Commun. Math. Phys. 132, 581–592 (1990)

Communications in Mathematical Physics

Implementation of Comparative Probability by Normal States. Infinite Dimensional Case

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Abstract. Let $\mathscr{H}$ be an infinite dimensional Hilbert space and $\mathscr{P}(\mathscr{H})$ the set of all (orthogonal) projections on $\mathscr{H}$. A comparative probability on $\mathscr{P}(\mathscr{H})$ is a linear preorder $\lesssim$ on $\mathscr{P}(\mathscr{H})$ such that $\mathbf{O} \lesssim P \lesssim \mathbf{1}$, $\mathbf{1} \not\lesssim \mathbf{O}$ and such that if $P \perp R$, $Q \perp R$, then $P \lesssim Q \Leftrightarrow P+R \lesssim Q+R$ for all P, Q, R in $\mathscr{P}(\mathscr{H})$. We give a sufficient and necessary condition for $\lesssim$ to be implemented in a canonical way by a normal state on $\mathscr{B}(\mathscr{H})$, the bounded linear operators on $\mathscr{H}$.

1. Introduction and Notation

Let $\mathscr{H}$ be a Hilbert space. $\mathscr{P}(\mathscr{H})$ denotes the set of all (orthogonal) projections on $\mathscr{H}$. If E is a closed subspace of $\mathscr{H}$, and $\phi \in \mathscr{H}$ then P_E and P_ϕ denote the corresponding projections. We drop the E and ϕ if no reference to the subspaces is required. $\mathscr{P}_1(\mathscr{H})$ is the subset of all one dimensional projections and $\mathscr{P}_\sigma(\mathscr{H})$ is the subset of all those projections P_E such that E is a separable (finite or infinite dimensional) subspace of $\mathscr{H}$. Lower case Roman subscripts as in P_j or P_{ϕ_k} will generally be used for indexing sequences and nets. $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ denote the natural numbers, the reals and the complex numbers respectively. All vectors of $\mathscr{H}$ appearing may be assumed to be normalised. $P_{\mathscr{H}}$ is denoted by $\mathbf{1}_{\mathscr{H}}$ or just $\mathbf{1}$ of no confusion arises and the zero vector is denoted by $\mathbf{O}$. The orthogonal complement of P (i.e. $\mathbf{1}-P$) is denoted by $P^\perp$. If $P, Q \in \mathscr{P}(\mathscr{H})$ and $P \leqq Q^\perp$ then we write $P \perp Q$.

Definition 1.1. Let $\mathscr{H}$ be a Hilbert space. A preorder relation $\lesssim$ on $\mathscr{P}(\mathscr{H})$ is called an elementary comparative probability (ECP) iff the following axioms are satisfied by all $P, Q, R \in \mathscr{P}(\mathscr{H})$:

A1 $P \lesssim Q$ or $Q \lesssim P$,

A2 $P \lesssim Q$ and $Q \lesssim R \Rightarrow P \lesssim R$,

A3 $\mathbf{O} \lesssim P \lesssim \mathbf{1}$, $\mathbf{1} \not\lesssim \mathbf{O}$.

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An ECP$\lesssim$ is called a comparative probability (CP) iff $\lesssim$ satisfies:

A4 If $P\perp R$, $Q\perp R$, then $P\lesssim Q \Leftrightarrow P+R\lesssim Q+R$ $\forall P, Q, R\in\mathscr{P}(\mathscr{H})$. □

Recall that a (normalized) Gleason measure on $\mathscr{P}(\mathscr{H})$ is a mapping $\mu:\mathscr{P}(\mathscr{H})\to[0,1]$ which is σ-orthoadditive and satisfies $\mu(\mathbf{I})=1$. If $\dim\mathscr{H}\geqq 3$, and $\mathscr{H}$ remains separable, then by Gleason's theorem [1], the set of all Gleason measures is exactly the set of all normal (i.e. σ-weakly continuous) states on $\mathscr{B}(\mathscr{H})$, the bounded linear operators on $\mathscr{H}$. If $\mathscr{H}$ is no longer separable then Gleason's theorem still holds providing the σ-orthoadditivity is strengthened to *complete* orthoadditivity or providing one assumes the generalized continuum hypothesis (see Kalmbach [2] Chapter 3) with only the σ-orthoadditivity.

It is clear that every normal state ω on $\mathscr{B}(\mathscr{H})$ induces a CP $\lesssim_\omega$ on $\mathscr{P}(\mathscr{H})$ according to the prescription:

$$P\lesssim_\omega Q\Leftrightarrow\omega(P)\leqq\omega(Q)\,. \tag{1}$$

The purpose of this paper is to investigate the inverse problem. Specifically we wish to find the sufficient and necessary conditions for a CP on $\mathscr{P}(\mathscr{H})$, where $\mathscr{H}$ is a real or complex *infinite dimensional* Hilbert space, to be implemented by a normal state (=completely additive Gleason measure) in the sense of (1) above. The special and somewhat pathological case $\dim\mathscr{H}=2$ was resolved in [3], where the question of uniqueness of implementation was also investigated. It is worth noting that the case $\dim\mathscr{H}=2$ is extra special in that not all the axioms can be brought to bear on $\lesssim$; in particular **A4** has no effect and is replaced in [3] by:

A5 $$P\lesssim Q\Leftrightarrow Q^\perp\lesssim P^\perp \quad \forall P, Q\in\mathscr{P}(\mathscr{H})\,.$$

We shall use the notation $P\cong Q$ to indicate $P\lesssim Q$ and $Q\lesssim P$; and $P\prec Q$ to denote $P\lesssim Q$ and $Q\not\lesssim P$. If E is a subspace of $\mathscr{H}$, the set $[P]_E$ is defined to be all $Q\in E$ such that $Q\cong P$ and $[\mathscr{P}(E)]$ to be the set of all $[Q]_E$ such that Q is in $\mathscr{P}(E)$, the subscript E being dropped if $E=\mathscr{H}$.

2. Preliminaries

The existence of states that are not normal shows that not every CP can be implemented by a normal state. In fact one can also show that for every dimension, finite or infinite, examples exist of CP's that cannot be implemented by states, normal or not. We need to impose some sort of "continuity" condition on $\lesssim$ for it to be implementable by a normal state.

Definition 2.1. Let $\lesssim$ be an ECP on $\mathscr{P}(\mathscr{H})$. We say that $\lesssim$ is weakly continuous (or just continuous if no confusion arises) if whenever the net P_j in $\mathscr{P}(\mathscr{H})$ converges weakly to P and $P_2\lesssim P_j\lesssim P_1$ $\forall j$, then $P_2\lesssim P\lesssim P_1$. □

Let $\lesssim$ be an ECP on $\mathscr{P}(\mathscr{H})$. Recall that the interval topology on $\mathscr{P}(\mathscr{H})$ induced by the linear preorder $\lesssim$ is generated by a neighbourhood base consisting of $\lesssim$ intervals of the form (P, Q), where $P\prec Q$, $[\mathbf{O}, P)$ where $\mathbf{O}\prec P$, and $(P, \mathbf{I}]$ where $P\prec\mathbf{I}$. Continuity of $\lesssim$ can be reformulated in terms of the strength of the $\lesssim$ (interval) topology relative to that of the weak topology on $\mathscr{P}(\mathscr{H})$ (induced by the weak topology on $\mathscr{B}(\mathscr{H})$). Before showing this we first explain some notation: $P_j\xrightarrow{w}P$ and $P_j\xrightarrow{\lesssim}P$ imply convergence in the weak and in the $\lesssim$ topologies respectively.

