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A New Contribution to Nonlinear Stability of a Discrete Velocity Model

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Abstract. For a class of discrete velocity models of kinetic theory we prove exponential nonlinear conditional stability of the constant basic state in the slab [0, 1].

1. Introduction and Main Results

In the context of kinetic theory of gases, discrete velocity models manifest several peculiarities in both the mechanical aspect and the mathematical feature. The conspicuous number of papers on this subject confirms the augmenting interest of scientists in this field [for a recent survey, see Illner and Platkowski (1988)].

Among the most interesting models, the generalized $2n$-velocity model seems to be appropriate because it preserves all mathematical difficulties and reflects the structure of the full Boltzmann system. In particular, the stability of a flow in a bounded domain is an open question, even for one-dimensional motions. It is well known that several papers provide existence and uniqueness of solutions with "large" initial data, in one space dimension, cf., e.g., Nishida and Mimura (1974), Tartar (1981), Beale (1986), Bony (1987), Cabannes and Kawashima (1988), Toscani (1989), Slemrod (1989). Moreover, recently Kawashima (1983) first proved, for the Cauchy problem an algebraic asymptotic decay in time of solutions toward a constant (non-zero) Maxwellian. However, the norm there used is a little complicated and, even though it is proved to be equivalent to the norm of $W^{1,2}$, the explicit constants cannot be given. On the other hand, it is known that explicit bounds in time for solutions to nonlinear prolems is of the utmost importance for numerical approaches, where the errors are computed in the norms in which the stability occurs, cf., e.g., Heywood and Rannacher (1982).

The objective of this paper is to propose a new method for studying nonlinear stability for a discrete velocity model in a slab. Such a method was inspired by the recent theory proposed by Galdi and Padula (1990) and can provide explicit bounds in time for solutions to nonlinear problems. One essential idea in the

work of Galdi and Padula (1990) is the construction of a Ljapounov functional for the perturbation such that
i) it is explicit and easily computable;
ii) it is suitable for studying the boundary value problem;
iii) it provides exponential decay in bounded domains.

In this paper, in order to not obscure the lines of the reasoning, we confine ourselves to study the stability of a constant basic state with respect to one dimensional perturbations. As a consequence, sometimes our assumptions are less important in the applications, but still they are a first attempt in the direction of finding an unifying approach for the study of the stability in the fields of continuum and discrete fluid dynamics.

Let us assume the mass density $F_i(x,t) \equiv F(x, v_i, t)$ of a gas particle at position x depends on one spatial direction only, say i, for all time t. We shall call x the space variable in this direction. Moreover, let us consider a discrete $2n$-velocity model in a bounded interval Ω which, after rescaling, we call $(0,1)$. Here we prove the exponential nonlinear conditional stability of the constant equilibrium state

$$F_i(x,t) = s, \qquad i = \pm 1, \dots, \pm n.$$

This is achieved into the very large range of boundary conditions:

unperturbed inflow data at $x = 0$ and $x = 1$; (1.1a)

periodic boundary conditions together with the conservation of the mass and of the first n odd momenta; (1.1b)

partially specular reflecting boundary conditions with ascribed inflow motion (the specular reflection is not included. (1.1c)

Of course, all the above boundary conditions must be compatible with the constant Maxwellian s. Concerning the admissible velocities we set, for some $n \geq 2$,

$$\begin{aligned} &v_j = -v_{-j}, \quad \text{and} \quad v_j \neq 0 \quad \text{for} \quad j = 1, \dots, n \\ &\text{and, in addition, } v_i \neq v_j \text{ for some } i, j, \text{ with } i, j = 1, \dots, n. \end{aligned} \tag{1.2}$$

Such a model, in particular, excludes the conservation of any mass density.

The main result we achieve is given by the following theorem:

Stability Theorem. *Any regular perturbation $\{f_i\}$ to the constant state verifying one of the conditions* (1.1) *decreases monotonically and exponentially decays in time to zero, in the norm of $W^{1,2}(0,1)$, provided that the initial data are "sufficiently small."*

Specifically, for any initial data, bounded by some value depending on s, n, v_i only the perturbation decays with a relaxation rate again depending on s, n, v_i.

As can be checked from the proof of the theorem, the stability region for the initial data is always bounded, even in the case of periodic boundary conditions. Such bounds represent to our opinion the influence on the stability of the nonlinearities (linked to the collision mechanism). Moreover, nonlinearities play a crucial role also in the choice of the region of the flow. As a matter of fact, for dealing with nonlinear terms we need to use the Poincaré inequality which fails in unbounded regions. This gives

Corollary. *The constant state $F_i(x,t) = s$ is linearly stable in any interval I in $\mathbb{R}$, with respect to any perturbation satisfying at the finite boundary planes (if any) one* *of the boundary conditions* (1.1). *Moreover, the spatial L^2-norm of the derivatives of the perturbation is in $L^2(0,\infty)$ in time.*

We would like to draw some other consequences of our approach. One interesting feature is that it explicitly furnishes the influence on stability of the boundary terms, which, in fact, act as stabilizing factors. Another immediate consequence is the pointwise exponential decay of the perturbation. However, in this case we deduce

$$\max_i \operatorname{esssup} |f_i| < A \exp(-ct),$$

with A a constant depending on the initial data and strictly greater than $\max_i \operatorname{esssup} |f_i^0|$, and $f_i(0) \equiv f_i^0$. This estimate does not imply the decay for such a norm from time $t = 0$, a result which is confirmed by the numerical analysis.

A third consequence is that the a priori estimates here deduced on perturbation can be used to prove global in time existence, by means, e.g., of the iteration scheme of Kaniel and Shinbrot (1978), cf. also, Babovsky (1984).

Finally, we like to notice that the Maxwellian state $s+\varepsilon$ cannot be considered as a perturbation to s. In fact, as will be seen in Sect. 2, the required compatibility conditions applied to the constant perturbation $f_i = \varepsilon$ imply $\varepsilon = 0$. Moreover, setting $F_i = s_1 + f_i$ and $F_i = s_2 + f_i$ the mass densities corresponding to the evolution of $s_1 + f_i^0$, $s_2 + f_i^0$ from our stability theorem we deduce

$$|F_i - F_i| \leq |f_i| + |f_i| + |s_1 - s_2| \leq |s_1 - s_2| + |f_i^0|\{\exp(-c_1 t) + \exp(-c_2 t)\}$$

with suitable constants c_i, c_i depending on s_1, s_2, n, v_i and the initial data only.

We feel optimistic that the method introduced here can provide successful results also for more general problems. In particular, we refer to that concerning unbounded intervals and the two dimensional case, preserving the basic state $F_i = s$. More general basic states or more general discrete velocity models [see Gatignol (1975, 1977), Monaco and Longo (1985)] are also to be worked out.

