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Relating the Approaches to Quantised Algebras and Quantum Groups

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Abstract. This paper constructs two representations of the quantum group $U_q g'$ by exploiting its quotient structure and the quantum double construction. Here the quantum group is taken as the dual to the quantised algebra $U_q g$, a one parameter deformation of the universal enveloping algebra of the Lie algebra g, as in Drinfel'd [6] and Jimbo [10]. From the two representations, the Hopf structure of the quantised algebra $U_q g$ is reexpressed in a matrix format. This is the very structure given by Faddeev et al. [7], in their approach to defining quantum groups and quantised algebras via the quantisation of the function space of the associated Lie group to g.

Introduction

A newcomer to the field of quantum groups will encounter four essential papers on the structure and definition of quantised algebras and quantum groups, namely those by Jimbo [9, 10], Drinfel'd [6] and Faddeev et al. [7]. These works define the concepts of quantised algebras and quantum groups using two alternative approaches. The first two authors use a more mathematical formulation for defining a quantised algebra, introducing a one parameter deformation of an universal enveloping algebra of a Lie (or Kac Moody) algebra. The concepts are rather intricate, and for this reason the approach of Faddeev et al. [7]—based on a quantisation of the function space of the accompanying Lie group—may well be more appealing initially, especially to the physics community. However the two approaches remain rather disjoint, the connection between the two being elusive, the reader only having claims of their equivalence in [7].

As discussed in Drinfel'd [6], the motivation for introducing the one parameter deformation of the universal enveloping algebra comes from the classical isomorphism between the function space of the (connected) Lie group G and the

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universal enveloping algebra of the associated Lie algebra g[1]:

$$Ug' \cong \mathrm{Fun}(G),$$

i.e. any function on the group G defines a linear mapping of the universal enveloping algebra Ug and vice versa. This isomorphism is between Hopf algebras. The subtleties of this isomorphism will be ignored in this paper, see [6]. Thus a deformation/quantisation[2] of the function space should also produce a deformation of the universal enveloping algebra. Hence on the quantum level we obtain the isomorphism [6, 7]:

$$U_q g' \cong \mathrm{Fun}_q(G) = \mathrm{Fun}(G_q), \tag{1}$$

where $U_q g$ denotes the quantised algebra, [6], $\mathrm{Fun}_q(G)$ the quantised functions on the Lie group G, [7], G_q the quantum formal group corresponding to G, [17] and $\mathrm{Fun}(G_q)$ the functions on G_q [17]. The quantised algebra, $U_q g$ and quantised function space, $\mathrm{Fun}_q(G)$ are defined as follows:

Quantised Algebra. We shall follow Drinfel'd [6] in defining the quantisation of a Lie algebra. For a general Lie algebra g, with a system of simple roots S, the quantised algebra $U_q g$ is an Hopf algebra over the ring $C[[h]]$, that is, a one parameter deformation of the universal enveloping algebra Ug. It is generated by $\{1, H_i, X_i^{\pm}\}$; three generators for each simple root α_i in S, the H_i corresponding to the coroots of g. These generators have the following Hopf structure:

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}, \quad [X_i^+, X_j^-] = \delta_{ij} \frac{\sinh\left(\frac{h}{2} H_i\right)}{\sinh\left(\frac{h}{2}\right)},$$

$$\Delta(H_i) = 1 \otimes H_i + H_i \otimes 1, \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{H_i/2} + q^{-H_i/2} \otimes X_i^{\pm}. \tag{2}$$

Here a_{ij} is the Cartan matrix of g. The generators $X_i^{\pm}$ are also required to satisfy the q-analogue Serre relations given in [6] and [10]. However by introducing generators for each root of g, the use of these Serre relations can be avoided [2, 14]. These quantisations are in fact quasi-triangular Hopf algebras [6], that is, there exists an universal R-matrix which will be denoted by R. This is an element of $U_q g \otimes U_q g$. The quantised algebra $U_q g$ as defined by Jimbo, [10] differs from the above construction in that it is an Hopf subalgebra of the above, only the combination $k_i = q^{H_i/2}$ occurring and not H_i itself. This Hopf algebra is only pseudo quasi-triangular [4, 6]. The approach of Faddeev et al. [7] reproduces the quantised algebra as defined by Jimbo [10].

The quantum group corresponding to the quantised algebra $U_q g$ is defined as

[1] A prime denotes the linear dual (Hopf). Fun is used to denote the C^∞ functions.

[2] The deformation parameter is called Planck's constant in analogy with the quantisation of classical mechanics, and denoted h. The quantity $q = e^{h/2}$ is found to be useful

the Hopf dual[3] to $U_q g$, this definition being natural in the framework of pseudo-groups as defined by Woronowicz [16]. From relation (1), the quantum group is $\mathrm{Fun}_q(G)$.

Quantised Function Spaces. Contrast the above definition of quantised algebras and quantum groups with that of Faddeev et al. [7] and Woronowicz [16]. This construction assumes the existence of a matrix R (not to be confused with the universal R-matrix), that is valued in $\mathrm{End}(V\otimes V)$, where V is some n dimensional vector space over the ring $C[[h]]$. R is assumed to satisfy the Quantum Yang Baxter equation (QYBE):

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{3}$$

An Hopf algebra $A(R)$ is then defined with generators $\{1,t_{ij}\}$ satisfying the following relations:

$$\begin{aligned} RT_1T_2 &= T_2T_1R, \\ \Delta(t_{ij}) &= \sum_k t_{ik}\otimes t_{kj}. \end{aligned} \tag{4}$$

Here the matrix T is a matrix of generators: $(T)_{ij}=t_{ij}$. For this algebra, the QYBE (3) corresponds to an associativity condition. For R in the fundamental representation of $U_q sl(n)$, the Hopf algebra $A(R)$ can be considered as the quantised function space of $Gl(n)$, i.e. $\mathrm{Fun}_q(Gl(n))$, [7]. An application of a quantum determinant condition reduces this to $\mathrm{Fun}_q(Sl(n))$.

From the algebra $A(R)$, an Hopf algebra $U(R)$ is defined, a subalgebra of the dual to $A(R)$. $U(R)$ is generated by $\{1,l_{ij}^{(\pm)}\}$, which are defined by the following evaluations:

$$(L^{(\pm)},T_1\dots T_k)=R_1^{(\pm)}\dots .R_k^{(\pm)}, \tag{5}$$

where the matrix of generators $L^{(\pm)}$ is defined as $(L^{(\pm)})_{ij}=l_{ij}^{(\pm)}$. The two matrices $R^{(\pm)}$ are: $R^{(+)}=PRP, R^{(-)}=R^{-1}$, with P being the transposition matrix on the two factors $V\otimes V$. In fact these generators do not freely generate $U(R)$, see [7] and Sect. 4. By manipulating duality and the evaluation structure in (5), it can be shown that the Hopf structure:

$$\begin{aligned} R_{21}L_1^{(\pm)}L_2^{(\pm)} &= L_2^{(\pm)}L_1^{(\pm)}R_{21}, \quad R_{21}L_1^{(+)}L_2^{(-)}=L_2^{(-)}L_1^{(+)}R_{21}, \\ \Delta(l_{ij}^{(\pm)}) &= \sum_k l_{ik}^{(\pm)}\otimes l_{kj}^{(\pm)}, \end{aligned} \tag{6}$$

is obtained for the generators $l_{ij}^{(\pm)}$ [7].

From the isomorphism (1), the Hopf algebra $U(R)$ should be the quantised algebra $U_q sl(n)$, if R is the R-matrix for $U_q sl(n)$ in the fundamental representation. In this paper this is proved by obtaining an explicit isomorphism between the

[3] Due to the problems of dualising a tensor product, the dual to $U_q g$ is not necessarily an Hopf algebra. We define the dual Hopf algebra in terms of the Hopf structure induced on a dual basis of the associated ring module

generators $\{l_{ij}^{(\pm)}\}$ and $\{q^{\pm H_k/2}, X_l^{\pm}\}$, i.e. $U(R)$ is the quantised algebra $U_q sl(n)$ as defined by Jimbo [10].

The essential observation of our construction is the fact that the Hopf structure of the T generators (4) follows from a representation of the quantised algebra $U_q g$, Sect. 1. On observing that the Hopf structure of the algebra $U(R)$, (6) is very similar to that in (4), it may be expected that this will follow from a representation of the dual to $U_q g$. In fact two representations are required, one for each of the matrices $L^{(\pm)}$. To reproduce the structure in (6), specific representations need to be chosen. These are chosen in Sect. 3 by observing that the matrices $L^{(\pm)}$ can be expressed in terms of the universal R-matrix of $U_q g$ (15), (16):

$$L_{ij}^{(+)} = \mathrm{Id} \otimes \rho_{ij}(R_{U_q g}) \in U_q b_+, \quad L_{ij}^{(-)} = \rho_{ij} \otimes \mathrm{Id}\,(R_{U_q g}^{-1}) \in U_q b_-.$$

The Hopf structure satisfied by these generators, (6) follows from the representations used, this being derived in Sects. 3 and 7; the mixed relation in $L^{(\pm)}$ being derived from the dual of the quantum double in Sect. 7. On restricting to the Lie algebra $sl(n)$, we prove that $\{t_{ij}\}$ and $\{l_{ij}^{\pm}\}$ do not generate $\mathrm{Fun}_q(Sl(n))$ and $U_q sl(n)$ freely, quantum determinant relations holding on the matrices $T, L^{(\pm)}$, and a diagonal relation between $L^{(\pm)}$. Section 6 illustrates the construction with $U_q sl(2)$. The dual to the quantum double is introduced in Sect. 7, this producing the possibility of constructing the quantum double from the quantum group $U_q g'$. Section 8 encodes the Hopf structure of the quantised algebras $U_q sl(n)$ into a matrix format by employing the construction of Sect. 3 and the universal R-matrices of $U_q sl(n)$, as derived in [2, 14]. This also allows a systematic construction of all the commutation relations of the generators of $U_q sl(n)$ as used in [2].

In the following sections, the notation $l_{ij}^{(\pm)}$ will not be employed for the generators of $U_q g$ in the matrix formulation. The symbol $\sigma_{ij}^{\pm}$ is preferred.

1. Algebra Representation Structures of Hope Algebras

In this section an Hopf subalgebra of the dual to an Hopf algebra A will be defined via an algebra representation of A. The commutation relations are expressed in a matrix form by defining a matrix valued in the dual to A. The resulting Hopf algebra is identical to that used to define a quantum group in [7].

