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# A Priori Estimates for $N=2$ Wess-Zumino Models on a Cylinder*

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**Abstract.** We establish bounds uniform in the ultraviolet cutoff (i.e., in the number of degrees of freedom) for a family of two-dimensional Wess-Zumino models. These estimates are useful in proving existence of the models, as well as in investigating their properties. For example, we require these estimates for the analysis of the supercharge and of the Hamiltonian. These are the fundamental a priori estimates for elliptic regularity in infinite dimensions.

## I. Introduction

In this paper we establish the fundamental, elliptic a priori estimates required for our analysis of two-dimensional Wess-Zumino models on a cylinder [1, 2]. These estimates are required for the construction of the models, as well as for the study of their detailed properties. We study the $N=2$ models in this paper as defined in [1, 2]. We follow the notation introduced in [1, 2]. These models are defined on the loop space of functions $\varphi: T^1 \to \mathbb{C}$.

The estimates here provide the first steps toward developing an analytic theory of Dirac operators on infinite dimensional manifolds. The extensions of these estimates to the $N=1$ and other frameworks, as well as to more general target spaces, are interesting questions under investigation.

We use the Feynman-Kac representations of [1, 2]. Our estimates generalize the methods used in the construction of the $Y_2$ and $P(\varphi)_2$ field theory models [3, 4]. The work here reduced the analysis of the models we study to standard estimates developed in Chap. 8 of [3]. Thus constructive field theory provides a suitable framework for this set of problems in infinite dimensional analysis.

It is useful to estimate operator norms of the heat kernel $\exp(-\beta H)$ using Schatten class norms $\|\cdot\|_p$ defined by the $l_p$ summability of the characteristic values. Thus if $\lambda_i$ are the eigenvalues of $(T^*T)^{1/2}$, the $I_p$ norm is

$$\|T\|_{I_p} = \|T\|_p = \Big(\sum_i \lambda_i^p\Big)^{1/p} = (\mathrm{Tr}(T^*T)^{p/2})^{1/p}.$$

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The $I_2$ or Hilbert-Schmidt norm is especially useful. The advantage of this estimate is that it automatically takes into account the dependence on both fermionic and bosonic states. A further advantage is the structure of the Feynman-Kac representations for the Hilbert-Schmidt norms; they all involve $L_1$ norms on path space of functions of the basic form

$$\det_3(1-K)\exp(-\mathscr{A}),$$

that have occurred already in many contexts. The fact that $I_p$ operator norms reduce to $L_1$ function-space estimates is essential for the preservation of the regularity properties which involve cancellations of local singularities. After such cancellations we can (and do) use $L_p$ estimates on path space.

It is convenient to rewrite certain functions of the bosonic field $\Phi$ as Wick-ordered expressions, since on a 2-cylinder the Wick-ordered polynomials have well understood regularity properties, see [3]. Let us mention here a general form of Wick's theorem for complex fields. This identity can be regarded as a definition of Wick monomials. This definition extends by linearity to polynomial functions of $\Phi$, and when convergent to limits of polynomials. The general algebraic structure is summarized by the identity:

**Wick's Theorem.** *For $F$ a function of $\Phi$ and $\Phi^*$,*

$$:F(\Phi, \Phi^*):_C = \exp(-\partial C\bar{\partial})\, F(\Phi, \Phi^*), \tag{1}$$

*where*

$$\partial C\bar{\partial} \equiv \tfrac{1}{4}\Delta_C = \int \frac{\partial}{\partial\Phi(x)}\, C(x,y)\, \frac{\partial}{\partial\Phi(y)^*}\, dx\, dy\,. \tag{2}$$

*The inverse to* (1) *is the transformation*

$$F(\Phi, \Phi^*) = \exp(\partial C\bar{\partial}) : F(\Phi, \Phi^*):_C\,. \tag{3}$$

We note that corresponding identities also hold for real fields and have the form

$$:F(\Phi):_C = \exp(-\tfrac{1}{2}\Delta_C)\, F(\Phi) \tag{4}$$

with inverse

$$F(\Phi) = \exp(\tfrac{1}{2}\Delta_C) : F(\Phi):_C\,,$$

where in this case

$$\Delta_C = \int \frac{\partial}{\partial\Phi(x)}\, C(x,y)\, \frac{\partial}{\partial\Phi(y)}\, dx\, dy\,.$$

In the following we suppress the subscript $C$. In estimating Gaussian integrals with respect to a measure $d\mu_C$ we perform Wick ordering with respect to the covariance $C$.

A second basic property that we use is *hypercontractivity* of the Gaussian path space measure $d\mu_C$, where $C$ is one of the covariance functions below. If $R$ is a polynomial in $\Phi$ of degree $\leqq n$, then for $p \geqq 2$, the following hypercontractivity bound holds:

$$\|R\|_{L_p} \leqq (p-1)^{n/2}\, \|R\|_{L_2}\,. \tag{5}$$

Here, and elsewhere, $L_p$ will denote path space norms. The case $n=1$ reduces estimates on moments of the measure $d\mu_C$ to Gaussian type estimates on the second moment. More generally, for any polynomial $R$ of degree $n$ in $\Phi$,

$$\int |R|^{2p}\, d\mu_C \leqq (2p-1)^{np}\, \|R\|_{L_2}^{2p}. \tag{6}$$

See Nelson [5] for a discussion of this bound. To establish the bound in our model, we view the complex field $\Phi = \mathrm{Re}\,\Phi + i\,\mathrm{Im}\,\Phi$ as a two-component real field $(\Phi_1, \Phi_2) = (\mathrm{Re}\,\Phi, \mathrm{Im}\,\Phi)$ and appeal to the general theorem. The hypercontractivity estimates could be replaced in our proofs by other explicit estimates on "Feynman graphs."

## II. Proof of the $N_\tau$ Estimate (Generalized Gårding Inequality)

The goal of this section is to prove a uniform estimate on the Hilbert-Schmidt norm of the heat kernel for $H_\tau(\kappa)$, with $\tau<1$ and with $\zeta$ sufficiently small. We fix $\tau$ and $\zeta$ in these estimates.

