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Localization in the Ground State of a Disordered Array of Quantum Rotators

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Abstract. We consider the zero-temperature behavior of a disordered array of quantum rotators given by the finite-volume Hamiltonian:

$$H_\Lambda = -\sum_{x\in\Lambda} \frac{h(x)}{2} \frac{\partial^2}{\partial\varphi(x)^2} - J \sum_{\langle x,y\rangle\in\Lambda} \cos(\varphi(x)-\varphi(y)),$$

where $x, y \in \mathbf{Z}^d$, $\langle\, ,\rangle$ denotes nearest neighbors in $\mathbf{Z}^d$; $J > 0$ and $\mathbf{h} = \{h(x) > 0, x\in\mathbf{Z}^d\}$ are independent identically distributed random variables with common distribution $d\mu(h)$, satisfying $\int h^{-\delta} d\mu(h) < \infty$ for some $\delta > 0$. We prove that for any $m > 0$ it is possible to choose $J(m)$ sufficiently small such that, if $0 < J < J(m)$, for almost every choice of $\mathbf{h}$ and every $x\in\mathbf{Z}^d$ the ground state correlation function satisfies

$$\langle\cos(\varphi(x)-\varphi(y))\rangle \leqq C_{x,\mathbf{h},J}\, e^{-m|x-y|}$$

for all $y\in\mathbf{Z}^d$ with $C_{x,\mathbf{h},J} < \infty$.

1. Introduction

Ferromagnetically coupled quantum rotators have been used in the physics literature to describe the effect of quantum fluctuations in granular superconductors [1]. In this paper we discuss the typical properties of a disordered array of such rotators with random moments of inertia. Apart from its intrinsic physical interest the study of this model is a natural step in the program initiated in [2] and [3] of understanding the effect of randomness in quantum spin systems. In [2], Klein and Perez studied the ground state of the one-dimensional quantum x–y model in the presence of a random transverse field: exponential decay of the correlation

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functions for any amount of disorder was shown to be a consequence of exponential localization for one-dimensional random Schrödinger operators. In [3] Campanino, Klein and Perez considered the d-dimensional Ising model in the presence of a random transverse field. A path space representation was used to map the original quantum model into a limit of a classical ferromagnetic Ising model in $(d+1)$-dimensions with d-dimensional disorder. This allowed the control of the Griffiths' type of singularities through the use of correlation inequalities and a multiscale analysis of the type used in the theory of localization for random Schrödinger operators [4, 5]. Their results were: exponential decay of correlation functions for high-disorder and any $d \geqq 1$ and long-range order in the low disorder regime for $d \geqq 2$. For $d = 1$ long range order at low disorder was established by Aizenman, Klein and Newman [13].

The ideas and techniques of this paper share much in common with those involved in [3], namely an approximate path space is used mapping the system into a limit of $(d+1)$-dimensional ferromagnetic classical rotators with d-dimensional disorder, allowing the use of correlation inequalities. The novel feature is the presence of a continuous symmetry which allows, through the use of McBryan–Spencer bounds, an easier control of the Griffiths' singularities. Our multi-scale analysis follows the strategy of von Dreifus and Spencer [4], with the role of the resolvent identity replaced by the Simon–Lieb–Rivasseau inequality [10] and the role of Wegner's estimate replaced by McBryan–Spencer bounds.

2. The Model and Results

In a finite volume $\Lambda \subset \mathbf{Z}^d$ the Hamiltonian is given by

$$H_\Lambda = -\sum_{x \in \Lambda} \frac{h(x)}{2} \frac{\partial^2}{\partial \varphi(x)^2} - J \sum_{\langle x, y \rangle \in \Lambda} \cos(\varphi(x) - \varphi(y)) \tag{2.1}$$

acting on the Hilbert space $\mathscr{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathscr{H}_x$, $\mathscr{H}_x = L^2[-\pi, +\pi]$, $x \in \mathbf{Z}^d$. The operator $-\dfrac{\partial^2}{\partial \varphi(x)^2}$ is taken with periodic boundary condition so that its spectrum is $\{n_x^2, n_x \in \mathbf{Z}\}$. The second sum in (2.1) is taken over all pairs of nearest neighbor sites $\langle x, y \rangle$ in Λ. The coupling between the rotators is ferromagnetic, i.e., $J > 0$. The inverse of the moments of inertia of the rotators, $\mathbf{h} = \{h(x) > 0, x \in \mathbf{Z}^d\}$, are taken to be independent identically distributed random variables with common distribution $d\mu(h)$. We shall allow $h(x)$ to take arbitrarily small, but positive values, with the condition that for some $\delta > 0$,

$$\frac{1}{\bar{h}^\delta} \equiv \int \frac{1}{h^\delta} d\mu(h) < \infty . \tag{2.2}$$

We shall denote by $\mathbf{P}$ and $\mathbf{E}$ the underlying expectation and probability measure induced by $d\mu$. The parameter $\bar{h}$ defined by (2.2) measures the amount of disorder in the system.