Consider a quasi-triangular Hopf algebra (A, R) [6] consisting of an Hopf algebra A over the ring $K = C[[h]]$, and an universal R-matrix $R \in A \otimes A$ that relates the two coalgebra maps Δ and $T \circ \Delta$:

$$T \circ \Delta(a) R = R \Delta(a), \quad \forall a \in A. \tag{7}$$

Let there be an algebra representation of A in an n dimensional K-module V, $\rho: A \to \mathrm{End}(V) \cong \mathrm{Mat}(n, K)$. The individual matrix elements ρ_{ij} define a mapping $\rho_{ij}: A \to K$ via the evaluation $a \to (\rho(a))_{ij}$, and hence $\rho_{ij} \in A'$, the dual of A. The ρ_{ij} will generate an Hopf subalgebra of A' which will be denoted by $A(R)$, following [7]. The Hopf structure of $A(R)$ is induced from the Hopf structure of A by duality, the coalgebra being given by:

$$(\Delta \rho_{ij}, a \otimes b) \underset{\mathrm{def}}{=} (\rho_{ij}, ab) = \sum_k \rho_{ik}(a) \rho_{kj}(b).$$

The fact that ρ is an algebra representation is necessary here, in order to expand the product of two elements of A. Since a, b are arbitrary, we deduce that:

$$\Delta\rho_{ij} = \sum_k \rho_{ik} \otimes \rho_{kj}. \tag{8}$$

To derive the algebra structure, we shall exploit the fact that A is quasi-triangular. From (7) the universal R-matrix relates the two coalgebras of A, the very structures that induce the commutation relations of the dual:

$$\rho_{ij}\rho_{kl}(a) \underset{\text{def}}{=} \rho_{ij} \otimes \rho_{kl}(\Delta a) = \rho_{kl} \otimes \rho_{ij}(T \circ \Delta a) = \rho_{kl} \otimes \rho_{ij}(R\Delta a R^{-1}).$$

Now use the coalgebra structure (8) to expand the multiplications:

$$\begin{aligned}\rho_{ij}\rho_{kl}(a) &= \rho_{ka} \otimes \rho_{ic} \otimes \rho_{ab} \otimes \rho_{cd} \otimes \rho_{bl} \otimes \rho_{dj}(R \otimes \Delta a \otimes R^{-1}) \\ &= (R^\rho)_{ka,ic}\rho_{ab}\rho_{cd}(a)(R^\rho)^{-1}_{bl,dj},\end{aligned}$$

where $R^\rho = \rho \otimes \rho(R)$, an $n^2 \times n^2$ matrix of End $(V \otimes V)$. A summation on repeated indices is implied. This simplifies on defining a matrix ρ which is valued in the dual, $\rho \in \mathrm{Mat}(n, A')$ by $(\rho)_{ij} = \rho_{ij}$. Then in an obvious notation, [7] we obtain:

$$R^\rho \rho_1 \rho_2 = \rho_2 \rho_1 R^\rho. \tag{9}$$

The coalgebra can be expressed in the form $\Delta(\rho_1) = \rho_1 \otimes \rho_1 \in \mathrm{Mat}(n, A' \otimes A')$, the subscript labelling the endomorphism space.

Consider the case when the Hopf algebra A is the quantised Lie algebra $U_q sl(n)$ and the representation ρ is the fundamental representation. From the Peter and Weyl theorem, the fundamental representation generates a dense subspace of the function space Fun $(Sl(n))$, with a determinant constraint on the generators. Thus we can define the quantised function space $\mathrm{Fun}_q(Gl(n))$ as the bialgebra $A(R)$, and $\mathrm{Fun}_q(Sl(n))$ as the quotient Hopf algebra, defined by introducing a quantum determinant constraint [6, 7, 13]. We denote the generators of $\mathrm{Fun}_q(Gl(n))$ by $\{1, t_{ij}\}$, reserving ρ for the representation.

2. Borel Subalgebras and the Quantum Double

The Borel Hopf subalgebras of the quantised algebras $U_q g$ will be discussed in this section, emphasising the various isomorphisms between the Borel subalgebras and their duals. This analysis introduces the important observation that the Hopf algebra[4] $U_q b^\circ_+$ used in the construction of the quantum double $D(U_q b_+)$, [6] can be realised as the Hopf algebra $U_q b'_-$. This essentially fixes the coalgebra anti-isomorphism $U_q b'_+ \to U_q b^\circ_+$ used in the quantum double construction [6]. Hence the quantum double $D(U_q b_+)$ is isomorphic, as a $C[[h]]$-module, to the tensor product $U_q b_+ \otimes U_q b'_-$.

First consider the following theorem that relates Hopf subalgebras and biideals[5]:

[4] Here A° denotes the dual Hopf algebra A' to A, with the comultiplication reversed relative to that induced by duality

[5] A multiplicative ideal I that is also a coideal, i.e. $\Delta(I) \subset I \otimes A + A \otimes I$

Theorem 1. *Given an Hopf subalgebra B of an Hopf algebra A over the ring K, then the annihilator space*:

$$B^{\perp}=\{\chi\in A':\chi(B)=0\}\subset A'$$

is an ideal and coideal with quotient:

$$\frac{A'}{B^{\perp}}\cong B'.$$

Proof. The annihilator space $B^{\perp}$ is obviously a K-submodule. The ideal structure of $B^{\perp}$ is considered first. Let $\chi\in B^{\perp}$, $\zeta\in A'$, then:

$$\chi\cdot\zeta(B)=\chi\otimes\zeta(\Delta B)=0 \quad \text{since} \quad \Delta B\subset B\otimes B.$$

Hence $\chi\cdot\zeta\in B^{\perp}$. Similarly for the reversed product $\zeta\cdot\chi$. So $B^{\perp}$ is an ideal. It is required to show that for $\chi\in B^{\perp}$, $\Delta(\chi)\in B^{\perp}\otimes A'+A'\otimes B^{\perp}$. So it is sufficient to evaluate this on $a\otimes b$, for $a,b\in B$. This is zero as required since B is a subalgebra, i.e. $a\cdot b\in B$. Since $B^{\perp}$ is a biideal, the quotient is well defined as an Hopf algebra. This can be proved to be isomorphic to the dual Hopf algebra B' by considering the evaluations on B.

The alternative situation where B is an Hopf ideal of A also follows by similar considerations.

This theorem can be applied to the quantised Lie algebras $U_q g$. The Hopf subalgebras of most interest are the Borel subalgebras. The Borel subalgebras are denoted $U_q b_{\pm}$, and are generated by $\{1,H_i,X_i^{\pm}\}$ respectively. They induce the following quotient structure on the dual:

$$U_q g' \xrightarrow{\pi^+} \frac{U_q g'}{U_q b_+^{\perp}} \cong U_q b'_+ \cong U_q b_+,$$

$$U_q g' \xrightarrow{\pi^-} \frac{U_q g'}{U_q b_-^{\perp}} \cong U_q b'_- \cong U_q b_-. \tag{10}$$

This quotient structure implies that if $U_q g'$ is quasi-triangular, then the Borel subalgebras $U_q b_{\pm}$ are also quasi-triangular. However, the Borel subalgebras are only pseudo quasi-triangular, i.e. there is an universal R-matrix but it is valued in an embedding Hopf algebra, $U_q g$ for example, [4, 6]. Thus we deduce that $U_q g'$ is not quasi-triangular. This has important implications in Sect. 3. Similar reasoning applies to the dual of the quantum double, Sect. 7, since the Hopf subalgebras $U_q b_+\subset D(U_q b_+)$, $U_q b'_-\subset D(U_q b_+)$ induce a similar quotient structure to that in (10).

Consider a general algebra anti-isomorphism, coalgebra isomorphism ϑ of $U_q g$ that interchanges the Borel subalgebras:

$$\vartheta: U_q b_{\pm}\to U_q b_{\mp}.$$

For example: $H_i\to H_i, X_i^{\pm}\to X_i^{\mp}$. On the dual Hopf algebra, the morphism ϑ induces a coalgebra anti-isomorphism, thus reversing the comultiplication:

$$\vartheta': U_q g'\to U_q g^{\circ}, \quad \vartheta': U_q b_{\pm}^{\perp}\cong U_q b_{\mp}^{\perp},$$

where: $\vartheta'\zeta(a)=\zeta(\vartheta a)$ for all $a\in U_qg$, $\zeta\in U_qg'$. On taking the quotient with $\pi^{\pm}$ we induce an Hopf isomorphism (anti-coalgebra) $\vartheta':U_qb'_{\pm}\to U_qb'_{\mp}$. Hence $U_qb^{\circ}_{+}\cong U_qb'_{-}$ as Hopf algebras. Thus the two Hopf subalgebras of the quantum double $D(U_qb_+)$ can be taken to be U_qb_+ and $\vartheta'(U_qb'_+)=U_qb'_-$. Hence, introducing the dual bases $\{\zeta_s\}\in U_qb_+,\{\eta^s\}\in U_qb'_+$, a basis for the quantum double $D(U_qb_+)$ is $\{\zeta_s\otimes\vartheta'(\eta^t)\}$, [6] with the universal R-matrix given by the canonical element of $U_qb_+\otimes U_qb'_-$, [6]:

$$R=\sum_s \zeta_s\otimes\vartheta'(\eta^s). \tag{11}$$

The inverse of the universal R-matrix is, [4]:

$$R^{-1}=S\otimes 1(R)=1\otimes S_{\circ}(R). \tag{12}$$

The symbols S and $S_{\circ}$ denote the antipode and skew-antipode respectively [1].

In the rest of this paper, the two maps ϑ,ϑ' will be understood as restricted to the Borel subalgebras:

$$\vartheta:U_qb_-\to U_qb_+,\quad \vartheta':U_qb'_+\to U_qb'_-. \tag{13}$$

Recall that the universal R-matrix for U_qg is obtained via a quotient mapping $\pi:D(U_qb_+)\to U_qg$ [6], that can be taken to satisfy $\pi|_{U_qb_+}=1$ [4]. The resultant R-matrix for U_qg is independent of the isomorphism ϑ', a change in ϑ' being compensated by a change in the quotient map π.

3. Representations of the Dual U_qg'

It is desired to reformulate the Hopf structure of the quantised algebra U_qg into a matrix form similar to that achieved for the quantum group in Sect. 1. Hence a representation of the quantum group U_qg' is required. This guarantees the coalgebra relation (8). However the formulation of the algebra into a matrix equation similar to (9) cannot be accomplished until an universal R-matrix is introduced. Since U_qg' does not possess an R-matrix, Sect. 2, it is necessary to find an homomorphism of U_qg' into a quasi-triangular Hopf algebra. The chosen algebra is U_qg, this giving us compatibility with the matrix formulation of the quantum group U_qg' in Sect. 1.

Given a representation $\rho:U_qg\to\mathrm{End}(V,K)$ we can construct a representation of U_qg' (more strictly a representation of the QUE algebra equivalent, [6]) by taking the quotient to the Hopf algebras of (10) and using representations of $U_qb_{\pm}$. However, since the Hopf algebras $U_qb_{\pm}$ are only pseudo-quasi-triangular, [4] it is necessary to use a representation of U_qg in which the Hopf algebras $U_qb_{\pm}$ are embedded, this giving access to an universal R-matrix. We define the two representations $\sigma^{\pm}=\rho\circ\phi^{\pm}\circ\pi^{\pm}$ of U_qg'. Here $\phi^{\pm}$ are algebra homomorphisms of $U_qb'_{\pm}$ into U_qg, with coalgebra properties to be determined. As before the individual matrix elements are elements of the dual, i.e. $\sigma^{\pm}_{ij}\in U_qg$. Since the maps $\sigma^{\pm}$ are representations we have the coalgebra structure (compare to (8)):

$$\Delta\sigma^{\pm}_{ij}=\sum_k\sigma^{\pm}_{ik}\otimes\sigma^{\pm}_{kj}.$$

To define the homomorphisms $\phi^{\pm}$, we observe that the generators $\sigma^{\pm}_{ij} \in U_q g$ can be expressed in a form reminiscent of the universal R-matrix of $U_q g$:

$$\sigma^{\pm}_{ij} = \sum_s \xi_s(\xi^s, \sigma^{\pm}_{ij}) = \sum_{\xi_s \in U_q b_\pm} \xi_s \rho_{ij} \circ \phi^{\pm} \circ \pi^{\pm}(\xi^s)$$

$$= \mathrm{Id} \otimes \rho_{ij} \circ \phi^{\pm} \left(\sum_{\xi_s \in U_q b_\pm} \xi_s \otimes \pi^{\pm} \xi^s \right). \tag{14}$$

Here the following dual bases have been defined:

$$\{\xi_s\} \in U_q g, \quad \{\xi^s\} \in U_q g',$$

for the index s in some suitable index set.