**Theorem II.1.** *Let $\beta>0$, $0\leqq\tau<1$ and let $\zeta\geqq 0$ be sufficiently small. Then there exists a constant $C=C(\tau,\zeta,\beta)<\infty$ (and independent of $\kappa$) such that for all $\kappa\geqq 0$,*

$$\|\exp(-\beta H_\tau(\kappa))\|_2 \leqq C. \tag{7}$$

It follows from (7) that $-H_\tau(\kappa)\leqq\beta^{-1}\log C$, since the Hilbert-Schmidt norm dominates the operator norm of the heat kernel. This establishes Theorem III.1 of [2].

We use Lemma VI.4 and Proposition VI.8 of [2] to write

$$\|\exp(-\beta H_\tau(\kappa))\|_2^2 = \Xi_{\tau,l,2\beta}\int \tilde{F}_{\tau,l,2\beta}(\Phi)\, d\mu_{C_{\tau,l,2\beta}}(\Phi). \tag{8}$$

We first establish the existence of the $L_p$ convergence of the integrand as $\kappa\to\infty$. Let

$$\tilde{\mathscr{A}}^{(\kappa)}_{\tau,l,\beta} = [A^{(\kappa)}_{l,\beta}(\Phi) + \mathrm{Tr}\,\tilde{K}^{(\kappa)}_{\tau,l,\beta}(\Phi) + \tfrac{1}{2}\mathrm{Tr}\,\tilde{K}^{(\kappa)}_{\tau,l,\beta}(\Phi)^2]. \tag{9}$$

**Proposition II.2.** (i) *Let $\beta>0$, $0\leqq\tau<1$, and $1\leqq p<\infty$. Then the limit*

$$\tilde{\mathscr{A}}_{\tau,l,\beta}(\Phi) = \lim_{\kappa\to\infty} \tilde{\mathscr{A}}^{(\kappa)}_{\tau,l,\beta}(\Phi)$$

*exists in $L_p(d\mu_{C_{\tau,l,\beta}})$.*

(ii) *The limiting action has the form*

$$\begin{aligned}\tilde{\mathscr{A}}_{\tau,l,\beta}(\Phi) = &\int_{T^2} :m\Phi\,\partial P(\Phi)^* + m\Phi^*\,\partial P(\Phi) + |\partial P(\Phi)|^2:dx\\ &+ \delta_k \int_{T^2} (\partial^2 P(\Phi)^* + \partial^2 P(\Phi))\,dx + \sum_{k=1}^{n-1} \alpha_k \int_{T^2} :|\partial^{k+1}P(\Phi)|^2:dx\\ &+ \sum_{k=1}^{n-1} \int_{T^2} \partial^{k+1}P(\Phi(x))^*\, \tilde{M}_k(x-y)\,\partial^{k+1}P(\Phi(y))\,dx\,dy,\end{aligned} \tag{10}$$

*where $\alpha_k$ are constants independent of $\chi$, where $\tilde{M}_k$ are functions independent of $\chi$, and where $\tilde{M}_k\in L_{1+\varepsilon}(T^2)$ for some $\varepsilon>0$. Here $\alpha_k$, $\delta_k$, and $\tilde{M}_k$ depend on $\tau$, $l$, and $\beta$.*

*Proof.* We establish here that $\tilde{\mathscr{A}}^{(\kappa)}_{\tau,l,\beta}$ has a representation of the form (10), and that the bounds on the corresponding cutoff coefficients $\delta_k^{(\kappa)}$, $\alpha_k^{(\kappa)}$, and $\tilde{M}_k^{(\kappa)}$ are uniform in $\kappa$ and convergent as $\kappa\to\infty$. Note that Wick polynomials of the form (10) are $L_p(d\mu_{C_{\tau,l,\beta}})$ for $1\leqq p<\infty$, as follows by standard constructive field theory estimates, see Chap. 8 of [3]. The $L_p$ convergence as $\kappa\to\infty$ then follows by the convergence of $\chi_\kappa$, of $\alpha_k^{(\kappa)}$ and of $\tilde{M}_k^{(\kappa)}$. The proposition therefore follows by establishing such bounds on $\delta$, $\alpha$, and $\tilde{M}$.

Using Wick's theorem, where we Wick order $A^{(\kappa)}_{l,\beta}$ with respect to the ($\tau$-dependent) covariance $C_{\tau,l,\beta}$,

$$\begin{aligned}A^{(\kappa)}_{l,\beta}(\Phi)=&\int_{T^2}(:m\Phi\,\partial P(\Phi_\kappa)^*+m\Phi^*\,\partial P(\Phi_\kappa)+|\partial P(\Phi_\kappa)|^2:)\,dx\\&+\tilde{C}^{(\kappa)}_{\tau,l,\beta}(0)\int_{T^2}(m\partial^2P(\Phi_\kappa)^*+m\partial^2P(\Phi_\kappa))\,dx\\&+\sum_{k=1}^{n-1}\frac{1}{k!}(C^{(\kappa)}_{\tau,l,\beta}(0))^k\int_{T^2}:|\partial^{k+1}P(\Phi_\kappa)|^2:dx\,,\end{aligned}\tag{11}$$

where $\tilde{C}^{(\kappa)}_{\tau,l,\beta}(x)=(\chi_\kappa * C_{\tau,l,\beta})(x)$, and where $C^{(\kappa)}_{\tau,l,\beta}(x)=\chi_\kappa * C_{\tau,l,\beta} * \chi_\kappa(x)$. The second and third terms on the right-hand side of (11) are singular in the limit $\kappa=\infty$, because $C^{(\kappa)}_{\tau,l,\beta}(0)$ and $\tilde{C}^{(\kappa)}_{\tau,l,\beta}(0)$ are $O(\log\kappa)$, for $\kappa$ large.