The operator H_Λ has a unique ground state Ω_Λ, with a normalized wave function $\Omega_\Lambda(\varphi) > 0$ for all $\varphi = \{\varphi(x), x \in \Lambda\}$, $\varphi(x) \in [-\pi, +\pi]$. This follows from the fact that H_Λ has discrete spectrum and generates a positivity improving semigroup (Sect. 3).

Moreover the correlation functions:

$$\langle \cos(\varphi(x) - \varphi(y)) \rangle_\Lambda \equiv (\Omega_\Lambda, \cos(\varphi(x) - \varphi(y))\Omega_\Lambda) \tag{2.3}$$

are monotonically increasing in Λ and $J > 0$ and monotonically decreasing in each $h(x)$, $x \in \mathbf{Z}^d$. This follows from correlation inequalities derived in the path space representation (see Sect. 3).

The Hamiltonian (2.1) in the deterministic homogeneous situation $h(x) \equiv h$, for all $x \in \mathbf{Z}^d$, has been used to describe quantum fluctuations in superconducting arrays [1]. In its ground state for $d \geqq 2$, it exhibits a phase transition with long range order for $\alpha = \frac{J}{h} > \alpha_c(d)$. This follows from the path space representation developed in Sect. 3 and standard techniques [6]. In $d = 1$, the system has a Kosterlitz–Thouless phase transition (see [14] for a similar gauge field model) with polynomial decay of correlation functions for $\alpha > \alpha_c(1)$. Mean field bounds (see Sect. 5) also guarantee the existence of $\bar{\alpha}_c(d)$, $\alpha_c(d) \geqq \bar{\alpha}_c(d) \geqq \frac{1}{4d}$, such that if $\alpha < \bar{\alpha}_c(d)$ the correlation functions decay exponentially.

In order to state our results we introduce the imaginary time correlation function

$$G_\Lambda((x, t), (y, s)) = \frac{(\Omega_\Lambda, e^{i\varphi(x)} e^{-|t-s|H_\Lambda} e^{-i\varphi(y)} \Omega_\Lambda)}{(\Omega_\Lambda, e^{-|t-s|H_\Lambda} \Omega_\Lambda)} . \tag{2.4}$$

Monotonicity in Λ (obtained from the path-space representation) ensures the existence of

$$G((x, t), (y, s)) = \lim_{\Lambda \to \mathbb{Z}^d} G_\Lambda((x, t), (y, s)) ,$$

in terms of which we state our main result.

It is important to notice that if for some $\varepsilon > 0$ we have $\alpha(x) = \frac{J}{h(x)} \geqq (1 + \varepsilon)\alpha_c(d)$ for all $x \in \mathbf{Z}^d$ with probability one, then the system will have long range order ($d \geqq 2$) or polynomial decay of correlations ($d = 1$), with probability one. This is a consequence of monotonicity of $G((x, t), (y, s))$ in each $\alpha(x)$, $x \in \mathbf{Z}^d$, and the corresponding result in the homogeneous deterministic case. Conversely if for some $\varepsilon > 0$, $\alpha(x) < (1 - \varepsilon)\bar{\alpha}_c(d)$, for all $x \in \mathbf{Z}^d$ with probability one, correlation functions will decay exponentially with probability one.

From the above it follows that non-trivial behavior is expected only when both $\alpha(x) > \alpha_c(d)$ and $\alpha(x) < \bar{\alpha}_c(d)$ may occur with non-zero probability.

Theorem 2.1. *Let* $d = 1, 2, \ldots, q > (1 + 3/\delta)d + 1$. *Then for any* $m > 0$ *there exists* $J(m)$ *such that, for any* $0 < J < J(m)$ *and almost every choice of* $\mathbf{h}$ *and every* $x \in \mathbf{Z}^d$, *we have*

$$G((x, t), (y, s)) \leqq C_{x, \mathbf{h}, J}\, e^{-m\|(x-y, |t-s|^{1/q})\|_\infty} , \tag{2.5}$$

where $\|(x, u)\|_\infty = \max(|x|, |u|)$ *and* $C_{x, \mathbf{h}, J} < \infty$.

It is important to notice the less than exponential decay in the time direction compared with the exponential decay in the space direction. This is a consequence of the Griffiths' singularities, i.e., the fact that with probability one there exists arbitrarily large regions Λ such that $\alpha(x) = \frac{J}{h(x)} > \alpha_c(d)$ for all $x \in \Lambda$. For these regions the energy gap between the ground state Ω_Λ and the first excited state gets arbitrarily small.

