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Adiabatic Limits of the η-Invariants The Odd-Dimensional Atiyah-Patodi-Singer Problem

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Abstract. We study the η-invariant of boundary value problems of Atiyah-Patodi-Singer type. We prove the formula for the spectral flow of the families over S^1. Assuming a product structure in a collar neighbourhood of the boundary, we show that the η-invariant behaves the same way as on a closed manifold. We also study the "adiabatic" limit of the η-invariant. In fact, we present a general method for the calculation of the "adiabatic" limits of the spectral invariants. In nice cases we are able to split them into a contribution from the interior, one from the cylinder, and an error term. Then we show that the error term disappears with the increasing length of the cylinder.

0. Introduction

Let $A: C^\infty(X; V) \to C^\infty(X; V)$ denote a generalized Dirac operator on an odd-dimensional manifold X with boundary Y, and $g: X \to U(N)$ denote a unitary gauge transformation equal to the identity on a certain neighbourhood of Y. We define the operators $D_0 = A \otimes \mathrm{Id}_{\mathbb{C}^N}$ and $D_1 = (\mathrm{Id}_V \otimes g) D_0 (\mathrm{Id}_V \otimes g)^{-1}$ and the family $\{D_r = rD_1 + (1-r)D_0\}$. We fix a self-adjoint boundary condition P for all D_r and as a result we obtain a self-adjoint Fredholm operator $(D_r)_P$. In fact, reduced modulo unitary equivalence $\{(D_r)_P\}_{r \in [0,1]}$ provides us with a family of self-adjoint Fredholm operators over the circle S^1.

The spectral flow is the only homotopy invariant of such families. It is the difference between the number of eigenvalues which change sign from $-$ to $+$ when r goes from 0 to 1 and the number of eigenvalues which change sign from $+$ to $-$. We want to compute this invariant. In the case of a closed manifold we use η-invariant. Let $\{B_r\}$ denote a family of self-adjoint elliptic operators of positive order on a closed manifold. The topological formula for the spectral flow is the result of the following equality:

$$\mathrm{sf}\{B_r\} = \int_0^1 d/dr(\eta_{B_r}(0))dr\,, \tag{0.1}$$

where $\eta_{B_r}(0)$ is the value of the η-function of the operator B_r at $s=0$. It is defined as:

$$\eta_{B_r}(s)=\left(1/\Gamma\left(\frac{s+1}{2}\right)\right)\cdot\int_0^\infty t^{(s-1)/2}\cdot\operatorname{Tr}(B_r e^{-tB_r^2})dt, \tag{0.2}$$

for $\operatorname{Re}(s)\gg 0$ and it has a meromorphic extension to the entire complex plane, which is regular at $s=0$ (see [2, 9]).

In the following, we assume a product structure near the boundary, which means that there exists a collar neighbourhood $N=[0,1]\times Y$ of the boundary Y such that in this neighbourhood A has the form $G(\partial_u+B)$, where G is an automorphism of $V|Y$, ∂_u denotes differentiation in the normal direction, and B is the corresponding Dirac operator on Y. Let us assume now that B is an invertible operator. Under these assumptions the boundary condition $\Pi_+=\Pi_+(B)$ of Atiyah, Patodi, and Singer (see [2]) is a self-adjoint condition. The η-function of such a boundary problem is well-defined and we can think about the following way of proving the formula for the spectral flow. We attach the cylinder $[-R,0]\times Y$ to X and we obtain a manifold $X_R=([-R,0]\times Y)\cup X$. A extends to X_R in a natural way. Let $\mathscr{E}_R$ denote the kernel of the operator Ae^{-tA^2}. It is natural to expect that as $R\to\infty$ $\mathscr{E}_R$ separates into an interior piece and a cylinder piece, plus an error term which disappears in the limit. The contribution to η, which comes from the cylinder equals 0 and only the interior contribution is left, equal to the integral from the "local" η-function on a closed manifold. Obviously there are technical problems, which have to be solved. In fact, to consider spectral flow, or more generally, the variation of the η-invariant, it is much easier to use the Duhamel principle directly on the manifold X.

Nevertheless, the following question arise:

(i) Can we compare, at least in the "adiabatic" limit (i.e. as $R\to\infty$), the η-invariant of A_{Π_+} with the corresponding object on a closed manifold?

and (this results from the non-locality of the η-invariant)

(ii) What is the "corresponding object"?

We found the answer for (ii) in the paper [8] (see also [18]), where we were constructing relative cycles in K-homology. The cycles for $K_1(X,Y)$ are represented by the operators on $\tilde{X}$, the closed double of X. These are operators of the form $\tilde{A}=A\cup(-A)$, by which we mean that $\tilde{A}$ is equal to A on one copy of X and $-A$ on the other copy. The η-invariant of $\tilde{A}$ is equal to 0, but in the case of a generalized Dirac operator we have a well-defined "local" η-invariant. Let $\tilde{\mathscr{E}}_R(t;x,y)$ denote the kernel of the operator $\tilde{A}e^{-t\tilde{A}^2}$. We define the "local" η-function by the formula:

$$\eta_{\tilde{A}}(s;x)=\Gamma\left(\frac{s+1}{2}\right)^{-1}\cdot\int_0^\infty t^{-(s-1)/2}\cdot\operatorname{tr}(\tilde{\mathscr{E}}(t;x,x))dt. \tag{0.3}$$

Once again, this is a well-defined holomorphic function of s for $\operatorname{Re}(s)$ large. Moreover, in the case of a generalized Dirac operator, it extends to a meromorphic function on the entire complex plane which is holomorphic in a neighbourhood of $s=0$ (see [4, 5]). For any value of s it equals 0 on the collar neighbourhood of Y. Therefore, the integral of this density provides an answer to the question (ii).

We offer also an answer to question (i). Let A_R denote the operator A on the manifold X_R. The main result of this paper is the following theorem:

Theorem 0.1. $\lim_{R\to\infty}\left\{\eta_{(A_R)_{\Pi_+}}(0)-\int_{X_R}\eta_{\tilde{A}_R}(0;x)dx\right\}=0$.

The proof of the Theorem 0.1 is an application of the Duhamel principle. We construct a parametrix for the kernel of the operator $Ae^{-t(A_R)^2_{\Pi_+}}$ from the corresponding heat kernel for the operator $\tilde{A}$ on $\tilde{X}$, and the heat kernel of the boundary problem on the cylinder $[0,+\infty)\times Y$. Then we use Levi's sum expansion (see [12]) to obtain a series which gives the heat kernel of the boundary problem on X_R. This gives us suitable pointwise estimates for the kernel of this operator *independent of* R.

The same method provides a proof of the corresponding estimates for the kernel of the operator $\tilde{A}_R e^{-t\tilde{A}_R^2}$. We take two copies of the heat kernels on $\tilde{X}$ restricted to X, and the kernel on the infinite cylinder and construct our parametrix. Then we show that the corresponding series is convergent and we give a pointwise estimate.

After we show the existence of the "nice" kernels on the manifolds X_R and $\tilde{X}_R$, we use a different parametrix, which is constructed from the heat kernel of $\tilde{A}_R$ on $\tilde{X}_R$ and the heat kernel on the cylinder. Using this parametrix we show that the "error" term (difference between the parametrix and the original heat kernel) disappears when $R\to\infty$.

The main technical achievements of this paper are: the precise construction of this parametrix, which allows us to get rid of the error term and the estimate, which establishes the exponential decay of the heat kernel on $\tilde{X}_R$ as $t\to\infty$, after scaling by the size of the manifold. The latter is discussed in the second part of Sect. 7, and enables us to consider, in the corresponding η-density only the integral from 0 to $\sqrt{R}$, which simplifies the calculation.

The "rescaling" technique can also be used in the case of our boundary problem. However, we would like to present a result kindly communicated to us by Werner Mueller instead. The lowest non-trivial eigenvalue of $(A_R)_{\Pi_+}$ is uniformly bounded away of 0. Under this circumstance we can apply the Cheeger-Gromov inequality in order to get rid of the large time contribution to the η-invariant. This approach gives us the possibility to define a spectral flow and more general analytic index for families of operators on the manifold $X_\infty=X\cup[0,+\infty)\times Y$. This will be discussed elsewhere.

Theorem 0.1 discusses only the limit of the difference of the η-invariants. Actually, we can show that $\lim_{R\to\infty}\int_{X_R}\eta_{\tilde{A}_R}(0,x)dx$ exists. We will discuss this result and its application elsewhere. The existence of this limit is a starting point of the discussion of the "cutting and pasting" of the η-invariant in the spirit of the earlier work on the "cutting and pasting" of the elliptic operators (and their indices) on even-dimensional manifold, by the second author and Bernhelm Booss (see [6, 17]). This agrees also with the philosophy of the beautiful note by Singer [15] and we acknowledge that this paper was greatly influenced by his work.

We have produced a general method for the calculation of the "adiabatic" limits of the spectral invariants of boundary problems. We are able to split them into a contribution from the interior and from the cylinder, plus the error term. Then we rigorously show that the error term disappears with the increasing length of the cylinder. In this paper we deal with the situation in which the boundary contribution is fixed (in fact equal 0), but we also apply this method in order to solve the problems in which the cylinder contribution is non-trivial. The important case of such a situation is discussed by Singer in [15] and lead the second author to a new proof of the Holonomy Theorem of Witten (see [20]). Another example is given by the computation of the phase of the determinant of the "Chiral Bag" model presented in [11].

The second goal of this paper is to give a formula for the spectral flow of the family of operators introduced at the beginning of the paper. It was pointed out before that the formula follows from Theorem 0.1. However, it is simpler to give the direct proof based on a straightforward application of the Duhamel principle. For simplicity we assume that X carries a Spin-structure and that $A=\partial\!\!\!/\otimes \mathrm{Id}_V : C^\infty(X; S\otimes V)\to C^\infty(X; S\otimes V)$ denotes the Dirac operator on X (S is a spinor bundle) with the coefficients in the auxiliary Hermitian vector bundle V. We assume once again that B, the tangential part of A, is invertible. Let $g: X\to U(N)$ denote a unitary gauge transformation, such that $g\equiv \mathrm{Id}$ in a certain open neighbourhood of the boundary Y. We have the family $\{(D_r)_{\Pi_+}\}_{r\in[0,1]}$ of self-adjoint Fredholm operators, where the operators $(D_r)_{\Pi_+}$ are given by the formula:

$$\begin{aligned}(D_r)_{N\cdot\Pi_+} &= D_r = r\cdot D_1 + (1-r)D_0\,,\\ \mathrm{dom}(A_r)_{N\cdot\Pi_+} &= \{s\in H^1(X; S\otimes V\otimes \mathbb{C}^N);\ \Pi_+(s|Y)=0\}\,.\end{aligned} \tag{0.4}$$

Here $D_0 = A\otimes \mathrm{Id}_{\mathbb{C}^N} = N\cdot A$ and $D_1 = (\mathrm{Id}_{S\otimes V}\otimes g)A_0(\mathrm{Id}_{S\otimes V}\otimes g)^{-1}$ $N\cdot\Pi_+ = \Pi_+\otimes \mathrm{Id}_{\mathbb{C}^N}$ and H^1 denotes the space of sections from the first Sobolev space.

Theorem 0.2. *The spectral flow of the family $\{(D_r)_{\Pi_+}\}$ is given by the formula:*

$$\mathrm{sf}\{(D_r)_{N\cdot\Pi_+}\} = \mathrm{rk}(V)\cdot \int_X \mathrm{ch}(g)\,, \tag{0.5}$$

where sf *denotes the spectral flow of the family and* $\mathrm{ch}(g)$ *denotes the "odd" Chern character of the element $[g]$ in $K^{-1}(X, Y)$, given by the formula:*

$$\mathrm{ch}(g) = \sum_k (i/2\pi)^k\cdot\frac{(k-1)!}{(2k-1)!}\cdot \mathrm{tr}(g^{-1}dg)\,. \tag{0.6}$$

This formula is well-known in the case of families on closed manifolds (see [3, 6, 16, 17]). In [7] the second author and Bernhelm Booss discussed the application of Theorem 0.2 to the spectral theory of the boundary problems. It was assumed in [7], that formula (0.6) holds also in case $\ker B \neq \{0\}$ and for the general boundary condition of the Atiyah-Patodi-Singer type. We present the proof of this general result in Appendix 1 to this paper. In the proof we use the fact that the space of boundary conditions we consider is path-connected. In Appendix 2 we present the calculations of the homotopy groups of this space.

In Sect. 1 we discuss estimates on the heat kernels of the Dirac operators on the closed manifolds. While completely elementary, they are nevertheless basic for our estimates of the large time contribution to the η-function.

In Sect. 2 we discuss $\tilde{A}$, the "double" of A. We review the elementary spectral properties of this operator. In the second part we discuss a uniform (with respect to R) estimate on the heat kernel of $\tilde{A}_R$. Here we use the Duhamel principle. As this is the model situation for all our applications of Duhamel's principle in this paper, we carefully explain all details.

Section 3 deals with the heat kernel of the odd-dimensional Atiyah-Patodi-Singer problem on the cylinder.

In the first part of Sect. 4, we discuss the heat kernel of the boundary problem $(A_R)_{\Pi_+}$. We take a parametrix constructed from the heat kernel on X and the heat kernel on the cylinder and use the Duhamel principle to show the existence of the heat kernel on the manifold X_R (see Theorem 4.1). In the second part of Sect. 4, we discuss the existence of the η-invariant of the operator A_{Π_+} on the manifold X. We also present the formula for the variation of the η-invariant, which allows us to reduce the calculation of the spectral flow to the case of the closed manifold. We close this section with a proof of Theorem 0.2.

Unfortunately, we cannot use the construction from Sect. 4 to show that the difference between the interior contribution and the trace of the operator $A_R e^{-t(A_R)^2_{\Pi_+}}$ approaches 0 when $R \to \infty$. Therefore, in Sect. 5, we construct another parametrix from the heat kernel on $\tilde{X}_R$ and the heat kernel on the cylinder. Here also the choice of the corresponding cut-off functions is crucial [see (5.1) and Lemma 5.1].