The singularities of the second term on the right-hand side of (11) and $\mathrm{Tr}(\tilde{K}^{(\kappa)}_{\tau,l,\beta}(\Phi))$ cancel. The remainder is the second term on the right-hand side of (10). The coefficient $\delta_k$ can be given in closed form as

$$\delta_k=\lim_{\kappa\to\infty}\delta_k^{(\kappa)}\,,$$

where $\delta_k^{(\kappa)}$ is just the difference $\frac{1}{2}\mathrm{Tr}(S^{(\kappa)}(0)-\tilde{S}^{(\kappa)}(0))$ of the regularized Green's function for $\not{D}$ between boundary conditions which are periodic and antiperiodic in time. The $\delta_k$ is exponentially small in the length $l$ of the circle $T^1$. Note that for boundary conditions which are periodic in time (rather than the antiperiodic conditions here) the cancellation between the second term in (11) and $\mathrm{Tr}(\tilde{K}^{(\kappa)}_{\tau,l,\beta}(\Phi))$ would be exact, i.e., $\delta_k^{(\kappa)}=\delta_k=0$. These boundary conditions enter estimates on the super (graded) trace. The same exact cancellation occurs for free boundary conditions in the time (i.e., for the vacuum functional, rather than the trace state).

The main fact about the cancellations is that

$$\lim_{\kappa\to\infty}\left[\frac{1}{2}\mathrm{Tr}\,\tilde{K}^{(\kappa)}_{\tau,l,\beta}(\Phi)^2+\sum_{k=1}^{n-1}\frac{1}{k!}(C^{(\kappa)}_{\tau,l,\beta}(0))^k\int_{T^2}:|\partial^{k+1}P(\Phi_\kappa)|^2:dx\right],\tag{12}$$

has the form of the last terms in (10). In other words, the cancellation of $\frac{1}{2}\mathrm{Tr}K^2$ in the sum (12) can be controlled. The cancellation of the logarithmic divergences takes place in such a fashion that the finite remainder terms are independent of the cutoff function $\chi$.

An explicit computation shows that the singularities of $\frac{1}{2}\mathrm{Tr}\,\tilde{K}^{(\kappa)}_{\tau,l,\beta}(\Phi)^2$ are contained in:

$$\frac{1}{2}\sum_{k=0}^{n-2}\frac{1}{k!}\int_{T^2\times T^2}C^{(\kappa)}_{\tau,l,\beta}(x-y)^k\,\sigma^{(\kappa)}_{\tau,l,\beta}(x-y):\partial^{k+2}P(\Phi_\kappa(x))^*\,\partial^{k+2}P(\Phi_\kappa(y)):dx\,dy\,,\tag{13}$$

where

$$\sigma^{(\kappa)}_{\tau,l,\beta}(x-y)=\int_{T^1\times T^1}\mathrm{Tr}(\tilde{S}_{\tau,l,\beta}(x-z)\,\chi_\kappa(z_1-y_1)\,\Lambda_+\tilde{S}_{\tau,l,\beta}(y-w)\,\chi_\kappa(w_1-x_1)\,\Lambda_-$$
$$+\tilde{S}_{\tau,l,\beta}(x-y)\,\chi_\kappa(y_1-z_1)\,\Lambda_+\tilde{S}_{\tau,l,\beta}(z-w)\,\chi_\kappa(w_1-x_1)\,\Lambda_-)\,dz_1\,dw_1\,. \tag{14}$$

The difference between (13) and $\frac{1}{2}\mathrm{Tr}(\tilde{K}^{(\kappa)}_{\tau,l,\beta})^2$ has the form of the last term in (10).

Next we write $P(\Phi_\kappa(y))=P(\Phi_\kappa(x))+\{P(\Phi_\kappa(y))-P(\Phi_\kappa(x))\}$. We extract the singular part of (13),

$$\frac{1}{2}\sum_{k=0}^{n-2}\frac{1}{k!}\left\{\int_{T^2}C^{(\kappa)}_{\tau,l,\beta}(x)^k\,\sigma^{(\kappa)}_{\tau,l,\beta}(x)\,dx\right\}\int_{T^2}:|\partial^{k+2}P(\Phi_\kappa(x))|^2:dx\,, \tag{15}$$

the remainder having the limit of the required form. In fact the remainder has the form

$$\frac{1}{2}\sum_{k=0}^{n-2}\frac{1}{k!}\int_{T^2\times T^2}\{C^{(\kappa)}_{\tau,l,\beta}(x-y)^k\,\sigma^{(\kappa)}_{\tau,l,\beta}(x-y)-D^{(\kappa)}_{\tau,l,\beta}\delta(x-y)\}$$
$$:\partial^{k+2}P(\Phi_\kappa(x))^*\,\partial^{k+2}P(\Phi_\kappa(y)):dx\,dy\,, \tag{16}$$

where

$$D^{(\kappa)}_{\tau,l,\beta}=\int_{T^2}C^{(\kappa)}_{\tau,l,\beta}(x)^k\,\sigma^{(\kappa)}_{\tau,l,\beta}(x)\,dx\,.$$

Let $H^{(\kappa)}(p)$ denote the Fourier transform of $C^{(\kappa)}_{\tau,l,\beta}(x-y)^k\,\sigma^{(\kappa)}_{\tau,l,\beta}(x-y)$. The Fourier transform of the expression in (16) in brackets is $H^{(\kappa)}(p)-H^{(\kappa)}(0)$. The function $H^{(\kappa)}(p)$ diverges as a power of $\log\kappa$ as $\kappa\to\infty$. It follows that the subtracted integral is convergent, uniformly in $\kappa$. The limit

$$\lim_{\kappa\to\infty}(H^{(\kappa)}(p)-H^{(\kappa)}(0))=H_{\mathrm{ren}}(p)$$

exists and is independent of $\chi$. Furthermore $H_{\mathrm{ren}}(p)$ is a bounded function of $p$, and hence its Fourier transform $M(x-y)$ is the kernel of a bounded operator. The singularity of this kernel on the diagonal is improved by the subtraction. It is $O(r^{-2+\varepsilon})$ for some $\varepsilon>0$, rather than $O(r^{-2}(\log r)^k)$. Hence $M(x-y)$ is $L_{1+\varepsilon}$ for some $\varepsilon>0$ and (16) has the form required by the lemma.