Proposition 2.2. *Let $\lesssim$ be an* ECP *on $\mathscr{P}(\mathscr{H})$. Then $\lesssim$ is continuous if and only if the $\lesssim$ topology on $\mathscr{P}(\mathscr{H})$ is weaker than the weak topology on $\mathscr{P}(\mathscr{H})$.*

Proof. Let $\lesssim$ be continuous. Suppose P_j is a net in $\mathscr{P}(\mathscr{H})$ such that $P_j \xrightarrow{w} P$, where $\mathbf{O} \prec P \prec \mathbf{1}$. Let P' and P'' be such that $P' \prec P \prec P''$ so that (P', P'') is a $\lesssim$ neighbourhood of P. Then for some j_0, $P_j \in (P', P'')$ $\forall j \geqq j_0$, lest we should find a subnet P_{j_k} such that $\forall j_k$, $P_{j_k} \lesssim P'$ or $P'' \lesssim P_{j_k}$. This would, by continuity of $\lesssim$ imply that $P \lesssim P'$ or $P'' \lesssim P$, a contradiction. Hence $P_j \xrightarrow{\lesssim} P$. Only trivial modification is required should $P \cong \mathbf{O}$ or $P \cong \mathbf{1}$. Thus the $\lesssim$ topology is weaker than the weak topology on $\mathscr{P}(\mathscr{H})$.

Now let the $\lesssim$ topology be weaker than the weak topology. Let $P_j \xrightarrow{w} P$ and let P', P'' be such that $P' \lesssim P_j \lesssim P''$ $\forall j$. Suppose that $P'' \prec P$, then there exist $R \in \mathscr{P}(\mathscr{H})$ such that $P \lesssim R$ and such that $(P'', R]$ is a $\lesssim$ neighbourhood, and hence by hypothesis, a weak neighbourhood of P. Thus there exists $j \in N$ such that $P_j \in (P, R]$ which implies $P \prec P_j$, a contradiction. Hence $P \lesssim P''$. A similar argument establishes that $P' \lesssim P$ and the result follows. □

Let $\mathscr{H}$ be an infinite dimensional Hilbert space. As is the case when dealing with normal states on $\mathscr{B}(\mathscr{H})$, separability of $\mathscr{H}$ is a natural condition when dealing with continuous CP's on $\mathscr{P}(\mathscr{H})$. This is obvious from the following:

Proposition 2.3. *Let $\mathscr{H}$ be any (not necessarily separable) Hilbert space and $\lesssim$ be a continuous* CP *on $\mathscr{P}(\mathscr{H})$.*
i) *Let $\mathscr{A} \subset \mathscr{P}_1(\mathscr{H})$ be any set of mutually orthogonal (one dimensional) projections such that $\forall P \in \mathscr{A}$, $\mathbf{O} \prec P$. Then $\mathscr{A}$ is at most a countably infinite set.*
ii) *If E is a subspace of $\mathscr{H}$ such that $P \in \mathscr{P}_1(E) \Rightarrow P \cong \mathbf{O}$, then $P_E \cong \mathbf{O}$.*

Proof. i) If $\mathscr{A}$ is a finite set there is nothing to prove; so assume that it is infinite. Now for any $P \in \mathscr{A}$ define $\mathscr{A}(P)$ to be $\{Q \in \mathscr{A} : P \lesssim Q\}$. We note that $\mathscr{A}(P)$ is a finite set for each $P \in \mathscr{A}$, otherwise we can find in $\mathscr{A}(P)$ a sequence Q_j such that $Q_j \xrightarrow{w} \mathbf{O}$ and hence, by continuity of $\lesssim$, $Q_j \xrightarrow{\lesssim} \mathbf{O}$. This contradicts the requirement that for all j, $\mathbf{O} \prec P \lesssim Q_j$.

Now let $\{P_j : j \in \mathbf{N}\}$ be a countably infinite subset of $\mathscr{A}$. Clearly $P_j \xrightarrow{\lesssim} \mathbf{O}$ as $j \to \infty$. Hence we must have $\mathscr{A} = \bigcup_{j \in \mathbf{N}} \mathscr{A}(P_j)$ and the result follows as each $\mathscr{A}(P_j)$ is finite.

ii) The result is clear for E separable, so assume that E is not separable. Let $\mathscr{Z}$ be the set of all projections P in $\mathscr{P}(E)$ such that $P \cong \mathbf{O}$. Then $\mathscr{Z}$ is a nonempty $\lesssim$ closed subset of $\mathscr{P}(\mathscr{H})$. Let P_{F_j} be an (operator theoretic) increasing net in $\mathscr{Z}$. Then P_{F_j} is evidently a weakly convergent net which weakly and hence $\lesssim$ converges to P_F, where F is the closure of the linear span of $\cup_j F_j$. Thus, by Zorn's lemma, $\mathscr{Z}$ has an operator theoretic maximal projection Q, say. But we cannot have $Q < P_E$ since $Q' \in \mathscr{Z}$ and $Q' \perp Q \Rightarrow Q + Q' \in \mathscr{Z}$ by axiom **A4**. Hence $P_E \cong \mathbf{O}$ as required. □

Weak continuity of a CP $\lesssim$ on $\mathscr{P}(\mathscr{H})$ immediately implies a number of useful topological properties for the $\lesssim$ topology. We list some of these below:

Proposition 2.4. *Let $\mathscr{H}$ be an infinite dimensional Hilbert space and $\lesssim$ be a continuous* CP *on $\mathscr{P}(\mathscr{H})$. Then the following are all true.*
i) *$\mathscr{P}(\mathscr{H})$ is $\lesssim$ connected.*
ii) *$\mathscr{P}(\mathscr{H})$ is order complete, i.e. every non-empty subset of $\mathscr{P}(\mathscr{H})$ has an* inf *and a* sup *with respect to $\lesssim$ and hence $\mathscr{P}(\mathscr{H})$ is $\lesssim$ compact.*
iii) *In the $\lesssim$ topology, $\mathscr{P}(\mathscr{H})$ is pseudometrizable and complete.*

Proof. i) Let $J=K^{\perp}$, where K is the set of all $\phi\in\mathscr{H}$ such that $P_{\phi}\cong\mathbf{O}$. Then, by Proposition 2.3 (i), J is separable. Thus, if J is infinite dimensional, $\mathscr{P}(J)$ is weakly and hence $\lesssim$ connected. By Proposition 2.3 (ii), $P_J\cong\mathbf{1}_{\mathscr{H}}$, and the result follows at once for J infinite dimensional. If J is finite dimensional the above argument may be used with J replaced by J_0, where J_0 is any separable infinite dimensional subspace of $\mathscr{H}$ containing J as a subspace.
ii) This follows from (i) see, for example, [7, Chap. 1 J].
iii) Let J be as in (i). Then, by Proposition 2.3, $\mathscr{P}(J)$ is weakly and hence $\lesssim$ second countable. Hence $[\mathscr{P}(J)]$ is a regular second countable topological space under the induced $\lesssim$ topology and is therefore $\lesssim$ metrizable by Urysohn's metrizability theorem. The corresponding metric pseudometrizes $\mathscr{P}(J)$ (or $\mathscr{P}(J_0)$) and hence also $\mathscr{P}(\mathscr{H})$. Since $\mathscr{P}(\mathscr{H})$ is order complete it is complete as a pseudometric space. □

3. Results

In this section, $\mathscr{H}_0$, ψ_0 and $\mathscr{H}$ are defined as follows: $\mathscr{H}_0$ is an infinite dimensional real or complex Hilbert space. $\psi_0\in\mathscr{H}_0$ is a fixed non-zero vector and $\mathscr{H}$ is its orthogonal complement in $\mathscr{H}_0$. $\mathscr{P}(\mathscr{H})$ is endowed with a continuous CP $\lesssim$. We start with a number of technical results.