Note. Since this paper was completed, one of us (M.P.) was kindly acquainted by Professor N. Bellomo with a preprint by Professor S. Kawashima entitled "Existence and stability of stationary solutions to the discrete Boltzmann equation." That paper deals with the full discrete velocity model and provides exponential in time decay for a norm of the perturbation to any steady state $F_i(x)$ which is supposed to be only close a Maxwellian. However, such an approach suffers from the same drawback of Kawashima (1983), since the norm with respect to which stability is proved, is equivalent to the norm of $W^{1,2}$ through constants whose explicit value can *not* be given.

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2. Mathematical Preliminaries

In the sequel we will denote by L^2 the usual Lebesgue space of square integrable functions ϕ on $[0,1]$ and by $\|\cdot\|$ the corresponding norm. $W^{1,2}$ denotes the Sobolev space of functions ϕ with

$$\|\phi\|^2 + \|\phi_x\|^2 < \infty .$$

While the symbols ϕ_i and ψ_i as well as σ_i and δ_i will always be used for scalar functions (in L^2, $W^{1,2}$ resp.), the symbols u, v and w will represent vector valued functions of the form

$$u = (v; w) = (\phi_1, \ldots, \phi_n; \psi_1, \ldots, \psi_n)$$

for fixed given $n \in N$. We define the L^2-norms for vector valued functions in the usual way:

$$\|u\|^2 = \|v\|^2 + \|w\|^2 := \sum_{i=1}^{n} \|\phi_i\|^2 + \sum_{i=1}^{n} \|\psi_i\|^2 .$$

Since it will not cause confusion, the corresponding Sobolev spaces are again denoted by L^2 and $W^{1,2}$. The scalar product in L^2 will be denoted by $\langle \cdot, \cdot \rangle$.

One of the main mathematical tools will be the following result which is a generalization of the well-known Poincaré inequality:

Theorem. *Suppose X and Y Banach spaces with $X \subset Y$. Suppose further $|\cdot|_K$ to be a seminorm on X and $K \subset X$ a subspace such that*
a) *for all $u \in K$, $|u|_K = 0$ implies $u = 0$;*
b) *K is close in the norm $|\cdot|_K$;*
(a) *and* b) *of course imply that K is a Banach space with norm $|\cdot|_K$)*
c) *the set*

$$\{u \in K : \|u\|_Y + |u|_K \leq 1\}$$

is compact in Y.
Then there exists a constant $\gamma_0 > 0$ such that for all $u \in K$

$$\|u\|_Y \leq \gamma_0 \cdot |u|_K . \tag{2.1}$$

Proof. Assume that (2.1) does not hold. Then for any $n \in N$ there exists a $u_n \in K$ such that

$$\|u_n\|_Y = 1, \tag{2.2}$$

and

$$|u_n|_K < \frac{1}{n} . \tag{2.3}$$

Obviously, u_n converges in the norm $|\cdot|_K$ to 0. Furthermore, we can choose a subsequence u_K converging weakly to some element $\hat{u} \in K$ and because of the compactness assumption c) in the norm $\|\cdot\|_Y$ to some element $u \in Y$. Therefore, $u = \hat{u} = 0$ which contradicts (2.2).

Remark. This inequality generalizes the Poincaré inequality of Coscia and Padula (1989), and Padula (1986), Lemma 3 p. 5.

However, a generalization of condition c) can also be provided, Sobolev (1963, Theorem 2, p. 64).

The Poincaré inequality as stated above will be applied in different versions, depending on the kind of boundary conditions we are using. We will distinguish three cases, all with

$$X = W^{1,2}, \qquad Y = L^2$$

but with different K:

Case 1. (zero boundary conditions)

$$K := \{u \in X : \phi_i(0) + \psi_i(0) = 0,\ \phi_i(1) - \psi_i(1) = 0,\ i = 1, \ldots, n\}$$
$$|u|_K^2 := \|u_x\|^2;$$

Case 2. (periodic boundary conditions)

$$K := \left\{u \in X : u(0) = u(1),\ \sum_{i=1}^{n} \int_0 \phi_i(x)dx = 0,\ \int_0^1 \Psi_i(x)dx = 0,\ i = 1, \ldots, n\right\}$$

$$|u|_K^2 := |u_x\|^2 + \sum_{i,j=1}^{n} \|\phi_i - \phi_j\|^2;$$

Case 3. (partially reflecting boundary conditions)

$$K := \{u \in X : (\phi_i + \Psi_i)\,(0) = \lambda_i(\phi_i - \Psi_i)\,(0),$$
$$(\phi_i - \Psi_i)\,(1) = \lambda_i(\phi_i + \Psi_i)\,(1),\ i = 1, \ldots, n\},$$

where λ_1 are constants such that $0 \leq \lambda_i < 1$,

$$|u|_K^2 := \|u_x\|^2.$$

We have to show in all cases that K and $|\cdot|_K$ satisfy the conditions a), b), c) of the theorem.

Condition a). In all cases,

$$|u|_K = 0$$

implies

$$\|u_x\| = 0$$

and thus

$$\phi_i \equiv c_i, \qquad \psi_i \equiv d_i$$

for appropriate constants $c_\imath$ and $d_\imath$. In case 2 it follows in addition

$$\sum_{i,j} \|\phi_i - \phi_j\|^2 = 0$$

so that

$$c_i = c_j \quad \text{for all } i, j\,.$$

From the definition of K we get immediately $c_1 = 0$, for all $i = 1, \ldots, n$, in cases 1, 2, 3. Moreover, in case 2, the conditions

$$\int_0^1 \Psi_\imath(x)dx = 0, \qquad \Psi_i(x) = d_\imath,$$

imply $d_\imath = 0$. Finally, in cases 1, 3, $c_i = 0$ and the boundary conditions imply $d_i = 0$.

Condition b). Choose any sequence $u^{(n)}$ in K converging with respect to $|\cdot|_K$. We deduce that then $u_x^{(n)}$ converges in L^2 to some element u_x. From this follows pointwise convergence of $u^{(n)}$ to some bounded function u with derivative u_x. u

is again an element of $W^{1,2}$, and because of the pointwise convergence it satisfies also the equations in the definition of K.

Condition c). We find

$$\{u : \|u\|_Y + |u|_K \leq 1\} \subset \{u : \|u\|_{L^2} + \|u_x\|_{L^2} \leq 1\} \subset \{u : \|u\|_{W^{1,2}} \leq 1\},$$

and the latter set is compact in L^2.