Let us return now to the singular expression in the bosonic action (15) which cancels the final term in (11). We assert that for some $\varepsilon>0$, and for each $k=1,2,\ldots,n-1$ there exists a constant $\alpha_k$ independent of the cutoff $\chi$ such that as $\kappa\to\infty$,

$$\frac{k}{2}\int C^{(\kappa)}_{\tau,l,\beta}(x)^{k-1}\,\sigma^{(\kappa)}_{\tau,l,\beta}(x)\,dx+C^{(\kappa)}_{\tau,l,\beta}(0)^k=\alpha_k k!+O(\kappa^{-\varepsilon})\,. \tag{17}$$

Given this bound, the sum of (16) and the final term in (11) has the $\kappa\to\infty$ limit

$$\sum_{k=1}^{n-1}\alpha_k\int_{T^2}:|\partial^{k+1}P(\Phi)|^2:dx\,,$$

as claimed in (10).

We illustrate the proof of (17) by a calculation which shows the important features of the cancellation: the reasons that the first term has a coefficient $k/2$, and the cutoff independence of the limit. We explain the case $\zeta=0$ and use continuum propagators in order to clarify the algebraic aspects of the cancellation and the regularity of the $\kappa\to\infty$ limit. We do not discuss the term proportional to $m^2$ (which gives rise to an absolutely convergent integral) nor the term involving $\hat{\chi}_\kappa(p_1+q_1)-\hat{\chi}(p_1)$ which can be bounded by $O(\kappa^{-1+\varepsilon})$). We then have

$$\hat{\sigma}^{(\kappa)}(q)=-2(2\pi)^{-2}\int\frac{p^2+pq}{[(p+q)^2+m^2](p^2+m^2)}\hat{\chi}_\kappa(p_1)^2\,dp+\dots.$$

Therefore

$$\frac{k}{2}\int C^{(\kappa)}(x)^{k-1}\,\sigma^{(\kappa)}(x)\,dx+C^{(\kappa)}(0)^k$$

$$=(2\pi)^{-2k}\int\Pi(p)\left\{-\frac{kp_k^2-kp_k\sum_{j=1}^{k-1}p_j}{\left(p_k-\sum_{j=1}^{k-1}p_j\right)^2+m^2}+1\right\}d^kp+c_k+O(\kappa^{-1+\varepsilon}),\tag{18}$$

where $c_k$ is the constant arising from the suppressed $m^2$ term and is independent of $\chi$. Also

$$\Pi(p)=\prod_{j=1}^{k}\frac{1}{p_j^2+m^2}\chi_\kappa(p_{j,1})^2.$$

Using the symmetries of the integral we find that (18) is equal to

$$(2\pi)^{-2k}\int\prod_{j=1}^{k}\frac{1}{p_j^2+m^2}\,\frac{m^2}{\left(p_k-\sum_{j=1}^{k-1}p_j\right)^2+m^2}\,d^kp+c_k+O(\kappa^{-1+\varepsilon}).$$

Since the above integral converges absolutely, the claim follows. This completes the proof of the proposition.

We set

$$\tilde{K}_{\tau,l,\beta}(\Phi)(x,y)=\tilde{S}_{\tau,l,\beta}(x-y)\,\Lambda_+\partial^2P(\Phi(y))+\tilde{S}_{\tau,l,\beta}(x-y)\,\Lambda_-\partial^2P(\Phi(y))^*.\tag{19}$$

We use the Sobolev space $\mathscr{K}_\alpha(T^2)=\mathscr{H}_\alpha(T^2)\oplus\mathscr{H}_\alpha(T^2)$, where $\mathscr{H}_\alpha$ is the Sobolev space of order $\alpha$ over $T^2$. We always regard $K\equiv\tilde{K}_{\tau,l,\beta}(\Phi)$ as a map $\mathscr{K}_{1/2}\to\mathscr{K}_{1/2}$. (If $K$ is considered as a map $K:\mathscr{K}_0\to\mathscr{K}_0$, and $n\geqq 2$, then there is no $p<\infty$ for which $K\in I_p$.) For instance, if we let $K^+$ denote the adjoint of $K$ on $\mathscr{K}_0$, then $\int(K^+K)^2\,d\mu_C$ has a singularity on the diagonal of order $(\log|x-y|)^{n-1}$. On the other hand, as an operator on $\mathscr{K}_{1/2}$,

$$K^*=C^{1/2}K^+C^{-1/2}$$

and

$$\int(K^*K)^2\,d\mu_C=\int C^{1/2}K^+C^{-1/2}KC^{1/2}K^+C^{-1/2}K\,d\mu_C$$

is trace class.

**Lemma II.3.** *As an operator on* $\mathscr{H}_{1/2}$, $\tilde{K}_{\tau,l,\beta}(\Phi)\in I_3$ *for almost all* $\Phi$, *with respect to* $d\mu_{C_{\tau,l,\beta}}$. *Moreover,*

$$\int \|\tilde{K}_{\tau,l,\beta}(\Phi)-\tilde{K}^{(\kappa)}_{\tau,l,\beta}(\Phi)\|_3^p \, d\mu_{C_{\tau,l,\beta}} \to 0, \quad as \quad \kappa\to\infty, \tag{20}$$

*for all* $1\leqq p<\infty$.