The projection operator $\pi^{\pm}$ reduces the sum over the basis of $U_q g'$ to that over $\mathrm{Im}(\pi^{\pm}) = U_q b'_{\pm}$, implying that $\sigma^{\pm}_{ij} \in U_q b_{\pm}$. In this expression the paired elements are dual: $\pi^{\pm} \xi^s(\xi_t) = \delta^s_t$. Thus the term in brackets is very similar to the universal R-matrix of $D(U_q b_+)$, (11). The representations are chosen to maximise this identification. If we choose $\phi^+ = \pi \circ \vartheta'$, we obtain from (11), (14):

$$\sigma^{+}_{ij} = \mathrm{Id} \otimes \rho_{ij}(R_{U_q g}) \in U_q b_+ . \tag{15}$$

Hence the representation ϕ^+ is equivalent to the sequence of maps:

$$U_q g' \xrightarrow{R} U_q g \xrightarrow{\rho} \mathrm{End}(V, K),$$

where the first map is given by evaluation on the first position of R.

The second representation σ^- is more difficult. It involves the algebra anti-homomorphism ϑ as follows:

$$U_q g' \xrightarrow{\pi^-} U_q b'_- \xrightarrow{S \circ \vartheta} U_q b_+ \xrightarrow{1} U_q g \xrightarrow{\rho} \mathrm{End}(V, K)$$

Fig. 1

i.e. $\phi^- = \pi \circ S \circ \vartheta$. The Hopf algebra $U_q b_+$ has been embedded in the quantum double, which allows us to use the quotient mapping $\pi: D(U_q b_+) \to U_q g$.This leads to the identity mapping in Fig. 1 since $\pi|_{U_q b_+} = 1$ [4]. The antipode has to be included to obtain an algebra homomorphism. The only other canonical anti-algebra homomorphism is the skew antipode. It is observed that σ^- is defined with the algebra anti-homomorphism ϑ (13) acting on the quotient Hopf algebra $U_q b'_-$ (10). However this Hopf algebra is isomorphic to $U_q b_-$. In fact, by using the dual to the quantum double, Sect. 7 the roles of $U_q b_-$ and $U_q b'_-$ are interchanged.

Since $(\vartheta'^{-1} \xi_s, \vartheta \xi^t) = (\xi_s, \xi^t) = \delta^t_s$, the matrix of generators σ^- can be verified to be (using (12), (14)):

$$\sigma^{-}_{ij} = \rho_{ij} \otimes \mathrm{Id}(R^{-1}_{U_q g}) \equiv \rho_{ij} \otimes S_\circ(R_{U_q g}) \in U_q b_- . \tag{16}$$

Note that the evaluation structure between the matrices t and σ is that of [7]:

$$(t, \sigma^+) = R^\rho, \qquad (\sigma^-, t) = R^{\rho^{-1}}. \tag{17}$$

The two homomorphisms $\sigma^\pm$ have now been constructed, with the embedding homomorphisms:

$$\phi^+ = \pi\circ\vartheta', \quad \phi^- = \pi\circ S\circ\vartheta. \tag{18}$$

Note that both $\phi^\pm$ are coalgebra anti-homomorphisms since they contain ϑ' and the antipode S respectively. The various homomorphisms can be summarised in the (non-commuting) diagram:

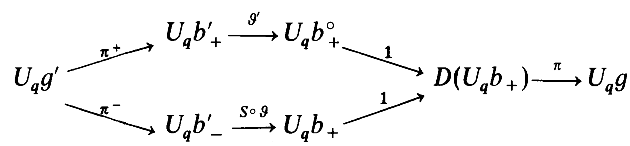

Fig. 2

All the maps are available from the quantum double construction. The map ϑ is arbitrary, it is only required to be an anti-algebra isomorphism and coalgebra isomorphism such that it exchanges the Borel subalgebras $\vartheta: U_q b_\pm \to U_q b_\mp$. The above combinations of maps are independent of ϑ up to the arbitrary isomorphism $\pi|_{U_q b_+}$. This is proved in Sect. 7. This can also be deduced from the expressions (15), (16). Note that since $U_q g'$ is not isomorphic to $U_q g$, any Hopf homomorphism $U_q g' \to U_q g$ has a non-zero kernel, which is a biideal. The possible biideals correspond to the following Hopf subalgebras of $U_q g$: $U_q h$, $U_q \tilde{g}$ ($\tilde{g}$ subalgebra of g) and the Borel subalgebras (of g and $\tilde{g}$). The Borel subalgebras $U_q b_\pm$ produce the maximal subalgebras of $U_q g$ that can be generated by the above construction involving representations.

The multiplication structure satisfied by the generators $\sigma^\pm$ can now be deduced. We can proceed in two ways. Using the definitions (15) and (16) in terms of the universal R-matrix, we can use the QYBE equation to derive (6). This is carried out in [3]. Alternatively these can be derived totally in terms of the representation structure of $\sigma^\pm$. This course is pursued here. Hence by duality we have:

$$(\sigma^\pm_{ij}\sigma^\pm_{kl},\zeta) = (\sigma^\pm_{ij}\otimes\sigma^\pm_{kl},\Delta\zeta) = (\rho_{ij}\otimes\rho_{kl}, T\circ\Delta(\phi^\pm\circ\pi^\pm\zeta)) \quad \forall\zeta\in U_q g'.$$

Since $\phi^\pm\circ\pi^\pm\zeta\in U_q g$, the existence of an R-matrix may now be used to deduce a matrix form of the algebra structure. The calculation is identical to that in Sect. 1, with the inverse of the R-matrix occurring where the R-matrix was used before. The Hopf structure of the generators $\sigma^\pm_{ij}\in U_q b_\pm$ can now be expressed as:

$$\Delta\sigma^\pm_{ij} = \sum_k \sigma^\pm_{ik}\otimes\sigma^\pm_{kj}, \quad R^\rho_{21}\sigma^\pm_1\sigma^\pm_2 = \sigma^\pm_2\sigma^\pm_1 R^\rho_{21}. \tag{19}$$

If the fundamental representation of $U_q sl(n)$ is used, then the Borel subalgebras $U_q b_\pm$ are generated[6] by $\{1, \sigma^\pm_{ij}\}$. This follows from (15), (16) since the fundamental representation is faithful on the Chevalley generators, and the universal R-matrix of $D(U_q b_+)$ is the canonical element of $U_q b_+ \otimes U_q b'_-$, [6]. The Borel subalgebras

[6] As defined by Jimbo [10], i.e. only the combination $k_i = q^{H_i/2}$ occurring

are generated as QUE algebras [6], since in the classical limit $\sigma^{\pm} \to I + h\eta^{\pm}$, where $\eta^{\pm}$ are matrices of generators for the Lie algebra $sl(n)$.

The commutation relations between the two sets of generators $\sigma^{\pm}_{ij}$ still has to be derived. These are more difficult than the calculation of those within the two sets, and their form depends upon the choice of the representations $\sigma^{\pm} = \rho \circ \phi^{\pm} \circ \pi^{\pm}$. Compare with the previous result where $R^{\rho}_{21}\sigma^{\pm}_1\sigma^{\pm}_2 = \sigma^{\pm}_2\sigma^{\pm}_1 R^{\rho}_{21}$ depends only on the morphisms $\phi^{\pm}$ (18) being algebra homomorphisms, coalgebra anti-homomorphisms. The following result is proved in Sect. 7:

$$R^{\rho}_{21}\sigma^{+}_1\sigma^{-}_2 = \sigma^{-}_2\sigma^{+}_1 R^{\rho}_{21}, \quad \text{where} \quad R^{\rho}_{21} = \rho \otimes \rho(T \circ R). \tag{20}$$

The equations in (19), (20) are the Hopf structure given in [7], for $U(R)$ (6). The duality structure in (5) is also reproduced from (17) and the coalgebra relations in (19).

4. Quantum Determinants and Constraints on Generators

In this section we restrict our analysis to the quantised algebras $U_q sl(n)$ with ρ its fundamental representation (42). We consider the definitions of the generators $\{t_{ij}\}$ and $\{\sigma^{\pm}_{ij}\}$, Sect. 1, (15), (16) and the constraints on these generators that are imposed due to the form of the universal R-matrix and the quantum determinant condition on the representation ρ. The occurrence of the constraints in the quantised function approach of [7] through the definition (5) is also considered, ultimately being a consequence of the algebraic relations satisfied by the R-matrix in the fundamental representation, (23). The determinant constraint on the quantum group must be imposed to obtain the correct duality. For the fundamental representation (42), ρ satisfies no further conditions; hence $A(R)$ and $U(R)$ are dual, [7].

The diagonal parts of σ^{+} and σ^{-} are only affected by the coroot prefactor of the universal R-matrix; this having a form [6, 2, 14]:

$$\exp\left(\frac{h}{2}\sum_{ij} a^{-1}_{ij} H_i \otimes H_j\right).$$

Since the diagonal parts (15), (16) only differ by an antipodal action, we obtain the constraint [7]: $\sigma^{-}_{ii} = \sigma^{+\,-1}_{ii}$, $i \in [1, n]$, since $S(H_i) = -H_i$.

All other constraints are consequences of the quantum determinant condition satisfied by the representation ρ. The quantum determinant can be expressed in terms of a quantum Levi-Civita symbol:

$$\varepsilon^{q}_{ijk\ldots} = \begin{cases} (-q^{-1})^{l(i,j,k..)} & \{i,j,..\} \in S_n \\ 0 & \text{otherwise.} \end{cases}$$

Here $\{ijk..\}$ denotes a permutation of the integers $1 \ldots n$, S_n the permutation group on n objects, and $l(i,j,k..)$ the length of the permutation, i.e. the number of inversions to reach $\{1,2,3..\}$. For example, if $n = 3$ then $l(3,2,1) = 3$.

The quantum determinant is expressed in the form:

$$\varepsilon^{q}_{i_1 i_2 \ldots i_n} t_{j_1 i_1} t_{j_2 i_2} \ldots t_{j_n i_n} = \varepsilon^{q}_{j_1 j_2 \ldots j_n} \det_q(t),$$

or

$$t_1 t_2 \ldots t_n \Omega = \det_q(t)\Omega, \tag{21}$$

by defining a vector Ω in $V^{\otimes(n)}$: $\Omega_{i_1 i_2 \ldots i_n} = \varepsilon^q_{i_1 i_2 \ldots i_n}$. We note that this definition for the quantum determinant is specific to the Hopf structure in (4) with the $U_q sl(n)$ R-matrix in the fundamental representation, (42), i.e. R-matrix (51). The appropriate definition for other representations and quantum groups are most easily treated using the comodule structures of [12, 13]. These will not be discussed.

