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Holonomy Groups, Complex Structures and $D = 4$ Topological Yang-Mills Theory

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Abstract. We study conditions for the existence of extended supersymmetry in topological Yang-Mills theory. These conditions are most conveniently formulated in terms of the holonomy group of the underlying manifold, on which the topological Yang-Mills theory is defined. For irreducible manifolds we find that extended supersymmetries are in 1-1 correspondence with covariantly constant complex structures. Therefore, the topological Yang-Mills theory on any Kähler manifold possesses one additional supersymmetry and on any hyperKähler manifold there are three additional supersymmetries. The Donaldson map, which plays a crucial role in the construction of the topological invariants, is generalized for Kähler manifolds, thus providing candidates for new invariants of complex manifolds.

1. Introduction

Recently, a quantum field theory method was proposed for constructing topological invariants of four dimensional manifolds [1]. It is based on $N = 2$, $D = 4$ Euclidean supersymmetric Yang-Mills theory minimally coupled to gravity. One might expect that the $N = 2$ supersymmetry of the theory is completely broken because the graviton's superpartners are absent. Nevertheless it turns out that the theory is invariant under a singlet rigid supersymmetry if one identifies the $SU(2)$ automorphism group of $N = 2$ supersymmetry with the $SU(2)_L$ subgroup of the Euclidean tangent group $SO(4) \simeq SU(2)_L \times SU(2)_R$. This singlet supersymmetry plays an important role in the construc-

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tion of topological invariants. Roughly speaking, its existence implies that the expectation values of supersymmetric observables do not depend on the choice of Riemannian metric of the four-manifold. So these expectation values are topological invariants (or, more precisely, invariants of a smooth structure) of the underlying four-manifold. For this reason the theory was called topological. The set of observables analyzed in [1] corresponds to the Donaldson invariants [2].

The study of supersymmetric theories has revealed many different remarkable relations between the structure of supersymmetry and geometry. For instance the target space of $D = 2$ supersymmetric σ-models is an arbitrary Riemannian manifold in $N = 1$ case, while in $N = 2$ (or, equivalently $N = 1$ $D = 4$) case the target space should be Kähler, and for $N = 4$ (or N $= 2$ $D = 4$) only hyperKähler manifolds are allowed [3]. Here, extended supersymmetry requires the existence of covariantly constant complex structures and hence reduces the holonomy group of the corresponding target space.

One may wonder whether extended supersymmetry in topological Yang-Mills theory is possible. The main purpose of the present paper is to study this problem. We restrict ourselves to the most interesting case of four-manifolds which are not locally metric products. We find that the additional supersymmetry requires the existence of a covariantly constant complex structure on the four-manifold. This complex structure reduces the holonomy group of the manifold. In four dimensions there are essentially two nontrivial possibilities: the holonomy group can be reduced to $U(2)$ or $SU(2)$. In the former case there is just one complex structure while in the latter one there are three complex structures. We prove that the additional supersymmetries are in 1-1 correspondence with covariantly constant complex structures. Therefore when the holonomy group is $U(2)$ (that is when the manifold is Kähler) the topological Yang-Mills theory has one additional supersymmetry, and when the holonomy group is $SU(2)$ (hyperKähler manifolds) there are three additional supersymmetries. This picture resembles amazingly the above mentioned situation with $D = 2$ σ-models. The only difference is that in the topological Yang-Mills theory under consideration all supersymmetry generators anticommute.

The original topological Yang-Mills theory [1] formulated on an arbitrary Riemannian four-manifold leads to invariants of a smooth structure of the manifold. In our case the topological field theory gives a way to construct invariants of complex of hypercomplex structures. We discuss supersymmetric observables specific to the Kähler case. Using topological Yang-Mills theory arguments we find also an interesting relation between complex structures on the four-manifold and on the modulus space of instantons on the manifold: it turns out that the modulus space of instantons on a Kähler (hyperKähler) manifold has a natural complex (respectively hypercomplex) structure.

The construction of topological invariants in [1] was based on the Donaldson map which relates the de Rham cohomology groups of the Riemannian manifold with the de Rham cohomology groups of the modulus space of instantons on this manifold. On the Kähler manifold the de Rham groups are decomposed into the Dolbeault groups. We propose a generalization of the Donaldson map which relates the Dolbeault groups of the Kähler manifold with the Dolbeault groups of its modulus space of instantons. This gives a way for construction of new invariants of complex structures on four-manifolds.

A natural explanation for the existence of the singlet supersymmetry in the topological field theory was given in [4]. It uses arguments of $N = 2$ conformal supergravity. We find it enlightening since it makes clear that any $N=2$ conformally invariant theory possesses a singlet supersymmetry on an arbitrary Riemannian background after twisting the $SU(2)_A$ and $SU(2)_L$ groups. Of course, $N = 2$ Yang-Mills is the most important example of such a theory. We use this approach in our study of extended supersymmetries in topological field theory. However a knowledge of conformal supergravity is not necessary for understanding this paper since the main results can be checked directly (see Sect. 4).

The paper is organised as follows. In Sect. 2 we review briefly some general facts about $N = 2$, $D = 4$ supersymmetric Yang-Mills theory and the topological Yang-Mills theory. Using $N = 2$ conformal supergravity arguments we establish in Sect. 3 the basic relation between holonomy groups and additional supersymmetries. In Sect. 4 we discuss properties of the topological Yang-Mills theory on Kähler manifolds, generalize the Donaldson map for this case and make several comments about the hyperKähler case.

This paper is a completed version of [5].

2. $N=2$ Supersymmetric Yang-Mills Theory and Topological Field Theory

In this section we review some basic facts about $N = 2$ Yang-Mills theory [6] and topological Yang-Mills theory [1] which are needed in the subsequent sections.

2.1. $N = 2$ Yang-Mills Theory

The Lorentz group of the Euclidean four-space is locally isomorphic to the group $SU(2)_L \times SU(2)_R$. We denote indices of fundamental representation of $SU(2)_L$ by $\alpha, \beta, \gamma, \ldots$, and of $SU(2)_R$ by $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \ldots$ The automorphism group of $N=2$ Euclidean supersymmetry is $SU(2)_A \times \mathbf{R}^*$, where $\mathbf{R}^*$ is the multiplicative group of nonzero real numbers. It replaces the $U(1)$ subgroup of the automorphism group of $N=2$ supersymmetry in Minkowski space. We denote indices of the fundamental representation of $SU(2)_A$ by $i, j, k, \ldots$ The supersymmetry parameters $\xi^{\alpha i}$, $\xi^{\dot{\alpha} i}$ are real in the following sense:

$$\overline{\xi^{\alpha i}} = \xi_{\alpha i} (\equiv \varepsilon_{\alpha\beta}\varepsilon_{ij}\xi^{\beta j}), \qquad \overline{\xi^{\dot{\alpha} i}} = \xi_{\dot{\alpha} i}, \tag{1}$$

where $\varepsilon_{\alpha\beta}$ is the standard antisymmetric tensor, $\varepsilon_{12} = -1$. The Yang-Mills $N = 2$ multiplet consists of a gauge field A_α, spinor fields $\psi_{\alpha i}$, $\psi_{\dot{\alpha} i}$ and scalar fields M, N. For the closure of supersymmetry algebra it is necessary to introduce an auxiliary field $T_{ij} = T_{ji}$. All these fields take values in the Lie algebra of some compact gauge group G. All fields are antihermitian,

$$\begin{aligned} &\overline{A_{\alpha\dot{\alpha}}} = -A^{\alpha\dot{\alpha}}, \qquad \overline{\psi_{\alpha i}} = -\psi^{\alpha i}, \qquad \overline{\psi_{\dot{\alpha} i}} = -\psi^{\dot{\alpha} i}, \\ &\overline{M} = -M, \qquad \overline{N} = -N, \qquad \overline{T_{ij}} = -T^{ij}. \end{aligned} \tag{2}$$