These arguments show that inequality (2.1) is applicable in all three cases of interest. In particular, the constant γ_0 can be computed numerically by solving the variational problem of finding

$$\max_{u \in K} \frac{\|u\|_Y}{|u|_K}.$$

There is another inequality which we are going to use in the sequel. It reads:

Lemma. *For any* $\phi \in W^{1,2}([0,1])$,

$$\sup_x |\phi(x)|^2 \leq (2 \cdot \|\phi_x\| + \|\phi\|) \cdot \|\phi\|. \tag{2.4}$$

Proof. Choose $\bar{x} \in [0,1]$ such that

$$\phi^2(\bar{x}) = \int_0^1 \phi^2(x)dx.$$

Then from Schwartz' inequality,

$$|\phi(x)|^2 = \left| \int_{\bar{x}}^{x} (\phi^2)_x(s)ds + \phi^2(\bar{x}) \right| \leq 2 \cdot \int_{\bar{x}}^{x} |\phi \cdot \phi_x| ds + \int_0^1 \phi^2(s)ds$$
$$\leq (2 \cdot \|\phi_x\| + \|\phi\|) \cdot \|\phi\|.$$

A consequence of (2.1) and (2.4) is

$$\sup_x |u(x)| \leq \gamma_1 \cdot |u|_x \tag{2.5}$$

for any $u \in K$ and any of the three cases considered above. Also in the unbounded case there is a pointwise estimate for functions in $W^{1,2}$. Suppose for example Ω to be unbounded from below, and $\phi \in W^{1,2}(\Omega)$. Then Schwartz' inequality yields

$$\phi^2(x) = \int_{-\infty}^{x} 2 \cdot \phi_x \cdot \phi \, ds \leq 2 \cdot \|\phi\| \cdot \|\phi_x\|. \tag{2.6}$$

3. Statement of the Problem and an Appropriate Ljapunov Functional

Depending on the situation of interest, numerous discrete velocity models have been treated. In this paper we use a generalized Broadwell model with $2n$ velocities $v_{\pm i}$, $i = 1, \ldots, n$, cf. Broadwell (1964), Gatignol (1975). The set of admissible velocities has been defined in Sect. 1. The set of equations describing the time evolution of this model of gas in a slab (which for simplicity is rescaled to $[0,1]$) is

$$\left(\frac{\partial}{\partial t}+v_i\cdot\frac{\partial}{\partial x}\right)F_i=\sum_{j=1}^{n}F_jF_{-j}-n\cdot F_iF_{-i},\qquad i=\pm 1,\ldots,\pm n. \tag{3.1}$$

Here, the symbol $\dfrac{\partial}{\partial x}$ denotes the partial spatial derivative in direction $\mathbf{i}$, and v_i is – as in Sect. 1 – the corresponding component of $\mathbf{v}_i$. One particular steady solution to these equations is represented by the homogeneous distribution

$$F_i\equiv s,\qquad i=\pm 1,\ldots,\pm n \tag{3.2}$$

for some arbitrary constant $s>0$. (However, this is not the only steady homogeneous solution.) In order to clarify the main ideas we restrict to this simplest solution as a basic flow.

Depending on the boundary conditions, we restrict ourselves to solutions satisfying – according to (1.1) –

Unperturbed inflow conditions:

$$F_{+i}(0,t)=s,\qquad F_{-i}(1,t)=s,\qquad i=1,\ldots,n; \tag{3.3a}$$

Periodic boundary conditions:

$$F_i(0,t)=F_i(1,t),\qquad i=\pm 1,\ldots,\pm n \tag{3.3b}$$

and the compatibility conditions

$$\sum_{i=1}^{n}\int_0^1(F_i(x,t)+F_{-i}(x,t))dx=2ns,$$

$$\sum_{i=\pm 1}^{n}v_i^{2k+1}\int_0^1 F_i(x)dx=M_k,\qquad k=0,\ldots,n-1, \tag{CC}$$

which ensure conservation of the mass and of the first n odd momenta. However, in order to include the solutions (3.2), the momenta have to be zero, this is equivalent to the condition

$$\int_0^1(F_i(x,t)-F_{-i}(x,t))dx=0,\qquad i=1,\ldots,n. \tag{CC$_3$}$$

Of course, the conservation laws associated to (3.1) imply that $(\mathrm{CC})_{1,3}$ are satisfied for all time $t>0$ once the initial data satisfy $(\mathrm{CC})_{1,3}$. This case, even though of minor relevance in applications is included here in order of completeness.

Partially reflecting boundary conditions:[1]

$$F_i(0,t)=\lambda_iF_{-i}(0,t)+b_i,$$

1 $$F_{-i}(1,t)=\lambda_iF_i(1,t)+b_i,\qquad i=1,\ldots,n, \tag{3.3c}$$

[1] Notice that, while (3.3a) coincides with $(1.4)_1$ p. 2 of Kawashima (preprint), the relations (3.3c) have not been considered before in such generality

where λ_i are constants such that $0 \leq \lambda_i < 1$, and b_i are non-negative constants. Furthermore, it is assumed that the basic flow (3.2) solves these equations, namely,

$$s = \lambda_i s + b_i, \qquad i = 1, \ldots, n.$$

Denote by f_i the perturbation of the flow:

$$F_i(x,t) = s + f_i(x,t).$$

The kinetic equations for

$$\sigma_i := f_i + f_{-i}; \qquad i = 1, \ldots, n$$

and

$$\sigma_i := f_i - f_{-i}; \qquad i = 1, \ldots, n$$

can be easily derived from (3.1) as

$$\sigma_{it} + v_i\delta_{ix} = 2s\sum_{j=1}^{n}(\sigma_j - \sigma_i) + \frac{1}{2}\sum_{j=1}^{n}(\sigma_j^2 - \sigma_i^2) - \frac{1}{2}\sum_{j=1}^{n}(\delta_j^2 - \delta_i^2),$$
$$\delta_{it} + v_i\sigma_{ix} = 0, \qquad i = 1, \ldots, n, \tag{3.4}$$

where the subscripts t and x denote partial derivatives. As one can check immediately, the perturbed flow F_i satisfies one of the three sets of conditions described in (3.3), if the vector-valued function $u(x,t)$,

$$u = (\sigma_1, \ldots, \sigma_n, \delta_1, \ldots, \delta_n)^T$$

is in the coresponding subspace K of $W^{1,2}$ as described in Sect. 1. In particular, the compatibility condition (CC) reads

$$\sum_{i=1}^{n}\int_0^1 \sigma_i(x,t)dx = 0, \qquad \int_0^1 \delta_i(x,t)dx = 0, \qquad i = 1, \ldots, n \tag{CC$'$}$$

and the boundary conditions are

unperturbed inflow conditions:

$$\sigma_i(0,t) + \delta_i(0,t) = 0,$$
$$\sigma_i(1,t) - \delta_i(1,t) = 0, \qquad i = 1, \ldots, n; \tag{3.5a}$$

periodic boundary conditions:

$$\sigma_i(0,t) = \sigma_i(1,t),$$
$$\delta_i(0,t) = \delta_i(1,t), \qquad i = 1, \ldots, n; \tag{3.5b}$$

partially reflecting boundary conditions:

$$\sigma_i(0,t) + \delta_i(0,t) = \lambda_i(\sigma_i(0,t) - \delta_i(0,t)),$$
$$\sigma_i(1,t) - \delta_i(1,t) = \lambda_i(\sigma_i(1,t) + \delta_i(1,t)), \qquad i = 1, \ldots, n. \tag{3.5c}$$

It is worth to notice that, for periodic boundary conditions (3.5b) the perturbation u belongs to K of case 2. Actually, from

$$|u|_K = 0$$

we deduce $\sigma_i = c_i$ and $\delta_i = d_i$, with c_i, d_i constants and $\sigma_i = \sigma_j$ for all $i, j = 1, \ldots, n$. From equation $(\mathrm{CC}')_2$ we deduce $d_i = 0$ for all $i = 1, \ldots, n$.

