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Fermionic Fields on $\mathbb{Z}_N$-Curves

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Abstract. The line bundles of degree $g-1$ on $\mathbb{Z}_N$-curves corresponding to $1/N$ nonsingular characteristics are considered. The determinants of Dirac operators defined on these line bundles are evaluated in terms of branch points. The generalization of Thomae's formula for $\mathbb{Z}_N$-curves is derived.

1. Introduction

In this paper we continue our investigations of conformal field theories that are induced in some specific way from algebraic curves. Our global strategy will be the following. Consider an algebraic curve represented as an N-sheeted ramified covering over CP^1. Such representation fixes the unique singular metric whose projection onto CP^1 is $dzd\bar{z}$. The determinants of different operators defined for line bundles of definite degree and corresponding to some characteristics can be represented as correlation functions of some conformal fields defined on a complex plane. We call such conformal fields twist operators or σ-fields. These operators simulate the proper monodromy behaviour and make the fields be multivalued fields defined not on a complex plane but on a ramified covering. This ideology takes its origin in the works of Sato et al. [1]. Our investigations of this problem were highly stimulated by a recent work by Zamolodchikov [2], who succeeded in representing the determinant of the scalar Laplacian defined on a hyperelliptic curve as a correlation function of some spin operators from the Ashkin-Teller model.

It turns out that the twist operators are actually the conformal fields with respect to the full stress-energy tensor

$$T(z)=\sum_{\text{sheets}} t^n(z). \tag{1.1}$$

$T(z)$ is a single-valued function on CP^1 that may only have poles at the branch points. To evaluate the conformal dimensions one has to analyze the behaviour of $T(z)$ in vicinities of the branch points. In a vicinity of the branch point the variable z defined on a complex plane is not a proper coordinate on the covering. The proper variable is $\zeta=(z-a)^{1/k}$, where k is the index of ramification. The stress energy tensor $t(\zeta)$ is a smooth function in terms of ζ. Thus, one can reexpress the stress-energy tensor in terms of z:

$$t(z)=t(\zeta(z))\left(\frac{d\zeta}{dz}\right)^2+\frac{c(j)}{12}\{\zeta,z\}, \tag{1.2}$$

where $\{\zeta,z\}$ is the Shwarz derivative, $c(j)$ is the conformal anomaly. One may easily see that $T(z)$ has poles of the first and the second order only. The coefficient of the pole of the second order is uniquely determined by the Shwarz derivative and coincide with the conformal dimension of the twist operator:

$$\Delta=c(j)\,\frac{k^2-1}{24k}\,. \tag{1.3}$$

In this paper we consider not the general case but the most simple one. The conformal field theory formulated below corresponds to the $\mathbb{Z}_N$-curves and the line bundles of degree $g-1$ and of definite characteristic. The $\mathbb{Z}_N$-curves are described by the equation

$$X=\left\{z,y\,|\,y^N=\prod_{i=1}^{L}(z-a_i)^{R_i}\right\},\qquad R_i\in\{1,2,\ldots,N-1\}\,. \tag{1.4}$$

We suppose that the covering is not ramified at infinity and thus $\sum R_i\equiv 0\,(\mathrm{mod}\,N)$. Each field $\psi(q)$ on a ramified covering can be represented as a vector valued field $\vec{\psi}(\zeta)$ on the complex plane according to the rule

$$\psi^i(\zeta)=\psi(q_i)\,, \tag{1.5}$$

where the projection of all the points $q_1,\ldots,q_N$ is a point $\zeta\in CP^1$. The vector field $\vec{\psi}(\zeta)$ is not single valued but is multivalued on CP^1. When ζ moves around the branch point the field $\vec{\psi}(\zeta)$ transforms via

$$\vec{\psi}(\zeta)\to M\vec{\psi}(\zeta)\,, \tag{1.6}$$

where M is the monodromy matrix. The determinant of $\bar{\partial}$ can be expressed as a path integral over sections of line bundles

$$\det\bar{\partial}=\int D\vec{\psi}D\vec{\chi}\exp-\int\psi^i\bar{\partial}\chi^i d^2\zeta\,. \tag{1.7}$$

The fields $\vec{\psi}(\zeta)$ and $\vec{\chi}(\zeta)$ are the sections of the line bundles L and L^+ such that $L\otimes L^+$ is the line bundle of holomorphic 1-forms. Only in this case the action is a well defined functional on a surface.

For $\mathbb{Z}_N$-curves there exist line bundles of degree $g-1$ corresponding to $1/N$ characteristics. Their divisors are integer combinations of the branch points. For these bundles the determinant of the $\bar{\partial}$-operator has the most simple form and is expressed as a Coloumb gas in terms of the branch points. Comparing this expression with the well known expression for $\det\bar{\partial}$ via the theta constants we derive the generalized Thomae's formula [3].