Lemma 3.1. *Let x and y be any positive reals such that $x+y\leqq 1$. Then for any $R\in\mathscr{P}_{\sigma}(\mathscr{H})$ there exist sequences P_j and Q_j in $\mathscr{P}_{\sigma}(\mathscr{H})$ satisfying:*

i) $P_j\perp Q_j\,\forall j$.

ii) $P_j\xrightarrow{w}xR$.

iii) $Q_j\xrightarrow{w}yR$.

Further, if R is infinite dimensional, then we may choose that for each j, $Q_j\leqq R$ and $P_j\leqq R$.

Proof. The case $R=\mathbf{O}$ is trivial. So assume that $R=P_{\phi}$ for some $\phi\in\mathscr{H}$. Let ξ_n be any orthonormal sequence in the orthogonal complement of ϕ. Clearly we can find another orthonormal sequence η_n, also in the orthogonal complement of ϕ satisfying $\langle\xi_n|\eta_n\rangle=\Gamma=\dfrac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}(1-y)^{\frac{1}{2}}}\,\forall n$. Note that $\Gamma\leqq 1$ for all allowed x and y. Thus $\langle a\xi_n+x^{\frac{1}{2}}\phi|b\eta_n+y^{\frac{1}{2}}\phi\rangle=0\,\forall n$, where $a=(1-x)^{\frac{1}{2}}$ and $b=-(1-y)^{\frac{1}{2}}$. Since $a\xi_n+x^{\frac{1}{2}}\phi\to x^{\frac{1}{2}}\phi$ and $b\eta_n+y^{\frac{1}{2}}\phi\to y^{\frac{1}{2}}\phi$ in the weak Hilbert space topology, $P_{a\xi_n+x^{\frac{1}{2}}\phi}\xrightarrow{w}xP_{\phi}$ and $P_{b\xi_n+y^{\frac{1}{2}}\phi}\xrightarrow{w}yP_{\phi}$. Hence the result is true for $R\in\mathscr{P}_1(\mathscr{H})$. Now let $R\in\mathscr{P}(\mathscr{H})$ be finite dimensional with $R=\sum_{j=1}^{N}P_{\phi_j}$, where $P_{\phi_j}\in\mathscr{P}_1(\mathscr{H})\,\forall j$ and $P_{\phi_j}\perp P_{\phi_k}$ if $j\neq k$. Since we can always find N mutually disjoint infinite orthonormal subsets of an infinite orthonormal set we can make use of the construction in the first part of the proof to erect, for each P_{ϕ_j}, sequences P_{jk} and Q_{jk} such that

i) $P_{jk}\xrightarrow{w}xP_{\phi_j}$, $Q_{jk}\xrightarrow{w}yP_{\phi_j}$ as $k\to\infty$,

ii) $P_{jk}\perp Q_{jk}\,\forall j,k$,

iii) $P_{jk}\perp Q_{j'k}\,\forall j,j',k$,

iv) $P_{jk}\perp P_{j'k'}$ and $Q_{jk}\perp Q_{j'k'}$ if $j\neq j'$ or $k\neq k'$.

Hence $\sum_{j=1}^{N} P_{jk} \xrightarrow{w} xR$ and $\sum_{j=1}^{N} Q_{jk} \xrightarrow{w} yR$ as $k \to \infty$. Further, $\sum_{j=1}^{N} P_{jk} \perp \sum_{j=1}^{N} Q_{jk} \forall k$, and the result follows for R finite dimensional.

Finally we consider the case where R is infinite dimensional. Let $R_n \xrightarrow{w} R$, where R_n is finite dimensional for all n. From the above case we can construct, for each n, sequences Q_{nj} and P_{nj} in $\mathscr{P}_\sigma(\mathscr{H})$ such that $Q_{nj} \perp P_{nj} \forall j$ with $Q_{nj} \xrightarrow{w} yR_n$ and $P_{nj} \xrightarrow{w} xR_n$ as $j \to \infty$. Since $yR_n \xrightarrow{w} yR$ and $xR_n \xrightarrow{w} xR$, we can clearly find sequences from the Q_{nj} and the P_{nj} with required property. The last statement of the lemma follows by application of this last case to the situation where $\mathscr{H} = E$ and $R = \mathbf{1}_E (= P_E)$; completing the proof of the lemma. □

Lemma 3.2. *Let P_j and Q_j be sequences in $\mathscr{P}_\sigma(\mathscr{H})$ both weakly converging to one limit in $\mathscr{B}(\mathscr{H})$. Then P_j and Q_j both $\lesseqq$ converge to one limit in $\mathscr{P}(\mathscr{H})$.*

Proof. There clearly exists a separable subspace $\mathscr{K}$ of $\mathscr{H}$ such that for all $j \in \mathbf{N}$, $P_j \in \mathscr{P}(\mathscr{K})$ and $Q_j \in \mathscr{P}(\mathscr{K})$. As $\mathscr{K}$ is separable the unit ball of $\mathscr{B}(\mathscr{K})$ is weakly metrizable, say with metric d_w. Let pseudometric $d_{\lesseqq}$ generate the $\lesseqq$ topology on $\mathscr{P}(\mathscr{H})$. Since P_j and Q_j are both d_w Cauchy, they are also both $d_{\lesseqq}$ Cauchy. By $d_{\lesseqq}$ completeness there exist P and Q in $\mathscr{P}(\mathscr{H})$ such that $P_j \xrightarrow{\lesseqq} P$ and $Q_j \xrightarrow{\lesseqq} Q$. We wish to show that $P \cong Q$. If this is not the case, then $d_{\lesseqq}(Q, P) > 0$, so that $d_{\lesseqq}(P_j, Q_j) \not\to 0 \Rightarrow d_w(P_j, Q_j) \not\to 0$ as $j \to \infty$ by continuity of $\lesseqq$. This contradicts the hypothesis of the lemma. Hence the result. □

The following result shows a limited form of joint $\lesseqq$ continuity of ortho-addition on $\mathscr{P}(\mathscr{H})$.