Remark. Our methods can actually prove a stronger result. We can admit random couplings $\{J_{\langle x, y \rangle}; \langle x, y \rangle \in \mathbf{Z}^d\}$ and relax our hypothesis on the probability distribution of $h(x)$. More precisely, let

$$H_\Lambda = -\sum_{x \in \Lambda} \frac{h(x)}{2} \frac{\partial^2}{\partial\varphi(x)^2} - J \sum_{\langle x, y \rangle \in \Lambda} J_{\langle x, y \rangle} \cos(\varphi(x) - \varphi(y)) ,$$

where $J > 0$, $\mathbf{h} = \{h(x) > 0, x \in \mathbf{Z}^d\}$ and $\mathbf{J} = \{J_{\langle x, y \rangle} > 0, \langle x, y \rangle \in \mathbf{Z}^d\}$ are independent sets of independent identically distributed random variables with

$$\mathbf{E}([\log(1 + 1/h(x))]^\delta) < \infty \quad \text{and} \quad \mathbf{E}([\log(1 + J_{\langle x, y \rangle})]^\delta) < \infty ,$$

where $\delta > 2d$. In this case Theorem 2.1 still holds, with the conclusion being true of almost every choice of $\mathbf{h}$ and $\mathbf{J}$, and (2.5) replaced by

$$G((x, t), (y, s)) \leqq C_{x, \mathbf{h}, \mathbf{J}}\, e^{-m\|(x - y, [\log(1 + |t - s|)]^q\|}$$

for some $q > 1$.

The modifications required in the proof are similar to the arguments in Klein [17].

3. The Approximation by Classical Rotators

Let us denote by h_0 the self-adjoint operator $-\frac{1}{2}\frac{d^2}{d\varphi^2}$ in $L^2[-\pi, +\pi]$ with periodic boundary condition. Our starting point is the formula

$$K_t(\varphi, \varphi') \equiv e^{-th_0}(\varphi, \varphi') = \frac{1}{\sqrt{2\pi t}} \sum_{m \in \mathbf{Z}} e^{\frac{-(\varphi - \varphi' + 2\pi m)^2}{2t}} \tag{3.1}$$

for the kernel of e^{-th_0}, $t > 0$.

Using (3.1) and the Trotter product formula we obtain the representation

$$(\Omega_\Lambda, F(\varphi(x_1), \dots, \varphi(x_n))\Omega_\Lambda) = \lim_{n \to \infty} \langle F(\varphi(x_1, 0), \varphi(x_2, 0), \dots, \varphi(x_n, 0)) \rangle_\Lambda^{(n)} , \tag{3.2}$$

where $\langle \cdot \rangle_\Lambda^{(n)}$ denotes the expectation for the classical plane rotator in $\Lambda \times \frac{\mathbf{Z}}{n} \subset \mathbf{Z}^d \times \frac{\mathbf{Z}}{n}$ $\left(\text{i.e., with lattice spacing } \frac{1}{n} \text{ in the "time" direction}\right)$ with the so-called Villain approximation taken in the "time" direction i.e., the Gibbs weight of a configuration $\varphi = \left\{\varphi(x, t), x \in \Lambda, t \in \frac{\mathbf{Z}}{n} \cap \left[-\frac{\beta}{2}, \frac{\beta}{2}\right]\right\}$ given by

$$e^{-H_{\Lambda,\beta}^{(n)}(\varphi)} \equiv e^{\sum_{\langle x, y \rangle, t} \frac{J}{n} \cos(\varphi(x, t) - \varphi(y, t))} \sum_{\mathbf{m}} e^{-\frac{n}{2} \sum_{x, t} \frac{1}{h(x)} (\varphi(x, t) - \varphi(x, t + 1/n) + 2\pi m(x, t))^2} , \tag{3.3}$$

where $\mathbf{m} = \left\{m(x, t); x \in \Lambda, t \in \left[-\frac{\beta}{2}, +\frac{\beta}{2}\right] \cap \frac{\mathbf{Z}}{n}\right\}$ and the summations are over $t \in \left[-\frac{\beta}{2}, +\frac{\beta}{2}\right]; x, y \in \Lambda$.

This approximation enables us to use Ginibre's correlation inequalities [9] taking into account the ferromagnetic nature of the model. This is made possible by the use of the remark [8] that

$$F_\beta(\varphi - \varphi') \equiv \frac{\sum_{k \in \mathbf{Z}} \exp\left[-\frac{\beta}{2}(\varphi - \varphi' + 2\pi k)^2\right]}{\sum_{k \in \mathbf{Z}} \exp -\frac{\beta}{2}(2\pi k)^2}$$

$$= \lim_{n \to \infty} c(n) \int_{[-\pi, +\pi]} d\theta_1 \dots \int_{[-\pi, +\pi]} d\theta_n \exp\{n\beta[\cos(\varphi - \theta_1) + \dots + \cos(\theta_n - \varphi')]\} \tag{3.4}$$

with suitably chosen $c(n) > 0$. Formula (2.4) allows the substitution of the Villain couplings by standard rotator (cosine) couplings for which Ginibre's inequalities apply.