2. Line Bundles of Degree $(g-1)$. Fermionic Fields in the Presense of External Vector Fields and Determinants

In this section we follow the discussion of paper [4]. We recall how the determinant of the Dirac operator depends on the point in the moduli space of line bundles. All the needed mathematics can be found in the books [3, 5].

To define a fermionic field on a curve one has to introduce a zweibein field (two orthogonal to each other vector field of the same length) $e^a_\alpha(\zeta)$ or, in other words, a pair of one-forms $e^a(\zeta) = e^a_\alpha(\zeta)d\zeta^\alpha$. The metric on a curve is defined as:

$$(ds)^2 = \lambda(\zeta)\delta_{ab}e^a \otimes e^b . \tag{2.1}$$

On the intersections of charts U_i and U_j the zweibein field transforms via

$$e^a_{(i)} = R^{ab}_{ij}e^b_{(j)}, \tag{2.2}$$

where $R^{ab} \in \mathbb{R} \times SO(2) \simeq \mathbb{C}^* = \mathbb{C}\backslash\{0\}$. In the conformal gauge the zweibein field can be chosen as follows: $e^+ = e^\varphi dz$, $e^- = e^\varphi d\bar{z}$, where (2φ) is the conformal factor. The field e^+ is a smooth section of the holomorphic bundle of 1-forms. In what follows we denote the holomorphic bundle of 1-forms by $\mathscr{K}$.

A spinor bundle is a bundle of the form $K_{1/2} \oplus \bar{K}_{1/2}$, where $K_{1/2}$ is a holomorphic line bundle such that $K^{\otimes 2}_{1/2} = \mathscr{K}$. Thus the degree of a spinor bundle equals $(g-1)$. The sections of spinor bundle transform as 1/2-forms, i.e. as $\psi(dz)^{1/2}$. Thus the transition functions are the square roots of the transition functions of the holomorphic bundle of 1-forms and the problem of the choice of signs arises. When the point ζ performs a full rotation along a cycle which is not homotopic to zero the spinor field acquires the sign factor. Let $\{a_i, b_i\}$ be a basis of $H_1(X, \mathbb{Z})$. The spinor bundle is uniquely determined by the set of sign factors $\{\mathrm{sign}(a_i), \mathrm{sign}(b_i)\}$. Thus there is a one-to-one correspondence between the spinor bundles and half integer characteristics $\vec{m}$ in the Jacobian variety $\mathrm{Jac}(X)$:

$$\vec{m} = \frac{1}{4}\sum_i (1 - \mathrm{sign}(a_i))\vec{e}_i + \frac{1}{4}\sum_j (1 - \mathrm{sign}(b_j))\vec{T}_j . \tag{2.3}$$

We will consider not only spinor bundles but arbitrary line bundles of degree $(g-1)$. Roughly speaking the difference between these bundles is the following: when the point ζ moves around a loop non-homotopic to zero, a section of an arbitrary line bundle acquires a phase factor $\exp i\varphi$ instead of sign. Let us represent the line bundle of degree $(g-1)$ in the form:

$$K_{1/2,m}(\vec{u}) = K_{1/2,m} \otimes L_0(\vec{u}), \tag{2.4}$$

where $K_{1/2,m}$ is a fixed bundle of 1/2-forms and $L_0(\vec{u})$ is the line bundle of degree zero. As is well known the line bundles of degree zero are classified by the points $\vec{u}$ of the $\mathrm{Jac}(X)$. Recall the connections between a vector $\vec{u} \in \mathrm{Jac}(X)$ and the monodromy properties of the sections $L_0(\vec{u})$. An arbitrary line bundle of degree zero admits a hermitian metric with a flat connection $d + A$ corresponding to the complex structure

$$d + A = (\partial_z + A_z)dz + (\bar{\partial}_z + A_{\bar{z}})d\bar{z}, \qquad A_{\bar{z}} = 0 . \tag{2.5}$$

An arbitrary line bundle of degree zero admits the global smooth trivialization, and thus one may consider the connection A defined on the trivial bundle $X \times \mathbb{C}$.

