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# Uniqueness of Gibbs States for General $P(\varphi)_2$-Weak Coupling Models by Cluster Expansion

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**Abstract.** We consider quantum fields with weak coupling in two space-time dimensions. We prove that the set of their ultraregular Gibbs states consists of only one point and this point is an extremal Gibbs state.

## 1. Introduction

In this paper we prove the uniqueness of Gibbs states for general $P(\varphi)_2$-weak coupling models. This extends to the case of the $P(\varphi)_2$-model results proven before for weak trigonometric interactions [AHK1] and exponential interactions [Ze1, Gie1].

The method we use is the method of cluster expansion [GJS1, 2]. In a companion paper [AHKZ] we used other methods (essentially FKG-order) to yield a uniqueness result and the global Markov property for the $\varphi_2^4$-models. Weak coupling $P(\varphi)_2$-models have been constructed by Glimm et al. [GJS1, 2], see also [GlJa], using their method of cluster expansion, starting from the models given by an interaction confined to a bounded space-time region.

In analogy with classical statistical mechanics, see [Do, LaRu] and also e.g. [Pr] one can define Gibbs states associated with quantum fields given in a bounded space-time region. This has been first discussed by Guerra et al. [GRS1, 2], see also [Si] and pursued e.g. in [FrSi, DoMi, DoPe, AHK1]. Roughly speaking, the construction of Gibbs states corresponds to Kolmogorov's construction of Markov processes from Markov kernels ("local specification"). For these general connections see [Fö2] (who also discusses the relations with Martin-Dynkin's boundary). The work on Gibbs states and their local specifications from a potential theoretical point of view, applied to the study of quantum fields, has been pursued in [AHK1, Gie1, 2, Ze1–5, Rö1–3, RöZ]. The structure of Gibbs states is rather well understood in classical statistical mechanics, both in specific models (like in 2-dimensional Ising ferromagnets, where one has a complete structure theory by work of [Aiz, Hig], see also [Me]) and in general models (Pirogov-Sinai theory [Sin]; this theory uses preceding work by Dobrushin, Minlos-Sinai, Gercik).

For an extension of Pirogov-Sinai results to the study of phase transitions in quantum field theory see [Im] (see also [GlJa, FrSi] and references therein). Structural results on the space of Gibbs states have been given for free fields ([HoSt, Rö3]) and for the case where the fields have trigonometric or exponential interaction [Ze4]. That the set of tempered Gibbs states reduces to a point ("uniqueness of Gibbs states") was first proven for trigonometric interactions in [AHK1]. The result was extended to general exponential interactions in [Ze1] (see also [Gie1]). Independence of classical boundary conditions was shown for $\varphi_2^4$-models in [GRS1, 2] and for weak coupling $P(\varphi)_2$-models in [FrSi, GJS1, 2].

For uniqueness results in statistical mechanics of lattice systems see [Do, AHKO, Föl, BePi, MaNi, Ge].

In this paper we give the first proof of uniqueness of Gibbs measures for general weak coupling $P(\varphi)_2$-models. Basic ingredients of our proof are:
a) estimates on the solutions of the Dirichlet problem with distributional data. Estimates of this type were first obtained in [AHK1], see also [DoMi2], and considerably refined in [Rö1–3]. These estimates yield a basic regularity of free conditional expectations;
b) an adaptation of the method of cluster expansion, originally developed in [GJS1, 2], see also [GlJa];
c) a representation of "ultraregular measures" in the sense of Fröhlich and Simon [FrSi]. Further discussions of related models and the related – but far from immediate! – proof of the global Markov property is given in [AHKZ].

The present paper is structured as follows. We start with some basic definitions, we then formulate the basic uniqueness theorem and prove it. Two technical lemmas are in the appendices.

During the final writing of this paper the terrible news of the sudden death of our dear friend and coworker Raphael Høegh-Krohn reached us. We deeply mourn his departure, with great sorrow and gratefulness for all he did for us.

S. Albeverio B. Zegarlinski

## 2. Uniqueness and Extremality

Let $\mathscr{F}$ be a family of open bounded sets $\Lambda \subset \mathbb{R}^2$ with piecewise $C^1$ boundaries $\partial\Lambda$. Let $\mathscr{F}_0 = \{\Lambda_n \in \mathscr{F}\}_{n\in N}$ be an increasing sequence consisting of squares centred at the origin and such that $\bigcup_n \Lambda_n = \mathbb{R}^2$. Let $G \equiv (-\Delta + m_0^2)^{-1}$, respectively, for $\Lambda \in \mathscr{F}$, let $G^{\partial\Lambda} \equiv (-\Delta^{\partial\Lambda} + m_0^2)^{-1}$, with a positive constant $m_0^2$ and $\Delta$ the two dimensional Laplacian respectively, $\Delta^{\partial\Lambda}$ the Laplacian with Dirichlet conditions on $\partial\Lambda$. For a real function $f$ on $\mathbb{R}^2$ we shall introduce the notation

$$\|f\|_{\pm 1} \equiv \|G^{\mp 1/2} f\|_{L_2}.$$

Let $(\mathscr{S}', \mathscr{B})$ be the space of real tempered distributions in $\mathbb{R}^2$ with Borel $\sigma$-algebra generated by the weak topology. For an open set $\Lambda \subset \mathbb{R}^2$ let $\mathscr{B}_\Lambda$ be the smallest $\sigma$-algebra such that all coordinate functions (evaluation functions) $\{\varphi(f) : f \in \mathscr{S}, \operatorname{supp} f \subset \Lambda\}$ are $\mathscr{B}_\Lambda$ measurable. For a closed set $\Lambda \subset \mathbb{R}^2$ we define $\mathscr{B}_\Lambda \equiv \{\cap \mathscr{B}_{\Lambda'} : \Lambda'$ open, $\Lambda \subset \Lambda'\}$. A probability measure $\mu$ on $(\mathscr{S}', \mathscr{B})$ is called regular if for any $\Lambda \in \mathscr{F}$

there is a constant $0<c_\Lambda<\infty$ such that

$$\mu e^{\varphi(f)} \leqq e^{c_\Lambda(\|f\|_{-1}+\|f\|_{-1}^p)} \tag{1}$$

for all $f\in\mathscr{S}$, $\operatorname{supp} f\subset\Lambda$ and some $p\in\mathbb{N}$ (we use the notations $\mu F$ for the expectation of $F$ with respect to $\mu$, i.e. $\mu F=\int F\,d\mu$). In particular the Gaussian measures $\mu_0$ and $\mu_0^{\partial\Lambda}$ with mean zero and covariances $G$ respectively $G^{\partial\Lambda}$ fulfill (1) with $c_\Lambda\equiv\frac{1}{2}$. Moreover all probability measures $\mu$ associated with Euclidean quantum fields in two space-time dimensions with exponential, trigonometric and polynomial interactions satisfy (1), as we shall discuss in App. 1. In fact such measures satisfy the estimate

$$\mu e^{\varphi(f)} \leqq \exp[a\|G*f\|_{L_1}+b\|f\|_{-1}^2+c\|G*f\|_{L_p}^p] \tag{1$'$}$$