The derivation of (3.2) starts with the fact that the operator

$$H_\Lambda^{(0)} = -\sum_{x \in \Lambda} \frac{h(x)}{2} \frac{\partial^2}{\partial \varphi(x)^2} \tag{3.5}$$

has a unique ground state, given by the function

$$\Omega_\Lambda^{(0)}(\varphi) = \frac{1}{(2\pi)^{|\Lambda|/2}} \tag{3.6}$$

its spectrum being $\left\{\sum_{x \in \Lambda} \frac{h(x)}{2} l_x^2, l_x \in \mathbf{Z}\right\}$. Moreover the operator H_Λ generates a positivity improving semigroup (this is true for $H_\Lambda^{(0)}$ from formula (2.1) and remains true for H_Λ since $V_\Lambda(\varphi) = -J \sum_{\langle xy \rangle \in \Lambda} \cos(\varphi(x) - \varphi(y))$ is a multiplication operator). Moreover, the spectrum of H_Λ is discrete, since $H_\Lambda^{(0)}$ has compact resolvent and V_Λ is bounded (e.g., [11, p. 113]). It follows from the Perron–Frobenius theory [11] that H_Λ has a unique ground state and $\Omega_\Lambda(\varphi)$ is a positive function. In particular

$$(\Omega_\Lambda(\varphi), \Omega_\Lambda^{(0)}(\varphi)) > 0 \,. \tag{3.7}$$

It then follows for any bounded operator A in $\mathscr{H}_\Lambda$:

$$\langle A \rangle_\Lambda = (\Omega_\Lambda, A\Omega_\Lambda) = \lim_{\beta \to \infty} \frac{(\Omega_\Lambda^{(0)}, e^{-\frac{\beta}{2}H_\Lambda} A e^{\frac{-\beta}{2}H_\Lambda} \Omega_\Lambda^{(0)})}{(\Omega_\Lambda^{(0)}, e^{-\beta H_\Lambda} \Omega_\Lambda^{(0)})} \,. \tag{3.8}$$

Using Trotter's product formula we obtain [6]:

$$\langle F(\varphi_\Lambda) \rangle_\Lambda \equiv (\Omega_\Lambda, F(\varphi_\Lambda)\Omega_\Lambda) = \lim_{\beta \to \infty} \lim_{n \to \infty} \langle F(\varphi_{\Lambda,0}) \rangle_{\Lambda,\beta}^{(n)} \,, \tag{3.9}$$

where $\varphi_A = \{\varphi(x), x \in A\}$, $A \subset \Lambda$; $\varphi_{A,t} = \left\{\varphi(x,t), x \in A, t \in \frac{\mathbf{Z}}{n}\right\}$. Here $\langle \cdot \rangle_{\Lambda,\beta}^{(n)}$ denotes the expectation in the classical rotator given by (3.3) and restricted to the region $\Lambda \times \left(\left[-\frac{\beta}{2}, \frac{\beta}{2}\right] \cap \frac{\mathbf{Z}}{n}\right)$ with free boundary conditions. Using Ginibre's inequalities (and the free boundary conditions) we get the monotonicity in the volume:

$$\langle F(\varphi_A)\rangle_{\Lambda,\beta}^{(n)} \leqq \langle F(\varphi_A)\rangle_{\Lambda',\beta'}^{(n)} \quad \text{if } \Lambda \subset \Lambda', \beta \leqq \beta' . \tag{3.10}$$

We may thus interchange the limits in (3.9):

$$\langle F(\varphi_A)\rangle_\Lambda \equiv \lim_{n\to\infty} \langle F(\varphi_{A,0})\rangle_\Lambda^{(n)} . \tag{3.11}$$

In particular we can take the thermodynamical limit

$$\langle F(\varphi_A)\rangle \equiv \lim_{\Lambda\to\mathbf{Z}^d} \langle F(\varphi_A)\rangle_\Lambda = \lim_{n\to\infty} (F(\varphi_A))^{(n)} \quad \text{for any } A \subset \mathbf{Z}^d . \tag{3.12}$$

Correlation functions involving time can also be obtained. For instance

$$\lim_{\Lambda\to\mathbf{Z}^d} (\Omega_\Lambda, F(\varphi_A)e^{-|t-s|H_\Lambda}G(\varphi_B)\Omega_\Lambda) = \lim_{n\to\infty} \langle F(\varphi_{A,t})G(\varphi_{B,s})\rangle^{(n)} . \tag{3.13}$$

4. An Estimate of the Energy Gap

The existence of a continuous symmetry plays an important role in the estimate of the energy gap $E_\Lambda^{(1)}$ between the ground state Ω_Λ and the excited states in the invariant subspace generated by $\{e^{-i\varphi(x)}\Omega_\Lambda, x \in \Lambda\}$.