The supersymmetry algebra is graded by dimension and by $\mathbf{R}^*$-charge. Denoting the dimension and the $\mathbf{R}^*$-charge of a field X by $[X]$ and $\deg X$ respectively, we have

$$[A_{\alpha\dot{\alpha}}] = [M] = [N] = cm^{-1}, \qquad [\psi_{\alpha i}] = [\psi_{\dot{\alpha} i}] = cm^{-3/2}, \qquad [T_{ij}] = cm^{-2}, \tag{3}$$

$$\begin{aligned} &\deg A_{\alpha\dot{\alpha}} = 0, \qquad \deg\psi_{\alpha i} = -1, \qquad \deg\psi_{\dot{\alpha} i} = 1, \qquad \deg M = 2, \\ &\deg N = -2, \qquad \deg T_{ij} = 0, \qquad \deg\xi_{\alpha i} = -1, \qquad \deg\xi_{\dot{\alpha} i} = 1, \end{aligned} \tag{4}$$

where $\xi_{\alpha i}$, $\xi_{\dot\alpha i}$ are the supersymmetry parameters. The supersymmetry transformations have the form

$$\begin{aligned}
\delta A_{\alpha\dot\alpha} &= i\xi_\alpha^{\ i}\psi_{\dot\alpha i} - i\xi_{\dot\alpha}^{\ i}\psi_{\alpha i}\,,\\
\delta\psi_{\alpha i} &= -\xi_\alpha^{\ j}T_{ij} + 2\xi^\beta_{\ i}F_{\alpha\beta} + \frac{1}{4}\xi_{\alpha i}[M,N] - \xi^{\dot\beta}_{\ i}\mathscr{D}_{\alpha\dot\beta}N\,,\\
\delta\psi_{\dot\alpha i} &= -\xi_{\dot\alpha}^{\ j}T_{ij} + 2\xi^{\dot\beta}_{\ i}F_{\dot\alpha\dot\beta} + \frac{1}{4}\xi_{\dot\alpha i}[M,N] + \xi^\beta_{\ i}\mathscr{D}_{\beta\dot\alpha}M\,,\\
\delta M &= 2i\xi^{\dot\alpha i}\psi_{\dot\alpha i}\,,\\
\delta N &= 2i\xi^{\alpha i}\psi_{\alpha i}\,,\\
\delta T_{ij} &= i\xi^\alpha_{\ i}\mathscr{D}_\alpha^{\ \dot\beta}\psi_{\dot\beta j} + i\xi^\alpha_{\ j}\mathscr{D}_\alpha^{\ \dot\beta}\psi_{\dot\beta i} - \frac{i}{2}\xi^\alpha_{\ i}[\psi_{\alpha j},M] - \frac{i}{2}\xi^\alpha_{\ j}[\psi_{\alpha i},M]\\
&\quad - i\xi^{\dot\alpha}_{\ i}\mathscr{D}^\beta_{\ \dot\alpha}\psi_{\beta j} - i\xi^{\dot\alpha}_{\ j}\mathscr{D}^\beta_{\ \dot\alpha}\psi_{\beta i} - \frac{i}{2}\xi^{\dot\alpha}_{\ i}[\psi_{\dot\alpha j},N] - \frac{i}{2}\xi^{\dot\alpha}_{\ j}[\psi_{\dot\alpha i},N]\,,
\end{aligned}\tag{5}$$

where $F_{\alpha\beta}$, $F_{\dot\alpha\dot\beta}$ are components of the curvature,

$$[\mathscr{D}_{\alpha\dot\alpha},\mathscr{D}_{\beta\dot\beta}] = \varepsilon_{\alpha\beta}F_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}F_{\alpha\beta}\,. \tag{6}$$

The Lagrangian of the theory equals, up to a total derivative,

$$\begin{aligned}
L = \operatorname{tr}(&4F_{\alpha\beta}F^{\alpha\beta} - \mathscr{D}_{\alpha\dot\alpha}M\mathscr{D}^{\alpha\dot\alpha}N - 4i\psi^{\alpha i}\mathscr{D}_{\alpha\dot\alpha}\psi^{\dot\alpha}_{\ i}\\
&- iM[\psi_{\alpha i},\psi^{\alpha i}] - iN[\psi_{\dot\alpha i},\psi^{\dot\alpha i}] - \frac{1}{8}[M,N]^2 - T_{ij}T^{ij})\,.
\end{aligned}\tag{7}$$

2.2. Topological N = 2 Yang-Mills Theory

The topological Yang-Mills theory can be obtained from $N = 2$ Yang-Mills theory by changing the action of the Lorentz group, namely by replacing $SU(2)_L$ by the diagonal subgroup in $SU(2)_L \times SU(2)_A$ (or, in other words, identifying the indices α, β, ... with the indices i, j, ...). We use the following notation

$$\begin{aligned}
\xi &= \frac{1}{2}\varepsilon^{\alpha i}\xi_{\alpha i}\,, &\quad \hat\xi_{\alpha i} &= \frac{1}{2}(\xi_{\alpha i} + \xi_{i\alpha})\,,\\
\psi &= \varepsilon^{\alpha i}\psi_{\alpha i}\,, &\quad \chi_{\alpha i} &= -\frac{1}{2}(\psi_{\alpha i} + \psi_{i\alpha})\,.
\end{aligned}\tag{8}$$

In what follows we will need formulas for supersymmetry transformations with parameters $\xi_{\alpha i}$ only. Rewriting them in terms of ξ, $\hat\xi_{\alpha\beta}$, we get

$$\begin{aligned}
\delta A_{\alpha\dot\alpha} &= -i\hat\xi_{\alpha\beta}\psi_{\dot\alpha}^{\ \beta} + i\xi\psi_{\dot\alpha\alpha}\,,\\
\delta M &= 0\,, \qquad \delta N = -2i\hat\xi^{\alpha\beta}\chi_{\alpha\beta} + 2i\xi\psi\,,\\
\delta\psi &= \hat\xi_{\alpha\beta}T^{\alpha\beta} + 2\hat\xi_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\xi[M,N]\,,\\
\delta\psi_{\dot\alpha\alpha} &= -\hat\xi_{\beta\alpha}\mathscr{D}^\beta_{\ \dot\alpha}M - \xi\mathscr{D}_{\alpha\dot\alpha}M\,,\\
\delta\chi_{\alpha\gamma} &= -\frac{1}{2}(\hat\xi_{\alpha\beta}T_\gamma^{\ \beta} + \hat\xi_{\gamma\beta}T_\alpha^{\ \beta}) + \xi T_{\alpha\gamma} + \hat\xi_{\beta\gamma}F_\alpha^{\ \beta} + \hat\xi_{\beta\alpha}F_\gamma^{\ \beta} + 2\xi F_{\alpha\gamma} - \frac{1}{4}\hat\xi_{\alpha\gamma}[M,N]\,,\\
\delta T_{\varrho\tau} &= -i\hat\xi_{\alpha\varrho}\mathscr{D}^{\alpha\dot\beta}\psi_{\dot\beta\tau} - i\hat\xi_{\alpha\tau}\mathscr{D}^{\alpha\dot\beta}\psi_{\dot\beta\varrho} - i\xi\mathscr{D}_\varrho^{\ \dot\beta}\psi_{\dot\beta\tau}\\
&\quad - i\xi\mathscr{D}_\tau^{\ \dot\beta}\psi_{\dot\beta\varrho} + \frac{i}{2}\hat\xi^\alpha_{\ \varrho}[\chi_{\alpha\tau},M] + \frac{i}{2}\hat\xi^\alpha_{\ \tau}[\chi_{\alpha\varrho},M] - i\xi[\chi_{\varrho\tau},M] - \frac{i}{2}\hat\xi_{\varrho\tau}[\psi,M]\,.
\end{aligned}\tag{9}$$

The commutator of two such transformations vanishes up to a gauge transformation:

$$[\delta, \delta'] = \tau, \tag{10}$$

where supersymmetry transformations δ and δ' have parameters $\xi, \hat{\xi}_{\alpha\beta}$ and $\eta, \hat{\eta}_{\alpha\beta}$, respectively, and τ is the gauge transformation with parameter $M(\hat{\xi}^{\alpha\beta}\hat{\eta}_{\alpha\beta} + 2\xi\eta)$.

It is useful to note that all fields of the theory now become Lorentz group tensors (and not spinors): $\psi_{\dot{\alpha}\alpha}$ corresponds to a vector $\psi_a = \sigma_a{}^{\alpha\dot{\alpha}}\psi_{\alpha\dot{\alpha}}$, ψ to a scalar and $\chi_{\alpha\beta}$ to a self-dual rank two tensor $\chi_{ab} = (\sigma_{ab})^{\alpha\beta}\chi_{\alpha\beta}$, $\chi_{ab} = \frac{1}{2}\varepsilon_{abcd}\chi^{cd}$ [1].