In Sect. 6 we show that in the case of an invertible operator B the lowest non-trivial eigenvalue of the operator $(A_R)_{\Pi_+}$ is uniformly bounded away from 0. Here we use the fact that for large R, we can compare the small eigenvalues of the operator $(A_R)_{\Pi_+}$ with the small eigenvalues of $\bar{A}$, the self-adjoint extension of A to the manifold $X \cup [0, +\infty) \times Y$. This extension has a discrete spectrum in a neighbourhood of 0 if and only if B is invertible.

In Sect. 7 we finish the proof of Theorem 0.1. First, we show that:

$$\lim_{R \to \infty} \left\{ \eta((A_R)_{\Pi_+}; 0) - (1/\sqrt{\pi}) \cdot \int_0^{\sqrt{R}} t^{-1/2} \cdot \int_{X_R} \operatorname{tr}(\tilde{A}_R e^{-t\tilde{A}_R^2}(t; x, x)) dx \right\} = 0 .$$

In the second part of this section we show that as $R \to \infty$ the second integral gives us $\int_{X_R} \eta_{\tilde{A}_R}(0; x) dx$. Here we rescale the metric on $\tilde{X}_R$ and show that this leads to better estimates for the kernels involved.

In the Appendix 1 we discuss the proof of the spectral flow formula for generalized Atiyah-Patodi-Singer problems.

Appendix 2 contains the calculation of the homotopy groups of the space of generalized Atiyah-Patodi-Singer problems.

1. Heat Kernel on Closed Manifold

In this section we state an elementary result which describes the behaviour of a heat kernel related to a Dirac operator on a closed manifold.

Let M denote a closed (compact, without boundary) Riemannian manifold of dimension d. Let $A: C^\infty(M; E) \to C^\infty(M; E)$ denote an elliptic differential self-adjoint operator of first order acting on sections of the Hermitian vector bundle E over M.

Proposition 1.1. *Let us assume that A is invertible and let λ_0^2 denote the lowest eigenvalue of the operator A^2. Let us denote by $e(t; x, y)$ the kernel of the operator e^{-tA^2} and by $\mathscr{E}(t; x, y)$ the kernel of the operator Ae^{-tA^2}. There exists a positive constant c such that the following inequalities hold for any $t > 8$ and $x, y \in M$:*

$$\|e(t; x, y)\| \leqq c \cdot e^{-(t/2) \cdot \lambda_0^2} \quad \textit{and} \quad \|\mathscr{E}(t; x, y)\| \leqq c \cdot e^{-(t/2) \cdot \lambda_0^2}, \tag{1.1}$$

which in our case give us the following inequalities for certain positive constants $c_1, c_2 > 0$ *and any positive* t:

$$\|e(t; x, y)\| \leqq c_1 t^{-d/2} \cdot e^{-(t/4)\cdot\lambda_0^2} \cdot e^{-c_2\frac{d^2(x,y)}{t}}, \tag{1.2}$$
$$\|\mathscr{E}(t; x, y)\| \leqq c_1 t^{-(1+d)/2} \cdot e^{-(t/4)\cdot\lambda_0^2} \cdot e^{-c_2\frac{d^2(x,y)}{t}}.$$

Proof. Let $\{\lambda_i; \varphi_i\}_{i\in\mathbb{Z}}$ denote the spectral decomposition of A. We have:

$$\|\mathscr{E}(t; x, y)\| \leqq \|\sum \lambda_i e^{-t\lambda_i^2} \cdot \varphi_i(x) \otimes \varphi_i(y)\|.$$

We use the Sobolev imbedding theorem (see for instance [9]) to get a pointwise estimate for the eigenfunction φ_i: There exists a constant $b > 0$ (which does not depend on i) such that:

$$\|\varphi_i(x)\| \leqq b \cdot (1 + \lambda_i^{2d}). \tag{1.3}$$

Therefore, we have for $t \geqq 8$:

$$\begin{aligned}\|\mathscr{E}(t; x, y)\| &\leqq b \cdot \{\sum |\lambda_i| e^{-t\lambda_i^2} \cdot (1 + \lambda_i^{2d})^2\} \\ &\leqq b \cdot \{((2d)!)^2 \cdot \sum e^{-(t-3)\lambda_i^2}\} \\ &\leqq b_1 \cdot \sum e^{-(t-4)\lambda_0^2} \cdot e^{-\lambda_i^2} \leqq b_2 \cdot e^{-(t/2)\lambda_0^2}.\end{aligned} \tag{1.4}$$

This proves (1.1) for the kernel of the operator Ae^{-tA^2}. Let us observe that there exist positive constants b_3, b_4, b_5:

$$t^{(1+d)/2} \cdot e^{-(t/4)\cdot\lambda_0^2} \leqq b_3 \quad \text{for any} \quad 0 \leqq t < +\infty, \tag{1.5}$$
$$b_4 \leqq e^{-c_2\frac{d^2(x,y)}{t}} \leqq b_5 \quad \text{for} \quad t \geqq 8,$$

hence, for $t \geqq 8$, (1.2) follows directly from (1.1). We prove (1.1) and (1.2) for the kernel of e^{-tA^2} in the same way. The corresponding results for the finite interval of time $0 < t < 8$ are well-known and they follow from the asymptotic expansion of the kernels on the "diagonal" (see [9]). This means that (1.1) and (1.2) holds for any positive t. $\square$

Now, let us assume that $\ker(A) \neq \{0\}$, and let λ_0^2 denote the lowest non-trivial eigenvalue of the operator A^2. The proof of the inequality (1.4) for $\mathscr{E}(t; x, y)$ is unchanged and (1.1) and (1.2) still hold in this case. This gives us the following theorem:

Theorem 1.2. *Let* λ_0^2 *denote the lowest non-trivial eigenvalue of the operator* A^2. *There exist positive constants* c_1 *and* c_2 *such that the following inequality holds for any* $t > 0$ *and any* $x, y \in M$:

$$\|\mathscr{E}(t; x, y)\| \leqq c_1 t^{-(1+d)/2} \cdot e^{-(t/4)\lambda_0^2} \cdot e^{-c_2\frac{d^2(x,y)}{t}}. \tag{1.6}$$

2. Spectral Properties of the "Double" of a Dirac Operator

We recall here the construction from [18] and [8]. Let X be an odd-dimensional compact manifold with boundary Y, and V denote a complex vector bundle over X. We fix a Riemannian structure on X and hermitian structure on V such that they are the product on some collar neighbourhood $N = [0, 1] \times Y$ of Y. We assume that V is a bundle of Clifford modules over TX (see [10] for the notation) and let A denote the generalized Dirac operator on V. A is a self-adjoint elliptic differential operator of first order. Moreover, it has the following form on N:

$$A = G \cdot (\partial_u + B), \tag{2.1}$$

where $G: V|Y \to V|Y$ is a bundle isomorphism (Clifford multiplication by the inward normal vector) and $B: C^\infty(Y; V|Y) \to C^\infty(Y; V|Y)$ is a corresponding Dirac operator on Y. G and B do not depend on the normal variable in N and they satisfy the following equalities:

$$\begin{aligned} G^2 = -\mathrm{Id}, \quad & G \cdot B = -B \cdot G, \\ G^* = -G \quad \text{and} \quad & B^* = B. \end{aligned} \tag{2.2}$$

We form $\tilde{X} = X \bigcup_Y X$, the closed double of X, and V^G, the double of V over $\tilde{X}$. We glue two copies of V along Y using the automorphism G. A section of V^G is a pair (s_1, s_2) such that s_1 is a section of V extended to $X \cup ((-1, 0] \times Y)$ and s_2 is a section of V (on the other copy of X) extended to $([0,1) \times Y) \cup X \subseteq \tilde{X}$ such that on $(-1, 1) \times Y$ we have:

$$G(y) s_1(u, y) = s_2(u, y). \tag{2.3}$$

We extend A to $X \cup ((-1, 0] \times Y)$ using formula (2.1) and then we define an elliptic operator $\tilde{A} = A \cup (-A): C^\infty(V^G) \to C^\infty(V^G)$ by the formula:

$$\tilde{A}(s_1, s_2) = (A s_1, -A s_2). \tag{2.4}$$

Lemma 2.1. $\eta(s, \tilde{A}) = 0$ *for any value of* s.

Proof. The equation $\tilde{A}(s_1, s_2) = \lambda(s_1, s_2)$ is equivalent to

$$A s_1 = \lambda s_1, \quad A s_2 = -\lambda s_2.$$

It is clear that $(s_2, -s_1)$ is a well-defined section of V^G and $\tilde{A}(s_2, -s_1) = -\lambda(s_2, -s_1)$. $\square$

Remark 2.2. (1) It follows from the Green's formula that $\ker \tilde{A} = \{0\}$ (see [14]), hence we have no 0-eigenvalue.
(2) Lemma 2.1 does not imply the vanishing of the local η-function of $\tilde{A}$. However, we are able to prove the vanishing of $\eta(s, \tilde{A}; x)$ in the collar neighbourhood of the boundary (see Lemma 2.3 below).
(3) In the simplest case, when we take $X = [0, \pi]$, $\tilde{X}$ is a circle and the double of $-i \cdot \frac{d}{dx}$ is unitarily equivalent to the operator $-i \cdot \frac{d}{dx} + \frac{1}{2}$, the natural generator of $K_1(S^1)$. This is the Dirac operator which corresponds to the non-trivial Spin-structure on S^1.

Lemma 2.3. $\eta_{\tilde{A}}(s; x) = 0$ *for any* s *and any* x *which belongs to* $\tilde{N} = N \cup N = (-1, 1) \times Y$.

Proof. Let $(s_1^\lambda, s_2^\lambda)$ denote an eigensection of $\tilde{A}$ which corresponds to the eigenvalue λ, and $x = (u, y)$ belongs to $\tilde{N}$. Then

$$\begin{aligned} \eta_{\tilde{A}}(s; (u, y)) &= \sum_\lambda \operatorname{sign}(\lambda) \cdot |\lambda|^{-s} \langle s_1^\lambda; s_1^\lambda \rangle\rangle(u, y) \\ &= \sum_{\lambda > 0} |\lambda|^{-s} \{\langle s_1^\lambda; s_1^\lambda \rangle\rangle - \langle G s_1^\lambda; G s_1^\lambda \rangle\}(u, y) \\ &= \sum_{\lambda > 0} |\lambda|^{-s} \{\|s_1^\lambda\|^2(u, y) - \|G s_1^\lambda\|^2(u, y)\} = 0. \quad \square \end{aligned}$$

In the rest of this section we investigate the behaviour of the heat kernel of the double of the Dirac operator, on $\tilde{X}_R$ the double of X in which we replace the bicollar N by the cylinder of the length R. By $\tilde{X}_R$ we denote the manifold $([-R,0]\times Y)\cup X$ and by $\tilde{X}_R$ the double of the manifold $X_{R/2}$. $\tilde{X}_R=(-X)\cup\{[-R,0]\times Y\}\cup X$ and we assume that collar N in the copy of $(-X)$ is parametrized by $[-R-1,R]\times Y$. A_R denotes the natural extension of A to X_R and $\tilde{A}_R=A_R\cup(-A_R)$. We have the following theorem:

Theorem 2.4. *Let $\mathscr{E}_R$ denote the kernel of the operator $\tilde{A}_R e^{-t\tilde{A}_R^2}$ on $\tilde{X}_R$. There exist positive constants c_1, c_2, c_3 which do not depend on R such that the following equality holds for any $t>0$ and any $R>0$:*

$$\|\mathscr{E}_R(t;x,y)\|\leqq c_1\cdot e^{c_2 t}\cdot t^{-(1+d)/2}\cdot e^{-c_3\frac{d^2(x,y)}{t}}. \tag{2.5}$$

Example 2.5. Let us consider the operator from the Remark 2.2(3). The double of $-i\cdot\frac{d}{dx}$ on $[0,\pi]$ is the operator $-i\cdot\frac{d}{dx}+\frac{1}{2}$ on the circle. Now let us consider the circle of length $2\pi R$. The double of the operator $-i\cdot\frac{d}{dx}$ on the interval of length πR is the operator $-i\cdot\frac{d}{dx}+\frac{1}{2R}$ on the circle of length $2\pi R$. We denote by $e_R(t;x,y)$ the corresponding heat kernel. Then we have:

$$\begin{aligned}(-i(d/dx)+(1/2R))e_R(t;x,y)&=\frac{1}{2\pi R}\cdot\left\{\sum_{k\in\mathbb{Z}}\frac{2k+1}{2R}\cdot e^{-t\frac{(2k+1)^2}{4R^2}}\cdot e^{\frac{ikx}{R}}\cdot e^{-\frac{iky}{R}}\right\}\\&=\frac{1}{R^2}\cdot\frac{1}{2\pi}\cdot\left\{\sum_{k\in\mathbb{Z}}\frac{2k+1}{2}\cdot e^{-(t/R^2)\cdot\frac{(2k+1)^2}{4}}\cdot e^{ik\frac{x-y}{R}}\right\}\\&=\frac{1}{R^2}\cdot\{(-i(d/dx)+1/2)e_1(t/R^2;x/R,y/R)\}.\end{aligned}$$

Therefore, if we fix constants c_1 and c_2 as in (1.2), such that:

$$|(-i(d/dx)+1/2)e_1(t;x,y)|\leqq c_1 t^{-1}\cdot e^{-t/16}\cdot e^{-c_2\cdot\frac{(x-y)^2}{t}},$$

then the following estimate holds for e_R:

$$\begin{aligned}|(-i(d/dx)+(1/2R))e_R(t;x,y)|&\leqq\frac{1}{R^2}\cdot c_1\cdot(t/R^2)^{-1}\cdot e^{-t/16R^2}\cdot e^{-c_2\frac{((x-y)/R)^2}{t/R^2}}\\&=c_1\cdot t^{-1}\cdot e^{-t/16R^2}\cdot e^{-c_2\cdot\frac{(x-y)^2}{t}}\end{aligned} \tag{2.6}$$