Such an estimate would give us a priori bounds on the decay of finite volume correlation functions, since it follows immediately from (2.4) that

$$G_\Lambda((x,t),(y,s)) \leqq e^{-E_\Lambda^{(1)}|t-s|} . \tag{4.1}$$

Let us introduce the total angular momentum operator

$$L_\Lambda = \sum_{x\in\Lambda} \frac{1}{i}\frac{\partial}{\partial\varphi(x)} .$$

The Hamiltonian can then be decomposed (e.g., [16, p. 77]) in the form of

$$H_\Lambda = I_\Lambda^{-1} L_\Lambda^2 + H_\Lambda^r , \tag{4.2}$$

where the first term is the "center of mass" Hamiltonian and the "relative" Hamiltonian H_Λ^r involves only the relative coordinates $\{\varphi(x) - \varphi(y); x, y \in \Lambda\}$, and

$$I_\Lambda = \sum_{x\in\Lambda} \frac{1}{h(x)} .$$

If we did not have periodic boundary conditions on $H_\Lambda^{(0)}$, our Hilbert space would be written as a tensor product with the two terms in (4.2) acting on different factors. It would then follow that $L_\Lambda\Omega_\Lambda = 0$ and

$$L_\Lambda e^{-i\varphi(x)}\Omega_\Lambda = e^{-i\varphi(x)}(L_\Lambda - 1)\varphi_\Lambda = -e^{-i\varphi(x)}\Omega_\Lambda .$$

Thus

$$L_\Lambda^2 e^{-i\varphi(x)}\Omega_\Lambda = e^{-i\varphi(x)}\Omega_\Lambda .$$

Therefore we would have

$$E_\Lambda^{(1)} \geqq I_\Lambda^{-1} \geqq \frac{h_{\min}^{(\Lambda)}}{|\Lambda|}, \tag{4.3}$$

where $h_{\min}^{(\Lambda)} = \min_{x\in\Lambda} h(x)$. It would then follow from (4.1) that

$$G_\Lambda((x,t),(y,s)) \leqq e^{-I_\Lambda^{-1}|t-s|} \leqq e^{\frac{-h_{\min}^{(\Lambda)}|t-s|}{|\Lambda|}} . \tag{4.4}$$

The above argument is not correct since the periodic boundary conditions do not allow us to write our Hilbert space as a tensor product where each term in (4.2) acts on a different factor. But if all the $h(x)$ are rational numbers, this can be done in a bigger space where we can prove (4.3). The result then follows for arbitrary $h(x)$ by a perturbation argument.

This estimate should be compared with the estimate

$$E_\Lambda^{(1)} \approx \prod_{x\in\Lambda}\left(\frac{h(x)}{J}\wedge 1\right) \geqq \left(\frac{h_{\min}^{(\Lambda)}}{J}\wedge 1\right)^{|\Lambda|}$$

obtained for the Ising model in the presence of a random transverse field [3].

We actually need more than (4.4), we need a uniform bound on the correlation functions of the classical rotators given by (3.3). This is given by

Lemma 4.1. *Let*

$$G_{\Lambda,\beta}^{(n)}((x,t),(y,s)) = \langle e^{i[\varphi(x,t)-\varphi(y,s)]}\rangle_{\Lambda,\beta}^{(n)} .$$

Then

$$G_{\Lambda,\beta}^{(n)}((x,t),(y,s)) \leqq e^{-I_\Lambda^{-1}|t-s|} \leqq e^{-\frac{h_{\min}^{(\Lambda)}|t-s|}{|\Lambda|}} \tag{4.5}$$

for all β and n.

Proof. We use a technique of McBryan and Spencer [7].

Let $\alpha(t)$ be a $C^{(2)}$-function on $\mathbf{R}$. We perform the imaginary shift

$$\varphi(x,t) \to \varphi(x,t) + i\alpha(t)$$

on the integration variables appearing in the numerator of the expression for the correlation functions to obtain:

$$\langle e^{i[\varphi(x,0)-\varphi(y,t)]}\rangle_{\Lambda,\beta}^{(n)} \leqq e^{-[\alpha(0)-\alpha(t)] + n\sum_{x\in\Lambda}\sum_{s\in[-\beta/2,\beta/2]\cap\frac{\mathbf{Z}}{n}} \frac{(\alpha(s)-\alpha(s+1/n))^2}{2h(x)}}$$

$$\leqq e^{-[\alpha(0)-\alpha(t)] + n(\sum_{x\in\Lambda n} 1/h(x))\sum_{s\in\frac{\mathbf{Z}}{n}} \frac{1}{n}\frac{(\alpha(s)-\alpha(s+1/n))^2}{2(1/n)^2}} .$$