*Proof.* For $\varepsilon>0$,

$$\begin{aligned}\|K\|_3^3 &= \mathrm{Tr}((K^*K)^{3/2}) = \mathrm{Tr}((K^*K)^{1/2}\, C_{l,\beta}^{\varepsilon/2}(C_{l,\beta}^{-\varepsilon/2}K^*K))\\ &\leqq \{\mathrm{Tr}(C_{l,\beta}^{\varepsilon}K^*K)\}^{1/2}\{\mathrm{Tr}(C_{l,\beta}^{-\varepsilon}(K^*K)^2)\}^{1/2}.\end{aligned}$$

To prove that $K\in I_3$ for almost all $\Phi$ it is sufficient to show that for some $\varepsilon>0$,

$$\begin{aligned}J_1 &= \int \mathrm{Tr}(C_{l,\beta}^{\varepsilon}K^*K)\, d\mu_{C_{\tau,l,\beta}} < \infty,\\ J_2 &= \int \mathrm{Tr}(C_{l,\beta}^{-\varepsilon}(K^*K)^2)\, d\mu_{C_{\tau,l,\beta}} < \infty.\end{aligned}$$

We separate the contributions from the two terms on the right-hand side of (19). The contribution to $J_1$ from the first term is bounded by

$$J_3 = 2\int_{T^2\times T^2} C_{l,\beta}^{1/2+\varepsilon}(x-y)\,\mathrm{Tr}(\mathscr{G}_{\tau,l,\beta}(y-x))\,\mathscr{H}^{(2)}_{\tau,l,\beta}(y-x)\,dx\,dy,$$

where

$$\begin{aligned}\mathscr{G}_{\tau,l,\beta}(y-x) &= \int_{T^2} \Lambda_+ S^*_{\tau,l,\beta}(u-y)(-\Delta+m^2)^{1/2} S_{\tau,l,\beta}(u-x)\,du,\\ \mathscr{H}^{(2)}_{\tau,l,\beta}(y-x) &= \int \partial^2 P(\Phi(y))^*\, \partial P(\Phi(x))\, d\mu_{C_{\tau,l,\beta}}(\Phi).\end{aligned}$$

Since $C_{l,\beta}^{1/2+\varepsilon}(x)\leqq O(1)|x|^{-1+2\varepsilon}$, $|\mathscr{G}_{\tau,l,\beta}(x)|\leqq O(1)|x|^{-1}$, and

$$\mathscr{H}^{(2)}_{\tau,l,\beta}(x)\leqq O(1)\left\{\log\frac{1}{|x|}\right\}^{n-2},$$

for $|x|<1$, it follows that the integral $J_3$ exists. The contribution from the second term in (19) is dealt with similarly.

The contribution to $J_2$ from the first term in (19) is bounded by

$$\begin{aligned}J_4 = 8\int_{(T^2)^4} & C_{l,\beta}^{1/2-\varepsilon}(x_1-x_2)\, C^{1/2}(x_3-x_4)\,\mathrm{Tr}(\mathscr{G}_{\tau,l,\beta}(x_2-x_3)\,\mathscr{G}_{\tau,l,\beta}(x_4-x_1))\\ &\times \mathscr{H}^{(4)}_{\tau,l,\beta}(x_1,x_2,x_3,x_4)\,dx_1\ldots dx_4,\end{aligned}$$

where

$$\mathscr{H}^{(4)}_{\tau,l,\beta}(x_1,\ldots,x_4) = \int \partial^2 P(\Phi(x_1))\,\partial^2 P(\Phi(x_2))^*\,\partial^2 P(\Phi(x_3))\,\partial^2 P(\Phi(x_4))^*\, d\mu_{C_{\tau,l,\beta}}(\Phi).$$

Since the singularities of $\mathscr{H}^{(4)}_{\tau,l,\beta}$ are logarithmic where points coincide, it follows that the integrand in $J_4$ is integrable and $J_4$ exists.

To prove (20) we notice that by means of Hölder's inequality or the hypercontractivity bound we reduce it to $p=3$. The proof follows by estimates similar to the ones above with $\tilde{K}_{\tau,l,\beta}$ replaced by $\tilde{K}_{\tau,l,\beta}-\tilde{K}^{(\kappa)}_{\tau,l,\beta}$.

We conclude from the previous two lemmas that

$$\tilde{F}_{\tau,l,\beta}(\Phi) = \exp(-\tilde{\mathscr{A}}_{\tau,l,\beta}(\Phi))\,\det{}_3(I-\tilde{K}_{\tau,l,\beta}(\Phi)) \tag{21}$$

is a random variable.

The integrability of $\tilde{F}_{\tau,l,\beta}$ depends crucially on properties of $K=\tilde{K}_{\tau,l,\beta}(\Phi)$ and of the determinant in (21). We set $\mathscr{A}=\tilde{\mathscr{A}}_{\tau,l,\beta}$ and define (following [4])

$$L=(I-K^*)(I-K)-I=-K-K^*+K^*K\,, \tag{22}$$

and

$$W=\mathscr{A}+\tfrac{1}{4}\|K\|_4^4-\mathrm{Tr}(K^2K^*)-\tfrac{1}{6}\,\mathrm{Re}\,\mathrm{Tr}\,L^3\,. \tag{23}$$

**Proposition II.4.** *The function W is real and*

$$|F_{\tau,l,\beta}|=|\det{}_3(I-K)|e^{-\mathscr{A}}=[\det{}_4(I+L)]^{1/2}e^{-W}\,. \tag{24}$$

*Proof.* We derive the above identity for regularized $K$, after which we may pass to the $\kappa=\infty$ limit. To study the question we write $S=(D+m)C$, where

$$D=i\gamma_0^E\partial_t+i\gamma_1^E\partial_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}\partial_t+\begin{pmatrix}0&-i\\i&0\end{pmatrix}\partial_x$$

is the Euclidean Dirac operator. With $\gamma_5=i\gamma_0^E\gamma_1^E=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$,

$$\gamma_5D+D\gamma_5=0\,.$$

It is convenient, as an intermediate step, to regularize $C$, replacing $C$ by $C^{(\kappa')}$. With these definitions, and $\partial^2P=u+iv$ with $u$ and $v$ real functions of $\Phi$,

$$K=Su+iS\gamma_5v\,.$$

With this representation, it is clear that $\mathrm{Tr}\,K=2m\,\mathrm{Tr}(Cu)$ is real. Here we have evaluated the trace over the spinor indices, using $\mathrm{Tr}\,\gamma_\mu=\mathrm{Tr}\,\gamma_5=0$. A straightforward computation, using $\mathrm{Tr}(\gamma_\mu\gamma_5)=0$, etc., yields reality of $\mathrm{Tr}(K^2)$ after taking the trace over the spinor indices. Therefore

$$\begin{aligned}|\det{}_3(I-K)|^2&=\det{}_3(I-K)\det{}_3(I-K^*)\\&=\det{}_3(I+L)\exp(-\tfrac{1}{2}\mathrm{Tr}(K^*KK^*K)+2\,\mathrm{Re}\,\mathrm{Tr}(K^2K^*))\,.\end{aligned}$$

Hence $W$ is real. Now (24) follows as an algebraic identity.