Now let us consider the operator $D_R=G(\partial_u+B)$, the Dirac operator we discussed before, but defined on the space $S^1_R\times Y$. Here S^1_R denotes the circle of length $2\pi R$. We have:

$$e^{-tD_R^2}=e^{t\partial_u^2}\cdot e^{-tB^2}.$$

Let us denote by e_R the kernel of the operator $e^{-tD_R^2}$ and by $\mathscr{E}_R$ the kernel of $D_R e^{-tD_R^2}$. Elementary calculations similar to those which gave (2.6) show that there exist positive constants b_1, b_2 such that the following estimates hold *for any* R:

$$\begin{aligned}\|e_R(t;x,y)\|&\leqq b_1 t^{-d/2}\cdot e^{-(t/4)\mu_0^2}\cdot e^{-b_2\frac{d^2(x,y)}{t}},\\\|\mathscr{E}_R(t;x,y)\|&\leqq b_1 t^{-(1+d)/2}e^{-(t/4)\mu_0^2}\cdot e^{-b_2\frac{d^2(x,y)}{t}},\end{aligned} \tag{2.7}$$

where $\mu_0^2>0$ denotes the lowest eigenvalue of B^2, d is dimension of $S_R^1\times Y$, and the constants b_1 and b_2 do not depend on R. We may also assume that this estimate holds for the kernel of the operator $e^{-t\tilde{A}^2}$ on $\tilde{X}$. Then we have positive constants b_1, b_2, b_3 such that the estimate:

$$\begin{aligned} \|e(t;x,y)\| &\leqq b_1 t^{-d/2}\cdot e^{-b_2 t}\cdot e^{-b_3\frac{d^2(x,y)}{t}}, \\ \|\mathscr{E}(t;x,y)\| &\leqq b_1 t^{-(1+d)/2}\cdot e^{-b_2 t}\cdot e^{-b_3\frac{d^2(x,y)}{t}}, \end{aligned} \tag{2.8}$$

where b_3 depends only on the lowest eigenvalue of B (and $\tilde{A}$), holds for all kernels involved.

Remark 2.6. Suppose that B is not invertible. We have to remove the factor $e^{-b_2 t}$ from (2.8). This follows from the fact that A acts as $G\cdot\partial_u$ on the infinite dimensional space $\ker(B)\otimes L^2(S_R^1)$, and Example 2.5 shows that b_2 has to approach 0 when $R\to+\infty$. The presence of a non-trivial constant b_2 (independent of R) is essential for the argument from the second part of Sect. 7 in order to avoid the large time contribution to the η-invariant.

Proof of Theorem 2.4. We use the Duhamel principle (see [12]). We define S_R, the parametrix for the operator $e^{-t\tilde{A}_R^2}$ on $\tilde{X}_R$, as the operator with the smooth kernel given by the formula:

$$S_R(t;x,y)=\sum_{i=1}^{3}\varphi_i(x)\cdot e_i(t;x,y)\psi_i(y). \tag{2.9}$$

In this formula e_1 and e_3 denote the heat kernel $e_1(t;x,y)$, but given on the different halves of the manifold $\tilde{X}$. e_2 stands for the kernel e_R on $S_{2R}^1\times Y$. $\{\psi_i\}_{i=1}^3$ is a smooth partition of unity on $\tilde{X}_R$ such that:

$$\begin{array}{ll} \text{(i)} & \operatorname{supp}\psi_2=[-R-(1/2);1/2]\times Y. \\ \text{(ii)} & \psi_2\equiv 1 \quad \text{on} \quad [-R-(1/4);1/4]\times Y. \\ \text{(iii)} & \psi_1 \text{ and } \psi_3 \text{ have supports in the corresponding copies of } X=X_1. \\ \text{(iv)} & \text{On the cylinder } \psi_i \text{ is a function of the normal variable.} \end{array} \tag{2.10}$$

φ_i is a smooth function with values in $[0,1]$ and slightly larger support than the support of the corresponding function ψ_i. Moreover, we assume that φ_i has the following properties:

$$\begin{array}{ll} \text{(a)} & \varphi_i\equiv 1 \quad \text{on the } \operatorname{supp}\psi_i. \\ \text{(b)} & \text{On the cylinder } \varphi_i \text{ is a function of the normal variable.} \\ \text{(c)} & \text{There exists } \delta>0 \text{ such that } \operatorname{dist}(\operatorname{supp}\partial\varphi_i/\partial u;\operatorname{supp}\psi_i)>\delta. \\ \text{(d)} & \text{There exists constant } c \text{ such that } |\partial^j\varphi_i/\partial u^j|\leqq c/\delta. \\ \text{(e)} & \operatorname{supp}\partial\varphi_i/\partial u \text{ is contained in a set of the form } [a,a+\delta]\times Y. \end{array} \tag{2.11}$$

It is obvious that estimate (2.8) holds also for the kernel S_R. Now we follow [12]:

$$e^{-t\tilde{A}_R^2}-S_R=\int_0^t d/ds(e^{-s\tilde{A}_R^2}\cdot S_R(t-s))ds=\int_0^t e^{-s\tilde{A}_R^2}(-\tilde{A}_R^2+d/ds)S_R(t-s)ds. \tag{2.12}$$

Let $C(t)$ denote the operator $(\tilde{A}_R^2+d/dt)S_R(t)$ and $C(t;x,y)$ denote the kernel of the operator $C(t)$. Equation (2.12) is equivalent to the equality:

$$e_R(t;x,y)=S_R(t;x,y)+\int_0^t ds\int_{X_R} dz e_R(s;x,z)C(t-s;z,y), \tag{2.13}$$

where $C(t;x,y)$, the kernel of $C(t)$, is given by the formula:

$$C(t;x,y) = -\sum_{i=1}^{3}\{(\partial^2\varphi_i/\partial u^2)(x)e_i(t;x,y)\psi_i(y)+2\cdot(\partial\varphi_i/\partial u)(x)(\partial e_i/\partial u)(t;x,y)\psi_i(y)\}, \tag{2.14}$$

where u denotes the normal coordinate of the variable x. This gives us the following result which follows from (2.11) and (2.14):

Lemma 2.7. *$C(t;x,y)=0$ for $d(x,y)<\delta$. The support of C with respect to the variable x is contained in the set $[-R-1,-R]\times Y\cup[0,1]\times Y$ and:*

$$\mathrm{vol}(\mathrm{supp}_x(C(t;x,y)))\leqq 2\delta\cdot\mathrm{vol}(Y). \tag{2.15}$$

Moreover, the following inequality holds:

$$\|C(t;x,y)\|\leqq(b_4/\delta)\cdot e^{-b_2t}\cdot e^{-b_5\frac{d^2(x,y)}{t}}. \tag{2.16}$$

We write (2.13) in the following way:

$$e_R(t;x,y)=S_R(t;x,y)+(e_R\#C)(t;x,y), \tag{2.17}$$

where

$$\alpha\#\beta(t;x,y)=\int_0^t ds\int_{X_R} dz\alpha(s;x,z)\beta(t-s;z,y).$$

Now, let us apply the operator $\tilde{A}_R$ to both sides of (2.17):

$$Ae_R(t;x,y)=AS_R(t;x,y)+(e_R\#AC)(t;x,y), \tag{2.18}$$

where for simplicity, we write A instead of $\tilde{A}_R$ and we remember that A acts on the first space variable of e_R, C, and S_R. Now we write the formal series for $Ae_R(t;x,y)$:

$$Ae_R(t;x,y)=AS_R(t;x,y)+S_R\#\mathscr{C}_R(t;x,y), \tag{2.19}$$

where

$$\mathscr{C}_R(t;x,y)=\sum_{k=1}^{\infty}C\#C_k \quad\text{and}\quad C_k=C\#C_{k-1} \quad\text{and}\quad C_1=AC(t;x,y).$$

It is not difficult to find an estimate for AC,

$$C(t;x,y)=A^2(\sum\varphi_ie_i\psi_i)-\sum G(\partial\varphi_i/\partial u)(Ae_i)\psi_i,$$

and it follows from (2.11) and (2.14) that we have:

$$\|AC(t;x,y)\|\leqq(b_1/\delta)\cdot e^{-b_2t}\cdot e^{-b_3\frac{d^2(x,y)}{t}}. \tag{2.20}$$

We can assume that S_R satisfies (2.8) and C and AC satisfy (2.20) for certain fixed positive constants b_1,b_2,b_3. Now we show that we have a nice estimate on $\mathscr{C}$. We start with the summand $C_2=C\#AC(t;x,y)$:

$$\|C_2(t;x,y)\|\leqq(b_1/\delta)^2\cdot e^{-b_2t}\cdot\int_0^t ds\int_{\mathrm{supp}_zC} dz e^{-b_3\left(\frac{d^2(x,z)}{s}+\frac{d^2(z,y)}{t-s}\right)}.$$

We use the "generalized triangle inequality:"

$$\frac{d^2(x,y)}{t} \leqq \frac{d^2(x,z)}{s} + \frac{d^2(z,y)}{t-s},$$

to obtain the final estimate for C_2:

$$\|C_2(t;x,y)\| \leqq (b_1/\delta)\cdot(t\cdot b_1\cdot \mathrm{vol}(Y))\cdot e^{-b_2 t}\cdot e^{-b_3\frac{d^2(x,y)}{t}}, \tag{2.21}$$

and in general:

$$\|C_k(t;x,y)\| \leqq (b_1/\delta)\cdot\frac{(b_1\cdot \mathrm{vol}(Y)\cdot t)^{k-1}}{(k-1)!}\cdot e^{-b_2 t}\cdot e^{-b_3\frac{d(x,y)}{t}}. \tag{2.22}$$

This gives the following estimate for $\mathscr{C}_R$:

$$\|\mathscr{C}_R(t;x,y)\| \leqq (b_1/\delta)\cdot e^{b_4 t}\cdot e^{-b_3\frac{d^2(x,y)}{t}}, \tag{2.23}$$

where the constant b_4 is given by:

$$b_4 = b_1\cdot \mathrm{vol}(Y) - b_2. \tag{2.24}$$

We estimate $S_R \# \mathscr{C}_R$ in the same way:

$$\begin{aligned}
\|S_R \# \mathscr{C}_R(t;x,y)\| &\leqq b_1^2\cdot\delta^{-2}\cdot e^{-b_2 t}\cdot\int_0^t ds\int dz s^{-d/2}\cdot e^{(b_4-b_2)(t-s)}\\
&\quad\times e^{-b_3\cdot\left(\frac{d^2(x,z)}{s}+\frac{d^2(z,y)}{t-s}\right)}\\
&= b_1^2\cdot\delta^{-2}e^{-b_2 t}\cdot e^{-b_3\frac{d^2(x,y)}{2t}}\cdot\int_0^t e^{(b_4-b_2)(t-s)}ds\int s^{-d/2}\cdot e^{-b_3\frac{d^2(x,y)}{s}}dz\\
&\leqq c_1 e^{c_2 t} e^{-c_3\frac{d^2(x,y)}{t}}. \quad \square
\end{aligned} \tag{2.25}$$

3. Computation on the Cylinder

In this section we consider the Dirac operator

$$A = G(\partial_u + B): C^\infty(Y\times[0,+\infty);V) \to C^\infty(Y\times[0,+\infty);V),$$

on the infinite cylinder. For simplicity, we assume that B is an *invertible* Dirac operator on the even-dimensional compact manifold Y (which does not depend on the normal variable u). It has symmetric spectrum and if $0<\lambda_1\leqq\lambda_2\leqq\dots$ denotes the sequence of positive eigenvalues, then B has the following spectral decomposition:

$$\{\lambda_k, \varphi_{\lambda_k}; -\lambda_k, G\varphi_{\lambda_k}\}_{k=1}^\infty. \tag{3.1}$$

Let $\Pi_+(B) = \Pi_+$ denote the spectral projection of B onto the subspace of $L^2(V|Y)$ generated by the eigenvectors corresponding to positive eigenvalues. We consider the operator A with the domain:

$$\{u\in C_0^\infty(Y\times[0,+\infty);V); \Pi_+(u|Y)=0\}. \tag{3.2}$$

A has a unique self-adjoint extension which we denote by A_{Π_+}. Then $A_{\Pi_+}^2$ is the operator A^2 with the domain:

$$\{u\in H^2(Y\times[0,+\infty);V); \Pi_+(u|Y)=0 \quad\text{and}\quad \Pi_-((\partial_u+B)(u|Y)=0\}, \tag{3.3}$$

where we put:

$$\Pi_-(B) = \mathrm{Id} - \Pi_+(B) = -G \cdot \Pi_+(B) \cdot G. \tag{3.4}$$

We have a direct formula for the heat kernel of $A^2_{\Pi_+}$ (see [2, pp. 52]) Let λ run through the positive eigenvalues of B and $e(t; u, x; v, y)$ denote the heat kernel $(u, v \in \mathbb{R}_+,\ x, y \in Y)$. The heat kernel e has the following form:

$$\begin{aligned}
&\sum_{\lambda > 0} (e^{-\lambda^2 t}/\sqrt{4\pi t}) \cdot \left\{ e^{-\frac{(u-v)^2}{4t}} - e^{-\frac{(u+v)^2}{4t}} \right\} \cdot \varphi_\lambda(x) \otimes \varphi^*_\lambda(y) \\
&\quad + (e^{-\lambda^2 t}/\sqrt{4\pi t}) \cdot \left\{ e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right\} \cdot G\varphi_\lambda(x) \otimes G\varphi^*_\lambda(y) \\
&\quad - \lambda e^{\lambda(u+v)} \cdot \operatorname{erfc}\{(u+v)/2\sqrt{t} + \lambda\sqrt{t}\} \cdot G\varphi_\lambda(x) \otimes G\varphi^*_\lambda(y).
\end{aligned} \tag{3.5}$$

It follows directly from (3.5) that:

$$\|e(t; u, x; v, y)\| < ct^{-(d/2)} \cdot e^{-\frac{(u-v)^2}{4t}}, \tag{3.6}$$

where d is the dimension of the cylinder (see [2, Proposition 2.21]).