We now reduce the problem of stability to the study of the evolution of some norm of σ_i and δ_i. One physically meaningful norm would be for example the sup-norm. In the sequel we prefer the stronger norm of $W^{1,2}$.

The problem is set in L^2 as follows: We want to study the time evolution of an appropriate Ljapunov functional related to the solution

$$u = (\sigma_1, \ldots, \sigma_n, \delta_1, \ldots, \delta_n)^T$$

in L^2 of

$$\frac{\partial}{\partial t} u = (S + M)u + Nu$$

with the linear symmetric operator S bounded in L^2:

$$Su = \left(-2s \sum_j (\sigma_1 - \sigma_j), \ldots, -2s \sum_j (\sigma_n - \sigma_j), 0, \ldots, 0 \right)^T,$$

the linear operator M bounded in $W^{1,2}$:

$$Mu = -(v_1 \delta_{1x}, \ldots, v_n \delta_{nx}, v_1 \sigma_{1x}, \ldots, v_n \sigma_{nx})^T$$

and the nonlinear operator N:

$$Nu = \left(-\sum_j (\sigma_1^2 - \sigma_j^2) + \sum_j (\delta_1^2 - \delta_j^2), \ldots, -\sum_j (\sigma_n^2 - \sigma_j^2) \right.$$
$$\left. + \sum_j (\delta_n^2 - \delta_j^2), 0, \ldots, 0 \right)^T.$$

We want to study the role on stability played by the operator M. This operator verifies always the condition

$$(Mu, u) \leq 0$$

(see second step of Sect. 4). Results obtained in Galdi and Padula (1990) suggest to look for possibly stabilizing effects coming from the coupling functional

$$\langle Su, Mu \rangle,$$

which here is a sum of terms of the form

$$\int_0^1 \delta_{ix} \cdot \sigma_j \, dx .$$

Partial integration of these terms yields terms of the form

$$\int_0^1 \delta_i \cdot \sigma_{jx} \, dx .$$

This motivates us to introduce the following two functionals:

$$F_1 := \sum_{i,j} \int_0^1 \left(\frac{\sigma_i}{v_i} - \frac{\sigma_j}{v_j} \right) \cdot (\delta_{ix} - \delta_{jx} \, dx$$

and

$$F_2 := \sum_{i,j} \int_0^1 \left(\frac{\delta_i}{v_i} - \frac{\delta_j}{v_j} \right) \cdot (\sigma_{ix} - \sigma_{jx})\, dx .$$

The Ljapunov function E to be investigated is a linear combination of the two norms in $W^{1,2}$:

$$E_0 := \frac{1}{2} \|u\|^2$$

and

$$E_1 := \frac{1}{2} \|u_x\|^2$$

and the two functionals F_1 and F_2. We end up with

$$E := E_0 + \mu \cdot E_1 + \lambda \cdot F_1 + \tau\lambda \cdot F_2 ,$$

where μ, λ and τ are positive coefficients. Since we want E to be positive, we impose the following sufficient restriction on μ, λ and τ:

$$\frac{2\lambda}{\sqrt{u}} \cdot (1 + \tau) \leq \frac{v_m}{2n} , \tag{3.6}$$

where

$$v_m = \min\{v_i,\ i = 1, \ldots, n\} .$$

4. The Evolution Equation for E

Taking scalar products from (3.4) and performing some elementary calculations leads to the following results:

$$\frac{dE_0}{dt} = -s \cdot X^2 + B_0 + N_0 ,$$
$$\frac{dE_1}{dt} = -s \cdot Y^2 + B_1 + N_1 ,$$
$$\frac{dF_1}{dt} = -Z^2 - S^2 + Y^2 + I_2 + B_2 + N_2, ,$$
$$\frac{dF_2}{dt} = -T^2 - Y^2 + Z^2 + I_3 + B_3 + N_3$$

with the nonnegative terms

$$X^2 = \sum_{i,j} \|\sigma_j - \sigma_i\|^2,$$

$$Y^2 = \sum_{i,j} \|\sigma_{jx} - \sigma_{ix}\|^2,$$

$$Z^2 = \sum_{i,j} \|\delta_{jx} - \delta_{ix}\|^2,$$

$$S^2 = \sum_i \left(\sum_j \frac{(v_i - v_j)^2}{v_i v_j} \right) \cdot \|\sigma_{ix}\|^2,$$

$$T^2 = \sum_i \left(\sum_j \frac{(v_i - v_j)^2}{v_i v_j} \right) \cdot \|\delta_{ix}\|^2$$

(X, Y, Z, S, and T being the nonnegative quantities related to these equations), the terms with indefinite sign

$$I_2 = 4s \cdot \sum_{i,j,k} \frac{1}{v_i} \langle \sigma_k - \sigma_i, \delta_{ix} - \delta_{jx} \rangle - \sum_{i,j} \frac{(v_i - v_j)^2}{v_i v_j} \langle \sigma_{ix}, \sigma_{jx} - \sigma_{ix} \rangle ,$$

$$I_3 = 4sn \cdot \sum_{i,j} \frac{1}{v_i} \langle \sigma_{jx} - \sigma_{ix}, \delta_i \rangle - \sum_{i,j} \frac{(v_i - v_j)^2}{v_i v_j} \langle \delta_{ix}, \delta_{jx} - \delta_{ix} \rangle ,$$

the boundary data

$$B_0 = - \sum_i v_i \cdot \sigma_i(x) \delta_i(x) \Big|_0^1 ,$$

$$B_1 = - \sum_i v_i \cdot \sigma_{ix}(x) \delta_{ix}(x) \Big|_0^1 ,$$

$$B_2 = -2 \cdot \sum_{i,j} (\sigma_i(x) - \sigma_j(x)) \cdot \sigma_{ix}(x) \Big|_0^1 + 2 \cdot \sum_{i,j} \frac{v_i - v_j}{v_j} \sigma_j(x) \cdot \sigma_{ix}(x) \Big|_0^1 ,$$

$$B_3 = -2 \cdot \sum_{i,j} (\delta_i(x) - \delta_j(x)) \cdot \delta_{ix}(x) \Big|_0^1 + 2 \cdot \sum_{i,j} \frac{v_i - v_j}{v_j} \delta_j(x) \cdot \delta_{ix}(x) \Big|_0^1 ,$$

and the third order terms originating from the nonlinearities on the right-hand side of (3.4):

$$N_0 = \frac{1}{2} \sum_{i,j} \langle \sigma_j^2 - \sigma_i^2, \sigma_i \rangle - \frac{1}{2} \sum_{i,j} \langle \delta_j^2 - \delta_i^2, \sigma_i \rangle ,$$