for some $b>0$, $a,c\geqq 0$, $p\geqq 4$, with $*$ for convolution, which implies (1) with $c_\Lambda=C|\Lambda|^{1/2}$ for some constant $C$ (where $|\cdot|$ means volume). The set of all regular measures is denoted by $M_\psi$. For $\Lambda\in\mathscr{F}$ let $\psi_z^{\partial\Lambda}(x)$, $(z\in\partial\Lambda,\ x\in\Lambda)$ be the Poisson kernel associated to the operator $(-\Delta+m^2)$ [i.e. $\psi_z^{\partial\Lambda}(x)$ is $-\Delta+m^2$-harmonic in $\Lambda$ and is $\delta_z(x)$ on $\partial\Lambda$]. Let $\{h_\kappa\in C^\infty(\mathbb{R}^2)\}$ be a sequence converging to the $\delta$-function as $\kappa\to\infty$ and for $\eta\in\mathscr{S}'$ denote $\eta_\kappa(x)\equiv\eta(h_\kappa(\cdot-x))$. Then for $\mu\in M_\psi$ the sequence $\psi_{\eta_\kappa}^{\partial\Lambda}(x)\equiv\int\psi_z^{\partial\Lambda}(x)\,\eta_\kappa(z)\,dz$ converges in $L_p(\mu)$ for any $1\leqq p<\infty$. Moreover its limit $\psi_\eta^{\partial\Lambda}(x)\equiv\lim_{\kappa\to\infty}\psi_{\eta_\kappa}^{\partial\Lambda}(x)$ is for $\mu$ a.e. $\eta\in\mathscr{S}'$ a solution of the following Dirichlet problem in $\Lambda$ (see [Rö1, AlHø, DoM]):

$$\left.\begin{aligned}(-\Delta+m_0^2)\,\psi_\eta^{\partial\Lambda}(x)&=0 \quad \text{for} \quad x\in\Lambda\\ \psi_\eta^{\partial\Lambda}(x)&=\eta \quad \text{for} \quad x\in\operatorname{int}\Lambda^c\end{aligned}\right\}. \tag{2}$$

Let $\Lambda\in\mathscr{F}$ be a log-normal set, i.e. such that: for any $x\in\Lambda$, $G^{\partial\Lambda}(x,y)\to 0$ as $d(y,\partial\Lambda)\to 0$ and $K^{\partial\Lambda}(x,x)\equiv\int_{\partial\Lambda}dz\,\psi_z^{\partial\Lambda}(x)\,G(z,x)$ is in $L_p(\Lambda,d_2x)$ for any $1\leqq p<\infty$. We remark that, in particular, all rectangles are log-normal (see e.g. [Si]). From now on we will restrict ourselves to the subfamily of log-normal sets, denoting it by the same letter $\mathscr{F}$. It follows from (1) and the definition of $\psi_\eta^{\partial\Lambda}(x)$ that for any log-normal set $\Lambda$ and $\mu\in M_\psi$,

$$\|\psi_\eta^{\partial\Lambda}(\cdot)\|_{L_p(\Lambda,d_2x)}<\infty,\ \mu-\text{a.e.} \tag{3}$$

For a semibounded polynomial $P(\cdot)$ and $\Lambda\in\mathscr{F}$ define an interaction functional by

$$U_\Lambda(\varphi):=\lambda\int_\Lambda :P(\varphi):_0(x)\,d_2x \tag{4}$$

with $0<\lambda<\infty$ and $:\ :_0$ the normal ordering with respect to $\mu_0$ (cf. e.g. [Si, GlJa]). [It is assumed that the polynomial $P(\cdot)$ is normalized in the sense of its constant term being equal to zero.] By basic estimates following from e.g. [GlJa] and [Si], using the assumed regularity property of $\mu$ and estimates on $\psi_\eta^{\partial\Lambda}$ given in the above references we obtain, for any $\mu\in M_\psi$,

$$U_\Lambda(\varphi+\psi_\eta^{\partial\Lambda})\in L_p(\mu_0^{\partial\Lambda}\otimes\mu) \tag{5}$$

and

$$0<\mu_0^{\partial\Lambda}e^{-U_\Lambda(\varphi+\psi_\eta^{\partial\Lambda})}<\infty\ \ \mu\text{-a.e.}\ \eta\,. \tag{6}$$

Here we denote by $\varphi$ the integration variable with respect to $\mu_0^{\partial\Lambda}$ and the $\mu$-integration is with respect to the variable $\eta$ in $\psi_\eta^{\partial\Lambda}$. This implies that the probability kernels

$$E^\eta_{\Lambda^c}(F) := \frac{\mu_0^{\partial\Lambda} e^{-U_\Lambda(\varphi+\psi_\eta^{\partial\Lambda})} F(\varphi+\psi_\eta^{\partial\Lambda})}{\mu_0^{\partial\Lambda} e^{-U_\Lambda(\varphi+\psi_\eta^{\partial\Lambda})}} \tag{7}$$

are, for any $\mu_0^{\partial\Lambda}\otimes\mu$-measurable bounded $F$, well defined for all $\eta\in\Omega$ with some Borel subset $\Omega\subset\mathscr{S}'$, $\mu(\Omega)=1$ $\forall\mu\in M_\psi$. Denote $\Sigma\equiv\mathscr{B}\cap\Omega$ and, for $\Lambda\subset\mathbb{R}^2$, $\Sigma_\Lambda\equiv\mathscr{B}_\Lambda\cap\Omega$. The family $\mathscr{E}\equiv\{E^\eta_{\Lambda^c}\}_{\Lambda\in\mathscr{F},\eta\in\Omega}$ is a local specification in the sense of ([Fö, Pr]), i.e. fulfills the following conditions:

a) $E^\eta_{\Lambda^c}(\cdot)$ are probability measures on $(\Omega,\Sigma)$ such that for any $F\in\Sigma_{\Lambda^c}$, $E^\eta_{\Lambda^c}(F) = \delta_\eta(F)$, (we write $F\in\Sigma_{\Lambda^c}$ to mean $F$ is $\Sigma_{\Lambda^c}$-measurable).

b) For any $F\in\Sigma$ the function $\Omega\ni\eta\mapsto E^\eta_{\Lambda^c}(F)$ is $\Sigma_{\Lambda^c}$-measurable.

c) The compatibility conditions: For any $F\in\Sigma$ if $\Lambda_1\subset\Lambda_2$ then $E^\eta_{\Lambda_2^c}E^{\cdot}_{\Lambda_1^c}(F) = E^\eta_{\Lambda_2^c}(F)$. For a detailed construction of local specifications, in this sense, for euclidean fields in two dimensions we refer to [Rö2] (see also [AHK, Ze2]).

A probability measure $\mu$ on $(\Omega,\Sigma)$ is called a *Gibbs measure* for $\mathscr{E}$ if for any $\Lambda\in\mathscr{F}$,

$$\mu E^{\cdot}_{\Lambda^c}(F) = \mu F \tag{8}$$

for all $F\in L_1(\mu)$. The set of all Gibbs measures for $\mathscr{E}$ is denoted by $\mathscr{G}(\mathscr{E})$. By $\partial\mathscr{G}(\mathscr{E})$ we denote the subset consisting of Gibbs measures which have no nontrivial convex linear representations in terms of other elements from $\mathscr{G}(\mathscr{E})$.

For the construction of Gibbs measures for euclidean fields with polynomial interactions in two dimensions see e.g. [GlJa, Si, GRS, FrSi] and references therein, see also [Rö2]. Denote by

$$\alpha_\infty \equiv \lim_{\mathscr{F}_0} \frac{1}{|\Lambda|} \ln \mu_0 e^{-U_\Lambda(\varphi)} \tag{9}$$

the infinite volume pressure for polynomial interactions (4) (this exists [Gu], see also e.g. the above references). Let $\mu\in\mathscr{G}(\mathscr{E})$. Following [FrSi] we call $\mu$ *ultraregular* if for any rectangle $\Lambda$,

$$\mu_{|\Sigma_\Lambda}(\cdot) = e^{-\alpha_\infty|\Lambda|}\mu_{0|\Sigma_\Lambda}(\varrho^{\partial\Lambda}e^{-U_\Lambda}\cdot), \tag{10}$$

with the boundary density $\varrho^{\partial\Lambda}\in\Sigma_{\partial\Lambda}$ fulfilling

a) $\varrho^{\partial\Lambda}>0$ $\mu_0$ – a.e.,

b) $\varrho^{\partial\Lambda}\in L_p(\mu_0)$ for $1\leq p<\infty$ and

$$\|\varrho^{\partial\Lambda}\|_p \leq \exp(c|\partial\Lambda|), \tag{11}$$

with a constant $0<c<\infty$ independent of the rectangle $\Lambda$ (for $|\partial\Lambda|>1$). By the notation $\mu_{|\mathscr{B}}$, we denote the restriction of $\mu$ to a sub $\sigma$-algebra $\mathscr{B}$. It was shown in [FrSi] that if $1<p<\frac{4}{3}$ then

$$\|\varrho^{\partial\Lambda}\|_p \leq 1 \tag{12}$$

for all sufficiently big rectangles $\Lambda$. The set of all ultraregular measures $\mu\in\mathscr{G}(\mathscr{E})$ will be denoted by $\mathscr{G}_{\mathrm{ur}}(\mathscr{E})$. The class of ultraregular $P(\varphi)_2$ measures is quite rich and contains in particular [FrSi, Theorem 7.2] all measures constructed

a) by the Glimm-Jaffe-Spencer (GJS) cluster expansion,

b) via monotonicity arguments [half-Dirichlet $=HD-$ states for $P=Q-h\varphi$ ($Q$ even)],

c) the maximal measures $\mu_P$ for any semibounded polynomial $P$ constructed in the cited paper.