**Proposition II.5.** (i) $\tilde{F}_{\tau,l,\beta}\in L_p(d\mu_{C_{\tau,l,\beta}})$, *for all* $1\leqq p<\infty$.

(ii) *Let* $1\leqq p<\infty$ *be given, and let* $\kappa\geqq\kappa_0(p)$. *Then there exists a constant C independent of* $\kappa$ *such that*

$$\|\tilde{F}^{(\kappa)}_{\tau,l,\beta}\|_{L_p}\leqq C\,.$$

*Proof.* Let $L_+$ and $L_-$ be the positive and negative parts of the self-adjoint operator $L=L_+-L_-$. Using (22), we infer $0\leqq L_-\leqq I$. Also, we define

$$R=\exp(-W)\{\det{}_4(I+L_+)\}^{1/2}\,. \tag{25}$$

Note that

$$\det{}_4(I+L)=\det{}_4(I+L_+)\det{}_4(I-L_-) \tag{26}$$

and

$$\det{}_4(I-L_-)\leqq 1\,.$$

It follows that

$$|\tilde{F}_{\tau,l,\beta}| \leqq R. \tag{27}$$

Hence, the proof of the proposition reduces to establishing:

**Proposition II.6.** *For* $1 \leqq p < \infty$, *and* $R$ *defined in* (25),

$$R \in L_p(d\mu_{C_{\tau,l,\beta}}). \tag{28}$$

*Proof.* To prove (28) first note that $R>0$ almost everywhere. Thus

$$v \equiv -\log R = W - \tfrac{1}{2}\log\det_4(I+L_+) \tag{29}$$

is a random variable. Let $v^{(\kappa)}$ be given by (29) with $\Phi$ replaced by $\Phi_\kappa$. By standard arguments (see e.g., [5, 3]) the proof of (28) follows if two bounds hold for all $\kappa < \infty$. The first bound is

$$v^{(\kappa)} \geqq -O((\log\kappa)^{n-1}). \tag{30}$$

The second bound asserts that on the space $L_2(d\mu_{C_{\tau,l,\beta}})$ there exists $\varepsilon > 0$ such that

$$\|v - v^{(\kappa)}\|_{L_2} \leqq O(\kappa^{-\varepsilon}). \tag{31}$$

Using the hypercontractivity bound (6), the estimate (31) yields associated $L_p$ estimates on $v - v^{(\kappa)}$.

To verify (30) we notice that

$$\log\det_4(I+L_+^{(\kappa)}) \leqq \tfrac{1}{2}\operatorname{Tr}(L_+^{(\kappa)})^2 - \tfrac{1}{3}\operatorname{Tr}(L_+^{(\kappa)})^3,$$

and this is bounded above by

$$\tfrac{1}{2}\operatorname{Tr}(L^{(\kappa)})^2 - \tfrac{1}{3}\operatorname{Tr}(L^{(\kappa)})^3.$$

Thus

$$\begin{aligned} v^{(\kappa)} &\geqq \mathscr{A}^{(\kappa)} + \tfrac{1}{4}\operatorname{Tr}(K^{(\kappa)}K^{(\kappa)})^2 - \operatorname{Re}\operatorname{Tr}((K^{(\kappa)})^2(K^{(\kappa)})^*) \\ &\quad - \tfrac{1}{4}\operatorname{Tr}(L^{(\kappa)})^2 = \mathscr{A}^{(\kappa)} - \tfrac{1}{4}\operatorname{Tr}(K^{(\kappa)} + (K^{(\kappa)})^*)^2, \end{aligned} \tag{32}$$

where $\mathscr{A}^{(\kappa)}$ is given by (10) with $\Phi$ replaced by $\Phi_\kappa$. We use the inequality

$$|\textstyle\int f(x)^* M(x-y)\, g(y)|\, dx\, dy \leqq \tfrac{1}{2}\int |M(x)|\, dx \,\{\int |f(x)|^2 dx + \int |g(x)|^2 dx\} \tag{33}$$

to bound the nonlocal, cutoff expressions in terms of local ones. In particular we estimate the $\partial^{k+2}P(\Phi_\kappa)^* \tilde{M}\, \partial^{k+2}P(\Phi_\kappa)$ terms in (10) by a multiple of $|\partial P(\Phi_k)|^2$. Then standard estimates, see e.g. [3], yield

$$\mathscr{A}^{(\kappa)} \geqq C \textstyle\int |\Phi_\kappa(x)|^{2(n-1)} dx - O((\log\kappa)^{n-1}). \tag{34}$$

A direct computation shows that $\operatorname{Tr}((K+K^*)^2)$ is bounded uniformly in the cutoff $\kappa$. This may be surprising, as $K \notin I_2$, but it is a consequence of a cancellation of singularities. This cancellation is not related to supersymmetry as it also holds in more general models; the property was discovered in the Yukawa model by Seiler, see Appendix A of [4]. We give here a simple argument in the case $\zeta = 0$. The claim then follows for $\zeta \neq 0$, as the leading asymptotics for the Green's functions $S_\tau$ in the vicinity of the diagonal are independent of $\tau$.

To study the question we use the representation of $D$, $S$, and $K$ used in the proof of Proposition II.4. Namely,

$$K = Su + iS\gamma_5 v = DCu + iD\gamma_5 Cv + \text{regular}.$$

Here the "regular" terms come from $m$ in $S = (D+m)C$. This decomposition yields $K = K_{\text{sing}} + K_{\text{reg}}$, where $K_{\text{reg}}$ has a logarithmic singularity on the diagonal (for $\zeta = 0$). Any term in $(K+K^*)^2$ has a trace which is bounded as $\kappa, \kappa' \to \infty$.