In this case A does not correspond to the complex structure on $X \times \mathbb{C}$ and the complex structure is determined by the following condition: the section ψ is holomorphic if and only if $D_{\bar{z}}(A)\psi = 0$. Obviously, different flat connections on $X \times \mathbb{C}$ may produce the same holomorphic line bundles L. To make this procedure unambiguous one may specify the connection A imposing the unitarity of the monodromy factors (Wilson factors):

$$W(\gamma) = P\exp \int_{\gamma} A_\alpha d\zeta^\alpha, \qquad |W(\gamma)| = 1, \tag{2.6}$$

where γ is a contour non-homotopic to zero. One may also prove that for an arbitrary set of $(2g)$ phase factors $e^{i\varphi_1}, \ldots, e^{i\varphi_{2g}}$ there exists a unique flat connection such that $W(\gamma_j) = \exp i\varphi_j$, where $(\gamma_1, \ldots, \gamma_{2g})$ is a basis in $H_1(X, \mathbb{Z})$. Thus we explained that there is a one-to-one correspondence between the line bundles of degree zero and the sets of $(2g)$ phase factors. Obviously, $(2g)$ Wilson factors $W(\gamma_i)$ determine a torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ which is a realization of the Jacobian variety. Now let us establish the correspondence with the standard definition. Let the cycles $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ be a canonical basis of $H_1(X, \mathbb{Z})$. The vector $\vec{u} = (u_1, \ldots, u_g) \in \mathrm{Jac}(X) = \mathbb{C}^g/(\mathbb{Z}^g + T\mathbb{Z}^g)$ (T is the period matrix) with the components

$$u_k = \alpha_k - T_{kl}\beta_l \tag{2.7}$$

corresponds to the set of Wilson factors $W(a_k) = \exp 2\pi i\alpha_k$ $W(b_k) = \exp 2\pi i\beta_k$ and the flat connection is determined by the formula:

$$A = 2\pi i \bar{u}_k (T - \bar{T})^{-1}_{kj} \omega_j(z) + \text{h.c.} \tag{2.8}$$

where $\omega_j(z)$ are the normalized holomorphic 1-forms.

Now consider a doublet of spinor fields χ_{sp} and ψ_{sp} of the same characteristics $\vec{m}$ coupled to the external vector field A with the following action:

$$S[\psi, \chi] = \int e d^2\zeta \bar{\chi}_{\mathrm{sp}} \gamma^a e^\alpha_a (D_\alpha + A_\alpha) \psi_{\mathrm{sp}}, \tag{2.9}$$

where D_α is the covariant spinorial derivative. We will also assume that the strength of A equals zero. Locally the vector field A may be gauged away and the action (2.9) takes the form:

$$S[\psi, \chi] = \int (\chi_m \bar{\partial} \psi_m + \text{h.c.}) d^2\zeta, \tag{2.10}$$

where χ_m and ψ_m are the sections of the line bundles $K_{1/2,m}(\vec{u})$ and $K_{1/2,m}(-\vec{u})$ respectively. Clearly, (2.10) may be written for an arbitrary pair of the line bundles L_j and L_{1-j} of degree $j(2g-2)$ and $(1-j)(2g-2)$ respectively, provided

$$L_j \otimes L_{1-j} = \mathscr{K}. \tag{2.11}$$

The determinant of the Dirac operator in the external vector field may be expressed in the form (see [4]):

$$\det(D + A) = |F(\vec{u})|^2 \times (\text{anomaly}). \tag{2.12}$$

The anomaly is the contribution to the determinant that is not represented as the square of the module of an analytic function in $\vec{u}$. The anomaly contribution can be easily evaluated (see [4]):

$$\text{anomaly} = \exp - \frac{i}{2\pi} \int A_z A_{\bar{z}} dz \wedge d\bar{z}. \tag{2.13}$$

Substituting the expression (2.8) for A into (2.13) and multiplying (2.13) by appropriate holomorphic and antiholomorphic functions one gets:

$$\text{anomaly} = \exp i\pi(u-\bar{u})_k(T-\bar{T})^{-1}_{k\ell}(u-\bar{u})_\ell \,. \tag{2.14}$$

The structure of the anomaly (2.14) allows us to determine the dependence of $F(\vec{u})$ on $\vec{u}$. The determinant (2.12) is invariant under the change of variables $\vec{u}\to\vec{u}+\vec{n}+T\vec{m}$ for $\vec{n}$, $\vec{m}\in\mathbb{Z}^g$. Under such changes the anomaly contribution is multiplied by the factor $\exp[i\pi\vec{m}(T-\bar{T})\vec{m}+2\pi i\vec{m}(u-\bar{u})]$. Thus $F(\vec{u})$ has the following transformation rules:

$$\begin{gathered} F(\vec{u}+\vec{n}) = \exp(2\pi i\vec{n}\vec{a})F(\vec{u}) \,, \\ F(\vec{u}+T\vec{m}) = \exp(-2\pi i\vec{m}(\vec{u}+\vec{b})-i\pi\vec{m}T\vec{m})\times F(\vec{u}) \end{gathered} \tag{2.15}$$

for some vectors $\vec{a}$, $\vec{b}\in\mathbb{R}^g$. These transformation laws determine $F(\vec{u})$ up to a constant factor:

$$F(\vec{u}) = \mathrm{const}_1\, \Theta(\vec{\delta}+\vec{u}\,|\,T) \,, \tag{2.16}$$