We prove the following result on uniqueness of Gibbs states.

**Theorem.** *Let $\mathscr{E}$ be the local specification corresponding to the $\lambda P(\varphi)_2$-interaction given by* (4) *with $0<\lambda<\lambda_0$ for some sufficiently small $\lambda_0$. Let $\mu=\lim_{\mathscr{F}_0} E^{\eta=0}_{\Lambda^c}$, with $E^\eta_{\Lambda^c}\in\mathscr{E}$. Then $\mathscr{G}_{\mathrm{ur}}(\mathscr{E})=\{\mu\}\subseteqq\partial\mathscr{G}(\mathscr{E})$.*

*Proof.* Let $\mu,\tilde{\mu}\in\mathscr{G}_{\mathrm{ur}}(\mathscr{E})$ and let $\varrho^{\partial\Lambda}$, respectively $\tilde{\varrho}^{\partial\Lambda}$, be the corresponding boundary density. Using GJS cluster expansion we will show that if the coupling constant $0<\lambda$ is sufficiently small, then for any polynomial function $F$

$$\lim_{\mathscr{F}_0}\mu\otimes\tilde{\mu}|E^\eta_{\Lambda^c}F-E^{\tilde{\eta}}_{\Lambda^c}F|=0 \tag{13}$$

(here $\eta$ respectively $\tilde{\eta}$ is $\mu$-integration variable respectively $\tilde{\mu}$-integration variable).

This equality implies uniqueness and extremality of the ultraregular Gibbs measures. Using ultraregularity (10) of $\mu$ and $\tilde{\mu}$ and the definition (7) of $E_{\Lambda^c}$ for any polynomial function $F\in\Sigma_{\Lambda_0}$, $\Lambda_0\subset\Lambda\in\mathscr{F}_0$ we have

$$\begin{aligned}\mu\otimes\tilde{\mu}|E^\eta_{\Lambda^c}(F)-E^{\tilde{\eta}}_{\Lambda^c}(F)|=e^{-2\alpha_\infty|\Lambda|}\mu_0\otimes\tilde{\mu}_0[\varrho^{\partial\Lambda}\tilde{\varrho}^{\partial\Lambda}|\mu_0^{\partial\Lambda}\otimes\tilde{\mu}_0^{\partial\Lambda}\\ \times e^{-[U_\Lambda(\varphi+\psi^{\partial\Lambda}_\eta)+U_\Lambda(\tilde{\varphi}+\psi^{\partial\Lambda}_{\tilde{\eta}})]}(F(\varphi+\psi^{\partial\Lambda}_\eta)-F(\tilde{\varphi}+\psi^{\partial\Lambda}_{\tilde{\eta}}))|],\end{aligned} \tag{14}$$

where $\mu_0$, $\tilde{\mu}_0$ respectively $\mu_0^{\partial\Lambda}$, $\tilde{\mu}_0^{\partial\Lambda}$ are the free field measures with free respectively Dirichlet boundary condition on $\partial\Lambda$, and $\varphi$ respectively $\eta$ are the integration variables with respect to $\mu_0^{\partial\Lambda}$ respectively $\mu_0$ and analogously $\tilde{\varphi}$ respectively $\tilde{\eta}$ are the corresponding integration variables of $\tilde{\mu}_0^{\partial\Lambda}$ respectively $\tilde{\mu}_0$. We can and do assume that $d(\Lambda_0,\partial\Lambda)>3$. For $\Lambda\in\mathscr{F}_0$ let $\chi^{\partial\Lambda}\in C^\infty(\Lambda)$ be such that $0\leqq\chi^{\partial\Lambda}(x)\leqq1$ $\forall x\in\Lambda$ and

$$\chi^{\partial\Lambda}(x)=\begin{cases}0 & \text{for}\quad d(x,\partial\Lambda)\leqq2\\ 1 & \text{for}\quad d(x,\partial\Lambda)\geqq3\,.\end{cases} \tag{15}$$

We will also assume that $|\Delta\chi^{\partial\Lambda}|$ and $|\nabla\chi^{\partial\Lambda}|$ are bounded by a constant independent of $\Lambda\in\mathscr{F}_0$.

By change of the integration variables

$$\varphi\to\varphi+\chi^{\partial\Lambda}\psi^{\partial\Lambda}_\eta,\qquad\tilde{\varphi}\to\tilde{\varphi}+\chi^{\partial\Lambda}\psi^{\partial\Lambda}_{\tilde{\eta}} \tag{16}$$

in the integration with $\mu_0^{\partial\Lambda}\otimes\tilde{\mu}_0^{\partial\Lambda}$ we get (using the Cameron-Martin type translation formula for Gaussian measures, see e.g. [Fr])

$$\begin{aligned}&\mu\otimes\tilde{\mu}|E^\eta_{\Lambda^c}(F)-E^{\tilde{\eta}}_{\Lambda^c}(F)|=e^{-2\alpha_\infty|\Lambda|}\mu_0\otimes\tilde{\mu}_0\{\varrho^{\partial\Lambda}\tilde{\varrho}^{\partial\Lambda}\\ &\quad\times|\mu_0^{\partial\Lambda}\otimes\tilde{\mu}_0^{\partial\Lambda}(\exp[-U_\Lambda(\varphi+(1-\chi^{\partial\Lambda})\psi^{\partial\Lambda}_\eta)-U_\Lambda(\tilde{\varphi}+(1-\chi^{\partial\Lambda})\psi^{\partial\Lambda}_{\tilde{\eta}})]\\ &\quad\times\exp[-\varphi((-\Delta+m^2)\chi^{\partial\Lambda}\psi^{\partial\Lambda}_\eta)-\tilde{\varphi}((-\Delta+m^2)\chi^{\partial\Lambda}\psi^{\partial\Lambda}_{\tilde{\eta}})]\cdot(F(\varphi)-F(\tilde{\varphi})))|\\ &\quad\times[\exp-\tfrac{1}{2}(\|\chi^{\partial\Lambda}\psi^{\partial\Lambda}_\eta\|^2_{+1}+\|\chi^{\partial\Lambda}\psi^{\partial\Lambda}_{\tilde{\eta}}\|^2_{+1})]\}\,.\end{aligned} \tag{17}$$