We then choose

$$\alpha = \frac{1}{\sum_{x\in\Lambda}\frac{1}{h(x)}}\left(-\frac{\partial^2}{\partial t^2}\right)_n^{-1}[\delta_0 - \delta_t] ,$$

where $\left(-\frac{\partial^2}{\partial t^2}\right)_n^{-1}$ is the inverse of the second difference operator (lattice Laplacian) in the lattice $\frac{\mathbf{Z}}{n}$, i.e.,

$$\left[\left(-\frac{\partial^2}{\partial t^2}\right)_n f\right](t) = n^2\left[2f(t) - f\left(t + \frac{1}{n}\right) - f\left(t - \frac{1}{n}\right)\right]$$

given by

$$\left(-\frac{\partial^2}{\partial t^2}\right)_n^{-1}(r, s) = \frac{1}{2}|r - s| \quad \text{for } r, s \in \frac{\mathbf{Z}}{n} .$$

It follows that

$$\langle e^{i[\varphi(x, t) - \varphi(y, s)]}\rangle_{\Lambda, \beta}^{(n)} \leqq e^{-\frac{|t-s|}{\sum_{x\in\Lambda}\frac{1}{h(x)}}} ,$$

proving the lemma.

5. Mean Field Bounds

An important feature of the classical path space model are the Simon–Lieb–Rivasseau [10] inequalities. Let us state them in a form that is suitable for our purposes.

Let

$$\Omega = \Lambda \times \left(\left[-\frac{\beta}{2}, \frac{\beta}{2}\right] \cap \frac{\mathbf{Z}}{n}\right) \subset \mathbf{Z}^d \times \frac{\mathbf{Z}}{n}, \quad W \subset \Omega .$$

We shall denote by $\partial_V W$ the "vertical" interior boundary of W i.e.,

$$\partial_V W = \{(z, u) \in W : \exists (z', u) \notin W, |z - z'| = 1\} ,$$

and by $\partial_H W$, the horizontal boundary of W, i.e.,

$$\partial_H W = \left\{(z, u) \in W : \left(z, u + \frac{1}{n}\right) \text{ or } \left(z, u - \frac{1}{n}\right) \notin W\right\} .$$

With the notation $X = (x, t)$, $Y = (y, s)$, $Z = (z, u) \in \mathbf{Z}^d \times \frac{\mathbf{Z}}{n}$ we have:

$$G_\Omega^{(n)}(X, Y) \leqq \sum_{Z \in \partial_H W} G_W^{(n)}(X, Z) G_\Omega^{(n)}(Z, Y) + \frac{J}{n} \sum_{\substack{Z \in \partial_V W \\ \langle Z, Z' \rangle}} G_W^{(n)}(X, Z) G^{(n)}(Z', Y) , \tag{5.1}$$

where the second summation is taken over all $\langle Z, Z' \rangle = \langle (z, u), (z', u) \rangle$ horizontal nearest neighbor pairs with $Z' \notin W$.

Remark. In its original form Simon–Lieb–Rivasseau's inequality reads simply

$$G_\Omega^{(n)}(X, Y) \leqq \sum_{Z \in \partial W} G_W^{(n)}(X, Z) G_\Omega^{(n)}(Z, Y) . \tag{5.2}$$

In order to shape it like in (5.1) one needs to apply Local Ward Identities [6, 16] to the bonds $\langle Z, Z'\rangle$ crossing $\partial_V W$. We need them in the present form in order to control the limit $n \to \infty$.

Inequalities (5.1) will serve here a double purpose: to produce mean field bounds on $\bar{\alpha}_c(d)$ and give decay in the multiscale expansion.

Mean-field bounds are obtained by taking W to be the thin set

$$W_l(x,t) = \left\{(x,s);\, s \in I_l^{(n)}(t) \equiv \left[t - \frac{l}{2}, t + \frac{l}{2}\right] \cap \frac{\mathbf{Z}}{n}\right\}.$$

In this case (5.1) reads

$$\begin{aligned} G_\Omega^{(n)}((x,t),(y,s)) \leqq & \sum_{|x-x'|=1} \sum_{u \in I_l^{(n)}(t)} \frac{J}{n} G_{W_l(x,t)}^{(n)}((x,t),(x,u))\, G_\Omega^{(n)}((x',u),(y,s)) \\ & + G_{W_l(x,t)}^{(n)}\left((x,t),\left(x, t+\frac{l}{2}\right)\right) G_\Omega^{(n)}\left(\left(x, t+\frac{l}{2}\right),(y,s)\right) \\ & + G_{W_l(x,t)}^{(n)}\left((x,t),\left(x, t-\frac{l}{2}\right)\right) G_\Omega^{(n)}\left(\left(x, t-\frac{l}{2}\right),(y,s)\right). \end{aligned} \tag{5.3}$$