Consider the bialgebra $\mathrm{Fun}_q(Gl(n))$ generated by $\{1, t_{ij}\}$ satisfying (4) with the matrix R in the fundamental representation of $U_q sl(n)$. Then the quantum determinant, as defined by (21), has the properties:

1. Group like [1]:

$$\Delta(\det_q(t)) = \det_q(t) \otimes \det_q(t) \in \mathrm{Fun}_q(Gl(n)) \otimes \mathrm{Fun}_q(Gl(n)).$$

2. Multiplicative, for commuting generators; $s_1 t_2 = t_2 s_1$:

$$\det_q(s \cdot t) = \det_q(s) \cdot \det_q(t) \in \mathrm{Fun}_q(Gl(n)) \otimes \mathrm{Fun}_q(Gl(n)).$$

3. Invariant under transposition:

$$\det_q(t^T) = \det_q(t).$$

4. Lies in the centre of the bialgebra $\mathrm{Fun}_q(Gl(n))$, i.e.:

$$[\det_q(t), t_{ij}] = 0.$$

Note that the matrices $s \cdot t$ and t^T satisfy Eq. (4) as required for the quantum determinant to be well defined. Properties 1 through 3 follow from the definition (21). Property 4 is a consequence of the quantum determinant condition on the representation ρ used for the R-matrix in (4):

$$\det_q(\rho) = 1. \tag{22}$$

This is the quantum generalisation of the determinant condition for a representation of $Sl(n)$. Note that ρ satisfies the algebra (4), Sect. 1. Equation (22) follows from observation [6], using (21), (42).

Relation (22) introduces four algebraic relations on the R-matrix in the fundamental representation. For example, consider the combination:

$$\begin{aligned} R_{1,n+1} R_{2,n+1} \ldots R_{n,n+1} \Omega_{1 \ldots n} &= \rho^{\otimes(n)} \Omega_{1 \ldots n} \otimes \rho(\Delta^{(n-1)} \otimes \mathrm{Id}(R_{U_q sl(n)})) \\ &= \Omega_{1 \ldots n} \det_q(\rho) \otimes \rho(R_{U_q sl(n)}) = \Omega_{1 \ldots n} I_{n+1}. \end{aligned}$$

Or:

$$\varepsilon^q_{i_1 i_2 \ldots i_n} R_{j_1 i_1, k_1 k_2} R_{j_2 i_2, k_2 k_3} \ldots R_{j_{n-1} i_n, k_n k_{n+1}} = \varepsilon^q_{j_1 j_2 \ldots j_n} \delta_{k_1 k_{n+1}}. \tag{23}$$

Here we have used the duality between multiplication in $U_q sl(n)'$ and comultiplication in $U_q sl(n)$, and the quasi-triangular and counit properties of the universal R-matrix [6]:

$$\Delta \otimes \mathrm{Id}(R) = R^{13} R^{23}, \quad \varepsilon \otimes \mathrm{Id}(R) = 1.$$

There are three similar equations with contraction of the ε tensor on other indices.

All the determinant constraints of [7] are consequences of these algebraic relations and definition (5).

Consider the commutation of the quantum determinant with the t generators:

$$\begin{aligned}\det_q(t)t_{n+1}\Omega_{1..n} &= t_1t_2\dots t_nt_{n+1}\Omega_{1..n}\\ &=(R_{1,n+1}R_{2,n+1}\dots R_{n,n+1})^{-1}t_{n+1}t_1t_2\dots t_n\\ &\quad\cdot(R_{1,n+1}R_{2,n+1}\dots R_{n,n+1})\Omega_{1..n}\\ &=t_{n+1}\det_q(t)\Omega_{1..n}.\end{aligned}$$

The second line follows on using the algebra (4), and the last line from the eigenvector equation (23). Hence we deduce the commutivity: $[\det_q(t), t_{ij}]=0$. Since $\det_q(t)$ is also group-like, $\det_q(t)-1$ defines a biideal, and hence the quotient is well defined as an Hopf algebra. This is $\mathrm{Fun}_q(Sl(n))$. The necessity for taking this quotient in the work of Faddeev et al. [7] is a consequence of the evaluation structure in (5), and the algebraic relation (23):

$$\begin{aligned}(\det_q(t)\Omega_{1\dots n},\sigma^+_{n+1}) &= (t_1t_2\dots t_n\Omega_{1\dots n},\sigma^+_{n+1})\\ &=R_{1,n+1}R_{2,n+1}\dots R_{n,n+1}\Omega_{1..n}=\Omega_{1..n}I_{n+1}.\end{aligned}$$

Hence $(\det_q(t)-1,\sigma^+)=0$. Similarly for σ^-. Thus we must set $\det_q(t)=1$ to obtain the correct duality structure. Similarly for the quantum determinant constraints on $\sigma^\pm$:

$$\begin{aligned}&(\det_{q^{-1}}(\sigma^+)\Omega^{q^{-1}}_{1\dots n}, t_{n+1}t_{n+2}\dots t_{n+k})\\ &=(\sigma_1^+\sigma_2^+\dots\sigma_n^+\Omega^{q^{-1}}_{1\dots n}, t_{n+1}t_{n+2}\dots t_{n+k})\\ &=(\sigma_1^+\otimes\sigma_2^+\otimes\dots\otimes\sigma_n^+,\Delta^{(n-1)}(t_{n+1}t_{n+2}\dots t_{n+k}))\Omega^{q^{-1}}_{1\dots n}\\ &=(\sigma_1^+,t_{n+1}\dots t_{n+k})(\sigma_2^+,t_{n+1}\dots t_{n+k})\dots(\sigma_n^+,t_{n+1}\dots t_{n+k})\Omega^{q^{-1}}_{1\dots n}\\ &=(R_{n+1,1}R_{n+1,2}\dots R_{n+1,n})\dots(R_{n+k,1}R_{n+k,2}\dots R_{n+k,n})\Omega^{q^{-1}}_{1\dots n}\\ &=\Omega^{q^{-1}}_{1\dots n}I_{n+1..n+k}.\end{aligned}$$

Similarly for the matrix σ^-. Hence definition (5) imposes the determinant constraints. In contrast, these determinant constraints are transparent from the viewpoint of our construction, since the generators $\{t_{ij}\}$ and $\{\sigma^\pm_{ij}\}$ are defined in terms of the representation ρ, Sect. 1, (15), (16) respectively. Since $\{t_{ij}\}$ are defined directly from ρ, we obtain $\det_q(t)=1$. By definition $\sigma^\pm=\rho\circ\phi^\pm\circ\pi^\pm$; hence we deduce:

$$\begin{aligned}(\det_{q^{-1}}(\sigma^\pm),a) &= \sum_{\{i_s\}\in S_n}(-q)^{1(\{i_s\})}(\rho_{1i_1}\otimes\rho_{2i_2}\otimes\dots\otimes\rho_{ni_n},\quad(\phi^\pm\circ\pi^\pm)^{\otimes(n)}\Delta^{(n-1)}(a))\\ &=\left(\sum_{\{i_s\}\in S_n}(-q)^{1(\{i_s\})}\rho_{ni_n}\rho_{n-1i_{n-1}}\dots\rho_{1i_1},\phi^\pm\circ\pi^\pm(a)\right)\\ &=(\det_q(\rho),\phi^\pm\circ\pi^\pm(a)),\quad\forall a\in U_qg'.\end{aligned}$$

Hence:

$$\det_{q^{-1}}(\sigma^\pm)=\prod_{i=1}^n\sigma^\pm_{ii}=1, \tag{24}$$

using in the second expression the fact that $\sigma^{\pm}$ are triangular (valid for a representation taking the Borel subalgebras to triangular matrices). Notice that the determinant is defined with a q^{-1} relative to (21). This is consistent with the differences in the algebra relations in (9), (19) since $R^{\rho}(q)^{-1} = R^{\rho}(q^{-1})$.

Finally, note that the antipode is intimately related to the determinant constraints, an antipodal mapping not existing in $\mathrm{Fun}_q(Gl(n))$. However $\mathrm{Fun}_q(Sl(n))$ has an antipode, since the dual is an Hopf algebra. In the matrix format, the antipode is defined by [7]:

$$S(t)\cdot t = t\cdot S(t) = I. \tag{25}$$

This implies[7] that $\det_q(t)^{-1} = \det_q(S(t))$. However since the only invertible element in $\mathrm{Fun}_q(Gl(n))$ is the identity, we must quotient by $\det_q(t) = 1$ if an antipode is to exist. Equation (25) also implies that $S(t) = t_{co}^T$, a relation given in [7], where:

$$\begin{aligned}(t_{co})_{kj} &= (-q)^{k-1} \varepsilon^q_{ji_1..i_{m-1}} t_{1i_1} \cdots t_{k-1 i_{k-1}} t_{k+1 i_k} \cdots t_{m i_{m-1}} \\ &= (-q)^{k-j} \det_q t_{\hat{k}\hat{j}}.\end{aligned}$$

Here $t_{\hat{k}\hat{j}}$ denotes the comatrix of t, i.e. t with row k and column j removed.

5. Borel Structure Isomorphisms

In this section the matrix format (6) of the Hopf structure of the quantised algebra $U_q sl(n)$ will be discussed, demonstrating the incorporation of the Borel structure of $U_q sl(n)$ and the quotient structure of $U_q sl(n)'$ in the matrix formulation. The representation is assumed to take the Borel subalgebras $U_q b_{\pm}$ to upper and lower triangular matrices respectively.

The generators $\sigma^+_{ij}, \sigma^-_{ij}$ generate isomorphic subalgebras. This is demonstrated by matrix transposition. Since σ^+ and σ^- are lower and upper triangular, transposing is expected to produce some type of isomorphism. We have $R_{12}^{T_1 T_2} = R_{21}$, where T_i is transposition in position i; hence:

$$(R_{21}\sigma_1^- \sigma_2^-)^{T_1 T_2} = \sigma_1^{-T} \sigma_2^{-T} R_{12} = R_{12} \sigma_2^{-T} \sigma_1^{-T}$$

or

$$R_{21} \sigma_1^{-T} \sigma_2^{-T} = \sigma_2^{-T} \sigma_1^{-T} R_{21}.$$

Since the determinant conditions, (24) are also invariant under transposition, matrix transposition induces an algebra isomorphism $\sigma^{-T} \to \sigma^+$. By considering the effect of transposing the coalgebra relation $\Delta\sigma^-_{ij} = \sum_k \sigma^-_{ik} \otimes \sigma^-_{kj}$, it can be proved that this is also a coalgebra anti-isomorphism. Due to this isomorphism, only the commutation relations for σ^+ need to be evaluated. The algebra relation $R_{21}\sigma_1^+ \sigma_2^- = \sigma_2^- \sigma_1^+ R_{21}$ is invariant under matrix transposition; hence the commutation relations must be invariant under the transformation of generators implied by $\sigma^{-T} \leftrightarrow \sigma^+$. This transformation corresponds to the algebra isomorphism, coalgebra anti-isomorphism $H_i \to -H_i$, $X_i^{\pm} \to -q^{\pm 1} X_i^{\mp}$.

[7] A proof of this fact is not as obvious as it would appear from (25), since $S(t)$ and t do not commute

By Theorem 1, we know that the Borel subalgebras induce a quotient structure on the quantum group, as given in (10). This structure can be demonstrated in the matrix format. The strictly lower triangular terms of the t matrix generate the biideal $U_q b_+^\perp$ (valid for any representation taking $U_q b_+$ to upper triangular matrices). The biideal property can also be deduced from the form of the R-matrix (51), and the coalgebra relation in (4). Taking the quotient with $U_q b_+^\perp$ sets the strictly lower triangular terms to zero. By transposing and acting with the antipode, (or skew antipode) we obtain the σ^+ Hopf structure (6), i.e. $U_q b'_+, U_q b_+$ are isomorphic as Hopf algebras (the homomorphism involves two coalgebra anti-homomorphisms). Similar reasoning applies to the strictly upper triangular terms.