where $\Theta(\ldots|T)$ is the Riemann theta function, $\vec{\delta}=\vec{a}+T\vec{b}$, const_1 does not depend on $\vec{u}$ but strictly depends on the point in the moduli space of the curves. One may also show that:

$$\vec{\delta}+\vec{u} = \vec{\Delta}-\vec{I}(D) \,, \tag{2.17}$$

where $\vec{\Delta}$ – is the Riemann constant; $\vec{I}(\ )$ is the Jacobian map, D is the divisor, corresponding to the line bundle $K_{1/2,m}(\vec{u})$ [see Eq. (2.4)]. Namely, $\Theta(\vec{\Delta}-\vec{I}(D))$ equals zero if and only if $D=\sum^{g-1}p_j$ due to the Riemann theorem. For the line bundles determined by such divisors the determinants of the Dirac operators are identically zero due to the existence of the holomorphic sections. The dependence of const_1 on the point of the moduli space of the curves may be easily determined. Recall a formula derived in [6]:

$$(\det\bar{\partial}_{1/2,m})(\det\bar{\partial}_0)^{1/2} = \mathrm{const}_2\,\Theta_m(0\,|\,T) \,, \tag{2.18}$$

where $\vec{m}$ is an even characteristic, const_2 is a number. Finally we get

$$F(\vec{u}) = \mathrm{const}_2(\det\bar{\partial}_0)^{-1/2}\Theta(\vec{\Delta}-\vec{I}(D)) \,. \tag{2.19}$$

On the other hand the determinant of the Dirac operator (2.12) may be represented as the square of the determinant of the $\bar{\partial}$-operator that acts on the sections of the line bundle of degree $(g-1)$:

$$\det(D+A) = |\det\bar{\partial}(\vec{u})|^2 \,. \tag{2.20}$$

The determinant of $\bar{\partial}$ is an analytic function on the moduli space of the curves and is determined by the following path integral:

$$\det\bar{\partial}(\vec{u}) = \int D\chi D\psi \exp - S[\chi,\psi] \,. \tag{2.21}$$

(For the definition of $S[\chi,\psi]$ see (2.10)). Comparing the formulas (2.12) and (2.20) we derive an expression for $\det\bar{\partial}(\vec{u})$ in terms of theta functions:

$$\det\bar{\partial}(\vec{u}) \sim \exp(i\pi\beta_k T_{k\ell}\beta_\ell)\times(\det\bar{\partial}_0)^{-1/2}\Theta(\vec{\Delta}-\vec{I}(D)) \,. \tag{2.22}$$

For the $\mathbb{Z}_N$ curves and non-singular $1/N$ characteristics (that will be defined below) $\det\bar{\partial}(\bar{u})$ can be expressed in terms of the branch points. Substituting this expression for $\det\bar{\partial}(\bar{u})$ in (2.22) we get a generalization of Thomae formula well known for the hyperelliptic curves [3].

3. Monodromy Matrices and Operator Product Algebra

For the scalar fields the monodromy matrices coincide with the permutation matrices that determine the order in which different sheets are glued together [1]. For the fermionic fields the situation is more subtle. We keep the term "fermionic fields" for the sections of arbitrary line bundle of degree $(g-1)$. First of all one has to specify a system of cycles $\{\gamma_i^\ell\}$ that is a basis of $H_1(X,\mathbb{Z})$. Let us make $(L-1)$ cuts (a_L, a_i), $i=1,2,\ldots,L-1$ on each sheet. The cycle γ_i^ℓ starts on the ℓ-th sheet near the right edge of the cut (a_L, a_i), passes around a_i and runs along the left edge to the point a_L. Then the cycle γ_i^ℓ make v rotations around a_L returning to the starting point. The number v satisfies the equation $vR_L+R_i\equiv 0 \pmod N$. This equation has an integer solution if $(R_i, N)\equiv 1$, $i=1,\ldots,L[(R,N)$ is the greatest common factor]. The cycles γ_i^ℓ are linearly dependent:

$$\sum_{\ell=1}^{N} \gamma_i^\ell = \sum_{i=1}^{L-1} \gamma_i^\ell = 0. \tag{3.1}$$

When the point ζ makes the total rotation along γ_i^ℓ the fermionic field is multiplied by the phase factor $\exp 2\pi i\varphi(\gamma_i^\ell)$. The phases $\varphi(\gamma_i^\ell)$ turn out to be linearly dependent due to (3.1):

$$\sum_{\ell=1}^{N} \varphi(\gamma_i^\ell) \equiv \sum_{i=1}^{L-1} \varphi(\gamma_i^\ell) \equiv 0 \quad (\operatorname{mod}\mathbb{Z}). \tag{3.2}$$

The monodromy matrices satisfy the following equations:

$$(M_j M_L^v)_{\ln} = (-1)^{N-1}\delta_{\ln}\exp 2\pi i\varphi(\gamma_j^\ell). \tag{3.3}$$