We want to find a formula for the trace of the operator $Ae^{-tA^2_{\Pi_+}}$. We have:

$$\begin{aligned}
&G(\partial_u + B_x) e(t; u, x; v, y)_{|u=v,\ x=y} \\
&= \sum_{\lambda > 0} (e^{-\lambda^2 t}/\sqrt{4\pi t}) \cdot \left\{ (u/t) e^{-\frac{u^2}{t}} + \lambda \left(1 - e^{-\frac{u^2}{t}} \right) \right\} \cdot G\varphi_\lambda(x) \otimes \varphi^*_\lambda(x) \\
&\quad - (e^{-\lambda^2 t}/\sqrt{4\pi t}) \cdot \left\{ -(u/t) e^{-\frac{u^2}{t}} + \lambda \left(e^{-\frac{u^2}{t}} - 1 \right) \right\} \cdot \varphi_\lambda(x) \otimes G\varphi^*_\lambda(y).
\end{aligned} \tag{3.7}$$

Terms which contain $(\lambda e^{-\lambda^2 t})/\sqrt{4\pi t}$ are responsible for the fact that we are not able to integrate with respect to the u variable, but pointwise we have:

$$\begin{aligned}
&\operatorname{Tr} Ae^{-tA^2_{\Pi_+}}(t; u, x; u, x) \\
&= \sum_{\lambda > 0} (e^{-\lambda^2 t}/\sqrt{4\pi t}) \cdot \left\{ \frac{u}{t} \cdot e^{-(u^2/t)} + \lambda(1 - e^{-(u^2/t)} \right\} \{\langle G\varphi_\lambda(x); \varphi_\lambda(x) \rangle \rangle \} = 0.
\end{aligned} \tag{3.8}$$

Therefore, we have the following lemma which is sufficient for our applications:

Lemma 3.1. *Let $\varphi : [0, +\infty) \to \mathbb{R}$ be a smooth function with compact support. Then the trace of the operator $\varphi(u) Ae^{-tA_{\Pi_+}}$ is equal to* 0.

4. Heat Kernel on the Manifold with Boundary. η-Invariant and the Spectral Flow Formula in the Invertible Case

In this section we come back to the set up from the beginning of Sect. 2. We have the generalized Dirac operator A on the odd-dimensional manifold X with boundary Y. A satisfies (2.1) and (2.2).

Our first task is to show that estimate (2.5) holds for the heat kernel of the boundary problem for the operator A_R. We repeat calculations from Sect. 2. We patch together the heat kernel on the manifold $\tilde{X}$, restricted to one copy of the manifold X, with the heat kernel of the boundary problem on the cylinder. Let $\mathscr{E}_1$ denote the kernel of the boundary problem on $[0, +\infty) \times Y$ introduced in Sect. 3. Actually, we consider here this kernel on the cylinder parametrized as

$[-R, +\infty)\times Y$. We denote this kernel by $\mathscr{E}_1^R$. Let $\mathscr{E}_2$ denote the kernel of the operator $\tilde{A}e^{-t\tilde{A}^2}$ on $\tilde{X}$. We patch $\mathscr{E}_1^R$ with $\mathscr{E}_2$ on $N\equiv[0,1]\times Y$, the normal neighbourhood of Y in X. Let $\varrho(a,b)$ denote an increasing smooth function of the real variable u which vanishes for $u\leqq a$ and is equal to 1 for $b\leqq u$. We fix a constant δ such that $0<\delta<1/4$ and we define:

$$\begin{aligned} \varphi_1(u)&=1-\varrho(3/4+\delta,1), & \psi_1(u)&=1-\psi_2,\\ \varphi_2(u)&=\varrho(1/4,1/2-\delta), & \psi_2(u)&=\varrho(1/2,3/4). \end{aligned} \tag{4.1}$$

These functions are defined as functions of the normal coordinate u on the collar N. They have obvious extensions to the whole manifold $\tilde{X}$ (or to the manifold $((-\infty,0]\times Y)\cup X$). We define Q_R a "parametrix" for $\mathscr{E}_R$ the kernel of the operator $A_R e^{-t(A_R)^2_{\Pi_+}}$ on the manifold $X_R=([-R,0]\times Y)\cup X$ by the formula:

$$Q_R(t;x,y)=\varphi_1(x)\cdot\mathscr{E}_1^R(t;x,y)\cdot\psi_1(y)+\varphi_1(x)\cdot\mathscr{E}_2(t;x,y)\cdot\psi_2(y). \tag{4.2}$$

We also introduce a similar parametrix for e_R the kernel of the operator $e^{-t(A_R)^2_{\Pi_+}}$. We repeat the calculation from the proof of the Theorem 2.4, and we obtain the following theorem:

Theorem 4.1. *Let e_R and $\mathscr{E}_R$ denote the heat kernel of the Atiyah-Patodi-Singer problem on the manifold X_R. There exist positive constants c_1,c_2,c_3 which do not depend on R such that for any $t>0$ and any $R>0$:*

$$\begin{aligned} \|e_R(t;x,y)\|&\leqq c_1\cdot e^{c_2t}\cdot t^{-d/2}\cdot e^{-c_3\cdot\frac{d^2(x,y)}{t}},\\ \|\mathscr{E}_R(t;x,y)\|&\leqq c_1\cdot e^{c_2t}\cdot t^{-(1+d)/2}\cdot e^{-c_3\cdot\frac{d^2(x,y)}{t}}. \end{aligned} \tag{4.3}$$

Now we can discuss the existence of the η-invariant for the Atiyah-Patodi-Singer problem. In this section we restrict ourselves to the case of fixed R. In fact, we put $R=0$ and consider the original manifold $X_0=X$. For convenience, let us state once again the simplest version of the Duhamel principle in our context:

Lemma 4.2. *The heat kernel $\mathscr{E}=\mathscr{E}_0$ on the manifold X is given by the formula:*

$$\mathscr{E}(t;x,y)=Q(t;x,y)+(\mathscr{E}\#C)(t;x,y), \tag{4.4}$$

where C denotes the "error" term (see (2.14)). There exist positive constants c_1,c_3,c_4 (we may assume that c_1 and c_3 are the constants from (4.3)) such that (see (2.16)):

$$\|C(t;x,y)\|\leqq c_1e^{-c_2t}\cdot e^{-c_3\frac{d^2(x,y)}{t}}. \tag{4.5}$$

Moreover, $C(t;x,y)$ equals 0, whenever $d(x,y)<\delta$.

Theorem 4.3. *If $\eta_{A_{\Pi_+}}(s)$ is defined by formula (0.2), then $\eta_{A_{\Pi_+}}$ is a well-defined meromorphic function on the entire complex plane. In fact, there exists a holomorphic function $h_1(s)$, such that we have the following equality:*

$$\eta_{A_{\Pi_+}}(s)=\int_X\eta_{\tilde{A}}(s;x)dx+h_1(s),$$

where $\eta_{\tilde{A}}(s;x)$ denotes the "local" η-function of the operator $\tilde{A}$ (see formula (0.4)).

Proof. We start with an elementary calculation, which shows that for any constant $K\geqq 1$ the integral:

$$\int_K^\infty t^{(s-1)/2}\cdot\operatorname{Tr}Ae^{-tA^2_{\Pi_+}}dt \tag{4.6}$$

is a holomorphic function of s. Let μ_0 denote the absolute value of lowest nontrivial eigenvalue of A_{Π_+}. The following estimate holds for any $t > 1$:

$$|\operatorname{Tr} A_{\Pi_+} e^{-tA_\Pi}| \leqq \sum_{\mu \neq 0} |\mu| e^{-t\mu^2} \leqq e^{-(t-1)\mu_0} \cdot \sum_{\mu \neq 0} |\mu| e^{-\mu^2} \leqq k_1 e^{-t\mu_0}. \tag{4.7}$$

We use the estimate (4.7) to show that (4.6) is bounded for any complex s ($K \geqq 1$) and now it is easy to show that it is in fact a holomorphic function of s. We are left with the integral:

$$\int_0^K t^{(s-1)/2} \operatorname{Tr}(Ae^{-tA^2_{\Pi_+}})dt = \int_0^K t^{(s-1)/2} \operatorname{Tr}(Q(t))dt + \int_0^K t^{(s-1)/2} \operatorname{Tr}((\mathscr{E} \# C)(t))dt. \tag{4.8}$$

We will show that the second term is non-singular. This will imply that $\eta_{A_{\Pi_+}}(s)$ has the same singularities as the integral of the local η-function of $\tilde{A}$. Once again, we follow the argument from Sect. 2. We have:

$$\|\mathscr{E} \# C(t; x, x)\| = \left\| \int_0^t dr \int_X dz \mathscr{E}(r; x, z) C(t-r; z, x) \right\|, \tag{4.9}$$

and we know that the integrand is 0, for $d(x,z) < \delta$. We apply (4.3) and (4.4):

$$\begin{aligned}
\|\mathscr{E} \# C(t; x, x)\| &\leqq c_1^2 e^{c_2 t} \cdot \int_0^t dr \int_X r^{-(1+d)/2} e^{-c_3(\delta^2/r)} e^{-c_3(\delta^2/t-r)} dx \\
&\leqq c_1^2 e^{c_2 t} \cdot \operatorname{vol}(X) \cdot \int_0^t r^{-(1+d)/2} \cdot e^{-c_3 \cdot \frac{t\delta^2}{r(t-r)}} \cdot dr \\
&\leqq c_5 \cdot \left\{ \int_0^{t/2} + \int_{t/2}^t \right\} \\
&\leqq c_5 \cdot \left\{ \int_0^{t/2} r^{-(1+d)/2} \cdot e^{-c_3(\delta^2/r)} dr + \int_{t/2}^t (t/2)^{-(1+d)/2} \cdot e^{-c_3(\delta^2/(t-r))} dr \right\} \\
&\leqq c_5 e^{-c_6(\delta^2/t)}.
\end{aligned} \tag{4.10}$$

Equation (4.10) shows that the only part which gives poles in the meromorphic extension of η, is the one which comes from $\operatorname{Tr}(Q(t))$. We know from Lemma 3.1 that

$$\operatorname{tr}(\varphi_1(u) \mathscr{E}_1(t; u, v) \psi(v))_{|u=v} = 0. \tag{4.11}$$

It follows from the Lemma 2.3 and from the definition of the cut-off functions φ_2 and φ_2 that

$$\operatorname{tr}(\varphi_2(x) \mathscr{E}_2(t; x, y) \psi_2(y))_{|x=y} = \operatorname{tr} \mathscr{E}_2(t; x, x), \tag{4.12}$$

hence we have just shown that:

$$\eta_{A_{\Pi_+}}(s) - \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \cdot \int_0^\infty t^{(s-1)/2} dt \int_X \operatorname{tr}(\mathscr{E}_2(t; x, x)) dx \tag{4.13}$$

is a holomorphic function of s. $\square$

Now we want to study the variation of the η-invariant. We assume here that we have given $\{A_r\}$, a family of generalized Dirac operators, such that:

$$A = A_0 \quad \text{and } A_r - A_0 \text{ is a bundle endomorphism equal 0 on } N. \tag{4.14}$$

In particular, this implies that for any r $\mathrm{dom}(A_r)_{\Pi_+} = \mathrm{dom}A_{\Pi_+}$. Following [9, Sect. 1.8] we proceed with the formal calculations:

$$\begin{aligned} d/dr&\left\{\int_0^\infty t^{(s-1)/2}\cdot \mathrm{Tr}(A_r e^{-t(A_r)^2_{\Pi+}})dt\right\} \\ &= \int_0^\infty t^{(s-1)/2}\cdot \mathrm{Tr}\{\dot{A}e^{-tA^2_{\Pi+}} - 2t\cdot \dot{A}A^2 e^{-tA^2_{\Pi+}}\}dt \\ &= \int_0^\infty t^{(s-1)/2}\cdot \mathrm{Tr}(\dot{A}e^{-tA^2_{\Pi+}})dt + 2\cdot \int_0^\infty t^{(s+1)/2}\cdot d/dt(\dot{A}e^{-tA^2_{\Pi+}})dt \\ &= (-s)\cdot \int_0^\infty t^{(s-1)/2}\cdot \mathrm{Tr}(\dot{A}e^{-tA^2_{\Pi+}})dt + 2\cdot \{t^{(s+1)/2}\,\mathrm{Tr}(\dot{A}e^{-tA^2_{\Pi+}})|_0^\infty. \end{aligned} \tag{4.15}$$

Here, as usual, $\dot{A} = d/dr(A_r)$. In order to eliminate the $\lim_{t\to\infty} \mathrm{Tr}\,\dot{A}e^{-tA^2_{\Pi+}}$ we must assume that A_{Π_+} is an invertible operator, which involves ultimately the deformation of the original family. Alternatively, we may use $\lim_{\varepsilon\to 0} \mathrm{Tr}\,\dot{A}e^{-\varepsilon t}e^{-tA^2_{\Pi+}}$, which is the correct regularization procedure. This leaves us with a formula

$$\dot{\eta}((A_r)_+; s) = -\frac{s}{\Gamma\left(\frac{s+1}{2}\right)}\cdot \int_0^\infty t^{(s-1)/2}\,\mathrm{Tr}(\dot{A}e^{-tA^2_{\Pi+}})dt. \tag{4.16}$$

Following the proof of Theorem 4.3 we may apply Duhamel's principle, to show that the difference

$$\begin{aligned} h_2(s) &= -\frac{s}{\Gamma\left(\frac{s+1}{2}\right)} \\ &\quad \times \left\{\int_0^\infty t^{(s-1)/2}\,\mathrm{Tr}(\dot{A}e^{-tA^2_{\Pi+}})dt - \int_0^\infty dt\cdot t^{(s-1)/2}\int_X \mathrm{tr}(\dot{A}e_2(t;x,x))dx\right\} \end{aligned} \tag{4.17}$$

is a holomorphic function of s. Here $e_2(t;x,x)$ denotes the kernel of the operator $e^{-t\tilde{A}^2}$.