$$N_1 = \frac{1}{2} \sum_{i,j} \langle (\sigma_j^2 - \sigma_i^2)_x, \sigma_{ix} \rangle - \frac{1}{2} \sum_{i,j} \langle (\delta_j^2 - \delta_i^2)_x, \sigma_{ix} \rangle ,$$

$$N_2 = \sum_{i,j,k} \frac{1}{v_i} \langle \sigma_k^2 - \sigma_i^2, \delta_{ix} - \delta_{jx} \rangle - \sum_{i,j,k} \frac{1}{v_i} \langle \delta_k^2 - \delta_i^2, \delta_{ix} - \delta_{jx} \rangle ,$$

$$N_3 = n \cdot \sum_{i,j} \frac{1}{v_i} \langle (\sigma_j^2 - \sigma_i^2)_x, \delta_i \rangle - n \cdot \sum_{i,j} \frac{1}{v_i} \langle (\delta_j^2 - \delta_i^2)_x, \delta_i \rangle .$$

Combining all these terms, we end up with

$$\frac{d}{dt} E = -D + I + I_\Sigma + N , \tag{4.1}$$

where

$$D = s \cdot X^2 + (\mu s - \lambda \cdot (1 - \tau)) \cdot Y^2 + \lambda \cdot (1 - \tau) \cdot Z^2 + \lambda \cdot S^2 + \lambda\tau \cdot T^2,$$
$$I = \lambda \cdot I_2 + \lambda\tau \cdot I_3 ,$$
$$I_\Sigma = B_0 + \mu \cdot B_1 + \lambda \cdot B_2 + \lambda\tau \cdot B_3 ,$$
$$N = N_0 + \mu \cdot N_1 + \lambda \cdot N_2 + \lambda\tau \cdot N_3 .$$

In order to show that the right-hand-side of (4.1) is strictly negative for appropriate choices of μ, λ and τ, we proceed in several steps:

First Step: Bounds for I. The following calculations don't make use of the Poincaré inequality and thus are valid also in unbounded domains.

We assume in advance

$$\tau \le \frac{1}{2} \quad \text{and} \quad \lambda \le \frac{\mu s}{2}.$$

Then

$$D \ge s \cdot X^2 + \frac{\mu s}{2} Y^2 + \frac{\lambda}{2} Z^2 + \lambda S^2 + \lambda\tau T^2 .$$

Define

$$a := \min\left\{ \sum_j \frac{(v_i - v_j)^2}{v_i v_j},\ i = 1, \dots, n \right\},$$

$$b := \max\left\{ \frac{(v_i - v_j)^2}{v_i v_j},\ i, j = 1, \dots, n \right\},$$

and recall that

$$v_m = \min\{v_i,\ i = 1, \dots, n\}.$$

From condition (1.2b) we obtain $a > 0$.

Employing the Cauchy Schwartz inequality and the inequality

$$2xy \le \varepsilon \cdot x^2 + \frac{1}{\varepsilon} y^2 \quad \text{for} \quad \varepsilon > 0$$

we conclude

$$\begin{aligned}
\left| \sum_{i,j,k} \frac{1}{v_i} \langle \sigma_k - \sigma_i, \delta_{ix} - \delta_{jx} \rangle \right| &\le n \cdot \left(\sum_{i,j} \frac{1}{v_i^2} (\sigma_j - \sigma_i)^2 \right)^{1/2} \cdot \left(\sum_{i,j} (\delta_{jx} - \delta_{ix})^2 \right)^{1/2} \\
&\le \frac{n}{v_m} XZ \\
&\le \frac{n}{\sqrt{2} v_m} \cdot \frac{1}{\sqrt{\lambda s}} \left(sX^2 + \frac{\lambda}{2} Z^2 \right) \\
&\le \frac{n}{v_m \cdot \sqrt{2\lambda s}} D .
\end{aligned}$$

Similar procedures lead to

$$\begin{aligned}
\left| \sum_{i,j} \frac{(v_i - v_j)^2}{v_i v_j} \langle \sigma_{ix}, \sigma_{jx} - \sigma_{ix} \rangle \right| &\le \frac{b}{\sqrt{2\lambda\mu s}} \left(\frac{\mu s}{2} Y^2 + \lambda S^2 \right) \\
&\le \frac{b}{\sqrt{2\lambda\mu s}} \cdot D
\end{aligned}$$

and

$$\begin{aligned}
\left| \sum_{i,j} \frac{(v_i - v_j)^2}{v_i v_j} \langle \delta_{ix}, \delta_{jx} - \delta_{ix} \rangle \right| &\le \frac{b}{\lambda \cdot \sqrt{2\tau}} \cdot (\lambda\tau T^2 + \lambda Z^2) \\
&\le \frac{b}{\lambda \cdot \sqrt{2\tau}} \cdot D .
\end{aligned}$$

Integrating by parts we obtain

$$\left|\sum_{i,j}\frac{1}{v_i}\langle\sigma_{jx}-\sigma_{ix},\delta_i\rangle\right| \leq \left|\sum_{i,j}\frac{1}{v_i}\langle\sigma_j-\sigma_i,\delta_{ix}\rangle\right| + |B|$$
$$\leq \frac{\sqrt{n}}{2v_m\sqrt{as\lambda\tau}}(sX^2+\lambda\tau T^2)+|B|$$
$$\leq \frac{\sqrt{n}}{2v_m\sqrt{as\lambda\tau}}\cdot D+|B|$$

with the boundary part

$$B=\sum_{i,j}\frac{1}{v_i}(\sigma_j-\sigma_i)\cdot\delta_i\Big|_0^1.$$

Collecting all these estimates, we end up with

$$|I| \leq \left(\frac{2n\cdot\sqrt{2\lambda s}}{v_m}+\frac{b\cdot\sqrt{\lambda}}{\sqrt{2\mu s}}+\frac{2\sqrt{n^3 s\lambda\tau}}{v_m\cdot\sqrt{a}}+\frac{b\cdot\sqrt{\tau}}{\sqrt{2}}\right)D+4ns\lambda\tau\cdot|B|$$
$$\leq c\cdot D+4ns\lambda\tau\cdot|B|$$

for

$$c=4\cdot\max\left\{\frac{2n\cdot\sqrt{2\lambda s}}{v_m},\ \frac{b\cdot\sqrt{\lambda}}{\sqrt{2\mu s}},\ \frac{2\sqrt{n^3 s\lambda\tau}}{v_m\cdot\sqrt{a}},\ \frac{b\cdot\sqrt{\tau}}{\sqrt{2}}\right\}.$$

We conclude that for any positive values c and ε and any positive numbers a, b, s, μ, the parameters λ and τ can be chosen small enough such that

$$|I| \leq c\cdot D+\varepsilon\cdot|B|.$$

For the time evolution of E follows then from (4.1):

$$\frac{d}{dt}E \leq -(1-c)D+\varepsilon\cdot|B|+I_\Sigma+N. \tag{4.2}$$

Of course we will fix c later on to be smaller than 1.