We consider the case of $\mathbb{Z}_N$-invariant characteristics. Then the phases $\varphi(\gamma_i^\ell)$ do not depend on ℓ $(\varphi(\gamma_i^\ell)=\varphi(\gamma_i))$. Equations (3.3) may be rewritten in form:

$$M_j M_L^v = (-1)^{N-1} I \exp 2\pi i\varphi(\gamma_j). \tag{3.4}$$

Now the monodromy matrices turn out to be commutative. This fact drastically simplifies the problem. In what follows we consider the case of prime N. In this case the system (3.4) has a solution

$$M_j = \pm M^{R_j}\exp 2\pi i k_j/N, \quad j=1,\ldots,L, \tag{3.5}$$

where $\varphi(\gamma_j)=k_j/N$, all k_j are integers, $k_L=0$. $M-$ is the matrix of cyclic permutation of N elements: $(1\to 2\to\ldots\to N\to 1)$.

Now we may introduce the fields σ_R^k that simulate the proper monodromy behaviour. As a point ζ rotates around a point a the correlation function transforms via

$$\langle\bar{\psi}(\zeta)\sigma_R^k(a)\ldots\rangle \to \exp(2\pi i\, k/N) M^R \langle\bar{\psi}(\zeta)\sigma_R^k(a)\ldots\rangle. \tag{3.6}$$

As two branch points a_i and a_j tend to each other a new branch point with $R \equiv R_i + R_j \pmod N$ arises. The fields σ_R^k obey the operator product algebra

$$\sigma_{R_i}^{k_i}(a_i)\sigma_{R_j}^{k_j}(a_j) \sim [\sigma_R^k], \tag{3.7}$$

where $R \equiv R_i + R_j \pmod N$ and $R \not\equiv 0 \pmod N$; $k \equiv k_i + k_j \pmod N$. Let us consider the case $R_i + R_j \equiv 0 \pmod N$, but $k_i + k_j \not\equiv 0 \pmod N$. In this case the branch point disappears but the marked point arises. A section of the bundle "remembers" that the surface had branch points. When ζ rotates around this marked point the fermionic field is multiplied by a non-trivial phase factor. We keep notation σ_0^k for the operators simulating the marked points. All the operators $\{\sigma_R^k\}$ obey the operator product algebra (3.7). Let us point out that there is an ambiguity in correspondence among the branch points and the twist operators σ_R^k due to the fact that the point a_L becomes the marked one after we choose a basis in $H_1(X, \mathbb{Z})$. This ambiguity produces a symmetry of the correlation functions of twist operators, that we are not going to discuss here. The determinant of $\bar\partial$ is determined as the path integral (2.21) over the fields $\chi(\zeta)$ and $\psi(\zeta)$ that are sections of the bundles $K_{1/2,m}(\vec u)$ and $K_{1/2,m}(-\vec u)$ respectively. Let us define the fields $\psi_\ell(\zeta)$ ($\chi_\ell(\zeta)$) by the formula:

$$\vec\psi(\zeta) = \sum \vec v_\ell \psi_\ell(\zeta), \tag{3.8}$$

where $\vec v_\ell$ are the eigenvectors of the monodromy matrix. We suppose that ℓ runs over the set

$$\mathscr{L} = \{-\tfrac{1}{2}(N-1), -\tfrac{1}{2}(N-1)+1, \ldots, \tfrac{1}{2}(N-1)\}. \tag{3.9}$$

The transformation properties of the correlation functions (3.6) uniquely determine the operator product expansion rules:

$$\psi_\ell(\zeta)\sigma_R^k(a) \simeq (\zeta - a)^{q_\ell(R,k)} \Sigma_R^k(a) + \ldots, \tag{3.10}$$

where

$$q_\ell(R,k) = \frac{1-N}{2N} + \left\{\frac{\ell R + k + (N-1)/2}{N}\right\}.$$

The operator product expansion rules for the field $\chi_\ell(\zeta)$ would coincide with (3.10) but for the substitution $k \to -k$ in the expression for $q_\ell(R,k)$. Note also that the composite fields $J_{\ell,n} = :\psi_\ell \chi_n:$ make the $GL(N)$ Kac-Moody algebra.