Proposition 3.3. *Let sequences P_{E_j} and P_{F_j} in $\mathscr{P}_\sigma(\mathscr{H})$ satisfy the following conditions:*

i) $P_{E_j} \xrightarrow{\lesseqq} P_E \in \mathscr{P}(\mathscr{H})$,

ii) $P_{F_j} \xrightarrow{\lesseqq} P_F \in \mathscr{P}(\mathscr{H})$,

iii) $P_{E_j} \perp P_{F_k} \forall j, k \in \mathbf{N}$.

Then there exist $P_{E'}$ and $P_{F'}$ in $\mathscr{P}(\mathscr{H})$ such that $P_{E'} \cong P_E$, $P_{F'} \cong P_F$, $P_{E'} \perp P_{F'}$ and such that $P_{E_j} + P_{F_j} \xrightarrow{\lesseqq} P_{E'} + P_{F'}$.

Proof. Let $\mathscr{L}$ be the minimal subspace of $\mathscr{H}$ such that $\forall j \in \mathbf{N}$, $P_{E_j}\mathscr{L} = E_j$ (that is $\mathscr{L} = \overline{Sp \bigcup_{j \in \mathbf{N}} E_j}$). If $\mathscr{L}$ is infinite dimensional, then $\mathscr{P}(\mathscr{L})$ is also complete under the $\lesseqq$ pseudometric. Hence there exists $P_{E'} \in \mathscr{P}(\mathscr{L})$ such that $P_{E'} \cong P_E$. The same conclusion can be reached if $\mathscr{L}$ is finite dimensional by taking $P_{E'}$ to be the norm $(=$weak$)$ limit of a suitable subsequence of the P_{E_j}. Similarly let $\mathscr{M}$ be the minimal subspace of $\mathscr{H}$ such that $\forall j \in \mathbf{N}$, $P_{F_j}\mathscr{M} = F_j$. Again there is a $P_{F'} \in \mathscr{P}(\mathscr{M})$ such that $P_{F'} \cong P_F$. Clearly $\mathscr{M} \perp \mathscr{L}$ so that $P_{E'} \perp P_{F'}$. Now we show that $P_{E_j} + P_{F_j} \xrightarrow{\lesseqq} P_{E'} + P_{F'}$. We examine three cases:

Case i. P_{E_j} and P_{F_j} are both $\lesseqq$ increasing.

Let $j \geqq k$, then $P_{E_k} + P_{F_k} \lesseqq P_{E_k} + P_{F_j} \lesseqq P_{E_j} + P_{F_j}$, where both inequalities follow from axiom **A4**. Hence $P_{E_j} + P_{F_j}$ is also $\lesseqq$ increasing. Thus $[P_{E_j} + P_{F_j}] \xrightarrow{\lesseqq} \lesseqq\sup_{j \in \mathbf{N}} (P_{E_j} + P_{F_j}) = [P_{E'} + P_{F'}]$ and the result follows.

Case ii. P_{E_j} and P_{F_j} are both $\lesssim$ decreasing.

From case (i) above, $[P_{E_j}+P_{F_j}] \xrightarrow{\lesssim} \lesssim\inf_{j\in\mathbf{N}} (P_{E_j}+P_{F_j}) = [P_{E'}+P_{F'}]$.

Case iii. One of P_{E_j} and P_{F_j} is $\lesssim$ increasing and the other $\lesssim$ decreasing.

Assume, without losing generality, that P_{E_j} is the $\lesssim$ increasing one. Then $P_{E'}+P_{F_j}$ and $P_{F'}+P_{E_j}$ are $\lesssim$ decreasing and $\lesssim$ increasing respectively and hence both $\lesssim$ converge to $P_{E'}+P_{F'}$. But by axiom **A4**, $P_{E'}+P_{F_j} \lesssim P_{E_j}+P_{F_j} \lesssim P_{E_j}+P_{F'}\,\forall j$, and the result follows.

Now let P_{E_j} and P_{F_j} satisfy the hypothesis of the proposition but otherwise be arbitrary. Let $P_{E_{j_k}}+P_{F_{j_k}}$ be a $\lesssim$ convergent subsequence of $P_{E_j}+P_{F_j}$. The subsequences $P_{E_{j_k}}$ and $P_{F_{j_k}}$ will in turn have subsequences which will fall in one of the three cases above. Thus every $\lesssim$ convergent subsequence of $P_{E_j}+P_{F_j}$ $\lesssim$ converges to $P_{E'}+P_{F'}$ and proof of the proposition follows. □

Lemma 3.4. *For any $P\in\mathscr{P}_\sigma(\mathscr{H})$ such that $P\prec\mathbf{1}$, the following statements are equivalent.*

i) $P\cong\mathbf{O}$.

ii) *For some $x>0$, there exists a sequence P_j in $\mathscr{P}_\sigma(\mathscr{H})$ such that $P_j \xrightarrow{w} xP$ and $P_j \xrightarrow{\lesssim} \mathbf{O}$.*

iii) *For each $y\in[0,1]$ there exists a sequence P_j in $\mathscr{P}_\sigma(\mathscr{H})$ such that $P_j \xrightarrow{w} yP$ and $P_j \xrightarrow{\lesssim} \mathbf{O}$.*

Proof. (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(i), and (iii)$\Rightarrow$(ii) are trivial. We prove the result by showing (ii)$\Rightarrow$(i) and then (i)$\Rightarrow$(iii)

(ii)$\Rightarrow$(i):

Condition (ii) implies the following:

(a) $Q_j \xrightarrow{w} yP$ and $y\leqq x \Rightarrow Q_j \xrightarrow{\lesssim} \mathbf{O}$.

(b) For P finite dimensional, $Q_j \xrightarrow{w} (x+y)P \Rightarrow Q_j \xrightarrow{\lesssim} \mathbf{O}$ if $0\leqq y\leqq x$ and $x+y\leqq 1$.

Result (a) is a straightforward consequence of Lemmas 3.1 and 3.2 but (b) needs more work because joint $\lesssim$ continuity of ortho-addition on $\mathscr{P}(\mathscr{H})$ has not been established. We now give a proof: By Lemma 3.1 there exist sequences P_{E_j} and P_{F_j} in $\mathscr{P}(\mathscr{H})$ such that $P_{E_j}\perp P_{F_j}\,\forall j$, $P_{E_j} \xrightarrow{w} xP$ and $P_{F_j} \xrightarrow{w} yP$. Lemma 3.2 and result (a) respectively give $P_{E_j} \xrightarrow{\lesssim} \mathbf{O}$ and $P_{F_j} \xrightarrow{\lesssim} \mathbf{O}$. Moreover, because P is a finite rank projection and $P\prec\mathbf{1}$, we can arrange that $K=(\cup_j E_j)^\perp$ is infinite dimensional and $\mathbf{O}\prec P_K$. Consequently, there exists a $\lesssim$ decreasing sequence P_{ϕ_j} in $\mathscr{P}_1(K)$ such that $\mathbf{O}\prec P_{\phi_j}\,\forall j$ and $P_{\phi_j} \xrightarrow{\lesssim} \mathbf{O}$. Since an appropriate subsequence of the P_{ϕ_j} can always be found, we may assume that $P_{F_j}\lesssim P_{\phi_j}\,\forall j$. Thus by **A4** and Lemma 3.3, $P_{E_j}+P_{F_j}\lesssim P_{E_j}+P_{\phi_j} \xrightarrow{\lesssim} \mathbf{O}$, giving the result (b). Repeated application (finitely many times) of result (b) eventually leads to $P\cong\mathbf{O}$ for P finite dimensional.