Now consider this theory minimally coupled to a gravitational background with a metric $g_{\mu\nu}$. Here μ, ν, ... are four-dimensional world indices. In vector notation the corresponding action and the Lagrangian read

$$S = \int d^4x \sqrt{g}\, L, \tag{11}$$

$$\begin{aligned} L = \operatorname{tr}\Big\{ &\frac{1}{2} F^+_{\mu\nu} F^{+\mu\nu} - \frac{1}{2}\mathscr{D}_\mu M \mathscr{D}^\mu N - i\psi_\mu \mathscr{D}^\mu \psi + i\mathscr{D}_\mu \psi_\nu \chi^{\mu\nu} \\ &- \frac{i}{2}\psi_\mu[\psi^\mu, N] - \frac{i}{2}\psi[\psi, M] - \frac{1}{8}[M,N]^2 + \frac{i}{8}[\chi_{\mu\nu}, M]\chi^{\mu\nu} - \frac{1}{8}T_{\mu\nu}T^{\mu\nu}\Big\}. \end{aligned} \tag{12}$$

Here $F^+_{\mu\nu}$ is the self-dual part of the Yang-Mills field strength,

$$F^+_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} + \tilde{F}_{\mu\nu}), \qquad \tilde{F}_{\mu\nu} = \frac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\lambda\varrho}F^{\lambda\varrho}, \tag{13}$$

and the fields $T_{\mu\nu}$ and $\chi_{\mu\nu}$ are self-dual,

$$T_{\mu\nu} = \frac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\lambda\varrho}T^{\lambda\varrho}, \qquad \chi_{\mu\nu} = \frac{1}{2}\sqrt{g}\,\varepsilon_{\mu\nu\lambda\varrho}\chi^{\lambda\varrho}. \tag{14}$$

Witten observed [1] that the singlet supersymmetry with parameter ξ of the twisted $N = 2$ Yang-Mills theory survives on an arbitrary gravitational background $g_{\mu\nu}$. This supersymmetry is remarkable since it exists in spite of the absence of $g_{\mu\nu}$ superpartners. Denoting the corresponding supersymmetry operator by Q (so that $\delta X = \xi Q X$), one finds

$$\begin{gathered} QA_\mu = i\psi_\mu, \qquad QM = 0, \qquad QN = 2i\psi, \\ Q\psi = \frac{1}{2}[M,N], \qquad Q\psi_\mu = -\mathscr{D}_\mu M, \\ Q\chi_{\mu\nu} = T_{\mu\nu} + 2F^+_{\mu\nu}, \\ QT_{\mu\nu} = -2i(\mathscr{D}_\mu\psi_\nu - \mathscr{D}_\nu\psi_\mu)^+ - i[\chi_{\mu\nu}, M], \\ Qg_{\mu\nu} = 0. \end{gathered} \tag{15}$$

The square of Q is zero modulo gauge transformations.

The Lagrangian can be written in the form

$$L = QV, \tag{16}$$

where

$$V = \operatorname{tr}\Big\{\frac{1}{4}F^+_{\mu\nu}\chi^{\mu\nu} + \frac{1}{2}\psi_\mu\mathscr{D}^\mu N - \frac{1}{4}\psi[M,N] - \frac{1}{8}T_{\mu\nu}\chi^{\mu\nu}\Big\}. \tag{17}$$

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[1] We use the following conventions: $\sigma_a{}^{\alpha\dot{\alpha}} = (1, i\vec{\sigma})$, $(\sigma_{ab})^{\alpha\beta} = \frac{1}{2}[\sigma_a{}^{\alpha\dot{\alpha}}\sigma_b{}^{\beta}{}_{\dot{\alpha}} + (\alpha\leftrightarrow\beta)]$

For constructing topological invariants it is very important to show that the energy-momentum tensor $\mathbf{T}^{\mu\nu}$ of the theory can also be represented as Q acting on some combination of fields. $\mathbf{T}^{\mu\nu}$ is defined as usual by the variation of the action under an infinitesimal change of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$,

$$\delta S = \int d^4 x \sqrt{g} \delta g_{\mu\nu} \mathbf{T}^{\mu\nu} . \tag{18}$$

This variation is not straightforward because the fields $\chi_{\mu\nu}$ and $T_{\mu\nu}$ are constrained by self-duality conditions. To preserve the latter, an arbitrary variation of the metric should be accompanied by some variation of the fields $\chi_{\mu\nu}$, $T_{\mu\nu}$. These variations are not uniquely defined: to a given variation we may add an arbitrary self-dual tensor, linear in $\delta g_{\mu\nu}$. However there is a distinguished way to define such variation. It consists simply in projecting the self-dual tensor to the three-dimensional subspace which is self-dual with respect to the new metric $g_{\mu\nu} + \delta g_{\mu\nu}$. This means that the variation should be orthogonal to the subspace of self-dual tensors, or in other words, it should be anti-self-dual. This defines the variation uniquely. In [1] the $\chi_{\mu\nu}$ variation was chosen exactly in this form. The corresponding formula reads

$$\delta \chi_{\mu\nu} = \frac{1}{4} g^{\sigma\tau} \delta g_{\sigma\tau} \chi_{\mu\nu} - \frac{1}{2} \varepsilon_{\mu\nu\lambda\varrho} \sqrt{g} g^{\lambda\sigma} \delta g_{\sigma\tau} \chi^{\tau\varrho} . \tag{19}$$

It is natural to require that the operators Q and $\delta/\delta g_{\mu\nu}$ commute,

$$\left[Q, \frac{\delta}{\delta g_{\mu\nu}} \right] = 0 . \tag{20}$$

Applying Q to (19) we find that the combination $T_{\mu\nu} + 2F^{+}_{\mu\nu}$ varies as in (19). This means that the variation of $T_{\mu\nu}$ itself is different from (19). It contains the anti-self-dual part $F^{-}_{\mu\nu}$ of the Yang-Mills field strength [2]:

$$\delta T_{\mu\nu} = \frac{1}{4} g^{\sigma\tau} \delta g_{\sigma\tau} (T_{\mu\nu} + 2F^{-}_{\mu\nu}) - \frac{1}{2} \varepsilon_{\mu\nu\lambda\varrho} \sqrt{g} g^{\lambda\sigma} \delta g_{\sigma\tau} (T^{\tau\varrho} - 2F^{-\tau\varrho}) . \tag{21}$$

Now one can easily check that (20) is satisfied. This useful property together with Eq. (16) imply that the energy-momentum tensor $\mathbf{T}^{\mu\nu}$ of the theory can indeed be represented in a form

$$\mathbf{T}^{\mu\nu} = Q \lambda^{\mu\nu} \tag{22}$$

with

$$\lambda^{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial g_{\mu\nu}} (\sqrt{g} V) . \tag{23}$$

(In [1], Q commuted with $\delta/\delta g_{\mu\nu}$ only on shell, so Eq. (22) needed a tedious verification which we have avoided by introducing the auxiliary field $T_{\mu\nu}$ and the transformation law (21).) This property of the energy-momentum tensor plays a crucial role in the construction of topological invariants.

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[2] In Sect. 3 we will discuss the singlet supersymmetry using conformal supergravity. We don't know how to explain this transformation law in the framework of conformal supergravity

2.3. Topological Invariants

The construction of topological invariants proposed in [1] used path integrals of the action (11) which is not positively defined. So this construction is more suggestive than rigorous. However if to play freely with ill-defined objects, it gives self-consistent results which can be verified by different methods. In our discussion of the complex case we make some similar conjectures, which need analytic proofs, about invariants of complex manifolds. Since in the complex case we follow the line of construction in [1], we briefly recall it here.

Consider an observable A which by definition is invariant under supersymmetry, $QA = 0$, and under changes of the metric, $\delta A/\delta g^{\mu\nu} = 0$. Then

$$\begin{aligned}\frac{\delta\langle A\rangle}{\delta g_{\mu\nu}} &= \frac{\delta}{\delta g_{\mu\nu}}\int(\mathscr{D}X)\exp\left(-\frac{1}{e^2}S\right)A\\ &= -\frac{1}{e^2}\int(\mathscr{D}X)\exp\left(-\frac{1}{e^2}S\right)\sqrt{g}\,\mathbf{T}^{\mu\nu}A\\ &= -\frac{1}{e^2}\int(\mathscr{D}X)\exp\left(-\frac{1}{e^2}S\right)\sqrt{g}\,AQ\lambda^{\mu\nu},\end{aligned}$$

where $(\mathscr{D}X)$ is the integration measure. Integrating by parts and taking into account that $Q\left(\exp\left(-\frac{1}{e^2}S\right)\sqrt{g}\right) = 0$, we obtain

$$\frac{\delta\langle A\rangle}{\delta g_{\mu\nu}} = 0\,. \tag{24}$$

Thus $\langle A\rangle$ does not depend on the choice of a metric and hence carries only topological information.