McBryan–Spencer bounds applied to $W_l(x,t)$ give:

$$G_{W_l(x,t)}^{(n)}((x,t),(x,s)) \leqq e^{-h(x)|t-s|}. \tag{5.4}$$

Therefore

$$G_\Omega^{(n)}((x,t),(y,s)) \leqq G_\Omega^{(n)}(\bar{Z},(y,s)) \left[2d\frac{J}{n} \sum_{u \in I_l^{(n)}(t)} e^{-h(x)|t-u|} + 2e^{-\frac{h(x)l}{2}}\right], \tag{5.5}$$

where $\bar{Z}$ is defined by

$$\begin{aligned} G^{(n)}(\bar{Z},(y,s)) = \max\{ & G_\Omega^{(n)}(Z,(y,s));\quad Z = (x, t \pm l/2) \\ \text{or} \quad & Z = (x',u), \quad |x-x'| = 1, \quad u \in I_l^{(n)}(t)\}. \end{aligned} \tag{5.6}$$

Suppose now $h(x) \geqq \frac{J}{\alpha} > 0$ for all $x \in \Lambda$. Then

$$\begin{aligned} 2d\frac{J}{n} \sum_{u \in I_l^{(n)}(t)} e^{-h(x)|t-u|} + 2e^{-\frac{h(x)l}{2}} & \leqq \left(\frac{4dJ}{h(x)} + 2e^{-\frac{h(x)l}{2}}\right) \\ & \leqq 4d\alpha + 2e^{-\frac{Jl}{\alpha}} < 1, \end{aligned} \tag{5.7}$$

provided $4d\alpha < 1$ and $l = l(\alpha)$ is sufficiently large. In this case, with

$$e^{-m} = (4d\alpha + 2e^{-\frac{Jl}{\alpha}}),$$

$$G_\Omega^{(n)}(X,Y) \leqq e^{-m} G_\Omega^{(n)}(\bar{Z},Y). \tag{5.8}$$

Iterating (5.8) we get

$$G_\Omega^{(n)}((x,t),(y,s)) \leqq e^{-m\|(x-y),(t-s)/l\|_\infty}. \tag{5.9}$$

6. Multiscale Analysis

For $x \in \mathbf{Z}^d$ and $L > 0$ let

$$\Lambda_L(x) = \{ y \in \mathbf{Z}^d ; |y - x| \leqq L \} , \tag{6.1}$$

and for $X = (x, t) \in \mathbf{Z}^d \times \frac{\mathbf{Z}}{n}$, $T > 0$,

$$B_{L,T}^{(n)}(X) = \Lambda_L(x) \times \left(\left[t - \frac{T}{2}, t + \frac{T}{2} \right] \cap \frac{\mathbf{Z}}{n} \right) . \tag{6.2}$$

With a choice of $q > 1$ to be later specified we set

$$B_L^{(n)} = B_{L,L^q}^{(n)}(X) .$$

Definition. A point $x \in \mathbf{Z}^d$ is m-regular at scale L (x is (m, L)-regular) if, for all $n = 1, 2, \ldots$,

$$G_{B_L((x,0))}^{(n)}((x, 0), Y) \leqq e^{-mL}, \quad \forall\, Y \in \partial B_L^{(n)}((x, 0)) . \tag{6.3}$$

Remarks. 1) By translation invariance in the time direction if x is (m, L)-regular then

$$G_{B_L}^{(n)}((x, t), Y) \leqq e^{-mL}, \quad \forall\, Y \in \partial B_L((x, t)), \quad \forall\, t \in \frac{\mathbf{Z}}{n} . \tag{6.4}$$

2) If we define, for $W \subset \mathbf{Z}^d \times \frac{\mathbf{Z}}{n}$,

$$G_W^{(n)}((x, t), \partial) \equiv \sum_{Y \in \partial W} \frac{1}{n} G_W^{(n)}((x, t), Y) , \tag{6.5}$$

it follows that if $x \in \mathbf{Z}^d$ is (m, L)-regular then

$$G_{B_L(x,t)}^{(n)}((x, t), \partial) \leqq e^{-m'L} \tag{6.6}$$

with

$$m' \geqq m - \frac{c \log L}{L} , \tag{6.7}$$

where the constant c depends on q.

Theorem 6.1. *Let $p > 2d$ and suppose that*

(H_0): *There exists $m_0 > 0$, $L_0 > 0$ such that $\mathbf{P}\{0$ is (m_0, L_0)-regular$\} \geqq 1 - 1/L_0^p$.*

Let $L_{k+1} = L_k^\alpha$, $k = 0, 1, 2, \ldots$ with $1 < \alpha < 2p/(p + 2d)$. Then for any $0 < m_\infty < m$ there exists $\bar{L} = \bar{L}(p, d, q, m_0, \alpha, m_\infty)$ such that if $L_0 > \bar{L}$ we have

$$\mathbf{P}\{0 \textit{ is } (m_\infty, L_k)\textit{-regular}\} \geqq 1 - \frac{1}{L_k^p}$$

for all $k = 1, 2, \ldots$.

Remark. Assumption H_0 can be satisfied for any choice of (L_0, m_0) by taking J sufficiently small. This follows from the mean-field bounds of Sect. 5.