Remark. The bound c for I was obtained by applying only the Schwartz inequality. Therefore it is valid also for unbounded domains. In the particular case $\Omega=\mathbb{R}$, the vanishing conditions at infinites yield:

$$I_\Sigma+\varepsilon\cdot|B|=0$$

since $u\in W^{1,2}$. In this case, (4.2) provides linear stability (cf. Corollary in Sect. 1).

Second Step: The Boundary Data. We want to prove that the boundary data we chose (i.e. u belonging to K) always have a *non-destabilizing* influence.

In the *periodic case* (case b), the condition (3.5b) and Eq. (3.4) assure that

$$I_\Sigma+\varepsilon\cdot|B|=0.$$

Therefore, the boundary data do not influence the linear stability condition

$$-(1-c)D+\varepsilon\cdot|B|+I_\Sigma<0.$$

The following calculations are related to the *unperturbed inflow case* (case a):

From (3.5a) we deduce

$$\sigma_i \cdot \delta_i\Big|_0^1 = \tfrac{1}{2}\,(\sigma_i^2(1) + \sigma_i^2(0)) + \tfrac{1}{2}\,(\delta_i^2(1) + \delta_i^2(0))\,.$$

This furnishes a dissipative term in (4.2). Furthermore, from

$$\begin{aligned} \sigma_{it}(0) &= -\,\delta_{it}(0)\,, \\ \delta_{it}(1) &= \quad \sigma_{it}(1)\,, \end{aligned}$$

and (3.4) follows at the boundary points $\bar{x} \in \{0, 1\}$:

$$\delta_{ix} = \mp\,\sigma_{ix} - \frac{2s}{v_i}\sum_j (\sigma_j - \sigma_i) - \frac{1}{2v_i}\sum_j (\sigma_j^2 - \sigma_i^2) + \frac{1}{2v_i}\sum_j (\delta_j^2 - \delta_i^2)\,.$$

Taking squares leads to

$$\begin{aligned} \pm\sigma_{ix} \cdot \delta_{ix} = &-\frac{1}{2}\,(\sigma_{ix}^2 + \delta_{ix}^2) + \frac{s^2}{v_i^2}\Big(\sum_j (\sigma_j - \sigma_i)\Big)^2 \\ &+ \frac{s}{v_i^2}\Big(\sum_j (\sigma_j - \sigma_i)\Big) \cdot \Big(\sum_j [(\sigma_j^2 - \sigma_i^2) - (\delta_j^2 - \delta_i^2)]\Big) \\ &+ \frac{1}{8v_i^2}\Big(\sum_j [(\sigma_j^2 - \sigma_i^2) - (\delta_j^2 - \delta_i^2)]\Big)^2 \end{aligned}$$

with the sign on the left-hand side being $+$ for $\bar{x} = 0$ and $-$ for $\bar{x} = 1$.
From this we obtain

$$I_\Sigma + \varepsilon \cdot |B| = -\,D_B + I_B + N_B\,,$$

where

$$D_B = \frac{1}{2}\,(V_1^2 + W_1^2) + \frac{\mu}{2}\,(V_2^2 + W_2^2)\,,$$

V_i, W_i being the nonnegative quantities defined by

$$\begin{aligned} V_1^2 &= \sum_i v_i(\sigma_i^2(0) + \sigma_i^2(1))\,, \\ W_1^2 &= \sum_i v_i(\delta_i^2(0) + \delta_i^2(1))\,, \\ V_2^2 &= \sum_i v_i(\sigma_{ix}^2(0) + \sigma_{ix}^2(1))\,, \\ W_2^2 &= \sum_i v_i(\delta_{ix}^2(0) + \delta_{ix}^2(1))\,; \end{aligned}$$

further

$$\begin{aligned} I_B = \mu s^2 \cdot \sum_i \frac{1}{v_i}\Big\{\Big(\sum_j (\sigma_j(1) - \sigma_i(1))\Big)^2 + \Big(\sum_j (\sigma_j(0) - \sigma_i(0))\Big)^2\Big\} \\ + 4ns\lambda\tau \cdot |B| + \lambda B_2 + \lambda\tau B_3 \end{aligned} \tag{4.3}$$

and

$$N_B = \mu s \cdot \sum_{\bar{x}=0}^{1} \left\{ \left(\sum_{i,j} \frac{1}{v_i} (\sigma_j - \sigma_i) \right)(\bar{x}) \cdot \left(\sum_{i,j} [(\sigma_j^2 - \sigma_i^2) - (\delta_j^2 - \delta_i^2)] \right)(\bar{x}) \right.$$
$$\left. + \sum_i \frac{1}{8v_i} \left(\sum_j [(\sigma_j^2 - \sigma_i^2) - (\delta_j^2 - \delta_i^2)]\,(\bar{x}) \right)^2 \right\}.$$

Again we have to estimate I_B in terms of D_B and D. To begin with the first term on the right-hand side of (4.3) we write, denoting by $\bar{x}$ simultaneously the boundary points 0 and 1:

$$\sum_i \frac{1}{v_i} \left(\sum_j (\sigma_j(\bar{x}) - \sigma_i(\bar{x})) \right)^2$$
$$= \sum_{i,j,k} \frac{1}{v_i} (\sigma_j(\bar{x}) - \sigma_i(\bar{x})) \cdot \sigma_k(\bar{x}) - n \cdot \sum_{i,j} \frac{1}{v_i} (\sigma_j(\bar{x}) - \sigma_i(\bar{x})) \cdot \sigma_i(\bar{x}).$$

Now

$$\left| \sum_{i,j,k} \frac{1}{v_i} (\sigma_j(\bar{x}) - \sigma_i(\bar{x})) \cdot \sigma_k(\bar{x}) \right| \le \frac{2n}{v_m} \sum_{i,k} |\sigma_i(\bar{x})| \cdot |\sigma_k(\bar{x})|$$
$$\le \frac{2n}{v_m} \left(n \cdot \sum_i |\sigma_i(\bar{x})|^2 \right) \le \frac{2n^2}{v_m^2} W_1^2$$

and also

$$\left| n \cdot \sum_{i,j} \frac{1}{v_i} (\sigma_j(\bar{x}) - \sigma_i(\bar{x})) \cdot \sigma_i(\bar{x}) \right| \le \frac{2n^2}{v_m^2} W_1^2,$$

so that

$$\sum_i \frac{1}{v_i} \left\{ \left(\sum_j (\sigma_j(1) - \sigma_i(1)) \right)^2 + \left(\sum_j (\sigma_j(0) - \sigma_i(0)) \right)^2 \right\}$$
$$= \frac{8n^2}{v_m^2} W_1^2 \le \frac{16n^2}{v_m^2} D_B.$$