Since $\psi_\eta^{\partial\Lambda}$ (and $\psi_{\tilde\eta}^{\partial\Lambda}$) fulfill in $\Lambda\in\mathscr{F}_0$ the equation

$$(-\Delta+m^2)\,\psi_\eta^{\partial\Lambda}(x)=0 \tag{18}$$

for $\mu$-a.a. $\eta\in\mathscr{S}'$ and any $\mu\in M_\psi$ [cf. (2)], so the change of integration variables (16) is possible. Moreover, the function

$$h_\eta^{\partial\Lambda}(x)\equiv(-\Delta+m^2)\,(\chi^{\partial\Lambda}(x)\,\psi_\eta^{\partial\Lambda}(x))=(-\Delta\chi^{\partial\Lambda})\psi_\eta^{\partial\Lambda}-2\nabla\chi^{\partial\Lambda}\cdot\nabla\psi_\eta^{\partial\Lambda}$$

is supported in the set $\{x\in\Lambda:2\leqq d(x,\partial\Lambda)\leqq 3\}$. Analogous statements hold for $h_{\tilde\eta}^{\partial\Lambda}\equiv(-\Delta+m^2)\chi^{\partial\Lambda}\psi_{\tilde\eta}^{\partial\Lambda}$. For fixed $\psi_\eta^{\partial\Lambda}$ and $\psi_{\tilde\eta}^{\partial\Lambda}$ (such that the above holds) and a subset $X\subseteqq\Lambda$ define

$$\begin{aligned}V_X(\varphi,\tilde\varphi)\equiv{}&U_X(\varphi+(1-\chi^{\partial\Lambda})\psi_\eta^{\partial\Lambda})+U_X(\tilde\varphi+(1-\chi^{\partial\Lambda})\psi_{\tilde\eta}^{\partial\Lambda})\\&-\int_X\varphi(x)\,h_\eta^{\partial\Lambda}(x)\,d_2x-\int_X\tilde\varphi(x)\,h_{\tilde\eta}^{\partial\Lambda}(x)\,d_2x\,.\end{aligned}\tag{20}$$

Note that if $X\subseteqq\{x\in\Lambda:d(x,\partial\Lambda)\geqq 3\}$ then we have the symmetry:

$$V_X(\varphi,\tilde\varphi)=V_X(\tilde\varphi,\varphi)\,. \tag{21}$$

Now we apply GJS cluster expansion ([GJS, GlJa]) to the unnormalized measure

$$\mu_0^{\partial\Lambda}\otimes\tilde\mu_0^{\partial\Lambda}\{e^{-V_\Lambda(\varphi,\tilde\varphi)}\cdot\} \tag{22}$$

[in $L^1(\mu_0\otimes\tilde\mu_0)$-sense, as specified below]. For that we define the family of interpolating covariances $\mathscr{C}(s)$ by iteration of the formula

$$\begin{aligned}\mathscr{C}(s_b)\equiv(I\otimes G+G\otimes I)\,(s_b)={}&s_bG^{\partial\Lambda}\otimes I+(1-s_b)\,G^{\partial\Lambda\cup b}\otimes I\\&+s_bI\otimes G^{\partial\Lambda}+(1-s_b)\,I\otimes G^{\partial\Lambda\cup b}\end{aligned}\tag{23}$$

with $0\leqq s_b\leqq 1$ and $b\in(\mathbb{Z}^2)^*$ a bond on the unit lattice $\mathbb{Z}^2$. Let $\mathscr{C}(s)$ (with $s$ a set of variables of the type $s_b$, only finitely many different from 1) be an interpolating covariance. Consider the measure defined as in (22) but with the free measure $\mu_0\otimes\tilde\mu_0|_{\mathscr{C}(s)}$ with covariance $\mathscr{C}(s)$ instead of $\mu_0^{\partial\Lambda}\otimes\tilde\mu_0^{\partial\Lambda}$. We see that, for any $A_1\in\Sigma_X$, $A_2\in\Sigma_{\Lambda\setminus X}$ if for all $b\in\partial X$ one has $s_b=0$, then our measure factorizes, i.e.

$$\begin{aligned}\mu_0\otimes\tilde\mu_{0|\mathscr{C}(s_{\partial X}=0)}(e^{-V_\Lambda(\varphi,\tilde\varphi)}A_1\cdot A_2)={}&\mu_0\otimes\tilde\mu_{0|\mathscr{C}(s_{\partial X}=0)}(e^{-V_X(\varphi,\tilde\varphi)}A_1)\\&\times\mu_0\otimes\tilde\mu_{0|\mathscr{C}(s_{\partial X}=0)}(e^{-V_{\Lambda\setminus X}(\varphi,\tilde\varphi)}A_2)\end{aligned}\tag{24}$$

(since in this case one has Dirichlet boundary conditions on $\partial X$). In particular for $X\subseteqq\tilde\Lambda$, $A_2\equiv 1$ and $A_1$ of the form

$$A_1=B(\varphi,\tilde\varphi)\,(F(\varphi)-F(\tilde\varphi)) \tag{25}$$

with $B$ a symmetric function

$$B(\varphi,\tilde\varphi)=B(\tilde\varphi,\varphi) \tag{26}$$

the right-hand side of (24) vanishes [since also $V_X(\varphi,\tilde\varphi)=V_X(\tilde\varphi,\varphi)$). Let us recall the formula for differentiation of measures ([GJS, GlJa]): for suitable $A$

$$\frac{d}{ds_b}\mu_0\otimes\tilde\mu_{0|\mathscr{C}(s)}(A)=\mu_0\otimes\tilde\mu_{0|\mathscr{C}(s)}\left(\frac{1}{2}\Delta_{\dot{\mathscr{C}}(s)}A\right), \tag{27}$$

where $\Delta_{\mathscr{C}(s)}$ is a well defined operator given by the symbolic notation

$$\Delta_{\dot{\mathscr{C}}(s)} \equiv \int d_2x\, d_2y \left[ (\dot{G}(s)\otimes I)(x,y) \frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)} + (I\otimes \dot{G}(s))(x,y) \frac{\delta}{\delta\tilde{\varphi}(x)} \frac{\delta}{\delta\tilde{\varphi}(y)} \right]$$
$$\equiv \dot{\mathscr{C}}(s)\cdot \Delta_{(\varphi,\tilde{\varphi})} \tag{28}$$

with $\dot{G}(s) \equiv \frac{d}{ds_b} G(s)$. What is important for us is that the operator $\Delta_{\mathscr{C}(s)}$ is symmetric with respect to $\varphi \leftrightarrow \tilde{\varphi}$.

Applying successively the fundamental theorem of calculus in the form

$$\mu_0^{\partial\Lambda}\otimes\tilde{\mu}_0^{\partial\Lambda}(A) = \mu_0\otimes\tilde{\mu}_{0|\mathscr{C}(s_b=0)}(A) + \int_0^1 ds_b \frac{d}{ds_b}\mu_0\otimes\tilde{\mu}_{0|\mathscr{C}(s_b)}(A), \tag{29}$$

we get the representation

$$\mu_0^{\partial\Lambda}\otimes\tilde{\mu}_0^{\partial\Lambda}(A) = \sum_{\gamma\subset\mathbb{Z}_2^*\cap\Lambda} \int ds_\gamma \partial_\gamma \mu_0\otimes\tilde{\mu}_{0|\mathscr{C}(s_\gamma)}(A), \tag{30}$$

with $\mathscr{C}(s_\gamma) \equiv \mathscr{C}(\{0 \leqq s_b \leqq 1 \text{ for } b\in\gamma,\ s_b \equiv 0 \text{ for } b\notin\gamma\})$ and $\partial_\gamma \equiv \prod_{b\in\gamma} \frac{d}{ds_b}$. We take

$$A \equiv e^{-V_\Lambda(\varphi,\tilde{\varphi})}(F(\varphi) - F(\tilde{\varphi})), \tag{31}$$

with a polynomial function $F$ localized in a union $X_0$ of unit cubes (contained in $\tilde{\Lambda}$). After the partial resummation of the expansion (30) we get

$$\mu_0^{\partial\Lambda}\otimes\tilde{\mu}_0^{\partial\Lambda} e^{-V_\Lambda(\varphi,\tilde{\varphi})}(F(\varphi)-F(\tilde{\varphi}))$$
$$= \sum_{(X,\Gamma)} \int ds_\Gamma (\partial_\Gamma \mu_0\otimes\tilde{\mu}_{0|\mathscr{C}(s_\Gamma)}(e^{-V_X(\varphi,\tilde{\varphi})}(F-\tilde{F})))\, \mu_0^{\partial\Lambda\cup\partial X}\otimes\tilde{\mu}^{\partial\Lambda\cup\partial X} e^{-V_{\Lambda\setminus X}(\varphi,\tilde{\varphi})}, \tag{32}$$

where the summation goes over the pairs $(X,\Gamma)$ with $X\subset\Lambda$, $X_0 \subseteqq X$ and $\Gamma\subset\mathbb{Z}_2^*\cap\Lambda$, $\Gamma\subset \operatorname{int} X$ and each component of $X\setminus\Gamma^c$ meets $X_0$.