6. An Example: $U_q sl(2)$

The quantised algebra $U_q sl(2)$ is generated by the generators $\{1, H, X^+, X^-\}$ with the Hopf structure given in (2). In this example we shall construct the quantum group and express the commutation relations of $U_q sl(2)$ in the matrix form (6). This involves constructing the matrices $\sigma^\pm \in \mathrm{Mat}(n, U_q b_\pm)$ (15), (16). The fundamental representation will be used [6]:

$$\rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(X^+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(X^-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The universal R-matrix has a form [6]:

$$R_{U_q sl(2)} = \exp\left(\frac{h}{4} H \otimes H\right)(1 + (1 - q^{-2}) q^{H/2} X^+ \otimes q^{-H/2} X^- + \ldots..). \tag{26}$$

The dots indicate irrelevant terms since they project to zero under the representation; $\rho(X^{\pm n}) = 0$ for $n > 1$. The universal R-matrix in this representation can be proved to be:

$$R^\rho = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & (q - q^{-1}) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{27}$$

This differs from the expression in [7] due to a different choice of representation, see (29). From Sect. 1 we have the following Hopf structure for the generators t_{ij} of the quantum group:

$$R^\rho t_1 t_2 = t_2 t_1 R^\rho, \quad \Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}.$$

Expressing the t matrix as: $t = \begin{pmatrix} k & W^+ \\ W^- & k' \end{pmatrix}$, the Hopf structure reads:

$$\begin{aligned} [k', k] &= (q - q^{-1}) W^+ W^-, \\ k W^\pm &= q^{-1} W^\pm k, \quad k' W^\pm = q W^\pm k', \\ W^+ W^- &= W^- W^+. \end{aligned}$$

$$\begin{aligned}
\Delta k &= k \otimes k + W^+ \otimes W^-, \\
\Delta k' &= k' \otimes k' + W^- \otimes W^+, \\
\Delta W^+ &= W^+ \otimes k' + k \otimes W^+, \\
\Delta W^- &= W^- \otimes k + k' \otimes W^-.
\end{aligned}$$

Compare this to the $U_q sl(2)$ Hopf structure in (2). The quantised function space $\mathrm{Fun}_q(Gl(2))$ is generated by $\{1, t_{ij}\}$, and the quantum determinant constraint quotients this to the Hopf algebra $\mathrm{Fun}_q(Sl(2))$: $\det_q(t) = kk' - q^{-1}W^+W^- = 1$.

The Hopf ideals $U_q b_\pm^\perp$ can now be constructed. Observe that the following elements generate the Borel ideals:

$$\begin{aligned}
&k \cdot k' - 1,\ k' \cdot k - 1,\ W^- \in U_q b_+^\perp, \\
&k \cdot k' - 1,\ k' \cdot k - 1,\ W^+ \in U_q b_-^\perp.
\end{aligned}$$

Hence the following quotient structure is obtained for $U_q b'_+$:

$$\begin{aligned}
k' &= k^{-1}, \\
kW^+ &= q^{-1} W^+ k, \\
\Delta k &= k \otimes k, \\
\Delta W^+ &= W^+ \otimes k^{-1} + k \otimes W^+.
\end{aligned}$$

This is isomorphic to the Hopf subalgebra $U_q b_+$ under the identification $k = q^{-H/2}$, $W^+ = X^+$. The combination k occurs instead of H, corresponding to the quantisation of the Lie algebra $sl(2)$ as defined in [9]. This isomorphism is transposition and antipodal action as discussed in Sect. 5. A similar analysis holds for $U_q b'_-$.

The matrices $\sigma^\pm$ will now be constructed from the definitions (15), (16). Using the universal R-matrix in (26), the following can be verified:

$$\sigma^+ = \begin{pmatrix} q^{H/2} & 0 \\ q^{-1/2}(q - q^{-1})X^+ & q^{-H/2} \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} q^{-H/2} & -q^{1/2}(q - q^{-1})X^- \\ 0 & q^{H/2} \end{pmatrix}. \tag{28}$$

The Hopf structure of $U_q sl(2)$ can be reproduced from the formulae (6), using (27). Observe that only the combination $q^{H/2}$ occurs and not H itself. It is interesting to observe that with this combination it is unnecessary to go to the enveloping algebra, since the commutation relations close. This is also true of $U_q sl(3)$, but not for the higher dimensional Lie algebras $U_q sl(n)$. This is due to the commutation relation, (52):

$$\mathrm{ad}_q e_\alpha \cdot e_\beta = e_\alpha e_\beta - e_\beta e_\alpha = (q - q^{-1}) e_\gamma e_{\bar{\alpha} + \beta},$$

which produces a product of generators. In this commutation relation the roots α, β are such that $\alpha + \beta$ is not a root and there exists a non-zero overlap of simple roots in α, β and no inclusion. Hence three or more simple roots are required. The notation is further explained in Sect. 8.

Instead of the fundamental representation ρ, any other representation may be used. Of greatest interest is the representation ρ' defined by:

$$\rho'(H) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho'(X^+) = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}, \quad \rho'(X^-) = \begin{pmatrix} 0 & \alpha^{-1} \\ 0 & 0 \end{pmatrix}, \tag{29}$$

where alpha is an arbitrary constant. This is related to the fundamental representation by $\rho' = \rho^T \circ S_\circ$ for the specific value $\alpha = -q^{-1}$. Notice that two algebra anti-homomorphisms are involved, transposition and the skew-antipode. A direct calculation of $\sigma^\pm$ is unnecessary since the results for ρ can be employed:

$$\sigma^{+\prime} = 1 \otimes \rho^T \circ S_\circ(R) = S \otimes \rho^T(R) = S(\sigma^+)^T.$$

Similarly $\sigma^{-\prime} = S(\sigma^-)^T$, and so the following matrices are obtained:

$$\sigma^{+\prime} = \begin{pmatrix} q^{-H/2} & -q^{3/2}(q-q^{-1})X^+ \\ 0 & q^{H/2} \end{pmatrix}, \quad \sigma^{-\prime} = \begin{pmatrix} q^{H/2} & 0 \\ q^{-3/2}(q-q^{-1})X^- & q^{-H/2} \end{pmatrix}.$$

The q factors can be removed by a renormalisation $X^\pm \to \beta^{\pm 1} X^\pm$, for some $\beta \in C[[h]]$. The R-matrix for this representation is:

$$R^{\rho'} = \rho^T \otimes \rho^T(R_{U_q sl(2)}) = \rho \otimes \rho(T \circ R),$$

since $S_\circ \otimes S_\circ(R_{U_q sl(2)}) = R_{U_q sl(2)}$.

This is the structure given in [7], under an appropriate change in normalisation of $X^\pm$.

7. The Dual to the Quantum Double

The dual to the quantum double is introduced in this section since there exists a basis of this Hopf algebra that is particularly suited to the quotient structure induced by the Borel subalgebras of $U_q g$. The dual to the quantum double is also important since it allows us to prove that the quantum double itself can be constructed from the dual algebra $U_q g'$. This establishes that the constructions of $\sigma^\pm$ are consistent, i.e. that they are independent of the arbitrary isomorphism ϑ'. The remaining commutation relation between σ^+ and σ^- (20) is derived in this section.

Recall that the quantum double $D(U_q b_+)$ has two Hopf subalgebras $U_q b_+, U_q b'_-$ and a quotient mapping to $U_q g$. If we denote the dual to the quantum double by $D'(U_q b_+)$, then Theorem 1 implies that $D'(U_q b_+)$ has the following structure:

Quotient Structure:

$$D'(U_q b_+) \xrightarrow{\pi^+} \frac{D'(U_q b_+)}{U_q b_+^\perp} \cong U_q b'_+ \cong U_q b_+,$$

$$D'(U_q b_+) \xrightarrow{\pi^-} \frac{D'(U_q b_+)}{U_q b'^\perp_-} \cong U_q b_-.$$

Hopf Subalgebra Structure:

$$U_q g' \subset D'(U_q b_+).$$

Here the quotient maps $\pi^\pm$ are understood as those corresponding to the ideals of the quantum double. These quotient homomorphisms will not be distinguished from those corresponding to $U_q g'$. The quotient structure of $U_q g'$, (10) is induced on restriction to the Hopf subalgebra $U_q g'$ of $D'(U_q b_+)$.

Note that due to the original isomorphism $U_q b_+ \otimes U_q b'_- \cong D(U_q b_+)$ (as a $C[[h]]$-module), Sect. 2, the projection π^- takes $D'(U_q b_+)$ to $U_q b_-$.

Define the following dual bases:

$$\{\zeta_s\} \in U_q b_+, \quad \{\eta^s\} \in U_q b'_+, \quad (\eta^s, \zeta_t) = \delta^s_t,$$
$$\{\eta_s\} \in U_q b_-, \quad \{\zeta^s\} \in U_q b'_-, \quad (\zeta^s, \eta_t) = \delta^s_t,$$

where

$$\eta_s = \vartheta^{-1}(\zeta_s), \quad \zeta^s = \vartheta'(\eta^s). \tag{30}$$

The quantum double $D(U_q b_+)$ has a basis $\{\zeta_t \zeta^s\}$[8], since it is isomorphic (as a $C[[h]]$-module) to the tensor product $U_q b_+ \otimes U_q b'_-$. $D'(U_q b_+)$ is isomorphic (as a $C[[h]]$-module) to the tensor product $U_q b'_+ \otimes U_q b_- \subset (U_q b_+ \otimes U_q b'_-)'$; hence a suitable basis for this algebra is $\{\eta^t \eta_s\}$, where the elements η^t, η_s have their definitions extended to:

$$\eta^u(\zeta_t \zeta^s) = \delta^u_t \delta^s_0, \quad \eta_p(\zeta_t \zeta^s) = \delta^0_t \delta^s_p.$$

We also note that $\eta^0 = \eta_0 = 1$ is the identity map. This definition is dual to the process of embedding $U_q b_+, U_q b'_-$ in the quantum double $D(U_q b_+)$ as Hopf subalgebras.

The chosen bases are dual, that is:

$$\eta^u \eta_v(\zeta_t \zeta^s) = \delta^u_t \delta^s_v.$$

In passing we note that the following Hopf structure holds in $D'(U_q b_+)$:

1. η_s, η^t have an identical algebra structure to the analogues in $U_q b_-, U_q b'_+$,
2. η_s, η^t commute,
3. the coalgebra structure of η_s, η^t is the same as that in $U_q b_-$, $U_q b'_+$ only on projection by the appropriate ideal.

Observe that the complications in the Hopf structure are now in the coalgebra, while those in the quantum double are in the algebra, (35) and [6].