Now let P be infinite dimensional. There exists a sequence of finite dimensional projections P_j', such that $P_j'<P\,\forall j$ and, such that $P_j' \xrightarrow{\lesssim} P$. It is clear that for any $t\in[0,1]$, $P_j \xrightarrow{w} tP$ and $P_j \xrightarrow{\lesssim} \mathbf{O}$ together imply that for each fixed j, if the sequence

$P'_{jk} \xrightarrow{w} tP'_j$ as $k \to \infty$, then $P'_{jk} \xrightarrow{\leqq} \mathbf{O}$ as $k \to \infty$. This gives $P'_j \cong \mathbf{O} \forall j$. Again this leads to $P \cong \mathbf{O}$ and so wwe have (ii)$\Rightarrow$(i).

(i)$\Rightarrow$(iii). Let $x, y \in [0, 1]$ be such that $x + y = 1$. Then by Lemma 3.1 there exist sequences S_j and T_j in $\mathscr{P}_\sigma(\mathscr{H})$ such that $S_j \perp T_j \forall j$ and such that $S_j \xrightarrow{w} xP$ and $T_j \xrightarrow{w} yP$. Hence $S_j + T_j \xrightarrow{w} P \cong \mathbf{O}$. This clearly implies that $S_j \xrightarrow{\leqq} \mathbf{O}$ and $T_j \xrightarrow{\leqq} \mathbf{O}$ and the result follows. Hence the lemma is proved. □

Proposition 3.5. *Let $R \in \mathscr{P}_\sigma(\mathscr{H})$ be such that* $\dim R$ *and* $\dim R^\perp$ *are both infinite and such that $\mathbf{O} \prec R$ and $\mathbf{O} \prec R^\perp$. Let P_j and Q_j be sequences in $\mathscr{P}_\sigma(\mathscr{H})$ such that $P_j \xrightarrow{w} sR$, $Q_j \xrightarrow{w} tR$, $P_j \xrightarrow{\leqq} P$ and $Q_j \xrightarrow{\leqq} Q$. Then $Q \precsim P \Leftrightarrow t \leqq s$.*

Proof. First we note that $s, t \in [0, 1]$ and that given any $x \in [0, 1]$ there always exists a sequence S_j in $\mathscr{P}_\sigma(\mathscr{H})$ which weakly converges to xR. If $s = t$ then by Lemma 3.2, $P \cong Q$. So we let $s > t$; we wish to show that $Q \prec P$. Now there exists sequences $\tilde{Q}_j$ and $\tilde{P}_j$ in $\mathscr{P}_\sigma(\mathscr{H})$ such that $\tilde{Q}_j \xrightarrow{w} tR$, $\tilde{P}_j \xrightarrow{w} (s-t)R$, $\tilde{P}_j \perp \tilde{Q}_j \forall j$ and such that for all j, $\tilde{P}_j < R$, $\tilde{Q}_j < R$. Lemma 3.2 gives $\tilde{P}_j + \tilde{Q}_j \xrightarrow{\leqq} P$. Furthermore, there clearly exist $\tilde{P} \leqq R$ and $\tilde{Q} \leqq R$ such that $\tilde{P}_j \xrightarrow{\leqq} \tilde{P}$ and such that $\tilde{Q}_j \xrightarrow{\leqq} \tilde{Q}$. As $\mathbf{O} \prec R$ and $s - t > 0$, Lemma 3.4 gives $\mathbf{O} \prec \tilde{P}$. Because $\mathbf{O} \prec \mathbf{R}^\perp$ and $\dim R^\perp$ is infinite, there exists $S < R^\perp$ such that $\mathbf{O} \prec S$ and such that for all j large enough, $S \precsim \tilde{P}_j$. Lemmas 3.2 and 3.3 give $\tilde{Q}_j + S \xrightarrow{\leqq} \tilde{Q} + S$ and **A4** gives $\tilde{Q} + S \precsim P$. Hence we have $Q \cong \tilde{Q} \prec \tilde{Q} + S \precsim P$ as required. Since $\precsim$ is a linear preorder, the proof of the lemma is complete. □

Lemma 3.6. *Suppose there exists an infinite dimensional projection $R \in \mathscr{P}(\mathscr{H})$ such that $R \cong \mathbf{O}$. Let P and Q be any projections in $\mathscr{P}(\mathscr{H})$ such that $P \perp R$ and such that $Q < R$ with* $\dim(R - Q)$ *infinite. Suppose that the following are true for any $x \in [0, 1]$:*

i) *$P_j \xrightarrow{w} x(P+Q)$ with $P_j \xrightarrow{\leqq} P_0$.*

ii) *$\tilde{P}_j \xrightarrow{w} xP$ with $\tilde{P}_j \xrightarrow{\leqq} \tilde{P}_0$.*

Then $P_0 \cong \tilde{P}_0$.

Proof. Since $\dim(R - Q)$ is infinite (required in case $\dim P$ is finite) we can set up sequences S_j in $\mathscr{P}(\mathscr{H})$ and T_j, where $T_j \leqq R \forall j$, such that $S_j \perp T_k \forall j, k$, and such that $S_j \perp T_k \forall j, k$, and such that $S_j \xrightarrow{w} xP$, $T_j \xrightarrow{w} xQ$. Hence $S_j + T_j \xrightarrow{w} x(P+Q)$ and Lemmas 3.2 and 3.3 give the required result. □

Proposition 3.7. *Let K be the set of all $\phi \in \mathscr{H}$ such that $P_\phi \cong \mathbf{O}$. Define J to be $K^\perp$, the orthogonal complement of K in $\mathscr{H}$. Let the sequences S_j and T_j in $\mathscr{P}_\sigma(\mathscr{H})$ satisfy:*

i) *$S_j \xrightarrow{w} sP_J$, $S_j \xrightarrow{\leqq} S$.*

ii) *$T_j \xrightarrow{w} tP_J$, $T_j \xrightarrow{\leqq} T$.*

Then $s \leqq t \Leftrightarrow S \precsim T$.

Proof. If $t = s$ then by Lemma 3.2, $S \cong T$. So we let $t < s$ and the proposition will have been proved if we can show that $S \prec T$. Define J_0 to be J if J is infinite dimensional and otherwise to be any fixed but arbitrary infinite dimensional separable subspace of $\mathscr{H}$ containing J as a subspace. Let P_{J_1} and P_{J_2} be both infinite dimensional projections such that $P_{J_0} = P_{J_1} + P_{J_2}$ and such that $\mathbf{O} \prec P_{J_1}$, $\mathbf{O} \prec P_{J_2}$. There exist sequences S_{1j}, S_{2j}, T_{1j} and T_{2j} and projections S_1, S_2, T_1 and T_2 such that

(a) $S_{1j} \xrightarrow{w} sP_{J_1}$ and $S_{1j} \xrightarrow{\lesssim} S_1$,

(b) $S_{2j} \xrightarrow{w} sP_{J_2}$ and $S_{2j} \xrightarrow{\lesssim} S_2$,

(c) $T_{1j} \xrightarrow{w} tP_{J_1}$ and $T_{1j} \xrightarrow{\lesssim} T_1$,

(d) $T_{2j} \xrightarrow{w} tP_{J_2}$ and $T_{2j} \xrightarrow{\lesssim} T_2$.