Since by (27) the differentiation gives us

$$\partial_\Gamma \mu_0\otimes\tilde{\mu}_{0|\mathscr{C}(s_\Gamma)}(e^{-V_X(\varphi,\tilde{\varphi})}(F-\tilde{F}))$$
$$= \mu_0\otimes\tilde{\mu}_{0|\mathscr{C}(s_\Gamma)}\left[\sum_{\pi\in\mathscr{P}(\Gamma)} \prod_{\gamma\in\pi} \partial_\gamma \mathscr{C}(s_\Gamma)\cdot \Delta_{(\varphi,\tilde{\varphi})} e^{-V_X(\varphi,\tilde{\varphi})}(F-\tilde{F})\right] \tag{33}$$

[with $\mathscr{P}(\Gamma)$ being the partitions of $\Gamma$], so using the symmetry of $\Delta_{(\varphi,\tilde{\varphi})}$, and $V_X(\varphi,\tilde{\varphi})$ together with antisymmetry of $F(\varphi)-F(\tilde{\varphi})$ with respect to $\varphi\leftrightarrow\tilde{\varphi}$ for $X \subseteqq \{d(x,\partial\Lambda) \geqq 3\}$, we get

$$\partial_\Gamma \mu_0\otimes\tilde{\mu}_{0|\mathscr{C}(s_\Gamma)} e^{-V_X(\varphi,\tilde{\varphi})}(F(\varphi)-F(\tilde{\varphi})) = 0. \tag{34}$$

Hence our sum on the right-hand side of (32) reduces to

$$\mu_0^{\partial\Lambda}\otimes\tilde{\mu}_0^{\partial\Lambda}(e^{-V_\Lambda(\varphi,\tilde{\varphi})})(F-\tilde{F})$$
$$= \sum_{\substack{(X,\Gamma) \\ X\cap\{d(x,\partial\Lambda)\leqq 3\}\neq\emptyset}} \int ds_\Gamma \partial_\Gamma \tilde{\mu}_{0|\mathscr{C}(s_\Gamma)}(e^{-V_X(\varphi,\tilde{\varphi})}(F-\tilde{F}))$$
$$\times\, \mu_0^{\partial\Lambda\cup\partial X}\otimes\tilde{\mu}_0^{\partial\Lambda\cup\partial X} e^{-V_{\Lambda\setminus X}(\varphi,\tilde{\varphi})}. \tag{35}$$

Let us set

$$\tilde{\Lambda} \equiv \{x \in \Lambda : d(x, \partial\Lambda) \geqq 4\}. \tag{36}$$

Let $\mu_0^{c(X)}$ denote the free measure with Dirichlet boundary conditions on $\partial(\Lambda\backslash X)$ and Neumann boundary conditions on $\partial\tilde{\Lambda}$.

Now we remark that (by conditioning inequalities of [GRS1, 2])

$$\begin{aligned} &\mu_0^{\partial\Lambda\cup\partial X} \otimes \tilde{\mu}_0^{\partial\Lambda\cup\partial X} e^{-V_{\Lambda\backslash X}(\varphi,\tilde{\varphi})} \\ &\leqq \mu_0^{c(X)} \otimes \tilde{\mu}_0^{c(X)} e^{-V_{\Lambda\backslash(\tilde{\Lambda}\cup X)}(\varphi,\tilde{\varphi})} \mu_0^{c(X)} \otimes \tilde{\mu}_0^{c(X)} e^{-V_{(\Lambda\backslash X)\cap\tilde{\Lambda}}(\varphi,\tilde{\varphi})}. \end{aligned} \tag{37}$$

From Lemma 1 proven in the Appendix 2 we have, using also the definition of $V$ for any $X$ such that $X \cap (\Lambda\backslash\tilde{\Lambda}) \neq \emptyset$, that

$$\begin{aligned} &e^{-2\alpha_\infty|(\Lambda\backslash X)\cap\tilde{\Lambda}|} \cdot \mu_0^{c(X)} \otimes \tilde{\mu}_0^{c(X)} e^{-V_{(\Lambda\backslash X)\cap\tilde{\Lambda}}(\varphi,\tilde{\varphi})} \\ &= (e^{-\alpha_\infty|(\Lambda\backslash X)\cap\tilde{\Lambda}|} \mu_0^{c(X)} e^{-U_{(\Lambda\backslash X)\cap\tilde{\Lambda}}(\varphi)})^2 \leqq e^{c|\partial(\Lambda\backslash X)|} \end{aligned} \tag{38}$$

with a constant $0 < c < \infty$ independent of $\Lambda, c(X)$ and $m_0, \lambda$ for $m_0 \geqq M_0$ and $0 < M_0 < \infty$ sufficiently big, and $0 < \lambda < \lambda_0$ with $0 < \lambda_0 < \infty$ sufficiently small.

Let us introduce the notation, for arbitrary $X$:

$$\partial_\Gamma \mu_0 \otimes \tilde{\mu}_{0|\mathscr{C}(s_\Gamma)} (e^{-V_X(\varphi,\tilde{\varphi})}(F - \tilde{F})) \equiv \mu_0 \otimes \tilde{\mu}_{0|\mathscr{C}(s_\Gamma)} (R_{X,\Gamma} e^{-V_X(\varphi,\tilde{\varphi})}) \tag{39}$$

for $R_{X,\Gamma}$ being the polynomial localized in $X \cup \Lambda_0$ defined in (27) and (33). Then using (35) and (37) together with triangle and Hölder inequalities we get the following bound for (17):

$$\begin{aligned} &\mu \otimes \tilde{\mu} |E_{\Lambda^c}^{\eta}(F) - E_{\Lambda^c}^{\tilde{\eta}}(F)| \\ &\leqq \sum_{\substack{X,\Gamma \\ X\cap(\Lambda\backslash\tilde{\Lambda}) \neq \emptyset}} \int ds_\Gamma \{(\mu_0 \otimes \tilde{\mu}_0 [\mu_0 \otimes \tilde{\mu}_{0|\mathscr{C}(s_\Gamma)} |R_{X,\Gamma}|^r])^{1/r} \cdot A_{X,\Gamma} \cdot B_{X,\Gamma} \\ &\quad \times \|\varrho^{\partial\Lambda}\|_{L_p(\mu_0)} \|\tilde{\varrho}^{\partial\Lambda}\|_{L_p(\mu_0)}\}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} A_{X,\Gamma} \equiv \mu_0 \otimes \tilde{\mu}_0 \Bigg[ & \mu_0 \otimes \tilde{\mu}_{0|\mathscr{C}(s_\Gamma)} e^{-qV_X(\varphi,\tilde{\varphi})} \cdot \mu_0^{c(X)} \otimes \tilde{\mu}_0^{c(X)} e^{-qV_{\Lambda\backslash(X\cup\tilde{\Lambda})}} \\ & \times \left\{ \exp - \frac{q}{2} (\|\chi^{\partial\Lambda} \psi_\eta^{\partial\Lambda}\|_{+1}^2 + \|\chi^{\partial\Lambda} \psi_{\tilde{\eta}}^{\partial\Lambda}\|_{+1}^2) \right\} \Bigg]^{1/q} e^{-2\alpha_\infty|X\cup(\Lambda\backslash X\cap\tilde{\Lambda})|} \end{aligned} \tag{41}$$

and

$$B_{X,\Gamma} \equiv [\mu_0^{c(X)} \otimes \tilde{\mu}_0^{c(X)} e^{-V_{(\Lambda\backslash X)\cap\tilde{\Lambda}}} e^{-2\alpha_\infty|(\Lambda\backslash X)\cap\tilde{\Lambda}|}] \tag{42}$$

with $1 < p < \dfrac{4}{3}$ and $\dfrac{1}{p} + \dfrac{1}{r} + \dfrac{1}{q} = 1$. By ultraregularity of $\mu$ and $\tilde{\mu}$ we have [FrSi]

$$\|\varrho^{\partial\Lambda}\|_{L_p(\mu_0)}, \|\tilde{\varrho}^{\partial\Lambda}\|_{L_p(\mu_0)} \leqq 1 \tag{43}$$

for $1 < p < \frac{4}{3}$.