The Hopf ideals can now be explicitly written down:

$$U_q b^{\perp}_+ = \{\eta^u \eta_v \in D'(U_q b_+) : u \geqq 0, v > 0\},$$
$$U_q b^{\perp}_- = \{\eta^u \eta_v \in D'(U_q b_+) : u > 0, v \geqq 0\}.$$

In particular: $\eta^s \notin U_q b^{\perp}_+, \eta_s \notin U_q b^{\perp}_-$. Thus the quotient structure is:

$$\pi^+(\eta^s \eta_t) = \delta_{0t} \eta^s, \quad \pi^-(\eta^s \eta_t) = \delta^{0s} \eta_t. \tag{31}$$

The following diagram is commutative:

$$\begin{array}{ccc} U_q b_+ & \xrightarrow{\ 1\ } & D(U_q b_+) \\ \downarrow{\scriptstyle \iota, \iota} & & \downarrow{\scriptstyle \iota} \\ U_q b'_+ & \xleftarrow{\ \pi^+\ } & D'(U_q b_+) \end{array}$$

[8] This is shorthand for the more clear expression $\{\zeta_t \otimes \zeta^s\}$

The map ι is a duality map, taking an element to its dual. For example, for the Borel Hopf algebra $U_q b_+$, $\iota:\zeta_s \to \eta^s$. Hence we obtain: $\iota\circ\pi^+(\eta^s)=\zeta_s$. But we also have: $\vartheta\circ\pi^-(\eta_s)=\zeta_s$ from (30) and (31). This implies that if we define an algebra anti-isomorphism on the dual $D'(U_q b_+)$ by:

$$\tilde{\imath}:\eta_s, \eta^t \to \eta^s, \eta_t$$

then the following diagram is commutative:

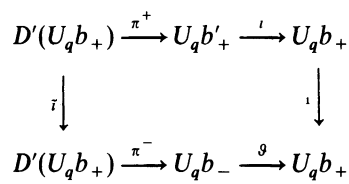

Fig. 3

Hence the following is equivalent to the quantum double construction:

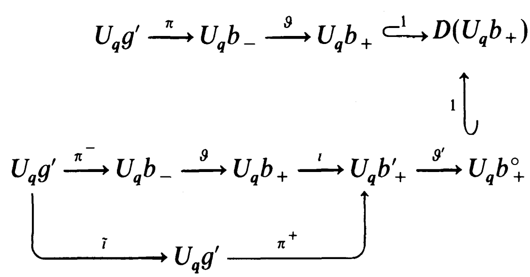

Fig. 4

The two mappings produce the dual bases $\{\vartheta\circ\pi^-(\eta_t)\}$, $\{\vartheta'\circ\pi^+\circ\tilde{\imath}(\eta_t)\}$ for the Hopf subalgebras $U_q b_+$ and $U_q b_+^\circ$ of $D(U_q b_+)$. Since the Hopf homomorphism $\pi:D(U_q b_+)\to U_q g$ is unique once $\pi|_{U_q b_+}$ is given, [4, 6] we deduce that the combinations $\pi_\vartheta\circ\vartheta\circ\pi^-$ and $\pi_\vartheta\circ\vartheta'\circ\pi^+$ are ϑ independent, (up to the arbitrary isomorphism $\pi|_{U_q b_+}$). The notation π_ϑ stresses the ϑ dependence of π. Recalling that $\sigma^+=\pi_\vartheta\circ\vartheta'\circ\pi^+$ and $\sigma^-=S\circ\pi_\vartheta\circ\vartheta\circ\pi^-$ implies that $\sigma^\pm$ are independent of ϑ, a conclusion also indicated in Eqs. (15), (16). This ϑ-independence (up to the arbitrary isomorphism $\pi|_{U_q b_+}$) is necessary for the construction to be reasonable, since ϑ is not intrinsic to the quantised algebra $U_q g$.

The algebra structure of the $\sigma^\pm$ generators will now be derived from $D'(U_q b_+)$. The representation ρ is extended to a representation of the quantum double $\rho:D(U_q b_+)\to \mathrm{End}(V,K)$ such that ρ quotients through $U_q g$. The starting point is the definition of multiplication in the dual:

$$\begin{aligned}(\sigma^\pm_{ij}\sigma^\pm_{kl},\eta) &= (\sigma^\pm_{ij}\otimes\sigma^\pm_{kl},\Delta\eta)=\rho_{ij}\otimes\rho_{kl}\circ\vartheta'\pi^+\otimes\vartheta'\pi^+(\Delta\eta), \quad (+)\\ &= \rho_{ij}\otimes\rho_{kl}\circ S\vartheta\pi^-\otimes S\vartheta\pi^-(\Delta\eta), \quad (-) \quad \forall\eta\in D'(U_q b_+).\end{aligned} \tag{32}$$

Hence the commutation structure of $\sigma^{\pm}$ can be obtained by relating $\Delta\eta$ and $T\circ\Delta\eta$ for a general element $\eta\in D'(U_q b_+)$. Since the action of the projectors $\pi^{\pm}$ is known on the base elements (31), we shall expand $\Delta\eta$ as:

$$\Delta\eta=(\Delta\eta,\zeta_q\zeta^p\otimes\zeta_t\zeta^s)\eta^q\eta_p\otimes\eta^t\eta_s. \tag{33}$$

By acting on the objects in (33) with the two homomorphisms $\vartheta'\circ\pi^+$, $S\circ\vartheta\circ\pi^-$, mapping $D'(U_q b_+)\rightarrow D(U_q b_+)$, we obtain:

$$\begin{aligned}\vartheta'\circ\pi^+\otimes\vartheta'\circ\pi^+(\Delta\eta)&=\eta(\zeta_q\zeta_t)\zeta^q\otimes\zeta^t=\eta(\zeta_v)m^v_{qt}\zeta^q\otimes\zeta^t,\\ \vartheta\circ\pi^-\otimes\vartheta\circ\pi^-(\Delta\eta)&=\eta(\zeta^p\zeta^s)\zeta_p\otimes\zeta_s=\eta(\zeta^u)\mu^{ps}_u\zeta_p\otimes\zeta_s,\\ \vartheta'\circ\pi^+\otimes S\circ\vartheta\circ\pi^-(\Delta\eta)&=\eta(\zeta_q\zeta^s)\zeta^q\otimes S(\zeta_s),\end{aligned} \tag{34}$$

because from (30), (31): $\vartheta'\circ\pi^+(\eta^s)=\zeta^s\in U_q b'_-$, $\vartheta\circ\pi^-(\eta_s)=\zeta_s\in U_q b_+$. The matrices m,μ are the matrices of multiplication and comultiplication for $U_q b_+$. Since $\Delta(\zeta^v)=m^v_{tq}\zeta^q\otimes\zeta^t$, $\Delta(\zeta_v)=\mu^{ps}_v\zeta_p\otimes\zeta_s$ (from duality), the first two relations can be manipulated into a coalgebra form:

$$\eta(\zeta_v)T\circ\Delta(\zeta^v),\quad \eta(\zeta^u)\Delta(\zeta_u).$$

Returning to (32), the universal R-matrix can now be used to obtain the results in (19). For the mixed algebra relation, (20) it is necessary to consider the commutation structure of the quantum double, since we are required to relate the elements $\zeta_q\zeta^s,\zeta^u\zeta_v$. The commutation relations of the quantum double, between elements of the Hopf subalgebras $U_q b_+$, and $U_q b^\circ_+$ can be expressed in terms of the matrices m, μ as [6]:

$$m^r_{up}\mu^{pv}_t\zeta^u\zeta_v=\mu^{qw}_t m^r_{ws}\zeta_q\zeta^s. \tag{35}$$

Hence we multiply the expression $\eta(\zeta_q\zeta^s)\zeta^q\otimes S(\zeta_s)$ in the third relation of (34) by appropriate terms in order to generate the m, μ matrices required in (35). So:

$$\begin{aligned}\eta(\zeta_q\zeta^s)\zeta^q\otimes S(\zeta_s)\cdot(\zeta^w\otimes S\zeta_w)&=\eta(\mu^{qw}_t m^r_{ws}\zeta_q\zeta^s)\zeta^t\otimes S(\zeta_r)\\ &=\eta(\zeta^u\zeta_v)\mu^{pv}_t\zeta^t\otimes m^r_{up}S(\zeta_r)\\ &=(\zeta^p\otimes S\zeta_p)\cdot\vartheta'\circ\pi^+\otimes S\circ\vartheta\circ\pi^-(T\circ\Delta\eta).\end{aligned}$$

Note that the occurrence of the antipode is necessary, such that the indices are in the correct order. On observing that $R^{-1}=S\otimes 1\left(\sum_s\zeta_s\otimes\zeta^s\right)$ (12), we return to (32) to obtain:

$$(\sigma^+_{ij}\sigma^-_{kl},\eta)=(\rho_{ij}\otimes\rho_{kl},T\circ R^{-1}(\vartheta'\circ\pi^+\otimes S\circ\vartheta\circ\pi^-(T\circ\Delta\eta))T\circ R).$$

Thus on rearranging as in Sect. 1, using the fact that ρ is a representation and that η is arbitrary, the structure:

$$R_{21}\sigma^+_1\sigma^-_2=\sigma^-_2\sigma^+_1R_{21},\quad R_{21}=\rho\otimes\rho(T\circ R),$$

is obtained, as quoted in (20).

8. The Quantised Lie Algebras A_n

The Universal R-matrices for the quantised Lie algebras[9] A_n are derived in [2] and [14]. Hence the Hopf structure of A_n can be expressed in the matrix form (6) by the construction of Sect. 3. In [2, 14] a system of generators for A_n based on the full root system of A_n is defined, this being contrasted to that of [6, 10] which only use the simple roots. This system of generators occurs as the matrix elements of $\sigma^{\pm}$, and the relations (6) allow all the commutation relations to be systematically derived. The fact that all the commutation relations are of the form of the adjoint structure, (36), as defined in [2], shows that this adjoint structure is sufficient to describe the algebra structure of A_n, and justifies the original definition.

The definitions of [2] will be summarised here for clarity:
Let $\boldsymbol{\Phi}^+$ denote the positive roots.

The positive roots are ordered by the length of the minimal word in the Weyl group needed to generate it from the end root of the Dynkin diagram; α_1. The length of the word for the root $\alpha = \sum_{s\in[i,j]} \alpha_s \in \boldsymbol{\Phi}^+, j \geqq i$, is $\mu(\alpha) = (j+i) - 2$. So $\alpha < \beta$ if $\mu(\alpha) < \mu(\beta)$.

The generators in each Borel subalgebra are ordered by the corresponding roots: $P_\alpha < O_\beta$ iff $\alpha < \beta, P_\alpha, O_\beta$ two arbitrary generators corresponding to the positive roots α, β.

Define the ordered products: $\prod^<, \prod^>$, where the $<$, $>$ denote an ascending and descending order of generators respectively, when read from left to right.

The adjoint map is defined by:

$$\mathrm{ad}_q P_\alpha \cdot O_\beta = P_\alpha O_\beta - q^{(\alpha,\beta)} O_\beta P_\alpha, \tag{36}$$

for $\alpha < \beta$, with the anti-symmetry condition $\mathrm{ad}_q P_\alpha \cdot O_\beta = -\mathrm{ad}_q O_\beta \cdot P_\alpha$. The generators e_i, f_i will be employed, the index i corresponding to the simple roots of g. These are related to the generators used in the introduction and most of the literature by [2]:

$$e_i = q^{H_i/2} X_i^+, f_i = q^{-H_i/2} X_i^- .$$

The following generators are then defined (for each positive root α) by:

$$e_\alpha = \prod_{s\in[i,j-1]}^< (\mathrm{ad}_q e_s) \cdot e_j, f_\alpha = \prod_{s\in[i+1j]}^> (\mathrm{ad}_q f_s) \cdot f_i. \tag{37}$$

The generators e_α, f_α corresponding to the root $\alpha = \sum_{s=i}^{j} \alpha_s$ (for $i \leqq j$, α_s simple) are denoted by e_{ij}, f_{ji} respectively. The order of the indices is suggestive of the definitions (37). Note that in this notation $e_{ii} = e_i, f_{ii} = f_i$. We shall also employ the definition $H_{ij} = \sum_{s=i}^{j} H_s$.

