.$$

Hence $CL(S^1)$ is a metrizable Lie algebra with a bracket $[a,b]=a\circ b-b\circ a$ and inner product

$$\langle a,b\rangle=\mathrm{Tr}(a\circ b). \tag{15}$$

Subalgebras $\mathrm{DOP}(S^1)$ and $CL_{-1}(S^1)$ are isotropic with respect to (15) hence we have

Proposition 7.1. *The data $(CL(S^1), \mathrm{DOP}(S^1), CL_{-1}(S^1))$ is a Manin triple.*

We are interested now in Poisson actions of this bialgebra.

7.3. We identify a *cotangent space* $T^*\mathscr{L}_m$ to $\mathscr{L}_m$ with the quotient $CL_{-1}(S^1)/CL_{-m}(S^1)$ via the pairing $T^*\mathscr{L}_m\times T^*\mathscr{L}_m\to\mathscr{F}(\mathscr{L}_m)$, $\langle X,V\rangle=\mathrm{Tr}(X\circ V)$.

Theorem 7.1 (Gelfand-Dikii). *The map $V\colon T^*\mathscr{L}_m\to T\mathscr{L}_m$,*

$$V(X)=V_X=L(XL)_+-(LX)_+L\,,\qquad X\in T^*\mathscr{L}_m\,,\qquad L\in\mathscr{L}_m$$

is Hamiltonian.

The corresponding Poisson bracket $\{F,G\}=V(dF)(G)$ is called the *second Gelfand-Dikii Poisson structure.* A Lie algebra of covector fields on $\mathscr{L}_n$ with the second Gelfand-Dikii structure:

$$\{X,Y\}=[(XL)_+Y+(YL)_-X-X(LY)_- - Y(LX)_+ + V_X(Y)-V_Y(X)]_-$$

will be called *Gelfand-Dikii algebra GD_n.*

7.4. Let us define a map $E\to W_E$, where $E\in\mathrm{DOP}(S^1)$, $W_E\in\mathrm{Vect}(\mathscr{L}_n)$ is a vector field on $\mathscr{L}_n$, by the following formula

$$W_E:=LE-(LEL^{-1})_+L=(LEL^{-1})_-L\,. \tag{16}$$

The first equation means that W_E is a differential operator and the second that $\mathrm{ord}(W_E)\leqq n-1$.

Theorem 7.2 [R]. *The map $E\to W_E$ is a homomorphism of Lie algebras.*

Actually, $W_E=V_{(EL^{-1})_-}$ and W_E defines the map $\mathrm{DOP}(S^1)\to GD_n$ which is a Lie algebra homomorphism.

7.5. Recall the definition of a *Poisson-Lie action* of a Lie algebra $\mathbf{g}$ on a Poisson manifold $(\mathscr{M},\{\ \})$. Let $\mathbf{g}$ be a Lie bialgebra. Then a Poisson-Lie action of $\mathbf{g}$ on a Poisson manifold $(\mathscr{M},\{\ \})$ is an action $\varphi\colon\mathbf{g}\to\mathrm{Vect}(M)$, $\varphi(X)=V_X$ such that

$$V_X\{f,g\}(m)=\{V_Xf,g\}(m)+\{f,V_Xg\}(m)-\langle X,[\varphi^*df(m),\varphi^*dg(m)]\rangle\,, \tag{17}$$

where $\varphi^*df(m)\in\mathbf{g}^*$ and $\langle X,\varphi^*df(m)\rangle=V_Xf(m)$.

Theorem 7.3 [R]. *The Lie algebra $\mathrm{DOP}(S^1)$ acts on the Poisson manifold $\mathscr{L}_n$ with the second Gelfand-Dikii structure in a Poisson-Lie way.*

We are interested now in a behaviour of the *first Gelfand-Dikii bracket* with respect to the described action of $\mathrm{DOP}(S^1)$ on $\mathscr{L}_n$. This bracket is associated with the Hamiltonian map

$$V(X)=V_X^{(1)}=[L,X]_+\,,\qquad X\in T^*\mathscr{L}_n\,,\qquad L\in\mathscr{L}_n\,,$$

which may be obtained from that for the second bracket as a cocycle by a well known procedure

$$V_X(L+1)=V_X(L)+[L,X]_+=V_X(L)+V_X^{(1)}. \qquad (18)$$

It is well known [GD, DS] that the first and the second Gelfand-Dikii brackets are compatible in $\mathscr{L}_n$. It would be desirable to prolong the analogy between $\mathscr{L}_n$ and finite-dimensional hermitian symmetric spaces and to find that the first Gelfand-Dikii structure is invariant with respect to the action of $\mathrm{DOP}(S^1)$, but it is *not* invariant. It is known only [LM] that the first Gelfand-Dikii bracket is invariant under the coadjoint action of *integral operators* $CL_{-1}(S^1)$.