*Remark.* What will be important for us afterwards is that the bound

$$\|\varrho^{\partial\Lambda}\|_{L_p(\mu_0)}, \|\tilde{\varrho}^{\partial\Lambda}\|_{L_p(\mu_0)} < e^{c|\partial\Lambda|} \tag{43'}$$

holds with a constant $0<c<\infty$ independent of $\Lambda\in\mathscr{F}_0$ and $0<\lambda<\lambda_0$, $m_0>M_0$ for sufficiently small $0<\lambda<\lambda_0$ and sufficiently big $0<M_0<\infty$.

From (43) and (38) it follows that the quantity $B_{X,\Gamma}\|\varrho^{\partial\Lambda}\|_{L_p(\mu_0)}\|\tilde{\varrho}^{\partial\Lambda}\|_{L_p(\mu_0)}$ appearing on the right-hand side of (40) is bounded by $\exp(c|\partial\Lambda\cup\partial X|)$ with a constant $0<c<\infty$ independent of $\Lambda, X$ and $0<\lambda<\lambda_0$, $m_0>M_0$, $\lambda_0$ sufficiently small, $M_0$ sufficiently big.

The polynomials $R_{X,\Gamma}$ differ from the corresponding polynomials coming from the cluster expansion for the interaction $U_\Lambda(\varphi)+U_\Lambda(\tilde{\varphi})$ nonessentially since $V_\Lambda(\varphi,\tilde{\varphi})$ differs from $U_\Lambda(\varphi)+U_\Lambda(\tilde{\varphi})$ only by a linear term $\varphi(h_\eta^{\partial\Lambda})+\tilde{\varphi}(h_{\tilde{\eta}}^{\partial\Lambda})$ localized in $\{2<d(x\,\partial\Lambda)\leqq 3\}$. Hence the $L_r$-norm of $R_{X,\Gamma}$ in (40) will have, after integration with respect to $\mu_0\otimes\tilde{\mu}_0$, as good estimations as in the usual case considered in Glimm-Jaffe-Spencer's cluster expansion [GJS], namely

$$(\mu_0\otimes\tilde{\mu}_0[\mu_0\otimes\tilde{\mu}_{0|\mathscr{C}(s_\Gamma)}|R_{X,\Gamma}|^r])^{1/r}\leqq e^{-2a|X|} \tag{44}$$

with a constant $a\equiv a(m_0^2,\lambda)>0$ increasing (to infinity) as the free mass $m_0$ is growing (to infinity) and the coupling constant $\lambda>0$ is decreasing (to zero).

Using Hölder and conditioning inequalities we estimate (41) by

$$\begin{aligned}A_{X,\Gamma}\leqq&(\mu_0\otimes\tilde{\mu}_0 e^{-w(U_{X\cap\tilde{\Lambda}}(\varphi)+U_{X\cap\tilde{\Lambda}}(\tilde{\varphi}))})^{1/w}\\&\times(\mu_0\otimes\tilde{\mu}_0(\mu_0^{c(X)}\otimes\tilde{\mu}_0^{c(X)}[e^{-v[V_{\Lambda\setminus\tilde{\Lambda}}(\varphi,\tilde{\varphi})]}\cdot e^{-\frac{v}{2}\{\|\chi^{\partial\Lambda}\psi_{\tilde{\eta}}^{\partial\Lambda}\|_2^2+1\}}]))^{1/v}\end{aligned} \tag{45}$$

with $w^{-1}+v^{-1}=q^{-1}$ (we also used the fact that $\alpha_\infty\geqq 0$). The first factor on the right-hand side of (45) is bounded with the use of standard arguments [GlJa] by $e^{2b|X|}$ with a constant $b\equiv b\left(\frac{\lambda}{m_0^2}\right)$ decreasing with $\frac{\lambda}{m_0^2}$. The second factor contains interactions concentrated at the boundary and is bounded in Lemma 2 in Appendix 2 by $e^{2b'|\partial\Lambda|}$ with a constant $b'>0$ decreasing as $m_0^2$ is increasing and $\lambda$ is decreasing (and independent of $s_\Gamma$). We recall that on the other hand the terms $B_{X,\Gamma}$ defined in (42) are bounded by $e^{c|\partial\Lambda\cup\partial X|}$ with a constant $0<c<\infty$ decreasing with $\frac{\lambda}{m_0^2}$ (and like the other constants is independent of $X,\Gamma$, and $\Lambda$). Combining the estimation on all the factors in each term of (40) we get the following estimate:

$$\mu\otimes\tilde{\mu}|E_{\Lambda^c}^\eta(F)-E_{\Lambda^c}^{\tilde{\eta}}(F)|\leqq\sum_{\substack{X,\Gamma\\X\cap(\Lambda\setminus\tilde{\Lambda})\neq\emptyset}} e^{-2a|X|+2b|X|+2b'|\partial\Lambda|}e^{c|\partial\Lambda\cup\partial X|} \tag{46}$$

(where we also used the $s_\Gamma$-independence of the estimates).

Taking into account that the entropy estimations for $\Gamma\subset X$ are the same as in usual case [GJS1, 2, GlJa], the fact that $0<a$ is increasing and all the constants $b, b'$ and $c$ are decreasing as $m_0^2$ increases and $\lambda$ decreases and also the fact that all $X\cap(\Lambda\setminus\tilde{\Lambda})\neq\emptyset$ are such that $|X|\geqq d(0,\partial\Lambda)-4$, we conclude with the bound

$$\mu\otimes\tilde{\mu}|E_{\Lambda^c}^\eta(F)-E_{\Lambda^c}^{\tilde{\eta}}(F)|\leqq e^{-\tilde{a}d(0,\partial\Lambda)} \tag{47}$$

with $0<\tilde{a}<\infty$ independent of $\Lambda\in\mathscr{F}_0$ for all $\infty>m_0^2>M_0^2$ and $0<\lambda<\lambda_0$, if $0<M_0^2<\infty$ respectively $0<\lambda_0<\infty$ are taken suitable big respectively small. By this we are finished with the proof of (13) and hence with the proof of the theorem.

## Appendix 1

**Lemma A1.1.** *For any* $2 \leqq p \leqq \infty$ *and in dimension* $d=2$ *we have* $\|G*f\|_{L_p} \leqq C\|f\|_{-1}$, *for any* $f \in H_{-1}(\mathbb{R}^2)$, *with a constant* $C>0$ *(independent of* $f$*) (with* $G \equiv (-\Delta + m_0^2)^{-1}$*).*

*Proof.* For $\infty > p \geqq 2$ by Hausdorff-Young inequalities we have (for arbitrary dimension $d$)

$$\|G*f\|_{L_p} \leqq (2\pi)^{d\left(\frac{1}{2}-\frac{1}{q}\right)} \|\widehat{G*f}\|_{L_q} = (2\pi)^{d\left(\frac{1}{2}-\frac{1}{q}\right)} \left\| \frac{|\hat{f}|}{(k^2+1)^{1/2}} \cdot \frac{1}{(k^2+1)^{1/2}} \right\|_{L_q} \tag{A1.2}$$

with $\frac{1}{p}+\frac{1}{q}=1$, $1<q<2$ with $\widehat{\ }$ denoting Fourier transform).