[9] With an abuse of notation, the symbol A_n will denote the quantised object corresponding to this Lie algebra, i.e. $U_q sl(n+1)$

The universal R-matrix for A_n takes the form [2]:

$$R_{A_n}=\exp\left(\frac{h}{2}\sum_{ij}a_{ij}^{-1}H_i\otimes H_j\right)\prod_{\alpha\in\Phi^+}^{<}E_{q^{-2}}(\lambda e_\alpha\otimes f_\alpha)$$

with:

$$E_q(x)=\sum_{r=0}^{\infty}\frac{x^r}{[r;q]!},\quad [u;q]!=\prod_{i=1}^{u}[i;q],\quad [i;q]=\frac{(1-q^i)}{(1-q)}. \tag{38}$$

Here a_{ij} is the Cartan matrix and $\lambda=1-q^{-2}$. E_q is a q-analogue exponential. The universal R-matrix can also be expressed with the generators in a reversed order [2]. This form will be important for calculating the matrix σ^- (50):

$$R_{A_n}=\prod_{\alpha\in\Phi^+}^{>}E_{q^{-2}}(\lambda S(e_\alpha)\otimes S(f_\alpha))\exp\left(\frac{h}{2}\sum_{ij}a_{ij}^{-1}H_i\otimes H_j\right). \tag{39}$$

This is proved from (38) by recalling that the antipode S is an algebra anti-homomorphism and using the relation $S\otimes S(R)=R$. The (skew) antipode does not preserve the adjoint structure (36); hence it is necessary to define an alternative definition of the adjoint action. Define [2]:

$$\mathrm{ad}'_q P_\alpha\cdot O_\beta=P_\alpha O_\beta-q^{-(\alpha,\beta)}O_\beta P_\alpha,\quad \alpha<\beta. \tag{40}$$

This differs by a $q\to q^{-1}$ transformation from the previous adjoint definition (36). The following system of generators is an alternative set to (37):

$$e'_\alpha=\prod_{s\in[i,j-1]}^{<}(\mathrm{ad}'_q e_s)\cdot e_j,\ f'_\alpha=\prod_{s\in[i+1,j]}^{>}(\mathrm{ad}'_q f_s)\cdot f_i.$$

The following can be proved by induction:

$$S(e_\alpha)=-q^2q^{-H_\alpha}e'_\alpha,\quad S(f_\alpha)=-q^{-2m}q^{H_\alpha}f'_\alpha, \tag{41}$$

for $\alpha\in\Phi^+$, the sum of $m+1$ simple roots.

The fundamental representation of $U_q sl(n+1)$ will be used [6]:

$$\rho(H_i)=E_{i,i}-E_{i+1,i+1},\quad \rho(e_i)=q^{1/2}E_{i,i+1},\quad \rho(f_i)=q^{1/2}E_{i+1,i}, \tag{42}$$

where E_{ij} is the matrix with value one at position i,j and zero elsewhere. It is necessary to evaluate the representations of e_α, f_α. Noting that $E_{st}\cdot E_{uv}=\delta_{tu}E_{sv}$, these can be calculated to be:

$$\rho(e_{ij})=\rho(e_ie_{i+1}\dots e_j)=q^{1/2(j-i+1)}E_{i,j+1},$$
$$\rho(f_{ij})=\rho(q^{-(j-i)}f_jf_{j-1}\dots f_i)=q^{1/2(i-j+1)}E_{j+1,i}.$$

The matrix $\sigma^+\in\mathrm{Mat}(n+1,U_qb_+)$ is defined as (15): $\sigma^+_{ij}=\mathrm{Id}\otimes\rho_{ij}(R_{A_n})$. Once this is evaluated the R-matrix in this representation can be obtained by $R^\rho=\rho(\sigma^+)$. The most difficult part of evaluating σ^+ is the treatment of the coroot dependent prefactor of R (38):

$$\mathrm{Id}\otimes\rho\left(\exp\frac{h}{2}\sum_{ij}a_{ij}^{-1}H_i\otimes H_j\right).$$

For this it is required to know the inverse of the Cartan matrix. The only property that we require is that:

$$a_{ij}^{-1} - a_{ij-1}^{-1} = \begin{cases} 1 - \dfrac{i}{n+1} & j \leq i \\ \dfrac{-i}{n+1} & j > i \end{cases}.$$

This is valid for all $i, j = 1$ to $n+1$. (Take $a_{i0}^{-1} = 0$ and $a_{i,n+1}^{-1} = 0$).

Using this equation we may proceed to calculate $\mathrm{Id} \otimes \rho\left(\exp \frac{h}{2} \sum_{ij} a_{ij}^{-1} H_i \otimes H_j \right)$:

$$\mathrm{Id} \otimes \rho\left(\sum_{ij} a_{ij}^{-1} H_i \otimes H_j \right) = \sum_{ij=1}^{n} a_{ij}^{-1} H_i (E_{jj} - E_{j+1\,j+1})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n+1} (a_{ij}^{-1} - a_{ij-1}^{-1}) H_i E_{jj} = \sum_{i=1}^{n} H_i \sum_{j=1}^{i} E_{jj} - \sum_{i=1}^{n} \frac{iH_i}{n+1} I.$$

I is the identity matrix. Define the matrices $I_i = \sum_{j=1}^{i} E_{jj}$. Hence we have:

$$\mathrm{Id} \otimes \rho\left(\exp \frac{h}{2} \sum_{ij} a_{ij}^{-1} H_i \otimes H_j \right) = \exp\left(\frac{h}{2} \sum_{i=1}^{n} H_i I_i \right) \exp\left(-\frac{h}{2} \sum_{i=1}^{n} \frac{iH_i}{n+1} I \right)$$

$$= \exp\left(-\frac{h}{2} \sum_{i=1}^{n} \frac{iH_i}{n+1} \right) \prod_{s=1}^{n} K_s(q^{H_s}). \qquad (43)$$

The matrices K_s have been defined as: $K_s(x) = I + (x-1) I_s$. They are diagonal with x in the first s positions, and 1 in those remaining.

Returning to the universal R-matrix, (38), we can now complete the calculation of the matrix σ^+ by projecting in the second position the non-coroot part to the representation space $\mathrm{End}(V)$:

$$\mathrm{Id} \otimes \rho\left(\prod_{\alpha \in \Phi^+}^{<} E_{q^{-2}}(\lambda e_\alpha \otimes f_\alpha) \right) = \prod_{1 \leq i \leq j \leq n}^{<} (I + \lambda q^{1/2(i-j+1)} e_{ij} E_{j+1,i})$$

$$= I + \sum_{1 \leq i \leq j \leq n} \lambda q^{1/2(i-j+1)} e_{ij} E_{j+1,i}. \qquad (44)$$

It is essential that the generators are in an ascending order such that no cross terms occur in expanding the product. Compare this to the R-matrix in the form (39), used in the calculation of σ^-, which would generate cross terms.

Collecting together the results (43) and (44), the matrix σ^+ can be evaluated. It takes the form:

$$\sigma^+ = q^{-\sum_{i=1}^{n} \frac{iH_i}{n+1}} \prod_{s=1}^{n} K_s(q^{H_s}) \left(I + \sum_{1 \leq i \leq j \leq n} \lambda q^{1/2(i-j+1)} e_{ij} E_{j+1,i} \right).$$

It is advantageous to define the following combinations of elements due to their occurrence in the matrix σ^+:

$$\omega_k = q^{-\sum_{i=1}^{n} \frac{iH_i}{n+1}} q^{\sum_{s=k}^{n} H_s}, \quad k = 1..n, \quad \text{and} \quad \omega_{n+1} = q^{-\sum_{i=1}^{n} \frac{iH_i}{n+1}}. \tag{45}$$

This allows the coroot dependent part to be written as:

$$q^{-\sum_{i=1}^{n} \frac{iH_i}{n+1}} \prod_{s=1}^{n} K_s(q^{H_s}) = \sum_{k=1}^{n+1} \omega_k E_{kk}. \tag{46}$$

The matrix σ^+ now takes the form:

$$\sigma^+ = \sum_{k=1}^{n+1} \omega_k E_{kk} + \sum_{1 \leqq i \leqq j \leqq n} \lambda q^{1/2(i-j+1)} \omega_{j+1} e_{ij} E_{j+1,i}. \tag{47}$$

The elements ω_k have the following algebra:

$$\begin{aligned} &\omega_k e_{kt} = q e_{kt} \omega_k, \quad \omega_{k+1} e_{sk} = q^{-1} e_{sk} \omega_{k+1}, \\ &\omega_k e_{st} = e_{st} \omega_k \quad \text{if} \quad s \neq k,\, t+1 \neq k, \end{aligned} \tag{48}$$

and a similar algebra under the transformation: $e_{st} \to f_{ts}$, $\omega_k \to \omega_k^{-1}$.

Now consider calculating the matrix σ^-. The starting point is the expression (39) for the universal R-matrix, since this avoids the production of cross terms when expanding the following product:

$$\begin{aligned} &\rho \otimes \mathrm{Id}\left(\prod_{\alpha \in \Phi^+}^{>} E_{q^{-2}}(-q^2 \lambda q^{-H_\alpha} e_\alpha \otimes S(f_\alpha)) \right) \\ &= \prod_{1 \leqq i \leqq j \leqq n}^{>} (I - \lambda q q^{1/2(j-i+1)} S(f_{ji}) E_{i,j+1}) \\ &= I - q\lambda \sum_{1 \leqq i \leqq j \leqq n} q^{1/2(j-i+1)} S(f_{ji}) E_{i,j+1}. \end{aligned} \tag{49}$$

The expression (16) for σ^- implies that the quantity of interest is $\rho \otimes \mathrm{Id}\,(R)$. Using the above formulae (49) and (46), this becomes:

$$\begin{aligned} \rho \otimes \mathrm{Id}\,(R) &= \left(I - q\lambda \sum_{1 \leqq i \leqq j \leqq n} q^{1/2(j-i+1)} S(f_{ji}) E_{i,j+1} \right) \sum_{k=1}^{n+1} \omega_k E_{kk} \\ &= \sum_{k=1}^{n+1} \omega_k E_{kk} - q\lambda \sum_{1 \leqq i \leqq j \leqq n} q^{1/2(j-i+1)} S(f_{ji}) \omega_{j+1} E_{i,j+1}. \end{aligned}$$

We now deduce by operating with the antipodal map that:

$$\sigma^- = \sum_{k=1}^{n+1} \omega_k^{-1} E_{kk} - q\lambda \sum_{1 \leqq i \leqq j \leqq n} q^{1/2(j-i+1)} \omega_{j+1}^{-1} f_{ji} E_{i,j+1}. \tag{50}$$

The only remaining calculation to complete the matrix formulation of $U_q sl(n+1)$ is to find the R-matrix in this representation. The easiest method is to use the expression (47) for σ^+ and $R^\rho = \rho(\sigma^+)$. The various components are:

$$\rho\left(\sum_{i=1}^{n} \frac{iH_i}{n+1}\right) = I - (n+1)E_{n+1,n+1}, \quad \rho\left(q^{-\sum_{i=1}^{n} \frac{iH_i}{n+1}}\right) = q^{-1/(n+1)}(I + (q-1)E_{n+1,n+1}),$$

and thus:

$$\rho\left(\prod_{s=1}^{n} K_s(q^{H_s})\right) = I \otimes I + (q-1)\sum_{k=1}^{n} E_{kk} \otimes E_{kk} + (q^{-1}-1)\sum_{k=1}^{n} E_{n+1,n+1} \otimes E_{kk}.$$

Throughout this calculation the element $E_{n+1,n+1}$ has to be treated separately. However this asymmetry cancels to give the final result:

$$R_{\rho\otimes\rho} = q^{-1/(n+1)}\left(I \otimes I + (q-1)\sum_{i=1}^{n+1} E_{ii} \otimes E_{ii} + (q-q^{-1})\sum_{i<j} E_{ij} \otimes E_{ji}\right). \quad (51)$$

This was quoted in [2] and also given in [6].