Furthermore, since J_1 and J_2 are both infinite dimensional, we can arrange that $S_{1j} \leqq P_{J_1}$, $T_{1j} \leqq P_{J_1}$, $S_{2j} \leqq P_{J_2} T_{2j} \leqq P_{J_2}$ for all j and also that $S_1 \leqq P_{J_1}$, $S_2 \leqq P_{J_2}$, $T_1 \leqq P_{J_1}$ and $T_2 \leqq P_{J_2}$. Now $S_{1j}+S_{2j} \xrightarrow{w} sP_J$ and $T_{1j}+T_{2j} \xrightarrow{w} tP_J$ so that Lemmas 3.2 and 3.6 give $S_{1j}+S_{2j} \xrightarrow{\lesssim} S$ and $T_{1j}+T_{2j} \xrightarrow{\lesssim} T$. By Proposition 3.3, $T \cong T_1+T_2$ and $S \cong S_1+S_2$. Proposition 3.5 gives $T_1 \prec S_1$ and $T_2 \prec S_2$ so that by **A4**, $T_1+T_2 \prec T_1+S_2 \prec S_1+S_2$. Hence $T \prec S$. This completes the proof. □

Lemma 3.8. *Let J be defined as in Proposition 3.7. Then for each $P \in \mathscr{P}(\mathscr{H})$ there exist a real $t \in [0,1]$ and a sequence P_k in $\mathscr{P}(\mathscr{H})$ such that $P_k \xrightarrow{w} tP_J$ and $P_k \xrightarrow{\lesssim} P$.*

Proof. Define J_0 as in Proposition 3.7 but with the extra requirement that $J_0^\perp$ be infinite dimensional if $J \neq J_0$. Then, where appropriate, Lemma 3.6 implies that the statement of Lemma 3.8 is true if and only if the same statement is true when J is replaced by J_0.

It is clear that for any $t \in [0,1]$, there is a sequence in $\mathscr{P}(J_0)$ weakly converging to tP_{J_0}. Moreover such a sequence is automatically $\lesssim$ convergent to a (unique) $f(t) \in [\mathscr{P}(J_0)]$ for some $[\mathscr{P}(J_0)]$ valued function f defined on $[0,1]$. We wish to show that f is $\lesssim$ continuous, where the $\lesssim$ topology on $[\mathscr{P}(J_0)]$ is the obvious one arising from the natural ordering induced by $\lesssim$. Let the metric d'_w generate the weak operator topology on the unit ball of $\mathscr{B}(J_0)$ and let the pseudo-metric $d'_{\lesssim}$ generate the $\lesssim$ topology on $\mathscr{P}(\mathscr{H})$. We will also, in an obvious sense, regard $d'_{\lesssim}$ as a metric generating the above mentioned $\lesssim$ topology on $[\mathscr{P}(J_0)]$. For each $s \in [0,1]$, let the sequence $Q_j(s)$ in $\mathscr{P}(J_0)$ satisfy $Q_j(s) \xrightarrow{w} sP_{J_0}$, $Q_j(s) \xrightarrow{\lesssim} Q(s) \in \mathscr{P}(J_0)$, and we may assume that $d'_w(Q_j(s), sP_{J_0}) < 1/j$ for all j by taking a suitable subsequence. For any $\varepsilon > 0$ there exists $\delta > 0$ such that $|s-t| < \delta$ implies that $d'_w(sP_{J_0}, tP_{J_0}) < \varepsilon/2$. Choosing j_0 such $j_0 \geqq 4/\varepsilon$, we have $d'_w(Q_j(s), Q_j(t)) < \varepsilon$ whenever $|s-t| < \delta$ and $j \geqq j_0$. Hence it follows from the weak continuity of $\lesssim$ that for any $\varepsilon > 0$, there exist $\delta' > 0$ and j'_0 such that $d'_{\lesssim}(Q_j(s), Q_j(t)) < \varepsilon$ whenever $|s-t| < \delta'$ and $j \geqq j'_0$. Now suppose the sequence $t_k \to t$ on $[0,1]$. Then there exists k_0 such that $|t_k - t| < \delta'$ for $k \geqq k_0$, and hence we have $d'_{\lesssim}(Q_j(t_k), Q_j(t)) < \varepsilon$ whenever $k \geqq k_0$ and $j \geqq j'_0$. Since $d'_{\lesssim}(Q_j(t_k), Q(t_k)) \to 0$ and $d'_{\lesssim}(Q_j(t), Q(t)) \to 0$ as $j \to \infty$, we have $d_{\lesssim}(f(t_k), f(t)) = d_{\lesssim}(Q(t_k), Q(t)) < \varepsilon$ whenever $k \geqq k_0$. This gives the desired continuity of f. The continuity of f implies that the range of f is $\lesssim$ connected and is a $\lesssim$ interval. Since $f(0) = [\mathbf{O}]$ and $f(1) = [P_{J_0}]$ and since $P_{J_0} \cong \mathbf{1}_{\mathscr{H}}$, the lemma is proved. □

We remark that by Proposition 3.7, the function f defined in the above lemma is strictly $\lesssim$ increasing and hence injective. We are therefore able to make the following definition:

Definition 3.9. Let J be defined as in Proposition 3.7. Define the real $\lesssim$ continuous function $\mu : \mathscr{P}(\mathscr{H}) \to [0,1]$ as follows:

$\mu(P) = t$ if an only if there exists a sequence P_k in $\mathscr{P}(\mathscr{H})$ such that $P_k \xrightarrow{w} tP_J$ and $P_k \xrightarrow{\lesssim} P$. □

We now wish to extend the preorder $\lesssim$ to all of $\mathscr{P}(\mathscr{H}_0)$. The extend preorder will not necessarily satisfy all of the axioms of a CP and, in particular, axiom **A4** will be replaced by a weaker one. First, we prove a lemma.

Lemma 3.10. *Let* $[P] = \lesssim \sup \mathscr{P}_1(\mathscr{H})$. *Then* $[P] \cap \mathscr{P}_1(\mathscr{H})$ *is not empty.*

Proof. Let $P_{\phi_j} \xrightarrow{\lesssim} P \in \lesssim \sup \mathscr{P}_1(\mathscr{H})$, and let $P_{\phi_{j_k}}$ be a weakly convergent subsequence of P_{ϕ_j}. Then, since the range of the weak limit of P_{ϕ_j} is one dimensional, $P_{\phi_{j_k}} \xrightarrow{w} tP_{\phi_0}$ for some $\phi_0 \in \mathscr{H}$, where $0 < t \leqq 1$. By Corollary 3.6, $t = 1$ and the proof is complete. $\square$

Let $P_F \in \mathscr{P}(\mathscr{H}_0) \backslash \mathscr{P}(\mathscr{H})$, then $P_F \psi_0 \neq 0$. Define ψ_F to be $P_F \psi_0 / \|P_F \psi_0\|$, then ψ_F may be written as $a\phi_0 + b\psi_0 + c\xi$, where $\xi \perp \psi_0$, $\xi \perp \phi_0$ and ϕ_0 is a fixed vector in $\mathscr{H}$ such that P_{ϕ_0} is $\lesssim$-maximal in $\mathscr{P}_1(\mathscr{H})$ (Lemma 3.10). Accordingly, any $P_F \in \mathscr{P}(\mathscr{H}_0) \backslash \mathscr{P}(\mathscr{H})$ may be "canonically" decomposed, with respect to the pair (ϕ_0, ψ_0), as follows:

$$P_F = P_E + P_{\psi_F} = P_E + P_{a\phi_0 + b\psi_0 + c\xi},$$

where $b \neq 0$, $P_E \in \mathscr{P}(\mathscr{H})$, $\xi \perp \psi_0$, $\xi \perp \phi_0$. Moreover, the decomposition is essentially unique, that is:

$P_E + P_{a\phi_0 + b\psi_0 + c\xi} = P_{E'} + P_{a'\phi_0 + b'\psi_0 + c'\xi'}$ are canonical decompositions implies that $E = E'$, $a = a'$, $b = b'$, and $c\xi = c'\xi'$.