Moreover $\langle A\rangle$ does not depend on the coupling constant:

$$\begin{aligned}\frac{\delta\langle A\rangle}{\delta\left(-\frac{1}{e^2}\right)} &= \int(\mathscr{D}X)\exp\left(-\frac{1}{e^2}S\right)AS\\ &= \int(\mathscr{D}X)\exp\left(-\frac{1}{e^2}S\right)A\left(Q\int V\right) = 0\,.\end{aligned} \tag{25}$$

(In the last equation we again integrated by parts.) Thus we may work in the limit $e\to 0$. In this limit the path integral is concentrated on the superspace $\tilde{\mathscr{M}}$ of configurations on which the action is equal to zero.

The kinetic term for the gauge field vanishes for self-dual connections (i.e. solutions of the equation $F_{\alpha\beta} = 0$). Witten argued that if the gauge group is chosen to be $G = SU(2)$ (in the following we always mean this choice) then in general the whole bosonic part $\mathscr{M}$ of $\tilde{\mathscr{M}}$ is the modulus space of self-dual connections. As for fermions, it is known that on the background of a typical self-dual $SU(2)$-connection the fields ψ, $\chi_{\mu\nu}$ do not have zero modes. When $M = N = 0$ the field equations for ψ_μ have the form

$$\begin{aligned}\mathscr{D}_\mu\psi^\mu &= 0\,,\\ (\mathscr{D}_\mu\psi_\nu - \mathscr{D}_\nu\psi_\mu)^+ &= 0\,.\end{aligned} \tag{26}$$

These coincide exactly with the equations describing infinitesimal deformations of self-dual connections. In other words, zero modes of ψ_μ correspond to tangent vectors to the modulus space of the instantons. Hence the fermionic part of $\tilde{\mathcal{M}}$ is described by the tangent sheaf to $\mathcal{M}$. It is clear that $\tilde{\mathcal{M}}$ is Q-invariant (the auxiliary field $T_{\mu\nu}$ vanishes on $\tilde{\mathcal{M}}$ also). Moreover the formula $QA_\mu = i\psi_\mu$ shows that the singlet supersymmetry operator Q reduces to the wedge derivative on $\mathcal{M}$ considered as a vector field on $\tilde{\mathcal{M}}$.

Consider now forms A_i, $i = 0, \ldots, 4$:

$$\begin{aligned} A_0 &= \operatorname{tr} M^2, \\ A_1 &= \operatorname{tr}(-2M\psi_\mu)\, dx^\mu, \\ A_2 &= \operatorname{tr}(-\psi_\mu\psi_\nu - iMF_{\mu\nu})\, dx^\mu dx^\nu, \\ A_3 &= \operatorname{tr}(i\psi_\lambda F_{\mu\nu})\, dx^\mu dx^\nu dx^\lambda, \\ A_4 &= \operatorname{tr}\left(-\frac{1}{4} F_{\varrho\lambda}F_{\mu\nu}\right) dx^\varrho dx^\lambda dx^\mu dx^\nu. \end{aligned} \tag{27}$$

One easily checks that

$$dA_j = QA_{j+1}, \tag{28}$$

where for $j = 4$ we put $A_5 = 0$. Now let ω be an arbitrary given j-form on M^4 which does not depend on fields. Assume that ω is closed, $d\omega = 0$. Then consider the observable

$$\tilde{\omega} = \int_{M^4} A_{4-j} \wedge \omega. \tag{29}$$

We have

$$Q\tilde{\omega} = \int QA_{4-j} \wedge \omega = \int dA_{3-j} \wedge \omega = \int d(A_{3-j} \wedge \omega) = 0. \tag{30}$$

Thus $\tilde{\omega}$ satisfies the conditions needed for constructing of topological invariants. Moreover, if $\omega = d\sigma$ then

$$\begin{aligned} \tilde{\omega} &= \int A_{4-j} \wedge d\sigma = (-1)^{j+1} \int dA_{4-j} \wedge \sigma \\ &= (-1)^{j+1} \int QA_{5-j} \wedge \sigma = Q((-1)^{j+1} \int A_{5-j} \wedge \sigma). \end{aligned} \tag{31}$$

It is clear that the expectation values of such observables vanish: if $A = QB$ then

$$\langle A \rangle = \int (\mathcal{D}X) \exp\left(-\frac{1}{e^2}S\right) QB = \int (\mathcal{D}X)\, Q\left(\exp\left(-\frac{1}{e^2}S\right)\right) B = 0. \tag{32}$$

Therefore we may obtain nontrivial observables only from nonzero de Rham cohomology classes of M^4. By construction, functions on $\tilde{\mathcal{M}}$ are differential forms on $\mathcal{M}$. Moreover, the $\mathbf{R}^*$-grading corresponds to the grading of differential forms by their degrees. The recipe for the construction of the Donaldson invariants proposed by Witten [1] is as follows. Each time when A_i contains M we replace it by $\langle M \rangle$, where $\langle M \rangle$ is the solution of the equation

$$\mathcal{D}^\mu \mathcal{D}_\mu \langle M \rangle = i[\psi_\mu, \psi^\mu]. \tag{33}$$

(Here $\mathcal{D}_\mu$ is a self-dual connection and ψ_μ is a zero-mode.) Let A_i' be the result of this substitution. Now let

$$\Phi_\omega = \int_M A_{4-j}' \wedge \omega. \tag{34}$$

Witten showed that Φ_ω is a well defined closed differential j-form on $\mathscr{M}$. Thus, closed j-forms on M^4 correspond to closed j-forms on $\mathscr{M}$ and exact forms correspond to exact ones. This gives a map

$$H^j(M^4, \mathbf{R}) \to H^j(\mathscr{M}, \mathbf{R}), \qquad \omega \mapsto \Phi_\omega, \tag{35}$$

which is called the Donaldson map. (We used here cohomology classes instead of cycles, used in [1], for the future convenience in the discussion of the Dolbeault groups in the Kähler case.) The space $\mathscr{M}$ consists of components $\mathscr{M}^n$, enumerated by the topological charge. If we take arbitrary closed forms $\omega_1, \ldots, \omega_k$ of degrees $j_1, \ldots, j_k$ so that $\Sigma j_\alpha = \dim \mathscr{M}_n$ for some n then $\Phi_{\omega_1} \wedge \ldots \wedge \Phi_{\omega_k}$ is a volume form on $\mathscr{M}_n$ and one can check that

$$\langle \tilde{\omega}_1 \ldots \tilde{\omega}_k \rangle = \int_{\mathscr{M}_n} \Phi_{\omega_1} \wedge \ldots \wedge \Phi_{\omega_k}. \tag{36}$$

Topological invariants obtained as expectation values of observables of the form $\tilde{\omega}_1 \ldots \tilde{\omega}_k$ precisely reproduce the Donaldson invariants.