From

$$\left| \sum_{i,j} (\sigma_i(\bar{x}) - \sigma_j(\bar{x})) \cdot \sigma_{ix}(\bar{x}) \right| \le 2 \cdot \left(\sum_{i,j} |\sigma_i(\bar{x}) - \sigma_j(\bar{x})|^2 \right)^{1/2} \cdot \left(\frac{n}{v_m} \sum_i v_i \sigma_{ix}^2(\bar{x}) \right)^{1/2}$$
$$\le 2 \cdot \left(2n \cdot \sum_i |\sigma_i(\bar{x})|^2 \right)^{1/2} \cdot \left(\frac{n}{v_m} \sum_i v_i \sigma_{ix}^2(\bar{x}) \right)^{1/2}$$
$$\le \frac{2n}{v_m} \cdot \sqrt{\frac{2}{\mu}} \cdot \left(\frac{1}{2} W_1^2 + \frac{\mu}{2} W_2^2 \right)$$
$$\le \frac{2n \cdot \sqrt{2}}{v_m \cdot \sqrt{\mu}} D_B$$

and

$$\left| \sum_{i,j} \frac{v_i - v_j}{v_j} \sigma_j(\bar{x}) \sigma_{ix}(\bar{x}) \right| \le \frac{\sqrt{bn}}{v_m} W_1 W_2 \le \frac{\sqrt{bn}}{v_m \cdot \sqrt{\mu}} D_B$$

we obtain as an estimate for B_2:

$$|B_2| \le \frac{8n}{v_m \cdot \sqrt{\mu}} \cdot \left(\sqrt{2} + \sqrt{b}\right) \cdot D_B .$$

Similarly,

$$|B_3| \le \frac{8n}{v_m \cdot \sqrt{\mu}} \cdot \left(\sqrt{2} + \sqrt{b}\right) \cdot D_B$$

and

$$|B| \le \frac{n}{v_m^2} (V_1^2 + W_1^2) \le \frac{2n}{v_m^2} D_B .$$

Collecting all terms, we end up with

$$|I_B| \le \frac{8n^2}{v_m^2} (2\mu s^2 + s\lambda\tau + \lambda \cdot (1+\tau) \cdot \left(\sqrt{2} + \sqrt{b}\right) \cdot \frac{v_m}{n\sqrt{\mu}} D_B .$$

Obviously we can for any $c > 0$ choose μ and (depending on μ) λ small enough such that

$$|I_B| \le c \cdot D_B .$$

For the case *c* we do not go through all calculations but indicate the main arguments.

In the *partially reflecting case*, the boundary conditions give

$$\sigma_i(0) = -r_i \delta_i(0), \qquad \sigma_i(1) = -r_i \delta_i(1) .$$

with

$$r_i = \frac{1 + \lambda_i}{1 - \lambda_i} .$$

Since $0 \le \lambda_i < 1$, we find a positive constant c_0 such that

$$B_0 \le -c_0 \sum_i (\sigma_i^2(0) + \sigma_i^2(1) + \delta_i^2(0) + \delta_i^2(1)) .$$

Furthermore, following the same arguments adopted in case *a*, from the evolution Eq. (3.4), we deduce

$$\begin{aligned} B_1 &= \sum_i \{v_i(-\sigma_{ix}(1)\delta_{ix}(1) + \sigma_{ix}(0)\delta_{ix}(0)\} \\ &= \sum_i \left\{ -\tfrac{1}{2} r_i [\delta_{ix}^2(1) + r_i^{-2} \sigma_{ix}^2(1)] - \tfrac{1}{2} r_i^{-1} [\delta_{ix}^2(0) + r_i^2 \sigma_{ix}^2(0)] \right\} \\ &\quad + \sum_i [\tilde{B}_i(1) - \tilde{B}_i(0)] , \end{aligned}$$

where

$$\tilde{B}_i(1) = r_i v_i^{-1} \sum_j \left\{ -2s(\sigma_j - \sigma_i) - \tfrac{1}{2} (\sigma_j^2 - \sigma_i^2) + \tfrac{1}{2} (\delta_j^2 - \delta_i^2) \right\}(1) ,$$

$$\tilde{B}_i(0) = r_i^{-1} v_i^{-1} \sum_j \left\{ -2s(\sigma_j - \sigma_i) - \tfrac{1}{2} (\sigma_j^2 - \sigma_i^2) + \tfrac{1}{2} (\delta_j^2 - \delta_i^2) \right\}(0) .$$

Hence, we easily obtain

$$B_1 \leq -c_1 \sum_i (\sigma_{ix}^2(0) + \sigma_{ix}^2(1) + \delta_{ix}^2(0) + \delta_{ix}^2(1)) + \hat{I} - \hat{N},$$

with a positive constant c_1, a scalar product $\hat{I}$ which – as well as the terms B_2 and B_3 – can be controlled as in the cases before, and a third oder term $\hat{N}$ which can be stimated like the other terms of N (compare the third step).

We conclude: In all cases, for proper choices of the Lagrange coefficients μ, λ and τ we have

$$\frac{d}{dt} E \leq -m \cdot D + N - n \cdot D_B + N_B \tag{4.4}$$

with suitable positive constants m and n. Actually D_B represents a dissipative effect due to the boundary data. In this regard, the boundary data we choose can also be stabilizing.

Remark. Since the inequality of (4.4) is still true for unbounded Ω, again linear stability is obtained and the Corollary in Sect. 1 is completely proven.

Third Step: The Nonlinearity. Here, we use the generalized energy method by Galdi and Padula (1990). Precisely, we shall prove that

$$N + N_B \leq f(E) \cdot (D + D_B)$$

with a strictly increasing function f on $\mathbb{R}_+$ with $f(0) = 0$. We recall the results from Sect. 2 stating that

$$\|u\| \leq \gamma_0 \cdot |u|_K, \tag{4.5}$$

$$\sup_x |u(x)| \leq \gamma_1 \cdot |u|_K \tag{4.6}$$

for any solution $u = (\sigma_1, \ldots, \sigma_n, \delta_1, \ldots, \delta_n)^T$ of (3.4) in K. Furhtermore, from inspection of $|\cdot|_K$ follows immediately,

$$|u|_K^2 \leq l_1 \cdot D, \tag{4.7}$$

$$|u|_K^2 \leq l_0 \cdot E, \tag{4.8}$$

where

$$l_1 = \max\left\{\frac{1}{\lambda\tau a}, \frac{1}{s}\right\},$$

$$l_0 = \max\left\{4n, \frac{2}{\mu}\right\}.$$

We will now investigate step by step all terms contained in N and N_B:

$$\begin{aligned}
|N_0| &\leq \tfrac{1}{2} \sum_{i,j} \left\{\int |\sigma_j - \sigma_i|\,|\sigma_j + \sigma_i|\,|\sigma_{ix}|\,dx + \int |\delta_j - \delta_i|\,|\delta_j + \delta_i|\,|\sigma_i|\,dx\right\} \\
&\leq \sum_{i,j} \sup|\sigma_i| \cdot \{(\|\sigma_j\| + \|\sigma_i\|)^2 + (\|\delta_j\| + \|\delta_i\|)^2\} \\
&\leq \gamma_1 \cdot |u|_K \cdot \sum_{i,j} (\|\sigma_i\|^2 + \|\sigma_j\|^2 + \|\delta_i\|^2 + \|\delta_j\|^2) \\
&\leq 2n\gamma_1 \cdot |u|_K \cdot \|u\|^2 \leq 2n\gamma_0\gamma_1 |u|_K^3 .
\end{aligned} \tag{4.9}$$