Proposition 4.4. *If* $\{A_r\}$ *denote a one-parameter family of generalized Dirac operators on the manifold* X, *which satisfies* (4.14), *then*

$$\dot{\eta}(A_{\Pi_+};0) = \dot{\eta}(A_r \cup (-A_0);0), \tag{4.18}$$

where $A_r\cup(-A_0)$ is a generalized Dirac operator on $\tilde{X}$ equals to A_r on one copy of the manifold X, and $(-A_0)$ on the other copy.

Proof. We have:

$$\dot{\eta}(A_{\Pi_+}; s) = -\frac{s}{\Gamma\left(\frac{s+1}{2}\right)}\cdot\left\{\int_0^\infty t^{(s-1)/2}dt\int_X \mathrm{tr}(\dot{A}e_2(t;x,x))dx + h_2(s)\right\},$$

and the second summand disappears as $s\to 0$. Now, we use the fact that the derivative of the η-invariant on a closed manifold is given by a local formula (see

[9, Sect. 1.8]). This gives us:

$$-\lim_{s\to 0}\frac{s}{\Gamma\left(\frac{s+1}{2}\right)}\cdot\int_0^\infty t^{(s-1)/2}dt\int_X \operatorname{tr}(\dot{A}e_2(t;x,x))dx=\int_X \dot{\eta}_{A_r\cup(-A_r)}(0;x)dx$$
$$=\int_X \dot{\eta}_{A_r\cup(-A_0)}(0;x)dx. \quad \square$$

Proof of Theorem (0.2). Let $\{(D_r)_{N\Pi_+}\}_{r\in[0,1]}$ denote the family defined in (0.5). We have the obvious equalities:

$$\begin{aligned}\operatorname{sf}\{(D_r)_{N\Pi_+}\}&=\int_0^1 \dot{\eta}_{(D_r)_{N\Pi_+}}dr=\int_0^1 \dot{\eta}_{D_r\cup(-D_0)}(0)=\operatorname{sf}\{D_r\cup(-D_0)\}\\&=\operatorname{rk}(V)\cdot\int_{\tilde{X}}\operatorname{ch}(\tilde{g})=\operatorname{rk}(V)\cdot\int_X\operatorname{ch}(g),\end{aligned}\tag{4.19}$$

where $\tilde{g}$ denotes the gauge transformation on $\tilde{X}$, which is equal to g on one copy of X and identity on the other copy. The formula for $\operatorname{sf}\{D_r\cup(-D_0)\}$ is proved in [3] (see also [5, 6, 16, 17]). $\square$

In the appendices we discuss the extension of this result to the more general boundary problems. There we also consider the case of the non-invertible B.

5. Heat Kernels on the Manifold X_R

Unfortunately, we cannot use Theorem 4.1 to show that the contribution to the η-invariant, which comes from the "error" term C vanishes with $R\to\infty$. Therefore, we introduce a different parametrix for the kernel $\mathscr{E}_R$.

Once again, let $\mathscr{E}_1$ denote the heat kernel of the Atiyah-Patodi-Singer problem on the cylinder, which was discussed in Sect. 3. Let $\mathscr{E}_2^R$ denote the heat kernel of the operator $\tilde{A}_R$ on the manifold $\tilde{X}_R$ restricted to one copy of X_R. We may assume that both $\mathscr{E}_1$ and $\mathscr{E}_2^R$ satisfy inequality (4.3). Let $\varrho(a,b)$ denote an increasing C^∞ function of the real variable u vanishing for $u\leqq a$ and equal to 1 for $u\geqq b$. We define:

$$\begin{aligned}&\imath_1=1-\varrho((5/7)R,(6/7)R), &&\jmath_1=1-\jmath_2,\\&\imath_2=\varrho((1/7)R,(2/7)R), &&\jmath_2=\varrho((3/7R,(4/7)R).\end{aligned}\tag{5.1}$$

Moreover, we may assume that

$$|\partial^k\imath_1/\partial u^k|\leqq c_0/R,\tag{5.2}$$

where c_0 denotes a certain positive constant. We extend functions $\imath_1$ and $\jmath_1$ to X_R in an obvious way and we define Q_R by the formula:

$$Q_R(t;x,y)=\imath_1(x)\cdot\mathscr{E}_1(t;x,y)\jmath_2(y)+\imath_2(x)\cdot\mathscr{E}_2^R(t;x,y)\jmath_2(y).\tag{5.3}$$

We have the following formula for $\mathscr{E}_R$, the kernel of the operator $A_Re^{-t(A_R)^2_{\Pi_+}}$:

$$\mathscr{E}_R(t)=Q_R(t)+\mathscr{E}_R\#C_R(t).\tag{5.4}$$

We know the formula for the kernel C_R [see (2.14)]. Equation (2.14) together with (5.1) gives us the following reformulation of Lemma 2.7.

Lemma 5.1. *$C_R(t;u,v)$ vanishes for u not in $[(1/7)R,R]$. Moreover, $C_R(t;u,v)=0$ whenever $|u-v|\leqq R/7$. We also have the following elementary estimate, which is a* *consequence of* (2.16):

$$\|C_R(t;x,y)\| \leqq c_1 \cdot e^{-c_2 t} \cdot e^{-c_3 \cdot \frac{R^2}{t}}. \tag{5.5}$$

As a consequence, we have the following proposition:

Proposition 5.2. $\operatorname{Tr}\mathscr{E}_R(t;x,x) = \operatorname{Tr}Q_R(t;x,x) + \operatorname{Tr}\mathscr{E}_R \# C_R(t;x,x)$, *and we have the following inequality for the "error" term:*

$$\|\mathscr{E}_R \# C_R(t;x,x)\| \leqq k_1 \cdot e^{k_2 t} \cdot e^{-k_3(R^2/t)}, \tag{5.6}$$

for certain positive constants $k_1, k_2, k_3 > 0$.

Proof. We estimate the error term:

$$\begin{aligned}\|\mathscr{E}_R \# C_R)(t;x,x)\| &\leqq c_1^2 e^{c_2 t} \cdot \int_0^t s^{-d/2} \int dz \cdot e^{-c_3 \frac{d^2(x,z)}{s}} \cdot e^{-c_3 \frac{d^2(z,x)}{t-s}} \\ &\leqq c_1^2 e^{c_2 t} \cdot \int_0^t ds \int_{\operatorname{supp}_z C(t-s;\, z,x)} dz s^{-d/2} \cdot e^{-c_3 \cdot \frac{t \cdot R^2}{s(t-s)}} \\ &\leqq c_4 e^{c_2 t} \cdot e^{-c_5(R^2/t)}. \quad \square\end{aligned} \tag{5.7}$$

6. The Estimate for the Lowest Non-Trivial Eigenvalue

Proposition 5.2 shows that, at least for t smaller than $\sqrt{R}$, the trace of the kernel of the operator $Ae^{-t(A_R^2)_{\Pi_+}}$ approaches $\operatorname{tr}Q_R(t;x,x)$ pointwise [and therefore $\operatorname{tr}(\tilde{A}_R e^{-t\tilde{A}_R^2})$] as $R \to \infty$. We want to show that we can neglect the contribution to the η-invariant which comes from the large time asymptotics. The natural way is to apply the Cheeger-Gromov estimate (see the proof of Lemma 7.1). In order to do this we have to show that the lowest non-trivial eigenvalue of A_Π is bounded away from 0. The following theorem was communicated to the authors by Werner Muller.

Theorem 6.1. *Let* $\lambda_1(R) > 0$ *denote the first positive eigenvalue of the operator* $(A_R)^2_{\Pi_+}$. *There exists a constant* $C > 0$ *such that*

$$\lambda_1(R) \geqq C, \tag{6.1}$$

for all $R \in [0, +\infty)$.

We start the proof with a discussion of an operator on the non-compact manifold. As before, let A_R denote the natural extension of A to X_R. We also define the open manifold X_∞ to be:

$$X_\infty = (-\infty, 0] \times Y \cup X. \tag{6.2}$$

Let $A_\infty : C_0^\infty(X_\infty; V_\infty) \to C_0^\infty(X_\infty; V_\infty)$ denote the natural extension of A to X_∞.

A_∞ has a unique self-adjoint extension on $L^2(V_\infty)$ (see [10]). We denote this extension by $\bar{A}$. Let μ_1 denote the smallest (in absolute value) non-zero eigenvalue of B.

Lemma 6.2. *The spectrum of* $\bar{A}$ *in the interval* $(-\mu_1, \mu_1)$ *consists of finitely many eigenvalues of finite multiplicity.*

Proof. Let us consider the operator $A = G(\partial_u + B)$ on $(-\infty, +\infty) \times Y$. Since $(A^2 s; s) \geqq \mu_1^2 \|s\|^2$, one has that A^2 (and therefore A) has bounded inverse in

$L^2((-\infty, +\infty)\times Y;\ V)$. More general, for any $\lambda\in(-\mu_1, +\mu)$ we have a well-defined bounded inverse of the operator $(A-\lambda)$, which we denote by $R_1(\lambda)$. The standard calculation shows that $R_1(\lambda)$ is a smooth family of elliptic pseudodifferential operators of order -1. Now we consider the operator $\tilde{A}=A\cup(-A)$ on the manifold $X\cup X$. We denote by $R_2(\lambda)$ a parametrix for the operator $\tilde{A}-\lambda$. In fact, we may assume that $R_2(\lambda)$ is a smooth family of elliptic operators of order -1 on $\tilde{X}$ such that:

$$(\tilde{A}-\lambda)R_2(\lambda)=\mathrm{Id}-K_1(\lambda) \quad \text{and} \quad R_2(\lambda)(\tilde{A}-\lambda)=\mathrm{Id}-K_2(\lambda),$$

where $K_1(\lambda)$, $K_2(\lambda)$ are operators with smooth kernels. Now we can construct a parametrix for the operator $\bar{A}$. We introduce the following auxiliary smooth function on X_∞:

$$\begin{aligned}
\psi_1&=\begin{cases}1-\varrho(3/8,1/2) & \text{on} \quad (-\infty,1]\times Y\\ 1 & \text{on} \quad X\backslash([0,1]\times Y)\end{cases}, \qquad \psi_2=1-\psi_1,\\
\varphi_1&=\begin{cases}1-\varrho(5/8,3/4) & \text{on} \quad (-\infty,1]\times Y\\ 0 & \text{on} \quad X\backslash([0,1]\times Y\end{cases}, \\
\varphi_2&=\begin{cases}\varrho(1/8,1/4) & \text{on} \quad (-\infty,1]\times Y\\ 1 & \text{on} \quad X\backslash([0,1]\times Y\end{cases}.
\end{aligned} \tag{6.3}$$

and we define the parametrix $R(\lambda)$ for the operator $\bar{A}$ by the formula:

$$R(\lambda)=\sum_{i=1}^{2}\varphi_i R_i(\lambda)\psi_i\,. \tag{6.4}$$

We will show that for any $\lambda\in(-\mu_1, +\mu_1)$ there exists the constant $c=c(\lambda)$ such that the space:

$$H_c(\lambda)=\{s\in L^2(X_\infty;\ V_\infty);\ \|(\bar{A}-\lambda)s\|\leqq c\|s\|\}$$

is finite dimensional. Let $s\in H_c(\lambda)$ then

$$\begin{aligned}
\|(\mathrm{Id}-R(\lambda)\cdot(\bar{A}-\lambda))s\| &\geqq \|s\|-\|R(\lambda)(\bar{A}-\lambda)s\|\\
&\geqq(1-c\|R(\lambda)\|)\cdot\|s\|\geqq\varepsilon\cdot\|s\|\,,
\end{aligned} \tag{6.5}$$

for c sufficiently small. It is enough to show that the operator $\mathrm{Id}-R(\lambda)(\bar{A}-\lambda)$ $=\sum_{i=1}^{2}\varphi_i R_i(\lambda)G(\partial\psi_i/\partial u)$ is compact. The operator $\varphi_2R_2(\lambda)G(\partial\psi_2/\partial u)$ is compact, so the only thing to investigate is the first summand, but this term has the following decomposition:

$$\varphi_1R_1(\lambda)G(\partial\psi_1/\partial u)=(\varphi_1R_1(\lambda)\varphi_1)\cdot(G(\partial\psi_1/\partial u))\,. \tag{6.6}$$

The first factor on the right side of (6.6) is a bounded operator (pseudodifferential operator of order -1), and the second has compact support, hence it is compact. The operator $(\mathrm{Id}-R(\lambda)(\bar{A}-\lambda))$ is compact and invertible when restricted to $H_c(\lambda)$, hence $H_c(\lambda)$ must be a finite-dimensional subspace of $L^2(X_\infty;\ V_\infty)$. This completes the proof of the lemma. □

Remark 6.3. 1. Let us assume that $A=G(\partial_u+B)$ is not invertible on $L^2((-\infty, +\infty)\times Y;\ V)$. We still can construct the parametrix R_1 for A, but this time:

$$\mathrm{Id}-\varphi_1R_1A\psi_1=\mathrm{Id}-\varphi_1(\mathrm{Id}-K_3)\psi_1-\varphi_1G(\partial\psi_1/\partial u)\,.$$

Here K_3 is an operator with a smooth kernel, but in general the support of this kernel is not a compact set, therefore the second summand provides us with a non-compact "error" term.
2. The proof we offer is elementary. Actually, using more advanced methods one can show that the essential spectrum of $\bar{A}$ is equal to $(-\infty, -\mu_1]\cup[\mu_1, +\infty)$ (see for instance [13, Chap. VI]).