3. Conformal Supergravity Approach to Topological Yang-Mills Theory

3.1. Singlet Supersymmetry

Karlhede and Rocek [4] gave the following explanation for the singlet supersymmetry in the topological Yang-Mills theory. Since topological Yang-Mills theory arises from $N = 2$ supersymmetric Yang-Mills theory, it is natural to use the language of $N = 2$ supersymmetry, or in other words to couple $N = 2$ supersymmetric Yang-Mills theory to $N = 2$ supergravity. We want to identify $SU(2)_L$ with $SU(2)_A$. Since $SU(2)_L$ becomes local in gravity, this enforces $SU(2)_A$ to be local also. This is a property of $N = 2$ conformal supergravity. So we are prompted to start with the coupling of $N = 2$ supersymmetric Yang-Mills theory to $N = 2$ conformal supergravity. We will work in the component formalism. The transformation laws of the fields of the supergravity multiplet are well known [7]. Let us consider configuration for which all fields except the vierbein $e_\mu{}^{\alpha\dot{\alpha}}$ and the gauge field $V_{\mu i}{}^j$ for the group $SU(2)_A$ (and, to be more precise, the auxiliary scalar field D) vanish. Then the formula for the transformation of the gravitino field $\psi_\mu^{\alpha i}$ reduces to

$$\delta \psi_\mu^{\alpha i} = \partial_\mu \xi^{\alpha i} + \omega_\mu{}^\alpha{}_\beta \xi^{\beta i} + V_\mu{}^i{}_j \xi^{\alpha j}, \tag{37}$$

where $\omega_\mu{}^\alpha{}_\beta$ is the spin-connection. After identifying the groups $SU(2)_L$ and $SU(2)_A$ the parameters $\xi^{\alpha i}$ break into a singlet ξ, which is antisymmetric in α, i, and a triplet $\hat{\xi}^{\alpha i}$, which is symmetric in α, i. Similarly, the gravitino field $\psi_\mu^{\alpha i}$ breaks into ψ_μ and $\hat{\psi}_\mu^{\alpha i}$. Moreover, after this identification we may put

$$\omega_\mu{}^\alpha{}_\beta = V_\mu{}^\alpha{}_\beta. \tag{38}$$

Then the transformation law for the gravitino field takes the form

$$\delta \psi_\mu = \partial_\mu \xi, \tag{39}$$

$$\delta \psi_\mu^{\alpha\beta} = \mathscr{D}_\mu \xi^{\alpha\beta} \equiv \partial_\mu \xi^{\alpha\beta} + \omega_\mu{}^\alpha{}_\gamma \xi^{\gamma\beta} + \omega_\mu{}^\beta{}_\gamma \xi^{\alpha\gamma}. \tag{40}$$

We want to find out which supersymmetry transformations preserve the background under consideration, or in other words for which values of ξ, $\xi^{\alpha\beta}$ the equalities

$$\delta\psi_\mu = 0, \qquad \delta\hat{\psi}_\mu^{\alpha\beta} = 0 \tag{41}$$

are satisfied. It is natural to call such transformations the Killing spinors of the field configuration considered. One possibility is obvious

$$\xi^{\alpha\beta} = 0, \qquad \xi = \text{const}. \tag{42}$$

This is precisely the Killing spinor discussed in [4]. This spinor leads to the existence of the residual singlet supersymmetry in the twisted $N = 2$ Yang-Mills theory on a general metric background. (The mechanism of the relation between Killing spinors and supersymmetries in Yang-Mills theory is as follows. After coupling of the Yang-Mills Lagrangian to conformal supergravity, the resulting Lagrangian is invariant under all supersymmetry transformations if we transform the fields of the supergravity multiplet simultaneously. Fixing the configuration of the supergravity multiplet, we obtain Yang-Mills theory on the background of this configuration. The corresponding Lagrangian is invariant only under those sypersymmetry transformations that preserve the chosen configuration.)

Remark. Suppose we are given some $N = 2$ supersymmetric theory. One may ask whether the existence of Killing spinors ensures that this theory will possess corresponding supersymmetries after twisting. In general, the answer is no. For instance, the twisted $N = 2$ σ-models with nontrivial target spaces do not seem to have supersymmetries. The reason lies in the absence of conformal invariance in such models. On the other hand, $N = 2$ Yang-Mills is conformally invariant in four dimensions. Hence the twisted Yang-Mills theory is indeed invariant under the supersymmetry transformations corresponding to the constructed Killing spinors. The invariance of the Yang-Mills Lagrangian under these supersymmetry transformations can be checked directly, however, without referring to conformal supergravity. We will discuss this in Sect. 4.

3.2. Additional Supersymmetries and Complex Structures

Let us go further. It is clear that in the case of a general metric there are no Killing spinors distinct from (42). We may ask what are the conditions for the metric that ensure the existence of other supersymmetry transformations satisfying (41)? Formula (40) shows that it requires the existence of real covariantly constant tensors of the form $H^{\alpha\beta}$. (Then we can take

$$\xi^{\alpha\beta} = \gamma H^{\alpha\beta}, \tag{43}$$

where γ is a real constant Grassmann parameter.)

In vector indices, a tensor $H_{\alpha\beta}$ becomes a self-dual tensor $H_{\mu\nu}$ of rank two,

$$H_{\mu\nu} = e_\mu{}^{\alpha\dot{\alpha}} e_\nu{}^\beta{}_{\dot{\alpha}} H_{\alpha\beta}. \tag{44}$$

Since $H_{\mu\nu}$ is covariantly constant, the tensor $L_\mu{}^\nu = g^{\nu\varrho} H_{\mu\varrho}$ is covariantly constant as well, $\mathscr{D}_\varrho L_\mu{}^\nu = 0$. Then it follows that the eigenvalues of L do not depend on the point x^μ in space-time. Indeed, the coefficients of the characteristic polynomial $\det(\lambda 1 - L)$ are simply functions on the manifold and since $\mathscr{D}_\varrho L_\mu{}^\nu = 0$ we have $\partial_\mu \det(\lambda 1 - L) = \mathscr{D}_\mu \det(\lambda 1 - L) = 0$. Hence the coefficients of the characteristic polynomial as well as the eigenvalues are constants. Since $H_{\mu\nu}$ is antisymmetric and real it follows that the eigenvalues of $L_\mu{}^\nu$ are imaginary and that non-zero eigenvalues constitute complex conjugate pairs. (To see this, let us choose an orthonormal basis. Then $L^t = -L$. Let v be an eigenvector, $Lv = \lambda v$. Then $v^+ L^+ = \bar{\lambda} v^+$. But L is real, so $L^+ = L^t = -L$. Thus $v^+ L = -\bar{\lambda} v^+$. Now

$$v^+ (Lv) = \lambda (v^+ v) = (v^+ L) v = -\bar{\lambda} v^+ v .$$

So $\bar{\lambda} = -\lambda$, i.e. λ is imaginary. Applying complex conjugation to $Lv = \lambda v$ we get $L\bar{v} = \bar{\lambda}\bar{v}$. Therefore, λ and $\bar{\lambda}$ are eigenvalues of L simultaneously.) The manifold is four-dimensional so L has four eigenvalues. Denote them by $i\lambda_1$, $-i\lambda_1$, $i\lambda_2$, $-i\lambda_2$, where λ_1, λ_2 are real. Let the corresponding eigenvectors be $v_1, \bar{v}_1, v_2, \bar{v}_2$. Then in the real basis $\operatorname{Re} v_1, \operatorname{Im} v_1, \operatorname{Re} v_2, \operatorname{Im} v_2$ the operator L will have the form

$$L = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_2 & 0 \end{bmatrix} \tag{45}$$

and the self-duality condition implies that $\lambda_1 = \lambda_2$. (Assuming that our four-manifold is not locally a metric product, we could obtain the same result in a different way, without using the self-duality of $H_{\mu\nu}$. To do this, note that the distribution corresponding to any eigenvalue is integrable: if

$$L_\mu{}^\nu v^\mu = \lambda v^\nu, \qquad L_\mu{}^\nu w^\mu = \lambda w^\nu,$$

then

$$\begin{aligned} L_\mu{}^\nu [v, w]^\mu &= L_\mu{}^\nu (v^\varrho \partial_\varrho w^\mu - w^\varrho \partial_\varrho v^\mu) \\ &= L_\mu{}^\nu (v^\varrho \mathscr{D}_\varrho w^\mu - w^\varrho \mathscr{D}_\varrho v^\mu) \\ &= v^\varrho \mathscr{D}_\varrho (L_\mu{}^\nu w^\mu) - w^\varrho \mathscr{D}_\varrho (L_\mu{}^\nu v^\mu) \\ &= \lambda (w^\varrho \mathscr{D}_\varrho w^\nu - w^\varrho \mathscr{D}_\varrho v^\nu) = \lambda [v, w]^\nu , \end{aligned} \tag{46}$$

since $\mathscr{D}_\varrho L_\mu{}^\nu = 0$ and $\lambda = \text{const}$ as proved above. This means that the two-dimensional real distribution corresponding to eigenvalues $i\lambda_1$, $-i\lambda_1$ is integrable. If $\lambda_1 \neq \lambda_2$ we get two orthogonal integrable distributions. They define on the manifold the structure of a local metric product, which contradicts the assumption. So again $\lambda_1 = \lambda_2$.) Since $\lambda_1 = \text{const}$ the operator

$$I_\mu{}^\nu = \frac{1}{\lambda_1} L_\mu{}^\nu \tag{47}$$

is covariantly constant and satisfies the identity

$$I^2 = -1 . \tag{48}$$

Hence any Killing spinor defines uniquely a covariantly constant complex structure. The expression (45) implies that the metric is Hermitian with respect to it. Thus, the existence of a Killing spinor is equivalent to the existence of a Kähler structure on M. Multiplying the Killing spinor by a non-zero real number, we obtain the same Kähler structure. However, we can from the very beginning fix the norm of $H_{\alpha\beta}$ in such a way that the resulting λ_1 equals ± 1. Let us make the convention that we consider only such normed $H_{\alpha\beta}$. We have proved the following proposition:

Proposition. *Additional Killing spinors are (up to a sign) in 1-1 correspondence with Kähler structures on M.*

The existence of a covariantly constant complex structure on the manifold reduces its holonomy group. In dimension four there are, in essence, two possibilities: the holonomy group reduces to $U(2)$ or to $SU(2)$. In the first case, there is only one covariantly constant complex structure (and the manifold is called Kähler), while in the second case there are three covariantly constant complex structures (and the manifold is called hyperKähler). We have come to the following corollary:

Corollary. *A Kähler manifold admits one additional Killing spinor and a hyperKähler manifold admits a triplet of additional Killing spinors.*

Remark. We could have tried to look for a Killing spinor of the opposite chirality (i.e. $\xi_{\dot{\alpha}i}$ before twisting). Then we would have had to worry about the gravitino field $\psi_{\mu\dot{\alpha}i}$ or, after twisting, $\psi_{\mu\dot{\alpha}\beta}$. Its transformation law is

$$\delta\psi_{\mu\dot{\alpha}\beta} = \mathscr{D}_\mu \xi_{\dot{\alpha}\beta}\,.$$

Demanding that $\delta\psi_{\mu\dot{\alpha}\beta} = 0$, we find that the existence of such a Killing spinor leads to the existence of a covariantly constant object $\xi_{\dot{\alpha}\beta}$ on the manifold. In world indices, this becomes a vector ξ_μ. A covariantly constant vector reduces the holonomy group to $SO(3)$. Indeed, since ξ_μ is covariantly constant, $\mathscr{D}_\mu \xi_\nu = 0$, it will be preserved by the operators of parallel transports, which therefore act nontrivially only in the hyperplane orthogonal to ξ^μ. This means that the operators of parallel transport lie in $SO(3)$. In this case the manifold is locally a metric product. The standard arguments showing this are as follows. Let us go to a coordinate system where $\xi^\mu(\equiv g^{\mu\nu}\xi_\nu)$ has a form $\xi^1 = \xi^2 = \xi^3 = 0$, $\xi^4 = 1$. (This is always possible locally since the length of ξ^μ is a non-zero constant and so the vector field ξ^μ has no zeroes on the manifold.) In this coordinate system, the condition $\mathscr{D}_\mu \xi^\nu$ amounts to $\Gamma_{\mu 4\nu} = 0$, or $\partial_\mu g_{4\nu} + \partial_4 g_{\mu\nu} - \partial_\nu g_{4\mu} = 0$. This equation is equivalent to ($i, j = 1, 2, 3$):

$$\partial_4 g_{ij} = \partial_4 g_{4j} = \partial_4 g_{44} = \partial_i g_{44} = 0\,,$$
$$\partial_i g_{4j} - \partial_j g_{4i} = 0\,.$$

Then $g_{4j} = \partial_j f$, where $f = f(x^1, x^2, x^3)$, and one can easily check that in the coordinate system $y^i = x^i$, $y^4 = x^4 + f$ the metric takes the form

$$g_{\mu\nu} = \begin{pmatrix} g_{ij}(y^k) & 0 \\ 0 & \text{const} \end{pmatrix}.$$

This means that the manifold decomposes into a local metric product, as stated above. So in this case we are back in three dimensions which was the starting point of the whole story for topological Yang-Mills theory [1, 8] and where the Floer groups arose [9]. However we are concerned in this paper with the irreducible case (i.e. with the case in which the manifold is not a local metric product) and so we will not pursue this line further.

4. Topological Yang-Mills Theory on Kähler and HyperKähler Manifolds

We have seen in the last section that covariantly constant tensors $H_{\alpha\beta}$ (corresponding to covariantly constant complex structures on the manifold) define additional Killing spinors. Using our knowledge of the conformal invariance of the $N = 2$ Yang-Mills theory we deduced that the topological Yang-Mills theory possesses corresponding additional supersymmetries. Now we want to check this directly.

The check of supersymmetry invariance of the $N = 2$ Yang-Mills theory on flat space is based on the fact that the supersymmetry parameters $\xi_{\alpha i}$, $\xi_{\dot{\alpha} i}$ are constant, i.e. $\partial_\mu \xi_{\alpha i} = 0$, $\partial_\mu \xi_{\dot{\alpha} i} = 0$. In curved space this holds true for covariantly constant supersymmetry parameters, $\mathscr{D}_\mu \xi_{\alpha\beta} = 0$. The only possible obstruction to the invariance of the action might be the appearance of the Riemann tensor in commutators of covariant derivatives. However, direct inspection shows that the Riemann tensor does not occur in the variation of the Lagrangian (12), as in the case of singlet supersymmetry [1]. This proves once more the following lemma.

Lemma. *Let the supersymmetry parameters in the formulas (9) have the form* $\xi_{\alpha\beta} = \lambda H_{\alpha\beta}(x)$, *where* $H_{\alpha\beta}(x)$ *is covariantly constant and* λ *is a constant Grassmann parameter. Then the Lagrangian* (12) *of the topological Yang-Mills theory is invariant under these transformations.*

We note that the commutator of two such transformations has the same form as in flat space.

Corollary. *The topological Yang-Mills theory has one additional supersymmetry operator on a Kähler manifold and three additional supersymmetry operators on a hyperKähler manifold.*

As mentioned above, the supersymmetry operator Q reduces to the wedge differential on the modulus space of instantons. The topological Yang-Mills theory on a Kähler manifold has two supersymmetry operators, Q and Q'. Later, we will write topological Yang-Mills theory in complex coordinates. It will be convenient to replace the operators Q, Q' by $q = (Q - Q')/2$, $\bar{q} = -(Q + Q')/2$. We will give explicit formulas for q, $\bar{q}$ in complex coordinates on a Kähler manifold. These operators are complex conjugated and $Q = q + \bar{q}$. This is reminiscent of the wedge differential on complex manifolds ($d = \partial + \bar{\partial}$). Analogously, on a hyperKähler manifold four operators d, $I_j d$ are defined, where I_j are complex structures. In topological Yang-Mills theory on a hyperKähler manifold precisely such a picture arises for the supersymmetry operators. Thus the modulus space of instantons on a Kähler (hyperKähler) manifold has a complex (hypercomplex) structure. The situation for general Kähler manifolds was studied in [10]. It was shown there that the standard metric on the modulus space of instantons on a Kähler manifold is itself Kähler. As far as we know, the situation for a general hyperKähler manifold has never been discussed. However it follows from the Kähler case that each covariantly constant complex structure on the manifold itself induces a Kähler structure on the modulus space. Thus in the hyperKähler case we conclude that there are three covariantly constant complex structures on the modulus space of instantons on a hyperKähler manifold. In other words, the modulus space of instantons on a hyperKähler manifold is itself hyperKähler. It would be tempting to interpret all these facts from the point of view of the topological Yang-Mills theory.

Remark. On Kähler (and similarly hyperKähler) manifolds, dotted and undotted indices are essentially different. A Kähler structure distinguishes one of chiralities: we defined the Kähler structure by the tensor $H_{\alpha\beta}$ with undotted indices. Therefore, the self-duality and anti-self-duality equations on a Kähler manifold have different geometric meanings. The action in topological Yang-Mills theory vanishes on configurations for which $F_{\alpha\beta} = 0$. So our discussion concerns the modulus spaces of instantons defined exactly by this equation.