To summarise: the universal R-matrix for $U_q sl(n+1)$ (38) is used to construct two triangular matrices valued in $U_q sl(n+1)$, defined by (15), (16):

$$\sigma_{ij}^{+} = Id \otimes \rho_{ij}(R_{A_n}), \quad \sigma_{ij}^{-} = \rho_{ij} \otimes \mathrm{Id}(R_{A_n}^{-1}).$$

These are found to have the form (47), (50):

$$\sigma^+ = \sum_{k=1}^{n+1} \omega_k E_{kk} + \sum_{1 \leq i \leq j \leq n} \lambda q^{1/2(i-j+1)} \omega_{j+1} e_{ij} E_{j+1,i},$$

$$\sigma^- = \sum_{k=1}^{n+1} \omega_k^{-1} E_{kk} - q\lambda \sum_{1 \leq i \leq j \leq n} q^{1/2(j-i+1)} \omega_{j+1}^{-1} f_{ji} E_{i,j+1},$$

where ω_k is defined in (45). These encode the Hopf structure of $U_q sl(n+1)$ in the matrix form (6). It is obvious from the above form for $\sigma^\pm$ that σ^+ and σ^- generate isomorphic subalgebras: the Borel subalgebras. An isomorphism was given in Sect. 5; matrix transposition inducing the following transformation of generators:

$$e_{ij} \leftrightarrow -q^{j+1-i} f_{ji} \quad \text{or} \quad e_\alpha \leftrightarrow -q^{m+1} f_\alpha.$$

This can easily be verified to be an algebra isomorphism and anti-coalgebra isomorphism as stated in Sect. 5.

The ordering of the generators required for the definition of the adjoint action and universal R-matrix used in [2] can be read off from the matrix σ^+. It is given by the orthogonal projection onto the main diagonal.

The results of expanding the matrices in (6), i.e. finding the commutation relations will now be stated. The proof is long and so omitted. Note that expressing the algebra in this matrix form allows all the commutation relations to be systematically derived. Only partial results were given in [2], only those that were required being calculated. All the commutation relations within a Borel subalgebra take the form of the adjoint structure; in particular there is no mixing between the two definitions of the adjoint (36), (40). This suggests that the adjoint structure (36) and generators (37) are intrinsic to the algebra structure of the quantised Lie algebra A_n. (Of course the alternative definition of adjoint (40) leads to a structure that is just as self consistent.)

In order to simplify the notation we shall treat the roots set theoretically with respect to the simple root decomposition using some base S, in addition to all previous interpretations and conventions. Hence the root $\alpha = \sum_{i=u}^{v} \alpha_i$, $v \geqq u$ will be considered as the set $\{\alpha_i\}_{i\in[u,v]}$.

Expanding $R_{21}\sigma_1^{\pm}\sigma_2^{\pm} = \sigma_2^{\pm}\sigma_1^{\pm}R_{21}$ gives:

For $\alpha+\beta\in\Phi^+, \alpha<\beta$,

$$e_{\alpha+\beta} = \mathrm{ad}_q\, e_\alpha \cdot e_\beta = e_\alpha e_\beta - q^{-1} e_\beta e_\alpha,$$
$$f_{\alpha+\beta} = -\mathrm{ad}_q\, f_\alpha \cdot f_\beta = -f_\alpha f_\beta + q^{-1} f_\beta f_\alpha.$$

For $\alpha+\beta\notin\Phi^+$ and $\alpha\cap\beta=\varnothing$ or $\alpha\subset\beta$ or $\beta\subset\alpha$,

$$\mathrm{ad}_q\, e_\alpha \cdot e_\beta = e_\alpha e_\beta - e_\beta e_\alpha = 0,$$
$$\mathrm{ad}_q\, f_\alpha \cdot f_\beta = f_\alpha f_\beta - f_\beta f_\alpha = 0, \quad \alpha<\beta.$$

For $\alpha+\beta\notin\Phi^+$ and $\alpha\cap\beta\neq\varnothing$, $\alpha=\bar{\alpha}+\gamma$, $\beta=\gamma+\bar{\beta}$,

$$\mathrm{ad}_q\, e_\alpha \cdot e_\beta = e_\alpha e_\beta - e_\beta e_\alpha = (q-q^{-1}) e_\gamma e_{\bar{\alpha}+\beta},$$
$$\mathrm{ad}_q\, f_\alpha \cdot f_\beta = f_\alpha f_\beta - f_\beta f_\alpha = (q-q^{-1}) f_\gamma f_{\bar{\alpha}+\beta}, \tag{52}$$

with $\gamma\in\Phi^+, \alpha<\gamma<\beta$. Note that the pair $e_\gamma, e_{\bar{\alpha}+\beta}$ commute, and similarly for the pair of f generators.

Expanding the relation $R_{21}\sigma_1^{+}\sigma_2^{-} = \sigma_2^{-}\sigma_1^{+}R_{21}$ produces:

For all roots $\alpha\in\Phi^+$,

$$[e_\alpha, f_\alpha] = \frac{1}{\lambda}(q^{H_\alpha} - q^{-H_\alpha}).$$

For $\alpha\cap\beta=\varnothing$ or $\alpha\subset\beta, \beta-\alpha\notin\Phi^+$ (i.e. β encloses α),

$$[e_\alpha, f_\beta] = 0, [e_\beta, f_\alpha] = 0.$$

For $\alpha\cap\beta\neq\varnothing$, $\alpha=\beta+\gamma$,

$$[e_\alpha, f_\beta] = -q^{-H_\beta} e_\gamma, \quad \beta<\gamma,$$
$$[e_\alpha, f_\beta] = q q^{H_\beta} e_\gamma, \quad \beta>\gamma.$$

For $\alpha\cap\beta\neq\varnothing$, $\beta=\alpha+\gamma$,

$$[e_\alpha, f_\beta] = -q^{H_\alpha} f_\gamma, \quad \alpha<\gamma,$$
$$[e_\alpha, f_\beta] = q q^{-H_\alpha} f_\gamma, \quad \alpha>\gamma.$$

For $\alpha\cap\beta\neq\varnothing$, $\alpha=\bar{\alpha}+\gamma, \beta=\gamma+\bar{\beta}$,

$$[e_\alpha, f_\beta] = -(q-q^{-1}) q^{H_\gamma} e_{\bar{\alpha}} f_{\bar{\beta}}, \quad \alpha<\beta,$$
$$[e_\alpha, f_\beta] = (q-q^{-1}) q^{-H_\gamma} e_{\bar{\alpha}} f_{\bar{\beta}}, \quad \alpha<\beta. \tag{53}$$

The algebra satisfied by the ω_k was given in (48). Alternatively the full Hopf structure with the generators H_i can be used. This is reproduced for completeness:

$$[H_\alpha, e_\beta] = (\alpha, \beta) e_\beta,$$
$$[H_\alpha, f_\beta] = -(\alpha, \beta) f_\beta.$$

The coalgebra structure for $U_q g$ is given by the relations $\Delta\sigma_{ij}^{\pm} = \sum_k \sigma_{ik}^{\pm} \otimes \sigma_{kj}^{\pm}$. It can be verified that the following structure is obtained:

$$\Delta e_\alpha = 1 \otimes e_\alpha + e_\alpha \otimes q^{H_\alpha} + (q - q^{-1}) \sum_{\substack{\beta > \beta' \\ \beta + \beta' = \alpha \\ \beta, \beta' \in \Phi^+}} e_\beta \otimes q^{H_\beta} e_{\beta'},$$

$$\Delta f_\alpha = q^{-H_\alpha} \otimes f_\alpha + f_\alpha \otimes 1 - (q - q^{-1}) \sum_{\substack{\beta > \beta' \\ \beta + \beta' = \alpha \\ \beta, \beta' \in \Phi^+}} q^{-H_\beta} f_{\beta'} \otimes f_\beta, \quad \forall \alpha \in \Phi^+. \tag{54}$$

This is the structure originally given in [2].

Conclusion

In this paper the Hopf structure of the quantised algebra $U_q g$ is expressed in a matrix form (6) by defining two representations of the quantum group $U_q g'$. In order to achieve this matrix formulation there are two requirements: we require a representation of the dual $U_q g'$, this giving the coalgebra expression (8), and an universal R-matrix such that the algebra can be constructed (9). However the dual $U_q g'$ is not quasi-triangular, Sect. 2, and hence it is necessary to embed $U_q g'$ into a quasi-triangular Hopf algebra, and use a representation of this algebra. The most obvious choice is the quantum double of $U_q g'$ [4], denoted by $D(U_q g')$. The construction in Sect. 3 exploiting the quotient structure of the quantum group $U_q g'$ (10), is interpretable in this fashion, that is, there exist two morphisms: $D(U_q g') \to U_q g$ which when restricted to $U_q g'$ map onto the Borel subalgebras; $U_q b_+$, $U_q b_-$. These will be algebra homomorphisms, coalgebra anti-homomorphisms. Our construction gives an expression for the matrices of generators $L^\pm$ defined in [7] in terms of the universal R-matrix of $U_q g$ and a representation of $U_q g$, (15), (16). When using the fundamental representation of the quantised algebra $U_q sl(n)$, we deduce that the Borel subalgebras $U_q b_\pm$ are generated by $L^\pm$, the quantum determinant condition arising naturally. The case for other quantised algebras is more complex; additional representations must be used since the fundamental representation no longer generates the function space of the associated Lie group. These cases are more easily treated by using the quantum double $D(U_q g)$, our construction deriving ultimately from the quasi-triangular structure of this Hopf algebra [4]. However the construction for general $U_q g$ will be very similar, matrices $L^\pm$, being defined by (15), (16) for each irreducible representation required to generate the quantised function space $\mathrm{Fun}_q(G)$. Alternatively it is expected that the quantum groups $\mathrm{Fun}_q(G)$ are attainable from $\mathrm{Fun}_q(Gl(n))$ by applying functional constraints.

The matrix formulation of [7] naturally leads to a system of generators based on the whole root system, as in [2, 14], contrasting to that based only on the simple roots as used in [6, 10]. The construction of the matrices $L^\pm$ for the general

Lie algebras, [5] will produce a system of generators based on the whole root system, and should reproduce the adjoint structure presented in [15], an extension of that for $U_q sl(n)$, [2, 14].

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On completion of this work, [8] and [11] came to the attention of the author. Paper [11] extends the analysis of [7], an universal R-matrix being defined via a homomorphism $A(R)$ to $U(R)$. Paper [8] considers the constructions of [7] from a universal aspect using quantum doubles. The relations (15) and (16) are obtained in both these works.

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