4. $\mathbb{Z}_N$-Invariant $1/N$ Characteristics

As was mentioned above, a line bundle is uniquely determined by the set of Wilson factors $W(\gamma_j^\ell) = \exp 2\pi i k_j/N$. Now we would like to describe a situation when all the k_j are integers and the arising line bundle does not admit a holomorphic section. This is just the case when $\det \bar\partial(\vec u)$ is not identically zero. A holomorphic section if it exists may be expressed in the form

$$f_\ell(z) = \prod_{i=1}^{L} (z - a_i)^{q_i} P(z), \tag{4.1}$$

where $P(z)$ is a polynomial. The obstruction for the existence of the zero mode (4.1) is the pole at infinity. Thus the line bundle determined by $\exp 2\pi i k_j/N$ does not admit a holomorphic section, if the numbers k_j are chosen so that

$$Q_\ell = \prod_{i=1}^{L} q_\ell(R_i, k_i) \geqq 0 \tag{4.2}$$

for arbitrary $\ell \in \mathscr{L}$ [see (3.9)]. Note that if $R_i \not\equiv 0 (\mathrm{mod}\, N)$ for all i and (4.2) is satisfied then all $Q_\ell \equiv 0 (\mathrm{mod}\, N)$ for all ℓ. This characteristic will be called a nonsingular $\mathbb{Z}_N$-invariant $1/N$ characteristic. Unfortunately this description is not explicit. We only succeeded to get an explicit description for nonsingular curves ($R_i = 1$ for all i). First of all we consider a simple example $N = 2$ (the hyperelliptic curves). For the hyperelliptic curves the indices k_i take only two values: $0, 1$, and the index ℓ runs over the set $\mathscr{L} = \{-\frac{1}{2}, \frac{1}{2}\}$. Let us divide all the branch points into two sets $\mathscr{K}_0$ and $\mathscr{K}_1$ with respect to the value of k,

$$\mathscr{K}_n = \{a_i, | k_i = n\}.$$

Denote by $\|\mathscr{K}_n\|$ the number of elements in $\mathscr{K}_n$. One may easily show that condition (4.2) implies that $\|\mathscr{K}_0\| = \|\mathscr{K}_1\|$. Thus there is one-to-one correspondence between nonsingular 1/2 characteristics and partitions of the branch points into two sets with the same number of elements. This is a well-known result of the theory of hyperelliptic curves. A divisor corresponding to such a bundle may be written explicitly. Namely

$$\psi_{1/2}(z) = \prod_{a_i \in \mathscr{K}_1} (z - a_i)^{1/2} (dz)^{1/2} / \sqrt{y(z)} \tag{4.3}$$

is a meromorphic section with first order poles at the infinities and zeros in $\mathscr{K}_1$. Thus the divisor of $\psi_{1/2}(z)$ equals

$$D(\mathscr{K}_1) = \sum_{\mathscr{K}_1} a_i - \infty_1 - \infty_2 . \tag{4.4}$$

Note that $D(\mathscr{K}_0)$ and $D(\mathscr{K}_1)$ are equivalent.

For $N = 3$ the situation is just the same. The indices k_i take three values: $0, 1, 2$. Thus all the branch points have to be decomposed onto three sets: $\mathscr{K}_0, \mathscr{K}_1, \mathscr{K}_2$. The conditions (4.2) lead to the following system of inequalities:

$$\|\mathscr{K}_0\| \geqq \|\mathscr{K}_1\|; \quad \|\mathscr{K}_1\| \geqq \|\mathscr{K}_2\|; \quad \|\mathscr{K}_2\| \geqq \|\mathscr{K}_0\| \tag{4.5}$$

that have only one solution: $\|\mathscr{K}_0\| = \|\mathscr{K}_1\| = \|\mathscr{K}_2\|$. Thus we get the same result. There is one-to-one correspondence between the nonsingular 1/3 characteristics and the partitions of the branch points on the three sets with the same number of elements. The divisor for this line bundle may be easily determined:

$$D = \sum_{\mathscr{K}_1} a_i + 2 \sum_{\mathscr{K}_2} a_i - \infty_1 - \infty_2 - \infty_3 . \tag{4.6}$$

This consideration is trivially generalized to the case of an arbitrary prime N. The $\mathbb{Z}_N$-invariant $1/N$ characteristics are uniquely determined by partitions of the branch points into N sets $\mathscr{K}_0, \mathscr{K}_1, \ldots, \mathscr{K}_{N-1}$ with the same number of elements. This partition is determined by the value of the index k. The divisor corresponding to this line bundle is

$$D = \sum_{\ell=0}^{N-1} \ell \left(\sum_{\mathcal{K}_\ell} a_i \right) - \sum_{i=1}^{N} \infty_i . \tag{4.7}$$

For nonsingular $1/N$ characteristics one may find N linearly independent meromorphic sections with the first order poles at infinities:

$$f_\ell(z, \vec{u}) = \prod_{i=1}^{L} (z - a_i)^{r_i} (dz)^{1/2} . \tag{4.8}$$

Here $r_i = q_\ell(R_i, k_i)$ [see expression (3.10)].