Let us now discuss the Kähler case in more detail. After choosing a gauge where $H_{11} = H_{22} = 0$, $H_{12} = \text{const}$, the transformations of the second supersymmetry take the form

$$\begin{gathered} Q'A_{1\dot\alpha} = i\psi_{\dot\alpha 1}\,, \qquad Q'A_{2\dot\alpha} = -i\psi_{\dot\alpha 2}\,, \\ Q'M = 0\,, \qquad Q'N = 4i\chi_{12}\,, \\ Q'\chi_{11} = T_{11} - 2F_{11}\,, \qquad Q'\chi_{22} = -T_{22} + 2F_{22}\,, \\ Q'\chi_{12} = -\frac{1}{4}[M,N], \qquad Q'\psi = -2T_{12} - 4F_{12}\,, \\ Q'\psi_{\dot\alpha 1} = \mathscr{D}_{1\dot\alpha}M\,, \qquad Q'\psi_{\dot\alpha 2} = -\mathscr{D}_{2\dot\alpha}M\,, \\ Q'T_{11} = 2i\mathscr{D}_1{}^{\dot\beta}\psi_{\dot\beta 1} + i[\chi_{11}, M], \qquad Q'T_{22} = -2i\mathscr{D}_2{}^{\dot\beta}\psi_{\dot\beta 2} - i[\chi_{22}, M], \\ Q'T_{12} = i\mathscr{D}_1{}^{\dot\beta}\psi_{\dot\beta 2} - i\mathscr{D}_2{}^{\dot\beta}\psi_{\dot\beta 1} - \frac{i}{2}[\psi, M]\,. \end{gathered} \tag{49}$$

It is useful to work in complex notation. Let z^m, $m = 1, 2$ be local complex coordinates. It is perhaps worthwhile to explain how spinor indices are related to world indices in the complex case. The complex structure $I_\mu{}^\nu$ in spinor indices becomes $I_{\alpha\dot\alpha}{}^{\beta\dot\beta}$ and the tensor $H_\alpha{}^\beta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ defines a complex structure by the formula

$$I_{\alpha\dot\alpha}{}^{\beta\dot\beta} = H_\alpha{}^\beta \delta_{\dot\alpha}^{\dot\beta}\,.$$

Two vector fields $X_{\dot\beta} = \partial_{1\dot\beta}$ satisfy the equality $IX_{\dot\beta} = iX_{\dot\beta}$. This means that in complex coordinates they are expressed in terms of $\partial/\partial z^m$ only (and not $\partial/\partial\bar z^m$). In other words, the vierbein $e_m{}^{\alpha\dot\alpha}$, $e_{\bar m}{}^{\alpha\dot\alpha}$ has in this case the properties

$$e_m{}^{2\dot\alpha} = 0\,, \qquad e_{\bar m}{}^{1\dot\alpha} = 0\,.$$

It is convenient to introduce fields $B = \psi + 2\chi_{12}$, $C = \psi - 2\chi_{12}$ and use operators $q = (Q - Q')/2$, $\bar q = -(Q + Q')/2$. Then

$$\begin{gathered} qA_m = 0\,, \qquad qA^m = i\psi^m\,, \\ qM = 0, \qquad qN = iC\,, \\ q\chi_{mn} = 2F_{mn}\,, \qquad q\chi^{mn} = T^{mn}\,, \\ qB = -\frac{1}{2}T - F + \frac{1}{2}[M,N], \qquad qC = 0\,, \\ q\psi_m = -\mathscr{D}_m M\,, \qquad q\psi^m = 0\,, \end{gathered}$$

$$
\begin{gathered}
qT_{mn} = 2i\mathscr{D}_n\psi_m - 2i\mathscr{D}_m\psi_n - i[\chi_{mn}, M],\\
qT^{mn} = 0, \qquad qT = -2i\mathscr{D}_m\psi^m - i[C, M],\\
\bar{q}A_m = -i\psi_m, \qquad \bar{q}A^m = 0,\\
\bar{q}M = 0, \qquad \bar{q}N = -iB,\\
\bar{q}\chi_{mn} = -T_{mn}, \qquad \bar{q}\chi^{mn} = -2F^{mn},\\
\bar{q}B = 0, \qquad \bar{q}C = -\frac{1}{2}T - F - \frac{1}{2}[M, N],\\
\bar{q}\psi_m = 0, \qquad \bar{q}\psi^m = \mathscr{D}^m M,\\
\bar{q}T_{mn} = 0, \qquad \bar{q}T^{mn} = -2i\mathscr{D}^n\psi^m + 2i\mathscr{D}^m\psi^n + i[\chi^{mn}, M],\\
\bar{q}T = -2i\mathscr{D}^m\psi_m = i[B, M].
\end{gathered}
\tag{50}
$$

Here we use the following notation. The components of the Yang-Mills curvature are given by

$$F_{mn} = [\mathscr{D}_m, \mathscr{D}_n],\ F^{mn} = [\mathscr{D}^m, \mathscr{D}^n],\ F^m_n = [\mathscr{D}^m, \mathscr{D}_n] + \tfrac{1}{2}\delta^m_n F,\ F = [\mathscr{D}_m, \mathscr{D}^m].$$

The fields χ_{mn}, T_{mn}, χ^{mn}, T^{mn} are antisymmetric,

$$\chi_{mn} = -\chi_{nm}, \qquad T_{mn} = -T_{nm}, \qquad \chi^{mn} = -\chi^{nm}, \qquad T^{mn} = -T^{nm}.$$

The reality conditions for the fields are

$$
\begin{gathered}
\bar{A}_m = -A_{\bar{m}}, \qquad \bar{M} = -M, \qquad \bar{N} = -N, \qquad \bar{\chi}_{mn} = -\chi_{\bar{m}\bar{n}},\\
\bar{T}_{mn} = -T_{\bar{m}\bar{n}}, \qquad \bar{\psi}_m = -\psi_{\bar{m}}, \qquad \bar{B} = -C, \qquad \bar{T} = T,\\
\bar{F}_{mn} = -F_{\bar{m}\bar{n}}, \qquad \bar{F} = F,
\end{gathered}
\tag{51}
$$

where $A_{\bar{m}} = g_{n\bar{m}}A^n$, $\psi_{\bar{m}} = g_{n\bar{m}}\psi^n$, $\chi_{\bar{m}\bar{n}} = g_{k\bar{m}}g_{l\bar{n}}\chi^{kl}$ etc., and g_{mn} is a Kähler metric.
The operators q, $\bar{q}$ satisfy the following commutation relations:

$$q^2 = 0, \qquad \bar{q}^2 = 0, \qquad \{q, \bar{q}\} = 2\tau_M, \tag{52}$$

where τ_M is a gauge transformation with parameter M.
The Lagrangian of the theory has the form

$$
\begin{aligned}
L = \operatorname{tr}\Big\{&F_{mn}F^{mn} - \frac{1}{2}F^2 + (\mathscr{D}^m\mathscr{D}_m M)N + \frac{1}{2}F[M, N] + i(\mathscr{D}_m\psi_n)\chi^{mn}\\
&- i\chi_{mn}\mathscr{D}^m\psi^n + i(\mathscr{D}_m\psi^m)B + i(\mathscr{D}^m\psi_m)C - \frac{i}{4}[\chi_{mn}, \chi^{mn}]M\\
&- i[\psi_n, \psi^n]N - \frac{i}{2}M[B, C] - \frac{1}{8}[M, N]^2 - \frac{1}{4}T_{mn}T^{mn} + \frac{1}{8}T^2\Big\}.
\end{aligned}
\tag{53}
$$

(The topological charge density reads

$$\operatorname{tr}\left(F_{mn}F^{mn} - \frac{1}{2}F^2 + F^m{}_n F^n{}_m\right)$$

in complex notation, so the kinetic term in the Lagrangian differs from the standard one by a total derivative, as above. The kinetic term vanishes on configurations for which $F_{mn} = 0$, $F^{mn} = 0$, $F = 0$. These are anti-self-dual fields. The self-duality equation is different. It has the form $F^{m}{}_{n} = 0$. This difference originates in the above-mentioned fact that on a Kähler manifold one of the chiralities is distinguished.)

The careful reader may ask where the Kähler condition for the metric comes into play, or how it is seen in complex notation that the Lagrangian (53) is not invariant under transformations (50) on the background of an arbitrary Hermitian metric. The answer is that in the Lagrangian (53) one meets expressions like $\mathscr{D}_m \psi^m = \partial_m \psi^m + \Gamma_{mk}{}^m \psi^k - \Gamma_{m\bar{k}}{}^m \psi^{\bar{k}}$, and supersymmetry transformations of the term containing $\Gamma_{m\bar{k}}{}^m \psi^{\bar{k}}$ does not cancel. The detailed analysis shows that this is the only obstruction. So the condition ensuring the invariance of the Lagrangian (53) is

$$g^{m\bar{n}} \Gamma_{m\bar{n}k} = 0 \tag{54}$$

(and complex conjugated). One can show that in four dimension this equation is precisely equivalent to the Kähler condition.