Now we compute

$$\operatorname{Tr} K_{\text{sing}}^2 = \operatorname{Tr}((DCu)^2 + (DCv)^2),$$

since the cross terms give

$$i \operatorname{Tr}(DCuD\gamma_5 Cv + D\gamma_5 CvDCu) = 0,$$

by cyclicity of the trace and $\{D, \gamma_5\} = 0$. Likewise, using $D^* = -D$, we have

$$\operatorname{Tr}(K_{\text{sing}}^2) = -\operatorname{Tr}(K_{\text{sing}} K_{\text{sing}}^*).$$

Thus

$$\operatorname{Tr}((K_{\text{sing}} + K_{\text{sing}}^*)^2) = 0 \tag{35}$$

and

$$|\operatorname{Tr}((K+K^*)^2)|$$

is bounded. We now take the $\kappa' \to \infty$ limit, which gives a uniform bound in $K$.

Thus

$$|\operatorname{Tr}(K^{(\kappa)} + K^{(\kappa)*})^2| \leqq C\left(\int |\Phi_\kappa(x)|^{2(n-2)} dx + 1\right), \tag{36}$$

which inserted in (32), (34) yields (30) as desired.
This proves (30).

To prove (31) we use the inequality

$$|\log \det_4(I + A_+) - \log \det_4(I + B_+)| \leqq \|A - B\|_4 \sum_{j=0}^{3} C_j \|A\|_4^j \, \|B\|_4^{3-j} \tag{37}$$

valid for self-adjoint $A, B \in I_4$ (see [4]). Using (36) with $A = L$, $B = L^{(\kappa)}$ we find that

$$\begin{aligned} |v - v^{(\kappa)}| \leqq {} & |\mathscr{A} - \mathscr{A}^{(\kappa)}| + \tfrac{1}{4} \big| \|K\|_4^4 - \|K^{(\kappa)}\|_4^4 \big| \\ & + |\operatorname{Tr}(K^2 K^* - (K^{(\kappa)})^2 (K^{(\kappa)})^*)| + \tfrac{1}{6} |\operatorname{Tr}(L^3 - (L^{(\kappa)})^3)| \\ & + \|L - L^{(\kappa)}\|_4 \sum_{j=0}^{3} C_j \|L\|_4^j \, \|L^{(\kappa)}\|_4^{3-j}, \end{aligned} \tag{38}$$

and the bound (31) follows by standard Feynman graph estimates (see Chap. 8 of [3]). This completes the proof of Proposition II.5 (i). The proof of (ii) is similar. The restriction on $\kappa$ is explained in Sect. VI.2 of [2].

The arguments of this section also apply to the case where we replaced the pair of covariances $(C_{\tau,l,\beta}, \tilde{S}_{\tau,l,\beta})$ by $(C_{l,\beta}, S_{l,\beta})$ or $(C_l, S_l)$. Denoting the corresponding heat kernel densities by $F_{l,\beta}(\Phi)$ and $F_l(\Phi)$ we obtain the following:

**Proposition II.7.** (i) $F_{l,\beta} \in L_p(d\mu_{C_{l,\beta}})$, *for* $1 \leqq p < \infty$.
(ii) $F_l \in L_p(d\mu_{C_l})$, *for* $1 \leqq p < \infty$.

*Remark.* The path integral representation of the index (II.19) of [2] is an immediate consequence of (i), the corresponding integrability and convergence of $F_{l,\beta}^{(\kappa)}$, and (II.18) of [2].

## III. Convergence of the Heat Kernel

In this section we prove the following

**Theorem III.1.** *Let $\beta>0$. Then*

$$\|\exp(-\beta H(\kappa))-\exp(-\beta H(\kappa'))\|_2=o(1), \tag{39}$$

*as $\kappa'\to\kappa$ or $\kappa,\kappa'\to\infty$.*

This establishes norm continuity and convergence of the heat kernels $\exp(-\beta H(\kappa))$ as functions of $\kappa$, and hence yields the proof of Theorem III.2 of [2], since $\|T\|\leqq\|T\|_2$ for $T\in I_2$. To prove (39) we notice that

$$\begin{aligned}\|\exp(-\beta H(\kappa))-\exp(-\beta H(\kappa'))\|_2^2=\mathrm{Tr}(&\exp(-2\beta H(\kappa))\\&+\exp(-2\beta H(\kappa'))-2\exp(-\beta H(\kappa))\exp(-\beta H(\kappa'))),\end{aligned}$$

which can be represented as

$$\begin{aligned}\Xi_{l,2\beta}\int(&\det(I-\tilde K_{l,2\beta}^{(\kappa)}(\Phi))\exp(-A_{l,2\beta}^{(\kappa)}(\Phi))\\&+\det(I-\tilde K_{l,2\beta}^{(\kappa')}(\Phi))\exp(-A_{l,2\beta}^{(\kappa)}(\Phi))\\&-2\det(I-\tilde K_{l,2\beta}^{(\kappa,\kappa')}(\Phi))\exp(-A_{l,2\beta}^{(\kappa,\kappa')}(\Phi))\,d\mu_{C_{l,2\beta}}(\Phi).\end{aligned} \tag{40}$$

Here

$$\tilde K_{l,2\beta}^{(\kappa,\kappa')}(\Phi)(x,y)=\tilde K_{l,2\beta}^{(\kappa)}(\Phi)(x,y)\,\chi_{[0,\beta]}(y_0)+\tilde K_{l,2\beta}^{(\kappa')}(\Phi)(x,y)\,\chi_{[\beta,2\beta]}(y_0), \tag{41}$$

where $\chi_{[a,b]}$ is the characteristic function of $[a,b]$, and

$$\begin{aligned}A_{l,2\beta}^{(\kappa,\kappa')}(\Phi)=&\int_{[0,\beta]\times T^1}[m\Phi\,\partial P(\Phi_\kappa)^*+m\Phi^*\,\partial P(\Phi_\kappa)+|\partial P(\Phi_\kappa)|^2]\,dx\\&+\int_{[\beta,2\beta]\times T^1}[m\Phi\,\partial P(\Phi_{\kappa'})^*+m\Phi^*\,\partial P(\Phi_{\kappa'})+|\partial P(\Phi_{\kappa'})|^2]\,dx.\end{aligned} \tag{42}$$