For $\mu N_1 + \lambda\tau N_3$ we obtain

$$
\begin{aligned}
&|\mu N_1 + \lambda\tau N_3| \\
&\quad \le \sum_{i,j} \int \left| \frac{\mu}{2}\sigma_{ix} + \frac{\lambda\tau n}{v_i}\delta_i \right| \cdot \{ |(\sigma_j^2 - \sigma_i^2)_x| + |(\delta_j^2 - \delta_i^2)_x| \}\, dx \\
&\quad \le \sum_{i,j} \left\| \frac{\mu}{2}\sigma_{ix} + \frac{\lambda\tau n}{v_i}\delta_i \right\| \cdot \{ \sup|\sigma_j + \sigma_i| \cdot (\|\sigma_{jx}\| + \|\sigma_{ix}\|) \\
&\qquad + sup|\sigma_j - \sigma_i| \cdot (\|\sigma_{jx}\| + \|\sigma_{ix}\|) + \sup|\delta_j + \delta_i| \cdot (\|\delta_{jx}\| + \|\delta_{ix}\|) \\
&\qquad + \sup|\delta_j - \delta_i| \cdot (\|\delta_{jx}\| + \|\delta_{ix}\|) \\
&\quad \le 4\gamma_1 \cdot |u|_K \cdot \sum_{i,j} \left\| \frac{\mu}{2}\sigma_{ix} + \frac{\lambda\tau n}{v_i}\delta_i \right\| \cdot \{ \|\sigma_{ix}\| + \|\sigma_{jx}\| + \|\delta_{ix}\| + \|\delta_{jx}\| \} \\
&\quad \le 2\gamma_1 \cdot |u|_K \cdot \sum_{i,j} \left\{ 4\left\| \frac{\mu}{2}\sigma_{ix} + \frac{\lambda\tau n}{v_i}\delta_i \right\|^2 + \|\sigma_{ix}\|^2 + \|\sigma_{jx}\|^2 + \|\delta_{ix}\|^2 + \|\delta_{jx}\|^2 \right\} \\
&\quad \le 2\gamma_1 \cdot 2n \cdot \left(\mu^2 + 4 \cdot \left(\frac{\lambda\tau n}{v_i} \right)^2 \cdot \gamma_0 + 1 \right) |u_K|^3 . \qquad (4.10)
\end{aligned}
$$

N_2 can be estimated similarly as N_0, and we find

$$
\begin{aligned}
|\lambda N_2| &\le \frac{\lambda}{v_m} \sup|u| \cdot \sum_{i,j,k} (\|\sigma_K\| + \|\sigma_i\| + \|\delta_K\| + \|\delta_i\|) \cdot (\|\delta_{ix}\| + \|\delta_{jx}\|) \\
&\le \frac{\lambda}{v_m} \sup|u| \cdot n^2 \sum_i (2\|\sigma_i\|^2 + 2\|\delta_i\|^2 + 2\|\delta_{ix}\|^2) \\
&\le \frac{\lambda n^2}{v_m} \cdot \gamma_1 \cdot (2\gamma_0 + 1)\, |u|_K^3 . \qquad (4.11)
\end{aligned}
$$

Equations (4.9)–(4.11) yield together with (4.7) and (4.8)

$$|N| \le \bar{c} \cdot |u|_K^2 \le \bar{c} \cdot l_1 \cdot \sqrt{l_0} D \cdot \sqrt{E} =: cD\sqrt{E}$$

with the constant $\bar{c}$ explicitly given above.

For N_B the following relations hold:

$$
\begin{aligned}
|N_B| &\le \frac{4n\mu s}{v_m} \cdot \sup|u| \cdot \sum_{\bar{x}=0}^{1} \sum_i (\sigma_i^2(\bar{x}) + \delta_i^2(\bar{x})) \\
&\quad + \frac{n^2\mu s}{v_m} \cdot (\sup|u|)^2 \cdot \sum_{\bar{x}=0}^{1} \sum_i (\sigma_i^2(\bar{x}) + \delta_i^2(\bar{x})) \\
&\le \left\{ \frac{4n\mu s\gamma_1}{v_m} \cdot |u|_K + \frac{n^2\mu s\gamma_1^2}{v_m} |u|_K^2 \right\} \cdot (V_1^2 + W_1^2) \\
&\le \left\{ \frac{4n\mu s\gamma_1\sqrt{l_0}}{v_m} \cdot \sqrt{E} + \frac{n^2\mu s\gamma_1^2 l_0^2}{v_m} E \right\} \cdot 2D_B \\
&=: g(E) \cdot D_B
\end{aligned}
$$

with $g(E)$ strictly increasing and $g(0) = 0$.

We want to mention that these estimates require the use of the Poincaré inequality and thus cannot be transferred to unbounded domains:

Fourth Step: Decay of E. The calculations of the previous steps allow us to state

$$\frac{d}{dt}E \leq -\left(m - c \cdot \sqrt{E}\right)D - (n - g(E))D_B\,. \tag{4.12}$$

Assume that the initial energy $E(0)$ satisfies

$$E(0) < \min\left\{\frac{m^2}{c^2},\ g^{-1}(n)\right\}. \tag{4.13}$$

Then $E(t)$ is initially decreasing. As a consequence, (4.13) is satisfied also by $E(t)$. This induces monotonicity:

$$E(t_2) \leq E(t_1) \quad \text{for} \quad t_2 > t_1\,,$$

and

$$\int_0^\infty D(t)\,dt < \infty\,, \qquad \int_0^\infty D_B(t)\,dt < \infty\,.$$

Moreover, from Poincaré's inequality follows

$$-D \leq -r \cdot E, \qquad r > 0\,. \tag{4.14}$$

Therefore

$$\frac{d}{dt}E \leq -r \cdot \left(m - c \cdot \sqrt{E(0)}\right) \cdot E\,.$$

Gronwall's lemma now yields

$$E(t) \leq E(0) \cdot \exp\left\{-r \cdot \left(m - c \cdot \sqrt{E(0)}\right) \cdot t\right\}$$

proving exponential decay of E. In particular, from (2.5) we obtain exponential decay in the ess-sup-norm for u and consequently for $f_i(x,t)$. However, in the latter norm we do not deduce, any more, the decay from the beginning because $E(0)$ strictly increases (equality holding only for constant data) such a norm. A further confirmation of such a fact might be numerical calculation.

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