In the discussion of topological invariants we used the fact that the Lagrangian can be written in the form (16). In the Kähler case, this formula can be refined:

$$L = q\bar{q}W, \tag{55}$$

where

$$W = \operatorname{tr}\left(\frac{1}{4}\chi_{mn}\chi^{mn} - \frac{1}{2}BC + iFN\right). \tag{56}$$

Having in mind the generalization of topological invariants for the complex case we have to show that the variation of the Lagrangian (53) under an infinitesimal change of the Kähler potential, $g_{m\bar{n}} \to {}_{m\bar{n}} + \delta g_{m\bar{n}}$, $\delta g_{m\bar{n}} = \partial_m \partial_{\bar{n}} \delta K$ can be represented as $q\bar{q}$ acting on some combination of fields. To this end we assume that the fields A_m, $A_{\bar{m}}$, ψ_m, $\psi_{\bar{m}}$, M, N, χ_{mn}, $\chi_{\bar{m}\bar{n}}$, B, C, T_{mn}, $T_{\bar{m}\bar{n}}$ do not vary, while the variation of T is as follows:

$$\delta T = -2F_{m\bar{n}}\delta g^{m\bar{n}}.$$

One can easily check that this variation commutes with the operators q, $\bar{q}$ (50).

Remarks. 1. The topological Yang-Mills theory on a hyperKähler manifold has supersymmetries Q, $Q_{\alpha\beta}$ and the Lagrangian can be written in the form

$$L = QQ_{\alpha\beta}Q_{\gamma}^{\beta}\overline{Q^{\gamma\alpha}}Y, \tag{57}$$

where

$$Y = \frac{1}{2}\operatorname{tr} N^2. \tag{58}$$

The formulas (16), (17), (55)–(58) are suggested by the superspace form of the Lagrangian for $N = 2$ Yang-Mills theory in the flat space. The action contains an integration over Grassmann variables which reduces to the application of supersymmetry transformations.

2. Let $\delta g_{\mu\nu}$ be a variation of the hyperKähler metric compatible with all three complex structures, δX be the corresponding variation of a field X. Fixing any complex structure from the S^2-family we return to the Kähler case where we know that $[\delta, Q'] = 0$ for Q' corresponding to this complex structure. This argument proves that $[\delta, Q_{\alpha\beta}] = 0$ for all $Q_{\alpha\beta}$. Hence, similarly to the Kähler case, the

variation of the action can be represented as $QQ_{\alpha\beta}Q^{\beta\gamma}Q^{\alpha}{}_{\gamma}$ acting on some field combination.

The construction of the topological invariants in Sect. 2.3 was based on the Donaldson map (35). As we have seen above, the modulus space $\mathscr{M}$ has in the Kähler case a natural complex structure. Therefore the space of differential forms on $\mathscr{M}$ is bigraded. In fact, in the Kähler case the whole space of fields of the theory is bigraded. This bigrading is given by formulas

$$\begin{gathered}
\deg' A_m = \deg' A^m = (0,0), \qquad \deg' \psi_m = (0,1), \qquad \deg' \psi_m = (1,0), \\
\deg' M = (1,1), \qquad \deg' N = (-1,-1), \\
\deg' \chi_{mn} = (-1,0), \qquad \deg' \chi^{mn} = (0,-1), \\
\deg' B = (-1,0), \qquad \deg' C = (0,-1), \\
\deg' T_{mn} = (-1,1), \qquad \deg' T^{mn} = (1,-1), \qquad \deg' T = (0,0), \\
\deg' q = (1,0), \qquad \deg' \bar{q} = (0,1),
\end{gathered} \tag{59}$$

where $\deg' X$ is the bidegree of X. Note that the degree $\deg X$ given by formulas (4) is simply the sum of bidegrees of X. It is easy to see that the bigrading on the space of differential forms on $\mathscr{M}$ is inherited from the bigrading (59). In the Kähler case we should consider forms $B_{r,s}$ instead of A_i [see (27)], where

$$\begin{aligned}
B_{0,0} &= \operatorname{tr} M^2, \\
B_{1,0} &= -2\operatorname{tr}(M\psi_m)\, dz^m, \\
B_{0,1} &= 2\operatorname{tr}(M\psi_{\bar{m}})\, d\bar{z}^m, \\
B_{2,0} &= \operatorname{tr}(\psi_m\psi_n)\, dz^m dz^n, \\
B_{1,1} &= 2\operatorname{tr}(\psi_m\psi_{\bar{n}} + iMF_{m\bar{n}})\, dz^m d\bar{z}^n, \quad \text{where} \quad F_{m\bar{n}} = [\mathscr{D}_m, \mathscr{D}_{\bar{n}}], \\
B_{0,2} &= \operatorname{tr}\left(-\psi_{\bar{m}}\psi_{\bar{n}} + \frac{i}{2} M T_{\bar{m}\bar{n}}\right) d\bar{z}^m d\bar{z}^n, \\
B_{2,1} &= 2i\operatorname{tr}(\psi_m F_{n\bar{k}})\, dz^m dz^n d\bar{z}^k, \\
B_{1,2} &= \operatorname{tr}\left(2i\psi_{\bar{k}} F_{m\bar{n}} + \frac{i}{2} M\mathscr{D}_m \chi_{\bar{n}\bar{k}} - \frac{i}{2}\psi_m T_{\bar{n}\bar{k}}\right) dz^m d\bar{z}^n d\bar{z}^k, \\
B_{2,2} &= \operatorname{tr}\left(\frac{1}{2} F_{mn} F_{\bar{k}\bar{l}} + F_{m\bar{l}} F_{n\bar{k}} - \frac{i}{2}\mathscr{D}_n(\psi_m \chi_{\bar{k}\bar{l}})\right) dz^m dz^n d\bar{z}^k d\bar{z}^l.
\end{aligned} \tag{60}$$

Note that

$$\deg' B_{r,s} = (2-p, 2-q), \qquad [B_{r,s}] = cm^{-2-\frac{1}{2}(p+q)}.$$

We have

$$\bar{q} B_{r,s} = \bar{\partial} B_{r,s-1}, \tag{61}$$

where $\bar{\partial} = d\bar{z}^m \frac{\partial}{\partial \bar{z}^m}$. [We put $B_{r,-1} = 0$ in the right hand side of (61) when $q = 0$.] The forms $B_{r,s}$ do not have good behavior with respect to ∂, q, since in general the forms $\bar{B}_{r,s}$ are not equal to $B_{s,r}$.

Formula (34) now takes the form

$$\Phi_\omega = \int_M B'_{2-r,2-s} \wedge \omega, \tag{62}$$

where ω is a $\bar{\partial}$-closed (r,s)-form on M, $\bar{\partial}\omega = 0$. Finally, we obtain the map

$$H^{r,s}(M) \to H^{r,s}(\mathcal{M}), \tag{63}$$

generalising for the Kähler case the Donaldson map.

In conclusion we note that in the Kähler case invariants of the complex structure can be constructed starting not only from observables A such that $QA = 0$. It is sufficient to demand the invariance of A under any non-zero linear combination of q and $\bar{q}$: if

$$(xq + y\bar{q})A = 0, \qquad \delta A/\delta K = 0,$$

then

$$\begin{aligned}\frac{\delta\langle A\rangle}{\delta K} &= \frac{\delta}{\delta K}\int(\mathscr{D}X)\exp\left(-\frac{1}{e^2}S\right)A\\ &= -\frac{1}{e^2}\int(\mathscr{D}X)\exp\left(-\frac{1}{e^2}S\right)gtA\\ &= -\frac{1}{e^2}\int(\mathscr{D}X)\exp\left(-\frac{1}{e^2}S\right)g(q\bar{q}\lambda)A,\end{aligned} \tag{64}$$

which vanishes after integrating by parts. Here $g = \det g_{m\bar{n}}$, $gt = \delta S/\delta K$, K is a Kähler potential. For $\tilde{\omega} = \int\limits_M B_{2-r,2-s} \wedge \omega$ we have $\bar{q}\tilde{\omega} = 0$. So we may use such observables to obtain invariants of the complex structure. This amounts, as in Sect. 2.3, to building volume forms on $\mathcal{M}_n$ from Φ_ω's and integrating them over $\mathcal{M}_n$.

Similarly, to obtain invariants of hypercomplex structure in the hyperKähler case, it is sufficient for observables to be invariant under arbitrary combination of Q, $Q_{\alpha\beta}$.

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