Let $\Phi \in L^2(V_R)$ denote an eigensection of $(A_R)_\Pi$ for the eigenvalue λ. We assume that:

$$0<|\lambda|<\mu_1/\sqrt{2}.$$

Let μ run over the positive eigenvalues of B with corresponding eigenfunction φ_μ. It follows from (2.2) that $-\mu$ is an eigenvalue of B with eigensection $G\varphi_\mu$. Φ has the following expansion on the cylinder $[-R,0]\times Y$:

$$\begin{aligned}\Phi(u,y)= \sum_{\mu>0} a_\mu \Big\{&\frac{\lambda}{(\mu^2-\lambda^2)^{1/2}}\cdot \operatorname{sh}((\mu^2-\lambda^2)^{1/2}(R+u))\varphi_\mu(y)\\ &+(\operatorname{ch}((\mu^2-\lambda^2)^{1/2}(R+u))+\frac{\mu}{(\mu^2-\lambda^2)^{1/2}}\operatorname{sh}((\mu^2-\lambda^2)^{1/2}(R+u)))G\varphi_\mu\Big\}.\end{aligned}$$

We assume that $\|\Phi\|_{L^2}=1$. Then we have:

$$\begin{aligned}1 &\geq \int_{[-R,0]\times Y} \|\Phi(u,y)\|^2 du dy\\ &= \sum_{\mu>0} a_\mu^2 \Big\{-\frac{\lambda^2}{\mu^2-\lambda^2}\cdot R+(1/4)\cdot\frac{\lambda^2}{(\mu^2-\lambda^2)^{3/2}}\cdot \operatorname{sh}(2(\mu^2-\lambda^2)^{1/2}R)\\ &\quad +(1/4)\cdot\left(1+\frac{\mu^2}{\mu^2-\lambda^2}\right)\cdot(\mu^2-\lambda^2)^{-1/2}\cdot\operatorname{sh}(2(\mu^2-\lambda^2)^{1/2}R)\\ &\quad +(1/2)\cdot\frac{\mu}{\mu^2-\lambda^2}\cdot\operatorname{ch}(2(\mu^2-\lambda^2)^{1/2}R)\Big\}.\end{aligned} \tag{6.7}$$

Hence, there is a positive constant C, such that

$$\sum_{\mu>0} a_\mu^2\cdot e^{\mu R}\leqq C, \tag{6.8}$$

for all $R\geqq R_0$ for some positive R_0.

Now we extend Φ to a section of V_∞ by the formula

$$\Phi_\infty(x)=\begin{cases}\Phi(x), & x\in X_R,\\ \sum_{\mu>0} a_\mu(R)e^{\mu(R+u)}G\varphi_\mu(y) & \text{for} \quad x=(u,y)\in(-\infty,-R]\times Y.\end{cases} \tag{6.9}$$

Lemma 6.4. *Φ_∞ is continuous on X_∞ and smooth on $X_\infty\backslash(\{-R\})\times Y)$. Moreover, Φ_∞ belongs to the 1st Sobolev space $H^1(X_\infty, V_\infty)$.*

We omit the easy proof of this result.

It follows from (6.8) that

$$\begin{aligned}\|\Phi_\infty\|^2=\|\Phi\|^2+\sum_{\mu>0} a_\mu^2(1/2\mu)&\leqq 1+\frac{1}{2\mu_1}\cdot\sum a_\mu^2\\ &\leqq 1+\frac{1}{2\mu_1}\cdot(\sum a_\mu^2\cdot e^{\mu R})\cdot e^{-\mu_1 R}\leqq 1+C\cdot e^{-\mu_1 R},\end{aligned} \tag{6.10}$$

for some $C>0$.

Next, let $\Psi \in \ker \bar{A}$ and assume that $\|\Psi\| = 1$. On $(-\infty, 0] \times Y$, Ψ has the form:

$$\sum_{\mu > 0} b_\mu e^{\mu u} G\varphi_\mu(y), \quad \text{with } \sum (1/2\mu) \cdot |b_\mu|^2 < +\infty .$$

Set $l = \Psi_{|X_R}$. Then l satisfies the equations:

$$A_R l = 0 \quad \text{and} \quad \Pi_+(l_{|\{-R\} \times Y}) = 0 . \tag{6.11}$$

Hence, l belongs to $\ker(A_R)_\Pi$. This implies the following equality:

$$\begin{aligned} \int_{X_R} \langle \Phi_\infty(x); \Psi(x) \rangle dx &= (1/\lambda) \cdot \int_{X_R} \langle A\Phi_\infty(x); l(x) \rangle dx \\ &= (1/\lambda) \cdot \int_{X_R} \langle \Phi_\infty(x); Al(x) \rangle dx \\ &\quad - (1/\lambda) \cdot \int_Y \langle G(\Phi_\infty(-R, y)); l(-R, y) \rangle dy = 0 . \end{aligned} \tag{6.12}$$

On the other hand,

$$\int_{(-\infty, -R] \times Y} \langle \Phi_\infty(x); \Psi(x) \rangle dx = \sum_{\mu > 0} a_\mu \bar{b}_\mu \cdot \frac{e^{-\mu R}}{2\mu} \leqq C_1 e^{-C_2 R} . \tag{6.13}$$

Therefore:

$$|\langle \Phi_\infty; \Psi \rangle| \leqq C_1 e^{-C_2 R} . \tag{6.14}$$

Let $\{\Psi_1, \dots, \Psi_m\} \subseteqq \ker \bar{A}$ denote an orthonormal basis of $\ker \bar{A}$ and put

$$\tilde{\Phi} = \Phi_\infty - \sum_{i=1} \langle \Phi_\infty; \Psi_i \rangle \Psi_i . \tag{6.15}$$

$\tilde{\Phi}$ is orthogonal to $\ker \bar{A}$ and belongs to $H^1(X_\infty; V_\infty)$. Furthermore,

$$\|\tilde{\Phi}\|^2 = \|\Phi_\infty\|^2 - \sum_{i=1}^{m} \langle \Phi_\infty; \Psi_i \rangle^2 .$$

It follows from (6.10) and (6.14) that

$$|\|\tilde{\Phi}\| - 1| \leqq C_3 e^{-C_2 R} . \tag{6.16}$$

Define $\hat{\Phi} = \tilde{\Phi} / \|\tilde{\Phi}\|$. Then:

$$\hat{\Phi} = \alpha \cdot \tilde{\Phi}, \quad \text{where} \quad |\alpha - 1| \leqq C_3 e^{-C_2 R} . \tag{6.17}$$

Set

$$\lambda_1 = \min_{\substack{\Psi \in H^1 \\ \Psi \perp \ker \bar{A}}} \|\bar{A}\Psi\|^2 / \|\Psi\|^2 . \tag{6.18}$$

$\hat{\Phi}$ belongs to H^1 and is orthogonal to $\ker \bar{A}$, hence

$$\|\bar{A}\hat{\Phi}\|^2 \geqq \lambda_1 . \tag{6.19}$$

We also have

$$\|\bar{A}\hat{\Phi}\|^2 = \alpha^2 \|\bar{A}\tilde{\Phi}\|^2 = \alpha^2 \|A\Phi\|^2_{X_R} = \alpha^2 \lambda^2 . \tag{6.20}$$

Since $\alpha \to 1$ as $R \to \infty$, we have:

$$\lambda^2 \geqq \lambda_1 / 2 , \tag{6.21}$$

for R sufficiently large. Assume that $0 < \lambda^2 < \mu_1^2/2$, then $\lambda_1 < \mu_1^2$. The essential spectrum of $(\bar{A})^2$ is equal to the interval $[\mu_1^2, +\infty)$ and it follows that $\lambda_1 > 0$ is an eigenvalue of $\bar{A}^2$. Now we have

$$\lambda^2 > \lambda_1/2, \tag{6.22}$$

which proves the theorem. $\square$

7. Adiabatic Limits of the η-Invariants

In this section we prove Theorem 0.1. $\operatorname{Tr}\mathscr{E}_R$ splits into the sum:

$$\operatorname{Tr}\mathscr{E}_R(t) = \operatorname{Tr}Q_R(t) + \operatorname{Tr}\mathscr{E}_R \# C_R(t), \tag{7.1}$$

where we follow the notation from Sect. 5. The following lemma tells us that in the "adiabatic" limit we can ignore the "large" time contribution to the η-invariant.

Lemma 7.1.

$$\lim_{R\to\infty} \int_{\sqrt{R}}^{\infty} t^{-1/2} \operatorname{tr}(A \cdot e^{-t(A_R)^2_\Pi}) dt = 0.$$

Proof. The lemma follows from the Cheeger-Gromov inequality. For $\mu > 0$ we have:

$$\int_{\sqrt{R}}^{\infty} t^{-1/2} \mu e^{-t\mu^2} dt = 2 \cdot \int_{\mu \cdot R^{1/4}}^{\infty} e^{-\gamma^2} d\gamma \leqq 2 \cdot e^{-\sqrt{R}\cdot \mu^2},$$

which gives:

$$\begin{aligned}\int_{R^{1/2}}^{\infty} t^{-1/2} \cdot \operatorname{Tr}(Ae^{-t(A_R)^2_\Pi}) dt &\leqq 2 \cdot \sum_{\mu \neq 0} e^{-R^{1/2}\cdot \mu^2} = 2 \cdot \sum_{\mu \neq 0} e^{-(R^{1/2}-1)\mu^2} \cdot e^{-\mu^2} \\ &\leqq 2 \cdot e^{-(R^{1/2}-1)\mu_0^2} \cdot \sum_{\mu \neq 0} e^{-\mu^2} \\ &\leqq l_0 e^{-R^{1/2}\mu_0^2} \cdot \operatorname{Tr} e^{-(A_R)^2_\Pi}.\end{aligned} \tag{7.2}$$

Here l_0 is a positive constant and μ_0 denotes the lowest non-trivial eigenvalue, which is bounded away from 0 by Theorem 6.1. It follows from Theorem 4.1 that the trace of the operator $e^{-(A_R)^2_\Pi}$ is bounded by a constant times the volume of the manifold X_R. Now we obtain the final estimate:

$$\begin{aligned}\int_{R^{1/2}}^{\infty} t^{-1/2} \cdot \operatorname{Tr}(Ae^{-t(A_R)^2_\Pi}) dt &\leqq l_1 e^{-R^{1/2}\mu_0^2} \cdot \operatorname{vol}(X_R) \leqq l_2 \cdot R \cdot e^{-R^{1/2}\mu_0^2} \\ &\leqq l_2 \cdot e^{-l_3 R^{1/2}} \xrightarrow[R\to\infty]{} 0. \quad \square\end{aligned} \tag{7.3}$$

We write $\eta_{(A_R)_{\Pi_+}} = \eta_{(A_R)_{\Pi_+}}(0)$ as the sum:

$$\begin{aligned}\eta_{(A_R)_{\Pi_+}} = (1/\sqrt{\pi}) \Bigg\{ &\int_0^{\sqrt{R}} t^{-1/2} \cdot \operatorname{Tr}(Q_R(t)) dt + \int_0^{\sqrt{R}} t^{-1/2} \cdot \operatorname{Tr}(\mathscr{E}_R \# C_R)(t) dt \\ &+ \int_{\sqrt{R}}^{+\infty} t^{-1/2} \cdot \operatorname{Tr}(\mathscr{E}_R(t)) dt \Bigg\}.\end{aligned} \tag{7.4}$$

The last summand is bounded by $l_2 e^{-l_3\sqrt{R}}$. The second summand also disappears as $R \to \infty$:

$$\begin{aligned}&\left| (1/\sqrt{\pi}) \cdot \int_0^{\sqrt{R}} t^{-1/2} dt \int dx \cdot \operatorname{tr}(\mathscr{E}_R \# C_R)(t; x, x) \right| \\ &\leqq k_4 e^{k_2 R} \cdot R \cdot e^{-k_3 \cdot R^{3/2}} \leqq k_5 e^{-k_6 \cdot R^{1/2}}.\end{aligned} \tag{7.5}$$

It follows from the results of Sects. 2 and 3 (Lemmata 2.3 and 3.1) that:

$$\operatorname{tr} Q_R(t; x, x) = \operatorname{tr} \mathscr{E}_2^R(t; x, x). \tag{7.6}$$

This gives us the following result:

Theorem 7.2.

$$\lim_{R\to\infty} \left\{ \eta((A_R)_\Pi; 0) - \pi^{-1/2} \cdot \int_0^{\sqrt{R}} \frac{dt}{\sqrt{t}} \int_{X_R} \operatorname{tr} \mathscr{E}_2^R(t; x, x) dx \right\} = 0.$$

Of course, we want to show that

$$\lim_{R\to\infty} \int_{\sqrt{R}}^{\infty} t^{-1/2} dt \int_{X_R} \operatorname{tr} \mathscr{E}_2^R(t; x, x) dx = 0. \tag{7.7}$$

The problem is that in estimate (2.5) we have the factor $e^{c_2 t}$ for a certain positive constant c_2.

In the second part of this section we will show that, in fact c_2 can be taken to be a negative constant, which gives us the desired result. The key is the estimate (2.24), which shows that if we increase the lowest eigenvalue of our operator, then constant b_4 becomes negative [see (1.1) and (1.2)]. Now, let g denote the Riemannian metric on the manifold $\tilde{X}$ and let us introduce the metric g_ε:

$$g_\varepsilon = \varepsilon^2 g. \tag{7.8}$$

We do not change any other structure involved. The corresponding Dirac operator A_ε can be described as follows (see [1, pp. 306, 315]). We define an isometry $T^*X \to T_\varepsilon^* X$ by the formula $(x, \xi) \to (x, \varepsilon \cdot \xi)$. This extends to isomorphism of the corresponding Spinor bundles $\Phi_\varepsilon : S \to S_\varepsilon$ and the corresponding Dirac operator A_ε is given by the formula:

$$A_\varepsilon = \frac{1}{\varepsilon} \cdot \Phi_\varepsilon A \Phi_\varepsilon^{-1}. \tag{7.9}$$

The heat kernel of this operator is equal to

$$A_\varepsilon e^{-tA_\varepsilon^2} = \frac{1}{\varepsilon} \cdot \Phi_\varepsilon A e^{-\varepsilon^{-2} t A^2} \Phi_\varepsilon^{-1}. \tag{7.10}$$

If we consider the case of the Dirac operator B on a closed manifold, then the corresponding "global" η-invariant is unchanged:

$$\eta_{B_\varepsilon}(s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda/\varepsilon) \cdot |\lambda/\varepsilon|^{-s} = \varepsilon^s \cdot \eta_B(s), \tag{7.11}$$

which gives $\eta_{B_\varepsilon}(0) = \eta_B(0)$ for any positive ε.