Lemma 3.11. *Let* P_{F_j} *be a net in* $\mathscr{P}(\mathscr{H}_0)$ *weakly converging to* $P_F \in \mathscr{P}(\mathscr{H}_0)$. *If* $P_{E_j} + P_{a_j\phi_0 + b_j\psi_0 + c_j\xi_j}$ *and* $P_E + P_{a\phi_0 + b\psi_0 + c\xi}$ *are the canonical decompositions of* P_{F_j} *and* P_F *respectively then*

i) $P_{E_j} \xrightarrow{w} P_E$ *and*

ii) $P_{a_j\phi_0 + b_j\psi_0 + c_j\xi_j} \xrightarrow{w} P_{a\phi_0 + b\psi_0 + c\xi}$.

Proof. Since we can always find appropriate subnets, we assume, without loss of generality, that $P_{E_j} \xrightarrow{w} A \in \mathscr{B}^+(\mathscr{H})$ and $P_{a_j\phi_0 + b_j\psi_0 + c_j\xi_j} \xrightarrow{w} kP_\phi$, for some positive k and some $\phi \in \mathscr{H}_0$. Thus we have

$$P_E + P_{a\phi_0 + b\psi_0 + c\xi} = A + kP_\phi \Rightarrow P_{a\phi_0 + b\psi_0 + c\xi}\psi_0 = kP_\phi \psi_0 \Rightarrow \phi = m(a\phi_0 + b\psi_0 + c\xi)$$

for some m such that $|m| = 1$. This in turn implies that $k = 1$, leading to the desired result. $\square$

Suppose that $a, b \in \mathbf{C}$ with $b \neq 0$. One easily shows that there exists a unique $t \geqq 0$ such that $|a|^2 + |b|^2 = |a + tb|^2$. Furthermore, t depends continuously on a and b. Now we define the function $\tilde{\mu}: \mathscr{P}(\mathscr{H}_0) \to \mathbf{R}$ as follows:

$$\tilde{\mu}(P_E) = \mu(P_E) \quad \text{if} \quad P_E \in \mathscr{P}(\mathscr{H}).$$

$$\tilde{\mu}(P_E + P_{a\phi_0 + b\psi_0 + c\xi}) = \mu(P_E) + \mu(P_{(a+tb)\phi_0 + c\xi}),$$

where

$$P_E + P_{a\phi_0 + b\psi_0 + c\xi} \in \mathscr{P}(\mathscr{H}_0) \backslash \mathscr{P}(\mathscr{H})$$

is in canonical decomposition and where $|a|^2 + |b|^2 = |a + tb|^2$, $t \geqq 0$.

Lemma 3.12. $\tilde{\mu}$ *satisfies the following:*

(a) $\tilde{\mu}$ *is weakly continuous on* $\mathscr{P}(\mathscr{H}_0) \backslash \mathscr{P}(\mathscr{H})$.

(b) *If* $P_\phi \in \mathscr{P}_1(\mathscr{H})$ *then there exists a sequence* P_{ϕ_j} *in* $\mathscr{P}(\mathscr{H}_0) \backslash \mathscr{P}(\mathscr{H})$ *such that* $P_{\phi_j} \xrightarrow{w} P_\phi$ *and such that* $\tilde{\mu}(P_{\phi_j}) \to \tilde{\mu}(P_\phi)$.

Proof. Let $P_{F_j} \xrightarrow{w} P_F \in \mathscr{P}(\mathscr{H}_0) \backslash \mathscr{P}(\mathscr{H})$ and let $P_{E_j} + P_{a_j\phi_0 + b_j\psi_0 + c_j\xi_j}$ and $P_E + P_{a\phi_0 + b\psi_0 + c\xi}$ be the canonical decomposition of P_{F_j} and P_F respectively. As $b \neq 0$, Lemma 3.11 implies that $P_{E_j} + P_{(a_j + t_j b_j)\phi_0 + c_j\xi_j} \xrightarrow{w} P_E + P_{(a+tb)\phi_0 + c\xi}$, where t and t_j have obvious meaning. The weak continuity of μ gives $\tilde{\mu}(P_{F_j}) \to \tilde{\mu}(P_F)$ as required. This proves part (a).

To prove part (b) let $P_{a_j\phi_0 + b_j\psi_0 + c_j\xi_j}$ and $P_{a\phi_0 + c\xi}$ be the "canonical forms" of P_{ϕ_j} and P_ϕ respectively. Explicit calculation shows that the choice $c_j\xi_j = c\xi$, $b_j = a\sqrt{1/j}$ and $a_j = a\sqrt{1 - 1/j}\,\forall j$, ensures the desired convergence for the case where $a \neq 0$. If $a = 0$ we make the choice $a_j = 0$, $\xi_j = \xi$, $c_j = c\sqrt{1 - 1/j}$ and $b_j = c\sqrt{1/j}\ \forall j$. □

Define the preorder $\trianglelefteq$ on $\mathscr{P}(\mathscr{H}_0)$ as follows:

$$P \trianglelefteq Q \Leftrightarrow \tilde{\mu}(P) \leqq \tilde{\mu}(Q).$$

$\trianglelefteq$ is clearly an ECP on $\mathscr{P}(\mathscr{H}_0)$ and as such, an extension of $\lesssim$. It also satisfies (cf. axiom **A4**):

A'4. Let P_E, $P_F \in \mathscr{P}(\mathscr{H})$ and let P_ϕ, $P_\psi \in \mathscr{P}_1(\mathscr{H}_0)$.

(a) If $P_E \perp P_\phi$ and $P_E \perp P_\psi$ then $P_\psi \trianglelefteq P_\phi \Leftrightarrow P_E + P_\psi \trianglelefteq P_E + P_\phi$.

(b) If $P_E \perp P_\phi$ and $P_F \perp P_\phi$ then $P_F \trianglelefteq P_E \Leftrightarrow P_F + P_\phi \trianglelefteq P_E + P_\phi$.

Lemma 3.13. *Let P_E, $P_F \in \mathscr{P}(\mathscr{H})$ be finite dimensional and let P_ϕ, $P_\psi \in \mathscr{P}_1(\mathscr{H})$ be such that*

i) $P_E \perp P_\phi$, $P_F \perp P_\psi$.

ii) $P_F \lesssim P_E$, $P_\psi \lesssim P_\phi$.