Corollary 6.2. *With the hypothesis of Theorem* 6.1, *given* $0 < m < m_\infty$, *there exists a constant* $C_{\chi, \mathbf{h}, J} < \infty$ *such that*

$$G^{(n)}((x, t), (y, s)) \leqq C_{\chi, \mathbf{h}, J}\, e^{-m\|(x-y, |t-s|^{1/q}\|_\infty} .$$

Proof of Corollary. It is the same as the proof of Corollary 3.2 in [12].

Proof of Theorem 6.1. Let $p > d$, $1 < \alpha < 2p/(p + 2\alpha)$ and for some $l > 0$ let

$$\mathbf{P}\{0 \text{ is } (m, l)\text{-regular}\} \geqq 1 - \frac{1}{l^p} .$$

Then for $L = l^\alpha$ we have

$$\mathbf{P}\{\exists\, x_1, x_2 \in \Lambda_L(0),\ x_1 \text{ and } x_2\ (m, l)\text{-singular with } \Lambda_l(x_1) \cap \Lambda_l(x_2) = \phi\}$$

$$\leqq \frac{L^{2d}}{l^{2p}} = \frac{1}{L^{\frac{2p}{\alpha} - 2d}} < \frac{1}{2L^p} \tag{6.8}$$

for l sufficiently large.

It is therefore sufficient to consider the situation where there exists at most one box of side $2l$ around some point $\mathbf{Z} \in \Lambda_L(0)$ such that X is (m, l) regular for all $x \in \Lambda_L(0) \setminus \Lambda_{2l}(Z)$. Let us first estimate $G^{(n)}_{B_L(0)}(0, X)$ for $X \in \partial_V B_L(0)$. From *SLR* inequalities we get

$$G_{B_L(0)}(0, X) = \prod_{i=0}^{n} G_{B_l(Z_i)}(Z_i, \partial) G_{B_L(0)}(Z_n, Z'_m) \prod_{j=0}^{m} G_{B_l(Z'_j)}(Z'_j, \partial) \tag{6.9}$$

for some $Z, \ldots, Z_n, Z'_1, \ldots, Z'_m$, where

$$Z_0 = (0, 0), Z'_0 = X; \quad Z_1, Z_2, \ldots, Z_{n-1}, Z'_1, \ldots, Z'_{m-1} \in B_L(0, 0) \setminus B_{2l, L^q}((z, 0))$$

and $Z_n, Z'_m \in B_L\,(0, 0)$, with $n + m \geqq \dfrac{\frac{L}{2} - 2l}{l/2} = \dfrac{L}{l} - 4$. Therefore,

$$G_{B_L(0)}((0, 0), X) \leqq (e^{-m'l})^{\frac{L}{l} - 4} \leqq e^{-ML} , \tag{6.10}$$

where $M = m' - o(l) = m - o(l)$, for large l. If now $X \in \partial_H B_L(0, 0)$ we now use the McBryan–Spencer bound (4.5), to get

$$\mathbf{P}\{G_{B_L(0,0)}((0, 0), (x, t)) \geqq e^{-\frac{|t|}{L^{\tau + d}}}\} \leqq \mathbf{P}\left\{h_L < \frac{1}{L^\tau}\right\} = \mathbf{P}\left\{\frac{1}{h_L} > L^\tau\right\} , \tag{6.11}$$

where $h_L = h_{\min}^{(B_L(0, 0))} = \min\{h(x), x \in B_L(0, 0)\}$. But

$$\mathbf{P}\left\{\frac{1}{h_L} > L^\tau\right\} \leqq L^d\, \mathbf{P}\left\{\frac{1}{h(0)} > L^\tau\right\} \leqq \frac{L^d\, \mathbf{E}\left(\dfrac{1}{h(0)^\delta}\right)}{L^{\tau\delta}} = \frac{1}{\bar{h}^\delta} \frac{1}{L^{\tau\delta - \alpha}} < \frac{1}{2L^p} \tag{6.12}$$

if $\tau\delta > p + d$, and L sufficiently large. Therefore, if $X \in \partial_H B_L(0, 0)$, i.e., $X = (x, t)$ with $|t| = \dfrac{L^q}{2}$ we have from (6.11) and (6.12)

$$\mathbf{P}\{G_{B_L(0, 0)}((0, 0), X) \leqq e^{-\frac{L^q}{2L^{\tau + \alpha}}}\} \geqq 1 - \frac{1}{2L^p} . \tag{6.13}$$

If we now take $q > \tau + d + 1$, and L sufficiently large,

$$\mathbf{P}\{G_{B_L(0,0)}((0,0), X) \leqq e^{-ML}\} \geqq 1 - \frac{1}{2L^p} \tag{6.14}$$

if $X \in \partial_H B_L(0,0)$. Putting together (6.14), (6.10) and (6.8) we find that

$$\mathbf{P}\{0 \text{ is } (M, l)\text{-regular}\} \geqq 1 - \frac{1}{L^p},$$

which concludes the proof.

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