Now by Hölder inequalities we get

$$\|\hat{G}\hat{f}\|_{L_q} \leqq \left\| \frac{|\hat{f}|}{(k^2+1)^{1/2}} \right\|_{L_2} \cdot \left\| \frac{1}{(k^2+1)^{1/2}} \right\|_{L_s} \equiv \|f\|_{-1} \, \|(k^2+1)^{-1/2}\|_{L_s} \tag{A1.3}$$

with $\frac{1}{2}+\frac{1}{s}=\frac{1}{q}$. Since $1<q\leqq 2$, so $s>2$ and for $d=2$ we have

$$\|(k^2+1)^{-1/2}\|_{L_s} < \infty, \tag{A1.4}$$

which then yields Lemma A1.1.

*Remark.* The same argument hold for $2 \leqq p < 6$ in dimension $d=3$, but for $d=4$ one needs $2 \leqq p < 4$.

**Lemma A1.2.** *For any* $f \in H_{-1}(\Lambda)$, $\Lambda \in \mathscr{F}$,

$$\|G*f\|_{L_1} \leqq c|\Lambda|^{1/2} \|f\|_{-1} \tag{A1.5}$$

*with a constant* $c>0$ *independent of* $\Lambda \in \mathscr{F}$ *and* $f$.

*Proof.* Let $\tilde{\Lambda} \equiv \{x \in \mathbb{R}^d : d(x,\Lambda) \leqq 2\}$. Let $\chi_\Lambda \in C^\infty(\mathbb{R}^d)$, $0 \leqq \chi_\Lambda \leqq 1$, $\operatorname{supp}\chi_\Lambda \subset \{x \in \mathbb{R}^d : d(x,\Lambda) \leqq 1\}$ and $\chi_\Lambda(x) \equiv 1$ for $x \in \Lambda$. We have with $f \in H_{-1}(\Lambda)$,

$$\int d_dx |G*f|(x) = \int_{\tilde{\Lambda}} d_dx |G*f| + \int_{\tilde{\Lambda}^c} d_dx |G*f|(x). \tag{A1.6}$$

For the first term from right-hand side (A1.6) we have the following estimation:

$$\begin{aligned}
\int_{\tilde{\Lambda}} d_dx |G*f|(x) &\leqq |\tilde{\Lambda}|^{1/2} \left( \int_{\tilde{\Lambda}} d_dx |G*f|^2(x) \right)^{1/2} \\
&\leqq |\tilde{\Lambda}|^{1/2} \left( \int_{\mathbb{R}^d} d_dx |G*f|^2(x) \right)^{1/2} \\
&= |\tilde{\Lambda}|^{1/2} \left( \int_{\mathbb{R}^d} d_dk \frac{|\hat{f}(k)|^2}{(k^2+1)^2} \right)^{1/2} \\
&\leqq |\tilde{\Lambda}|^{1/2} \|f\|_{-1} \leqq C_1 |\Lambda|^{1/2} \|f\|_{-1}
\end{aligned} \tag{A1.7}$$

with a constant $C_1>0$ independent of $\Lambda\in\mathscr{F}$ (for sufficiently big $|\Lambda|$). Consider now the second term from the right-hand side of (A1.5). We have

$$\begin{aligned}\int_{\tilde{\Lambda}^c} d_dx|G*f|(x)&=\int_{\tilde{\Lambda}^c} d_dx\left|\int d_dy G(x-y)f(y)\right|\\&=\int_{\tilde{\Lambda}^c} d_dx\left|\int d_dy G(x-y)\chi_\Lambda(y)f(y)\right|,\end{aligned}\tag{A1.8}$$

since $f\in H_{-1}(\Lambda)$ and $\chi_\Lambda(y)\equiv 1$ for $y\in\Lambda$. The integrand in (A1.8) can be further estimated as follows:

$$\begin{aligned}&\left|\int d_dy G(x-y)\chi_\Lambda(y)f(y)\right|\equiv(G(x-\cdot)\chi_\Lambda(\cdot),f(\cdot))_{L_2(\mathbb{R}^d)}\\&=|((-\Delta_y+1)^{1/2}G(x-\cdot)\chi_\Lambda(\cdot),(-\Delta_y+1)^{-1/2}f(\cdot))_{L_2(\mathbb{R}^d)}|\\&\leqq\|(-\Delta_y+1)^{1/2}G(x-\cdot)\chi_\Lambda(\cdot)\|_{L_2(\mathbb{R}^d)}\cdot\|(-\Delta+1)^{-1/2}f\|_{L_2(\mathbb{R}^d)}\\&\equiv\|(-\Delta_y+1)^{1/2}G(x-\cdot)\chi_\Lambda(\cdot)\|_{L_2(\mathbb{R}^d)}\|f\|_{-1}.\end{aligned}\tag{A1.9}$$

We have also

$$\begin{aligned}&\int_{\tilde{\Lambda}^c} d_dx\|(-\Delta_y+1)^{1/2}G(x-\cdot)\chi_\Lambda(\cdot)\|_{L_2(\mathbb{R}^d)}\\&=\int_{\tilde{\Lambda}^c} d_dx\left[\int d_dy G(x-y)\chi_\Lambda(y)(-\Delta+1)(G(x-y)\chi_\Lambda(y))\right]^{1/2}\\&=\int_{\tilde{\Lambda}^c} d_dx\left[\int d_dy G(x-y)\chi_\Lambda(y)(-2\nabla_y G(x-y)\cdot\nabla_y\chi_\Lambda(y)-G(x-y)\Delta_y\chi_\Lambda(y))\right]^{1/2}\\&\leqq\sup_{\substack{x\in\tilde{\Lambda}^c\\ y\in\operatorname{supp}\chi_\Lambda}}|2\nabla_y G(x-y)\cdot\nabla_y\chi_\Lambda(y)+G(x-y)\Delta_y\chi_\Lambda(y)|^{1/2}\cdot|\tilde{\Lambda}\setminus\Lambda|^{1/2}\\&\quad\times\int_{\tilde{\Lambda}^c} d_dx\left(\sup_{y\in\operatorname{supp}\chi_\Lambda}G(x-y)\right)^{1/2}\leqq C_2|\tilde{\Lambda}\setminus\Lambda|\leqq C_3|\Lambda|^{1/2}\end{aligned}\tag{A1.10}$$

with some constants $C_2,C_3>0$ independent of $\Lambda\in\mathscr{F}$ (for sufficiently large $\Lambda$). Combining (A1.8)–(A1.10) we get

$$\int_{\tilde{\Lambda}^c} d_dx|G*f|(x)\leqq C_3|\Lambda|^{1/2}\|f\|_{-1}.\tag{A1.11}$$

From (A1.7) and (A1.11) we get the statement (A1.5) of our lemma. □

## Appendix 2

Let $\tilde{\Lambda}\equiv\{x\in\Lambda:d(x,\partial\Lambda)\geqq 4\}$, $\Lambda\in\mathscr{F}$.