Then $P_F + P_\psi \lesssim P_E + P_\phi$.

Proof. If $P_\psi \cong \mathbf{O}$ then the result is clearly true. Now let $\mathbf{O} \prec P_\psi$. There exists $P_\xi \in \mathscr{P}_1(\mathscr{H})$ such that $P_\xi \perp P_F$, $P_\xi \perp P_E$ and $P_\xi \lesssim P_\psi$. Using the continuity properties of Lemma 3.12 we can find $P_\chi \in \mathscr{P}_1(\mathscr{H}_0)$, where χ is a linear combination of ξ and ψ_0 and satisfies $P_\psi \trianglelefteq P_\chi \trianglelefteq P_\phi$. By axiom **A'4** we have: $P_F + P_\psi \trianglelefteq P_F + P_\chi \trianglelefteq P_E + P_\chi \trianglelefteq P_E + P_\phi$. Hence the result. □

Proposition 3.14. *Let $P_E \in \mathscr{P}(\mathscr{H})$ be finite dimensional and let $P_\phi \in \mathscr{P}_1(\mathscr{H})$ be such that $P_E \perp P_\phi$. Let sequences P_{E_j} and P_{ϕ_j} in $\mathscr{P}(\mathscr{H})$ and in $\mathscr{P}_1(\mathscr{H})$ respectively be both $\lesssim$ increasing with P_{E_j} finite dimensional for all j. Suppose $P_{E_j} \xrightarrow{\lesssim} P_E$ and $P_{\phi_j} \xrightarrow{\lesssim} P_\phi$ with $P_{E_j} \perp P_{\phi_j} \forall j$, then $P_{E_j} + P_{\phi_j} \xrightarrow{\lesssim} P_E + P_\phi$.*

Proof. By Lemma 3.13, $P_{E_j} + P_{\phi_j}$ is $\lesssim$ increasing, hence $\lesssim$ convergent. P_E is finite dimensional, hence there exist strictly $\lesssim$ increasing sequences $P_{\tilde{E}_j}$ and $P_{\tilde{\phi}_j}$ in $\mathscr{P}(\mathscr{H})$ and $\mathscr{P}_1(\mathscr{H})$ respectively such that $P_{\tilde{E}_j}$ is finite dimensional for all j, $P_{\tilde{E}_j} \xrightarrow{\lesssim} P_E$, $P_{\tilde{\phi}_j} \xrightarrow{\lesssim} P_\phi$ and such that $P_{\tilde{E}_j} \perp P_{\tilde{\phi}_k}$ for all j and k. By Proposition 3.3, $P_{\tilde{E}_j} + P_{\tilde{\phi}_j} \xrightarrow{\lesssim} P_E + P_\phi$; moreover $P_{\tilde{E}_j} + P_{\tilde{\phi}_j}$ is strictly $\lesssim$ increasing. Clearly the sequences $P_{\tilde{E}_j} + P_{\tilde{\phi}_j}$ and $P_{E_j} + P_{\phi_j}$ converge to the same limit and the result follows. □

Proposition 3.15. *Let $P_{\phi_j} \in \mathscr{P}_1(\mathscr{H})$: $1 \leqq j \leqq N < \infty$ be such that $P_{\phi_j} \perp P_{\phi_k}$ if $j \neq k$. Then*

$$\mu\left(\sum_{j=1}^{N} P_{\phi_j}\right) = \sum_{j=1}^{N} \mu(P_{\phi_j}).$$

Proof. Suppose that the result is true for some $N \geqq 1$. Let $P_{\psi_j}: 1 \leqq j \leqq N+1$ satisfy the hypothesis of the proposition. Let $P_E = \sum_{j=1}^{N} P_{\psi_j}$, $\mu(P_E) = t$ and $\mu(P_{\psi_{N+1}}) = s$, then $\mu(P_E) = \sum_{j=1}^{N} \mu(P_{\phi_j})$. Clearly we can find $\lesssim$ increasing sequences P_{E_j} and P_{ϕ_j}, with the P_{E_j} finite dimensional, such that

(i) $P_{E_j} \xrightarrow{w} tP_J$, $P_{\phi_j} \xrightarrow{w} sP_J$ (J as defined in Proposition 3.7)

(ii) $P_{E_j} \xrightarrow{\lesssim} P_E$, $P_{\phi_j} \xrightarrow{\lesssim} P_{\psi_{N+1}}$ and

(iii) $P_{E_j} \perp P_{\phi_j} \forall j$.

By Proposition 3.14, $P_{E_j} + P_{\phi_j} \xrightarrow{\lesssim} P_E + P_{\psi_{N+1}}$, so that $\mu(P_E + P_{\psi_{N+1}}) = s + t$. Hence

$$\mu\left(\sum_{j=1}^{N+1} P_{\psi_j}\right) = \mu(P_{\psi_{N+1}}) + \mu\left(\sum_{j=1}^{N} P_{\psi_j}\right) = \sum_{j=1}^{N+1} \mu(P_{\psi_j}),$$

and the result is true for $N+1$. But it is trivially true for $N=1$ and the proof is complete. □

Corollary 3.16. *Let the sequence P_{ϕ_j} in $\mathscr{P}_1(\mathscr{H})$ be such that $P_{\phi_j} \perp P_{\phi_k}$ if $j \neq k$. Then*

$$\mu\left(\sum_{j \in \mathbb{N}} P_{\phi_j}\right) = \sum_{j \in \mathbb{N}} \mu(P_{\phi_j}).$$

Proof. Define $P_{E_n} = \sum_{j=1}^{n} P_{\phi_j}$. Then $\forall n$, $\sum_{j=1}^{n} \mu(P_{\phi_j}) = \mu(P_{E_n}) \leqq \mu\left(\sum_{j=1}^{\infty} P_{\phi_j}\right)$. Hence the sum $\sum_{j=1}^{\infty} \mu(P_{\phi_j})$ is absolutely convergent. Since μ is $\lesssim$ continuous and $P_{E_n} \xrightarrow{\lesssim} \sum_{j=1}^{\infty} P_{\phi_j}$, the result follows. □

4. The Main Result

Corollary 3.16 together with Proposition 2.3 imply that μ is in fact a completely additive Gleason measure on $\mathscr{P}(\mathscr{H})$. Hence we have the main result:

Theorem 4.1. *Let $\mathscr{H}$ be an infinite dimensional (not necessarily separable) Hilbert space and $\lesssim$ a* CP *on $\mathscr{P}(\mathscr{H})$. Then the following statements are equivalent:*

i) *$\lesssim$ can be implemented by a normal state.*

ii) *$\lesssim$ is weakly continuous.* □

This result shows that, subject to the usual condition that quantum expectation values should respect the weak operator topology on the appropriate algebra of observables, there is nothing gained by departing from the traditional formulation in which expectation values are represented by normal states.

Acknowledgements. The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency, UNESCO and the Swedish Development Cooperation for their support and hospitality at the International Centre for Theoretical Physics, Trieste, Italy.

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Communicated by H. Araki

Received August 19, 1989; in revised form Februar 27, 1990