Now we want to consider the corresponding "local" invariant. We have the following lemma:

Lemma 7.3. *Let B denote a Dirac operator with coefficients in an auxiliary vector bundle on the odd-dimensional manifold. The "local" η-invariant of B is independent of the scaling of the Riemannian metric in the following sense:*

$$\eta_{B_\varepsilon}(0; x) d_\varepsilon x = \eta_B(0; x) dx, \tag{7.12}$$

where $d_\varepsilon x$ denotes the volume form of the metric $\varepsilon^2 g$.

Proof. We have the following sequence of equalities:

$$\eta_{A_\varepsilon}(0;x)\cdot d_\varepsilon x=\left\{(1/\sqrt{\pi})\cdot\int_0^\infty t^{-1/2}\cdot\varepsilon^{-1}\operatorname{tr}\{\Phi_\varepsilon(x)\mathscr{E}(t/\varepsilon^2;x,x)\Phi_\varepsilon^{-1}(x)\}dt\right\}d_\varepsilon x$$
$$=\varepsilon^d\cdot\left\{(\pi)^{-1/2}\cdot\int_0^\infty v^{-1/2}\cdot\operatorname{tr}\{\Phi_\varepsilon(x)\mathscr{E}(v;x,x)\Phi_\varepsilon^{-1}(x)\}dt\right\}dx,\quad(7.13)$$

where $\mathscr{E}$ once again denotes the kernel of the operator $\tilde{A}e^{-t\tilde{A}^2}$. Let $\{e_i\}_{i=1}^d$ denote an orthonormal basis of T_xX. $\mathscr{E}$ is in fact a sum:

$$\mathscr{E}(v;x,x)=\sum_J K_J(v;x,x)\otimes e_J,\qquad(7.14)$$

where the sum runs through all multi-indices $J=(j_1,\ldots,j_k)$, $1\leqq k\leqq d$ and $\{e_J\}$ denotes the corresponding orthonormal basis of the Clifford bundle. The trace we consider is a product of the trace of the matrix $K_J(v;x,x)$ and of the trace of e_J in the spinor representation. Since d is odd, the only non-trivial spinor trace comes from the scalar 1 and $e_1\cdot\ldots\cdot e_d$ the Clifford product of all elements of the basis (see for instance [4, Sect. 1]). We deal with the trace of the "odd" operator $\tilde{A}e^{-t\tilde{A}^2}$, hence we have:

$$\operatorname{tr}\Phi_\varepsilon(x)\mathscr{E}(v;x,x)\Phi_\varepsilon^{-1}(x)=\operatorname{tr}(K_{(1,\ldots,d)}(v;x,x))\cdot\operatorname{tr}(\Phi_\varepsilon(x)(e_1\ldots e_d)\Phi_\varepsilon^{-1}(x))$$
$$=\varepsilon^{-d}\cdot\operatorname{tr}(K_{(1,\ldots,d)}(v;x,x))\cdot\operatorname{tr}(e_1\ldots e_d)$$
$$=\varepsilon^{-d}\cdot\operatorname{tr}\mathscr{E}(v;x,x).\quad\square\qquad(7.15)$$

It follows from (7.9) and (7.10) that, in order to estimate the heat kernel for $(\tilde{A}_R)_\varepsilon$, we have to estimate the kernel of the operator $(\tilde{A}_R)e^{-\varepsilon^{-2}t\tilde{A}_R}$ on the manifold $\tilde{X}_R$. Once again we patch together the corresponding kernels on the cylinder and on the double of X. We reason in an obvious way. The estimate from Sect. 1 gives

$$\|Ae^{-\varepsilon^{-2}tA^2}(t;x,y)\|\leqq b_1t^{-(1+d)/2}\cdot e^{-\varepsilon^{-2}b_2t}\qquad(7.16)$$

on $\tilde{X}$ and on the cylinder. We use the Duhamel principle as in Sect. 2, using (7.16) instead of (2.8) [we can ignore the factor which involves the exponent of $d^2(x,y)/t$]. This gives us a negative constant b_4 in (2.24) and now the error term (the difference between parametrix and the heat kernel) is bounded by $c_1e^{-c_2t}$ for some positive constants. We have just proved the following lemma:

Lemma 7.4. *If $\varepsilon>0$ is sufficiently small, then the heat kernel of the operator $(\tilde{A}_R)_\varepsilon$ decays exponentially as $t\to\infty$. In other words the following inequality holds for any R:*

$$\|\tilde{A}_Re^{-\varepsilon^{-2}t\tilde{A}_R^2}(t;x,y)\|\leqq c_1e^{-c_2t}.\qquad(7.17)$$

Now we split the integral from the "local" η-density into two parts:

$$\int_{X_R}\eta(\tilde{A}_R,0;x)dx=(1/\sqrt{\pi})\cdot\int_{X_R}\int_{0}^\infty t^{-1/2}\cdot\operatorname{tr}\mathscr{E}_1(t;x,x)dtdx$$
$$=(1/\sqrt{\pi})\cdot\int_{X_R}\int_0^{\sqrt{R}}\cdot+(1/\sqrt{\pi})\cdot\int_{X_R}\int_{\sqrt{R}}^\infty.$$

We estimate the second summand using (7.17):

$$\begin{aligned}\left|\int_{X_R} dx \int_{\sqrt{R}}^{\infty} t^{-1/2}\operatorname{tr}(Ae^{-tA^2})(t;x,x)dt\right| &\leqq \int_{X_R} dx \int_{\sqrt{R}}^{\infty} t^{-1/2} c_1 e^{-c_2\varepsilon^2 t}dt\\ &\leqq c_3 R\cdot \int_{\sqrt{R}}^{\infty} t^{-1/2}\cdot e^{-c_2\varepsilon^2 t}dt\\ &\leqq 2\varepsilon^{-1}c_2 R\cdot \int_{\varepsilon R^{1/4}}^{\infty} e^{-v^2}dv\\ &\leqq 2\varepsilon^{-1}c_3 R\cdot e^{-\varepsilon^2\sqrt{R}} \xrightarrow[R\to\infty]{} 0\,. \end{aligned} \tag{7.18}$$

This proves (7.7) and ends the proof of Theorem 0.1. □

Appendix 1. Spectral Flow Formula for the Families of Generalized Atiyah-Patodi-Singer Problems

In this appendix we discuss the spectral flow formula for the boundary problems introduced in [7]. For simplicity, we assume that X is a Spin-manifold and that $A=\not{\partial}\otimes \mathrm{Id}_V: C^\infty(X;S\otimes V)\to C^\infty(X;S\otimes V)$ is a Dirac operator with coefficients in an auxiliary Hermitian bundle V. We assume product structure in N, the collar neighbourhood of the boundary. Therefore, A has the form (2.1) in N.

1. The "Ideal" Boundary Conditions. We consider here the case $\ker B \neq \{0\}$. Green's formula [14, Chap. XVII, Proposition 1] gives the following equality for smooth sections f and g:

$$(Af,g)-(f,Ag)=\int_Y \langle G(f|Y);(g|Y)\rangle dy\,. \tag{A1}$$

This shows that the operator A_{Π_+} is symmetric if and only if:

$$-G\Pi_+G=\mathrm{Id}-\Pi_+\,. \tag{A2}$$

It follows from (2.2), that this is the case *if and only if* $\ker B=\{0\}$. However, we can still define a nice self-adjoint boundary condition for the case of non-invertible B. This is guaranteed by the following deep result:

Theorem A1. *The operator $B=\not{\partial}_Y\otimes\mathrm{Id}_{V|Y}$ "bounds" the operator A and, therefore:*

$$\dim\ker B_+=\dim\ker B_-\,, \tag{A3}$$

where as usual $B_\pm=1/4\cdot(\mathrm{Id}\pm iG)B(\mathrm{Id}\mp iG)$ acts between the spinors of different chirality. In particular, the kernel of the operator B is even-dimensional.

This is Theorem 2 from Chap. XVII of Palais book [14]. $G|_{\ker B}$ is still an antiinvolution, and Theorem A1 implies the existence of an orthogonal splitting of $\ker B$, $\ker B=W_+\oplus W_-$, such that:

$$G(W_\pm)=W_\mp\,. \tag{A4}$$

Let $S_\pm$ denote the orthogonal projection of $\ker B$ onto $W_\pm$. We extend $S_\pm$ to an operator with a smooth kernel, which acts on $C^\infty(Y;(S\otimes V)|Y)$. Let us denote by $\tilde{\Pi}_+$ the spectral projection of B onto the subspace of $L^2(Y;(S\otimes V)|Y)$ spanned by the eigensections of B corresponding to the positive eigenvalues of B. We define the boundary condition Π_W

$$\Pi_W=\tilde{\Pi}_+ + S_+\,, \tag{A5}$$

and the operator A_{Π_W} by the formula

$$\begin{cases} A_{\Pi_W} = A\,, \\ \operatorname{dom} A_{\Pi_W} = \{s \in H^1(X; S \otimes V);\ \Pi_W(s|Y) = 0\}\,. \end{cases} \tag{A6}$$

Remark A2. 1. Π_W is a pseudodifferential operator with principal symbol p_+ equal to the principal symbol of the projection Π_+.
2. We do not have a canonical choice for the orthogonal splitting $W_\pm$. The set of the orthogonal splittings of $\ker B$ which satisfy (A4) provides us with a special class of the boundary conditions, the so-called "ideal" boundary conditions. It seems to be an interesting problem to investigate the topological structure of the space of "ideal" conditions.

We repeat the argument from Sect. 4, to show that $\eta_{A_{\Pi_W}}(s)$ is a well-defined meromorphic function, regular at $s=0$. The point is that we still have a nice representation of the heat kernel on the cylinder (see Sect. 3), and we apply the Duhamel principle to construct the kernel of the operator $Ae^{-tA^2_{\Pi_W}}$ on X. Theorem 4.3 and Proposition 4.4 hold also in this case. Now, we study the family (0.5), where we replace the boundary condition Π_+ by the condition Π_W. We repeat the consideration from the end of Sect. 4 to obtain:

Theorem A3. $\operatorname{sf}\{(D_r)_{N\Pi_W}\} = \operatorname{rk}(V) \cdot \int_X \operatorname{ch}(g)$, where D_r is the operator defined in (0.5) and Π_W is any ideal boundary condition.

2. *Generalized Atiyah-Patodi-Singer Problems.* Identity (A2) suggests that we may consider as a boundary condition any orthogonal pseudodifferential projection P, which satisfies the following conditions:

$$\begin{cases} -GPG = \mathrm{Id} - P\,, \\ \sigma_L(P) = p_+\,, \end{cases} \tag{A7}$$

where $\sigma_L(P)$ denotes the principal symbol of the pseudodifferential operator P (see [7]). The operator $A_P : \operatorname{dom} A_P \to L^2(X; S \otimes W)$ is a self-adjoint Fredholm operator, and $\ker A_P$ consists of smooth sections. We denote by $ELL^*(A)$ the space of all pseudodifferential projections, which satisfy (A7). We put aside the question of the existence of the η-invariant of the operator A_P, where $P \in ELL^*(A)$. We concentrate here only on the spectral flow formula.

Let us consider a family $\{(D_r)_{NP}\}$, where $P \in ELL^*(A)$. It is the family of self-adjoint Fredholm operators over circle, and hence it has a well-defined spectral flow. We want to show that there exists a continuous deformation of this family to a family $\{(D_r)_{N\Pi}\}$, where Π is an "ideal" boundary condition for A. In Appendix 2 we show that $ELL^*(A)$ is path-connected. More precisely, we know that there exists a family $\{\mathscr{I}_t\}_{t \in I}$ of elliptic pseudodifferential operators of order 0, such that:

$$\begin{cases} \text{(a) } \mathscr{I}_t : L^2(Y; (S \otimes V)|Y) \text{ is a unitary operator for any } t\,. \\ \text{(b) The principal symbol of } \mathscr{I}_t \text{ is equal to the identity}\,. \\ \text{(c) } \mathscr{I}_1 = \mathrm{Id} \text{ and } \mathscr{I}_0 P \mathscr{I}_0^{-1} = \Pi, \text{ where } \Pi \text{ is an "ideal" boundary condition}\,. \\ \text{(d) } \mathscr{I}_t \text{ commutes with } G \text{ for any } t\,. \end{cases} \tag{A8}$$

The family $\{(D_r\}_{N(\mathscr{I}_t P \mathscr{I}^{-1})}\}_{r \in I,\, t \in I}$ provides us with a continuous deformation of the family $\{(D_r)_{NP}\}$ into a family defined by the "ideal" boundary condition. At this point, it is unclear whether the spectral flow remains unchanged under this deformation. The problem is that we vary the domain of the unbounded operator.