5. The Determinant of the $\bar{\partial}$ Operator

In this section we evaluate the determinant of the $\bar{\partial}$-operator acting on $K_{1/2,m}(\vec{u})$. Vector $\vec{u}$ corresponds to the $\mathbb{Z}_N$-invariant nonsingular $1/N$ characteristic. A vector field A is determined by the expression (2.8). First of all one has to determine the Green function $\langle \psi_n(z) \chi_\ell(w) \rangle$, and then to evaluate the average of the stress-energy tensor. The expectation value of the stress-energy tensor determines the variation of the correlation function under small variations of branch points. The Green function $\langle \psi_\ell(z) \chi_n(w) \rangle$ is a meromorphic section of the line bundle $E_{1/2,m}(\vec{u})$ ($E_{1/2,m}(-\vec{u})$) in the variable $z(w)$ with the only first order pole as z tends to w and the residue equals $\delta_{\ell+n,0}$. If $\ell + n \neq 0$, there is no pole and the Green function is a holomorphic section in either variable z or w, and thus it is identically zero since $\vec{u}$ corresponds to a nonsingular characteristic. Hence, $\langle \psi_n(z) \chi_\ell(w) \rangle \sim \delta_{\ell+n,0}$. One may find the expression for the Green function terms of meromorphic sections $f_n(z \,|\, \vec{u})$ [see (4.8)]:

$$\langle \psi_n(z) \chi_\ell(w) \rangle = \frac{\delta_{\ell+n,0}}{z - w} f_n(z \,|\, \vec{u}) f_\ell(w \,|\, -\vec{u}) . \tag{5.1}$$

Now one may easily find the expectation value for the currents

$$\langle J_{n,\ell}(z) \rangle = \delta_{n+\ell,0} \sum_i \frac{q_n(R_i, k_i)}{z - a_i} . \tag{5.2}$$

Hence the numbers $q_n(R, k)$ [see (3.10)] are the charges of the twist operators σ_R^k with respect to the currents $J_{n,-n}(z)$. The expectation value for the stress-energy tensor is:

$$\langle T(z) \rangle = \sum \frac{\Delta_i}{(z - a_i)^2} + \sum \frac{1}{z - a_i} \sum \frac{\gamma_{ij}}{a_i - a_j} , \tag{5.3}$$

where $\Delta_i = (N^2 - 1)/24N$ for a prime N. The coefficients γ_{ij} are determined by the expression:

$$\gamma_{ij} = \sum_{\ell \in \mathcal{L}} q_\ell(R_i, k_i) q_\ell(R_j, k_j) . \tag{5.4}$$

For $N = 2$ the expression for γ_{ij} can be easily computed and γ_{ij} turns out to be $\gamma_{ij} = {}_{(-)}^{+} 1/8$ for $k_i = k_j$ (respectively $k_i \neq k_j$). If $N = 3$, then $\gamma_{ij} = 2/9$ for $k_i = k_j$ and $\gamma_{ij} = -1/9$ for $k_i \neq k_j$.

One may easily integrate the expression (5.3) for the expectation value of the stress-energy tensor and obtain the following result for the determinant of the Dirac operator

$$\det(D+A) = \prod_{i<j} |a_i - a_j|^{2\gamma_{ij}(\vec{u})}. \tag{5.5}$$

The vector field A determined by a vector $\vec{u}$ [see (2.8)] corresponds to a $\mathbb{Z}_n$-invariant $1/N$ characteristic.

Now let us formulate the bosonisation rules. The currents $J_{\ell,-\ell}$ form a Cartan subalgebra of $GL(N)$, and thus we may define the currents $J_{\ell,-\ell}$ as $J_{\ell,-\ell} = \partial_z \psi_\ell$, where $\varphi_\ell(z)$ are free scalar fields. The charges of the fermionic fields ψ_n, χ_{-n} are $\delta_{n,\ell}$, $-\delta_{n,\ell}$ correspondingly with respect to $J_{\ell,-\ell}$. Thus, they may be bosonized by the expressions:

$$\psi_n(z) = :\exp \varphi_n(z):, \tag{5.6}$$

$$\chi_{-n}(z) = :\exp - \varphi_n(z):. \tag{5.6'}$$

The fields σ_R^k may also be represented as exponents of free scalar fields

$$\sigma_R^k = :\exp \sum_n q_n(R,k)\varphi_n(z):. \tag{5.7}$$

These bosonization rules reproduce all the correlation functions.