7.6. A. Rosly suggested to make the analogy more close by describing the action of $\widehat{\mathrm{DOP}}(S^1)$ on $\mathscr{L}_n$ in terms of dressing action. It may be done if we use the central extension $\widehat{\mathrm{DOP}}(S^1)$ of $\mathrm{DOP}(S^1)$ by means of *logarithmic cocycle* [KK, R]:

$$c(a,b)=\int \mathrm{res}([a,\log\partial]\circ b)=\mathrm{Tr}([a,\log\partial]\circ b),$$

where $\log\partial$ is an exterior derivation of the Lie algebra $CL(S^1)$ which can be written in coordinates as

$$[\log\partial, a]=\sum_{n\geqq 0}\frac{(-1)^{n-1}}{n}a_x^{(n)}\cdot\partial^{-n}.$$

To compose the extended Manin triple $(\widehat{CL},\widehat{\mathrm{DOP}}(S^1),\widehat{CL}_{-1})$ we have to add the central element c to $\mathrm{DOP}(S^1)$ and the dual cocentral element $\log\partial$ to CL_{-1}.

Let us consider a *formal Volterra group of integral operators*

$$G^*:=G(CL_{-1})=\{1+a_{-1}\xi^{-1}+a_{-2}\xi^{-2}+\dots\}$$

and a formal exponential $\hat{G}^*$ of the extended Lie algebra $\widehat{CL}_{-1}$. An element of $\hat{G}^*$ is a formal expression

$$\exp[(\alpha\log\partial)(1+a_{-1}\partial^{-1}+a_{-2}\partial^{-2}+\dots)]=\partial^{\alpha}(1+a_{-1}\partial^{-1}+a_{-2}\partial^{-2}+\dots),$$
$$\alpha\in\mathbf{C}.$$

Choosing α to be a positive integer $\alpha=n$ we may consider $\mathscr{L}_n$ as a submanifold in $\hat{G}^*$.

Proposition 7.2[1] (A. Rosly) (cf. [STS]). *The restriction of the dressing action of the Lie algebra* $\mathrm{DOP}(S^1)$ *on* $\hat{G}^*$ *to* $\mathscr{L}_n$ *coincides with* (16).

7.7. On the other hand, the formal groups G^* and $\hat{G}^*$ have the properties analogous to the properties of the Borel subgroup in finite-dimensional situation.

Proposition 7.3. *Coadjoint orbits of* G^* *in* $\mathrm{DOP}(S^1)$ *are not Poisson-Lie cosets for the bialgebra* $CL_{-1}(S^1)$.

Example. Let $L=\partial_x^4+u_2\partial_x^2+u_1\partial_x+u_0$. $X\in\mathrm{Stab}\,L$ iff $[L,X]_+=0$. Then $(\mathrm{Stab}\,L)^{\perp}$ $=\{$set of diff. operators $M=[L,Y]_+,\ Y\in CL_-\}$. Indeed, $\langle X,M\rangle=\langle X,[L,Y]_+\rangle$ $=\langle X,[L,Y]\rangle=\langle [X,L],Y\rangle=\langle [X,L]_+,Y\rangle=0$.

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[1] When our article was finished I. Zakharevich and B. Khesin informed us of their forthcoming text on the same subject

And it is easy to see that $(\mathrm{Stab}L)^{\perp}$ is not a subalgebra in $\mathrm{DOP}(S^1)$ (just as in the finite-dimensional case).

8. Concluding Remarks

The natural problem of a quantization of the family π_λ arises. It is treated in [Ra]. Similar questions were considered in [GRZ, DGM]. It would be interesting to investigate the relations of quantization of our results with the construction of quantum flags by Soibelman [S] and quantum Schubert schemes by [LR]. We are going to discuss corresponding families of (pseudo) hermitian metrics, and Kahler geometry elsewhere. We will consider in our forthcoming papers integrable systems associated with the family of Poisson brackets (see [Ma] for the idea applied to the KdV equation), KP hierarchy and Poisson-Lie structures on infinite-dimensional Sato grassmannian [SW].

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