However, there is an easy way of showing that the spectral flow is constant. We may assume that the family $\{\mathscr{J}_t\}$ is constant for t close to 0 and 1, and we define an automorphism of the bundle $S\otimes V\otimes\mathbb{C}^N$ by the formula:

$$\mathscr{J}=\begin{cases}\mathscr{J}_u\otimes\mathrm{Id}_{\mathbb{C}^N} & \text{on} \quad \{u\}\times Y\subseteqq N,\\ \mathrm{Id} & \text{on} \quad X\backslash N.\end{cases} \tag{A9}$$

The operator $(D_r)_{NP}$ is unitarily equivalent to the operator $(\mathscr{J}D_r\mathscr{J}^{-1})_{N\Pi}$. We have also the equality:

$$\mathscr{J}D_r\mathscr{J}^{-1}=D_r+[\mathscr{J},D_r]\mathscr{J}^{-1}=D_r+\mathscr{T}_r, \tag{A10}$$

where $\mathscr{T}_r$ is a bounded operator on $L^2(X;S\otimes V\otimes\mathbb{C}^N)$, and thus we may consider the spectral flow of the family $\{(\mathscr{J}D_r\mathscr{J}^{-1})_{N\Pi}\}$. Now, we can define a continuous deformation of the family $\{(\mathscr{J}D_r\mathscr{J}^{-1})_{N\Pi}\}$ into the family $\{(D_r)_{N\Pi}\}$:

$$\{(D_{r,\mu}=D_r+\mu\mathscr{T}_r)_{N\Pi}\}_{r,\mu\in I}. \tag{A11}$$

This gives us the most general variant of the Theorem 0.2:

Theorem A4. $\mathrm{sf}\{(D_r)_{NP}\}=\mathrm{rk}(V)\cdot\int_X \mathrm{ch}(g)$ for any generalized Atiyah-Patodi-Singer boundary condition $P\in ELL^*(A)$.

Appendix 2. The Homotopy Groups of $ELL^*(A)$

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To make the paper self-contained, we present here the calculation of the homotopy groups of the space $ELL^*(A)$. First, we compute the homotopy groups of the space $ELL(A)$ of all pseudodifferential projections with principal symbol equal to p_+, the principal symbol of Π_+. Then we show how to modify this argument in order to get the homotopy groups of $ELL^*(A)$.

1. The Homotopy Groups of ELL(A). In this section we study the space $ELL(A)$, which consists of all pseudodifferential projections on $S\otimes V$, with the principal symbol p_+. We use the norm topology coming from the space $L^2(Y;(S\otimes V)|Y)$.

Theorem B1.

$$\pi_i(ELL(A))=\begin{cases}\mathbb{Z} & \textit{for} \quad i=2k,\\ 0 & \textit{for} \quad i=2k+1.\end{cases}$$

Remark B2. 1. This theorem is well-known for the closure of $ELL(A)$ in the space of all bounded operators acting on $L^2(Y;(S\otimes V)|Y)$ (see for instance [B2, B3]). A slight difficulty here comes from the fact that we have considered only pseudodifferential projections.

2. The proof we present here is slightly different from the proofs presented for instance in [B2] and [B3]. In these references the homotopy groups of the group $GL_{\Pi_+}=\{T\in GL(L^2);[T;\Pi_+]$ is compact$\}$ were computed, and then it was shown that $\overline{ELL(A)}$ and GL_{Π_+} were homotopy equivalent. Here we compute the homotopy groups of the Grassmannian directly and, in fact, we can use this result to compute the homotopy of GL_{Π_+}.

3. The idea of using the Grassmannian in the index theory appeared first in the work of B. Bojarski (see [B1]).

Let us introduce a group of invertible elliptic operators on $(S\otimes V)|Y$:

$$\mathscr{E}\mathscr{U}^{-1}=\{\mathscr{g};\ \mathscr{g} \text{ is an invertible pseudodifferential operator of order 0 with the principal symbol Id}\}. \tag{B1}$$

Using a standard deformation argument we can show that the homotopy groups of $\mathscr{E}\mathscr{U}^{-1}$ are equal to the homotopy groups of GL_c, which consists of all operators $\mathscr{g}$, such that $\mathscr{g}$ is invertible on $L^2(Y;(S\otimes V)|Y)$ and $\mathscr{g}-\mathrm{Id}$ is a compact operator. Thus we have:

$$\pi_i(\mathscr{E}\mathscr{U}^{-1})=\begin{cases}0 & \text{for } i=2k,\\ \mathbb{Z} & \text{for } i=2k+1.\end{cases} \tag{B2}$$

We show that each connected component of $ELL(A)$ is a base of a principal fibre bundle with the total space $\mathscr{E}\mathscr{U}^{-1}$. This is a consequence of the following lemma:

Lemma B3 (see [B3]). *Let $P_0, P_1\in ELL(A)$ and $\|P_0-P_1\|<1$. The operator:*

$$T=\mathrm{Id}+(P_1-P_0)(2P_0-\mathrm{Id}) \tag{B3}$$

belongs to the space $\mathscr{E}\mathscr{U}^{-1}$ and satisfies:

$$T\cdot P_0\cdot T^{-1}=P_1\,. \tag{B4}$$

Corollary B4. *The space $ELL(A)$ is locally contractible. In particular, connected components of $ELL(A)$ are path-connected.*

Now let us fix a projection P from $ELL(A)$. We denote by $ELL_P(A)$ the connected component of $ELL(A)$, such that P belongs to $ELL_P(A)$. Let us introduce the subgroup $\mathscr{E}\mathscr{U}_P^{-1}$, consisting of all elements of $\mathscr{E}\mathscr{U}^{-1}$, such that they commute with P. It is obvious that we can identify $ELL_P(A)$ with a homogeneous space $\mathscr{E}\mathscr{U}^{-1}/\mathscr{E}\mathscr{U}_P^{-1}$. Since Lemma B2 provides us with a local section in a neighbourhood of the class of identity in this space, we have the following theorem:

Theorem B5. *Let $f:\mathscr{E}\mathscr{U}^{-1}\to ELL_P(A)$ denote the continuous map given by the formula $f(\mathscr{g})=\mathscr{g}\cdot P\cdot\mathscr{g}^{-1}$. $\mathscr{E}\mathscr{U}^{-1}\xrightarrow{f} ELL_P(A)$ is a principal fibre bundle with structure group equal to $\mathscr{E}\mathscr{U}_P^{-1}$ ($\mathscr{E}\mathscr{U}_P^{-1}$ acts from the right on $\mathscr{E}\mathscr{U}^{-1}$).*

Proof of Theorem B1. P fixes a splitting of $L^2(Y;(S\otimes V)|Y)$ as the direct sum of $\mathrm{Ran}(P)$ and $\mathrm{Ran}(\mathrm{Id}-P)$. $\mathscr{E}\mathscr{U}_P^{-1}$ decomposes with respect to this splitting into two factors, such that each of them has the homotopy groups of GL_c. Therefore, the homotopy groups of $\mathscr{E}\mathscr{U}_P^{-1}$ are given by:

$$\pi_i(\mathscr{E}\mathscr{U}_P^{-1})=\begin{cases}0 & \text{for } i=2k,\\ \mathbb{Z}\oplus\mathbb{Z} & \text{for } i=2k+1.\end{cases} \tag{B5}$$

We employ the exact homotopy sequence for fibre bundles to compute the homotopy groups of $ELL_P(A)$. □

Remark B6. 1. We have calculated the homotopy groups $\pi_i(ELL(A))$ for $i\geqq 1$. It is easy to show that $\pi_0(ELL(A))=\mathbb{Z}$ and that the index classifies connected components of $ELL(A)$ (see for instance [B3]). More precisely, $P_1, P_2\in ELL(A)$

belong to the same connected component of $ELL(A)$ *if and only if* $\operatorname{index}(P_2 \cdot P_1 : \operatorname{Ran} P_1 \to \operatorname{Ran} P_2) = 0$.
2. We describe a local trivialization in a neighbourhood of P, which is a consequence of Lemma B3. Let $U_P \subseteqq ELL_P(A)$ denote the open set $\{S \in ELL(A);\ \|P-S\| < 1\}$. If $S \in U_P$ and $g \in f^{-1}(S)$, then $T_S \cdot g^{-1}$ is an element of $f^{-1}(P)$. T_S is the operator from Lemma B3:

$$T_S = \mathrm{Id} + (S-P)(2P - \mathrm{Id}).$$

The local trivialization in U_P is given by the map $\Phi_P : f^{-1}(U_P) \to U_P \times \mathscr{E}\mathscr{U}_P^{-1}$:

$$\Phi_P(g) = (S = g \cdot P \cdot g^{-1};\ T_S^{-1} \cdot g). \tag{B6}$$

2. The Homotopy Groups of $ELL^*(A)$. Now we fix an "ideal" boundary condition Π. Π is an element of the space $\widetilde{ELL}^*(A) = \{R \in ELL(A);\ -GRG = \mathrm{Id} - R\}$. $ELL^*(A)$ is a subspace of $\widetilde{ELL}^*(A)$, which consists of orthogonal projections. We have the following lemma:

Lemma B7. $\widetilde{ELL}^*(A) \subseteqq ELL_\Pi(A)$.

Proof. Let $P \in \widetilde{ELL}^*(A)$. We will show that the index of the operator $P \cdot \Pi : \operatorname{Ran}(\Pi) \to \operatorname{Ran}(P)$ equals 0 [see Remark B6(1)]. We have:

$$\begin{aligned}&\operatorname{index}(P\Pi : \operatorname{Ran}(\Pi) \to \operatorname{Ran}(P))\\ &\quad = \operatorname{index}(GPGG\Pi G : \operatorname{Ran}(-GPG) \to \operatorname{Ran}(-GPG)\\ &\quad = \operatorname{index}((\mathrm{Id}-P)(\mathrm{Id}-\Pi) : \operatorname{Ran}(\mathrm{Id}-\Pi) \to \operatorname{Ran}(\mathrm{Id}-P)),\end{aligned} \tag{B7}$$

which implies

$$2 \cdot \operatorname{index}(P \cdot \Pi) = \operatorname{index}(P\Pi + (\mathrm{Id}-P)(\mathrm{Id}-\Pi)). \tag{B8}$$

The operator on the right side of (B8) is an elliptic operator with the principal symbol equal to the identity; hence its index is 0. □

Lemma B8. *Let* $S \in \widetilde{ELL}^*(A)$ *and* $\|S - \Pi\| < 1$. *The operator* $T_S = \mathrm{Id} + (S-\Pi)(2\Pi - \mathrm{Id})$ *commutes with* G.

Now we repeat some considerations from the previous section. We introduce the group $\mathscr{E}\mathscr{U}_G^{-1}$:

$$\mathscr{E}\mathscr{U}_G^{-1} = \{g \in \mathscr{E}\mathscr{U}^{-1};\ g \cdot G = G \cdot g\}, \tag{B9}$$

and define the map $f_G : \mathscr{E}\mathscr{U}_G^{-1} \to \widetilde{ELL}^*(A)$, by the formula $f_G(g) = g\Pi \cdot g^{-1}$.

Theorem B9. f_G *is a principal fibre bundle with structure group* $\mathscr{E}\mathscr{U}_{G,\Pi}^{-1}$, *where* $\mathscr{E}\mathscr{U}_{G,\Pi}^{-1} = \{g \in \mathscr{E}\mathscr{U}_G^{-1};\ g \cdot \Pi = \Pi \cdot g\}$.

Proof. We repeat the proof of Theorem B5.
Let $g \in \mathscr{E}\mathscr{U}_{G,\Pi}^{-1}$. We have:

$$(\mathrm{Id}-\Pi) \cdot g \cdot (\mathrm{Id}-\Pi) = G \cdot \Pi \cdot G \cdot g \cdot G \cdot \Pi \cdot G = -G \cdot \Pi \cdot g \cdot \Pi \cdot G. \tag{B10}$$

Therefore, if we fix a decomposition of $L^2(Y; (S \otimes V)|Y)$ into $\operatorname{Ran}(\Pi)$ and $\operatorname{Ran}(\mathrm{Id}-\Pi)$ (G identifies these two subspaces), we see that g has the following form with respect to this decomposition:

$$\begin{bmatrix} g_1 & 0 \\ 0 & -g_1 \end{bmatrix}, \tag{B11}$$

where $\mathscr{g}_1$ denotes the operator $\Pi \cdot \mathscr{g} \cdot \Pi$. Now we can calculate the homotopy groups of $E\tilde{L}L^*(A)$. Thanks to (B10 and (B11) we know the homotopy groups of $\mathscr{E}\mathscr{U}_{G,\Pi}^{-1}$:

$$\pi_i(\mathscr{E}\mathscr{U}_{G,\Pi}^{-1}) = \begin{cases} 0 & \text{for} \quad i=2k, \\ \mathbb{Z} & \text{for} \quad i=2k+1. \end{cases} \tag{B12}$$

We use the exact homotopy sequence for fibre bundles to conclude the proof of the main theorem:

Theorem B10.

$$\pi_i(E\tilde{L}L^*(A)) = \begin{cases} 0 & \textit{for} \quad i=2k, \\ \mathbb{Z} & \textit{for} \quad i=2k+1. \end{cases} \tag{B13}$$

In particular, $E\tilde{L}L^*(A)$ is a path-connected space.

Remark. B11. 1. It is obvious that $ELL^*(A)$ has the same homotopy groups as $E\tilde{L}L^*(A)$. This follows from the fact, that the subgroup of $\mathscr{E}\mathscr{U}_G^{-1}$, which consists of unitary operators, has the same homotopy groups as $\mathscr{E}\mathscr{U}_G^{-1}$. In fact, the operator A_P, for $P \in E\tilde{L}L(A) \setminus ELL(A)$ still has a discrete spectrum and we can prove the spectral flow formula for the families defined by such a condition P.
2. The closure of $ELL^*(A)$ in the space of all bounded operators in $L^2(Y; (S \otimes V)|Y)$ is a classifying space for the functor K^{-1}. We omit the proof.
3. Let us consider the groups:

$$\begin{aligned} GL_\Pi &= \{\mathscr{h} \in GL(L^2);\ \mathscr{h} \cdot \Pi - \Pi \cdot \mathscr{h} \text{ is a compact operator}\}, \\ GL_{\Pi,G} &= \{\mathscr{h} \in GL_\Pi;\ \mathscr{h} \cdot G = G \cdot \mathscr{h}\}. \end{aligned} \tag{B14}$$

Now, we can use the group GL_c instead of $\mathscr{E}\mathscr{U}^{-1}$. We repeat our argument to show that GL_Π and $GL_{P,G}$ are homotopy equivalent to the spaces $\overline{ELL(A)}$ and $\overline{ELL^*(A)}$, respectively.

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