Now we would like to analyze the expression (5.7) for the determinant. Consider the simplest case of the hyperelliptic curves, the torus $N=2$, $L=4$. For this case there are three nonsingular 1/2 characteristics corresponding to different boundary conditions on the torus $(-,-)$, $(+,-)$, $(-,+)$, and the singular one, $(+,+)$. There are also three different partitions of the branch points on two sets with two branch points in each set. The determinant of the Dirac operator is

$$\det(D+A) = \prod_{i<j} |a_i - a_j|^{\pm 1/4} = \prod_{i<j} |a_i - a_j|^{-1/12} F(x,\bar{x}), \tag{5.8}$$

where x is the anharmonic ratio. One may easily get for different partitions:

$$F(x,\bar{x}) = \begin{cases} |x|^{1/3}|1-x|^{-1/6} & \text{for} \quad (a_1a_2)\cup(a_3a_4) \\ |x|^{-1/6}|1-x|^{-1/6} & \text{for} \quad (a_1a_3)\cup(a_2a_4) \\ |x|^{-1/6}|1-x|^{1/3} & \text{for} \quad (a_4a_1)\cup(a_2a_3). \end{cases} \tag{5.9}$$

The expression (5.8) for the determinant on the torus looks very unfamiliar. One usually works on the torus represented not as a two-sheeted covering but as a parallelogram $(1,\tau)$ with the identified boundaries. Let us compare the results for the Dirac determinant arising in these representations. The answer of the Dirac determinant computed on the parallelogram is as follows:

$$(\det D)(\tau,\bar{\tau}) = \begin{cases} |\Theta_0(\tau)\Theta_2(\tau)\Theta_3^{-2}(\tau)|^{-2/3} & \text{for} \quad (-,-) \\ |\Theta_0(\tau)\Theta_2^{-2}(\tau)\Theta_3(\tau)|^{-2/3} & \text{for} \quad (-,+) \\ |\Theta_0^{-2}(\tau)\Theta_2(\tau)\Theta_3(\tau)|^{-2/3} & \text{for} \quad (+,-). \end{cases} \tag{5.10}$$

These expressions are written in terms of theta constants. The formulas (5.10) may be easily rewritten in terms of the anharmonic ratio $x = (\Theta_3/\Theta_2)^4$:

$$(\det D)(\tau,\bar{\tau}) = F(x,\bar{x}). \tag{5.11}$$

At first glance we have got a very strange result. The Dirac determinant computed in the representation of the branch points differs from the Dirac determinant computed on the parallelogram by the scale factor $\prod_{i<j} |a_i - a_j|^{-1/12}$. The resolution of the problem is rather simple. Due to the conformal anomaly the determinant explicitly depends on the metric on the surface. The determinant (5.8) is evaluated at a singular metric on a surface that projects into $dzd\bar{z}$ on CP^1. The formulas (5.9) are written under the assumption that there is a flat metric $dzd\bar{z}$ on the torus. Hence, the derived results have to be different by a scale factor that is nothing other than the Liouville factor. The direct calculations of the Liouville factor give the desired result: $\prod_{i<j} |a_i - a_j|^{-1/12}$.

6. Thomae's Formula

In this section we rewrite the expression (2.22) in terms of the branch points. The determinant of $\bar{\partial}_0$ was computed in terms of the branch points in our previous paper [7]:

$$(\det \bar{\partial}_0)^{-1/2} = \prod_{i<j} (a_i - a_j)^{-\mu_{ij}} (\det \hat{A})^{-1/2}, \tag{6.1}$$

where $\mu_{ij} = \frac{1}{2} \sum_{\ell=0}^{N-1} \{\ell R_i/N\} \{\ell R_j/N\}$, $\hat{A}$ is the matrix of a-periods of the non normalized holomorphic 1-forms:

$$f_{k,\ell}(z) = \frac{z^{k-1} dz}{y_\ell(z)}, \tag{6.2}$$

$y_\ell(z) = \prod (z - a_i)^{\{\ell R_i/N\}}$. Combining the formulas (6.1), (5.5), and (2.22), we get the generalized Thomae's formula

$$\begin{aligned}(\det \hat{A})^{1/2} \simeq & \prod_{i<j} (a_i - a_j)^{-\gamma_{ij} - \mu_{ij}} \\ & \times \exp(i\pi \beta_k T_{kl} \beta_l) \Theta(\vec{\Delta} - \vec{I}(D)),\end{aligned} \tag{6.3}$$

where the divisor D is determined by (4.7). For $N = 2$ the formula (6.3) reduces to the ordinary Thomae's formula for the hyperelliptic curves [3]. The classical Thomae's formula allows us to get the Frobenius relations [8] that are satisfied by the period matrices of hyperelliptic curves. We believe that there are analogues of Frobenius relations for $\mathbb{Z}_N$-curves that allow one to describe the structure of the period matrices for $\mathbb{Z}_N$-curves. We also think that there exists a generalized Thomae's formula for arbitrary curves that may turn out to be useful for the Shottky problem.

7. Historical Comments and Acknowledgements

We would like to note the previous works by Zamolodchikov [2] and Fateev, who considered the cases of hyperelliptic curves and some cases of $\mathbb{Z}_3$-theory. The works [9] by Dixon et al. and [10] by Knizhik, are devoted to the related problems.

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