**Lemma A2.1.** *For any set $X\subset\Lambda$ defined in the cluster expansion* (32),

$$\mu_{0|_{N\text{ on }\partial\tilde{\Lambda}}}^{\partial\Lambda\setminus X}e^{-U_{\Lambda\setminus X\cap\tilde{\Lambda}}(\varphi)}e^{-\alpha_\infty|\Lambda\setminus X\cap\tilde{\Lambda}|}\leqq e^{c|\partial\Lambda\setminus X|}\tag{A2.1}$$

*with a constant $C>0$ independent of $\Lambda$, $X$, and $\tilde{\Lambda}$ decreasing to zero as $\frac{\lambda}{m_0^2}$ is decreasing to zero.*

*Proof.* First we note that by conditioning inequalities [GRS] we have with $Y\equiv\Lambda\setminus X\cap\tilde{\Lambda}$,

$$\mu_0^{c(X)}e^{-U_Y}\leqq\mu_0^{N(\partial\tilde{\Lambda})}e^{-U_Y}\tag{A2.2}$$

(with $\mu_0^{N(\partial\tilde{\Lambda})}$ meaning the free field measure with Neumann boundary condition on $\partial\tilde{\Lambda}$). Let $\mu_{0,s}$ be the gaussian measure with mean zero and covariance $sG^{N(\partial\tilde{\Lambda})}+(1-s)G\equiv G(s)$, $s\in[0,1]$. Then we have [GlJa]

$$\begin{aligned}\frac{d}{ds}\mu_{0,s}e^{-U_Y}&=\mu_{0,s}(\tfrac{1}{2}\Delta_{\dot G}e^{-U_Y})\\&=\left(\frac{\mu_{0,s}\frac{1}{2}\Delta_{\dot G}e^{-U_Y}}{\mu_{0,s}e^{-U_Y}}\right)\mu_{0,s}e^{-U_Y}\\&\equiv B_Y(s)\mu_{0,s}e^{-U_Y}.\end{aligned}\tag{A2.3}$$

Hence we get, by Eq. (A2.3) and using reflection positivity, a bound

$$\mu_{0,s}e^{-U_Y}\leqq(\mu_0e^{-U_Y})e^{B_Y}\leqq e^{\alpha_\infty|Y|+B_Y}\tag{A2.4}$$

with

$$B_Y\equiv\sup_{s\in[0,1]}B_Y.\tag{A2.5}$$

Now by definition of $\Delta_{\dot G}$ we have

$$B_Y(s)=\frac{1}{2}\left(\mu_{0,s}e^{-U_Y}\left((-1)\left\langle\dot G\frac{\delta}{\delta\varphi},\frac{\delta}{\delta\varphi}U_Y\right\rangle+\left\langle\dot G\frac{\delta}{\delta\varphi}U_Y,\frac{\delta}{\delta\varphi}U_Y\right\rangle\right)\right)\tag{A2.6}$$
$$[\mu_{0,s}e^{-U_Y}]^{-1},$$

where $\dot G\equiv\frac{d}{ds}G(s)=G^{N(\partial\tilde{\Lambda})}-G$. Applying the standard cluster expansion [GJS1, 2, GlJa] to (A2.6) and using the fact that $\dot G$ decays exponentially fast with distance from $\partial\tilde{\Lambda}$ we get the estimate

$$|B_Y(s)|\leqq c|(\partial\Lambda\setminus X)|\tag{A2.7}$$

with a constant $c>0$ decreasing to zero as $\frac{\lambda}{m_0^2}$ is decreasing to zero. This together with (A2.4) gives us the estimate (A2.1). □

**Lemma A2.2.** *For any $p\in\mathbb{N}$ and sufficiently big $m_0>0$,*

$$\mu_0\left(\mu_0^c\exp\left\{-pU_{\Lambda\setminus\tilde{\Lambda}}(\varphi+(1\setminus\chi^{\partial\Lambda})\psi_\eta^{\partial\Lambda})-p(\varphi((-\Delta+m_0^2)\chi^{\partial\Lambda}\psi_\eta^{\partial\Lambda}))\right.\right.$$
$$\left.\left.-\frac{p}{2}\|\chi^{\partial\Lambda}\psi_\eta^{\partial\Lambda}\|_{+1}^2\right\}\right)\leqq e^{b'|\partial\Lambda|}\tag{A2.8}$$

*with a constant $b'>0$ decreasing to zero as $\frac{\lambda}{m_0^2}$ decreases to zero. (Here $\mu_0^c$ is the free measure with Dirichlet boundary condition on $\partial\Lambda$ and Neumann boundary condition on $\partial\tilde{\Lambda}$).*

*Proof.* By the Hölder inequality we have with $\frac{1}{s}+\frac{1}{t}=1$ that the left-hand side of (A2.8) is less or equal to

$$[\mu_0\otimes\mu_0^c\exp(-s\cdot pU_{\Lambda\setminus\tilde{\Lambda}}(\varphi+(1\setminus\chi^{\partial\Lambda})\psi_\eta^{\partial\Lambda}))]^{1/s}$$
$$\times\left[\mu_0\otimes\mu_0^c\exp\left(-t\cdot p[\varphi(-\Delta+m_0^2)\chi^{\partial\Lambda}\psi_\eta^{\partial\Lambda}]-\frac{tp}{2}\|\chi^{\partial\Lambda}\psi^{\partial\Lambda\eta}\|_{+1}^2\right)\right]^{1/t}\tag{A2.9}$$

The first factor in (A2.9) has an estimation of the type (A2.8), by the usual arguments using Duhamel's expansion (e.g. [DiGl, GRS1]) and the fact that $|\Lambda \backslash \tilde{\Lambda}| \leqq 4|\partial \Lambda|$. Concerning the second factor we first observe that performing first the $\mu_0^c$-integration with respect to the variable $\varphi$, it is enough to bound $\mu_0 \exp\left[\frac{(tp)^2-tp}{2} \|\chi^{\partial\Lambda}\psi_\eta^{\partial\Lambda}\|_{+1}^2\right]$, with some fixed $t$, $p$ independent of $m_0, \Lambda, \tilde{\Lambda}$. We also observe that $[(-\Delta+m_0^2)\chi^{\partial\Lambda}\psi_\eta^{\partial\Lambda}](x)$ is supported in the set $\{x \in \Lambda, 2 \leqq d(x, \partial\Lambda) < 3\}$ [by definition of $\chi^{\partial\Lambda}$ in (15) and the fact that $(-\Delta+m_0^2)\psi_\eta^{\partial\Lambda}(x)=0$ in $\Lambda$]. Moreover the kernel $\psi^{\partial\Lambda}(x)$ is exponentially decaying with $m_0 d(x, \partial\Lambda)$. The exponential in the above $\mu_0$-expectation has the form $\exp\left\{\int\limits_{\partial\Lambda}\int\limits_{\partial\Lambda} dz\,dz'\eta(z)\,k(z,z')\,\eta(z')\right\}$ with a kernel $k(z,z')$ which is Hilbert-Schmidt on $L^2(\partial\Lambda \times \partial\Lambda)$, since it is of the form $C \int\limits_{\Lambda\backslash\tilde{\Lambda}} \chi^{\partial\Lambda}\psi_z^{\partial\Lambda}(x)(-\Delta+m_0^2)\chi^{\partial\Lambda}(x)\,\psi_{z'}^{\partial\Lambda}(x)\,d_2x$ for some constant $C>0$. That $k(z,z')$ is Hilbert-Schmidt follows from its exponential decay in $m_0|z-z'|$ and its regularity.

[We also use the fact that $d(\operatorname{supp}(-\Delta+m_0^2)\chi^{\partial\Lambda}\psi_\eta^{\partial\Lambda}, \partial\tilde{\Lambda}) \geqq 1$, so we have no logarithmic singularity coming from Neumann boundary conditions on $\partial\tilde{\Lambda}$ in the region $\{2 \leqq d(x, \partial\Lambda) \leqq 3\}$ which is relevant for our estimates.]

It also follows that $G^{-1/2}kG^{-1/2}$ is also Hilbert-Schmidt, since $k$ is smooth. By known results on Gaussian measures the $\mu_0$-expectation is then finite, see e.g. [GlJa], and estimated by $e^{c|\partial\Lambda|}$ with $c$ going to zero for $m_0 \to \infty$ as $e^{-2m_0}$. This ends the proof of the lemma. □

*Remark.* Using the fact that the kernel $k(x,x')$ is small and exponentially decaying one can get the estimate for $\mu_0 e^{c\|\chi^{\partial\Lambda}\psi_\eta^{\partial\Lambda}\|_{+1}^2}$ by applying the checker-board estimate, Hölder's inequality and analyticity arguments.

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