Let

$$\tilde F_{l,\beta}(\Phi)=\det{}_3(I-\tilde K_{l,\beta}(\Phi))\exp(-\tilde{\mathscr{A}}_{l,\beta}(\Phi)),$$

and let

$$\tilde F_{l,\beta}^{(\kappa)}(\Phi)=\det(I-\tilde K_{l,\beta}^{(\kappa)}(\Phi))\exp(-A_{l,\beta}^{(\kappa)}(\Phi))\equiv\det{}_3(I-\tilde K_{l,\beta}^{(\kappa)}(\Phi))\exp(-\tilde{\mathscr{A}}_{l,\beta}^{(\kappa)}(\Phi)).$$

Similarly, we write

$$\begin{aligned}\tilde F_{l,2\beta}^{(\kappa,\kappa')}(\Phi)&=\det(I-\tilde K_{l,2\beta}^{(\kappa,\kappa')}(\Phi))\exp(-A_{l,2\beta}^{(\kappa,\kappa')}(\Phi))\\&\equiv\det{}_3(I-\tilde K_{l,2\beta}^{(\kappa,\kappa')}(\Phi))\exp(-\tilde{\mathscr{A}}_{l,2\beta}^{(\kappa,\kappa')}(\Phi)).\end{aligned}$$

The convergence statement (39) for $\kappa,\kappa'\to\infty$ is an immediate consequence of (40), a $2\varepsilon$ argument, and the following

**Lemma III.2.** (i) $\|\tilde{F}_{l,2\beta}-\tilde{F}^{(\kappa)}_{l,2\beta}\|_{L_1}=o(1)$, *as* $\kappa\to\infty$.
(ii) $\|\tilde{F}_{l,2\beta}-\tilde{F}^{(\kappa,\kappa')}_{l,2\beta}\|_{L_1}=o(1)$, *as* $\kappa,\kappa'\to\infty$.

*Proof*. We give a detailed proof of (i). The proof of (ii) is similar, and we do not present it. The basic strategy of our bound is to study a well-behaved interpolation between $\tilde{F}_{l,2\beta}$ and $\tilde{F}^{(\kappa)}_{l,2\beta}$. Set $\tilde{K}(s)\equiv s\tilde{K}_{l,\beta}(\Phi)+(1-s)\tilde{K}^{(\kappa)}_{l,\beta}(\Phi)$, $\tilde{\mathscr{A}}(s)\equiv s\tilde{\mathscr{A}}_{l,\beta}(\Phi)+(1-s)\tilde{\mathscr{A}}^{(\kappa)}_{l,\beta}(\Phi)$, and $\tilde{F}(s)\equiv\det_3(I-\tilde{K}(s))\exp(-\tilde{\mathscr{A}}(s))$. Then using

$$\frac{d}{ds}\log\det_3(I-\tilde{K}(s))=-\operatorname{Tr}(\tilde{K}'(s)\tilde{K}(s)^2(I-\tilde{K}(s))^{-1}),$$

we have

$$\begin{aligned}\tilde{F}_{l,\beta}(\Phi)-\tilde{F}^{(\kappa)}_{l,\beta}(\Phi)&=\int_0^1\frac{d}{ds}\tilde{F}(s)\,ds\\&=-\int_0^1\tilde{\mathscr{A}}'(s)\tilde{F}(s)\,ds-\int_0^1\operatorname{Tr}(\tilde{K}'(s)\tilde{K}(s)^2(I-\tilde{K}(s))^{-1})\tilde{F}(s)\,ds\\&\equiv I_1+I_2.\end{aligned}$$

First we estimate $\int|I_1|\,d\mu$. By means of methods explained in Sect. II, we can prove that

$$\sup_{0\leqq s\leqq 1}\|\tilde{F}(s)\|_{L_p}\leqq\sup_{0\leqq s\leqq 1}\|\tilde{R}(s)\|_{L_p}\leqq C \tag{43}$$

uniformly in $\kappa$, where the meaning of $\tilde{R}(s)$ is clear. Since $\tilde{\mathscr{A}}'(s)=\tilde{\mathscr{A}}_{l,\beta}(\Phi)-\tilde{\mathscr{A}}^{(\kappa)}_{l,\beta}(\Phi)$, it follows from (43) that

$$\int|I_1|\,d\mu_{C_{l,\beta}}\leqq C\|\tilde{\mathscr{A}}_{l,\beta}-\tilde{\mathscr{A}}^{(\kappa)}_{l,\beta}\|_{L_2}.$$

We infer from Proposition II.2, that this is $o(1)$ as $\kappa\to\infty$. To prove that $\int|I_2|\,d\mu=o(1)$ as $\kappa\to\infty$ we notice that

$$\begin{aligned}&|\operatorname{Tr}(\tilde{K}'(s)\tilde{K}(s)^2(I-\tilde{K}(s))^{-1})\tilde{F}(s)|\\&\quad\leqq\|\tilde{K}'(s)\|_3\|\tilde{K}(s)\|_3^2\|(I-\tilde{K}(s))^{-1}\det_3(I-\tilde{K}(s))\|\exp(-\tilde{\mathscr{A}}(s)).\end{aligned}$$

Now we use $\|T\|=\|T^*T\|^{1/2}$ and (22) to obtain

$$\begin{aligned}&\|(I-\tilde{K}(s))^{-1}\det_3(I-\tilde{K}(s))\|\exp(-\tilde{\mathscr{A}}(s))\\&\quad=\|(I+\tilde{L}(s))^{-1}\det_4(I+\tilde{L}(s))\|^{1/2}\exp(-\tilde{W}(s)),\end{aligned}$$

with self-explanatory notation. This can be bounded by

$$C\{\det_4(I+\tilde{L}(s)_+)\}^{1/2}\exp(-\tilde{W}(s))\equiv C\tilde{R}(s),$$

and thus

$$\begin{aligned}\int|I_2|\,d\mu_{C_{l,\beta}}&\leqq C\{\int\|\tilde{K}_{l,\beta}(\Phi)-\tilde{K}^{(\kappa)}_{l,\beta}(\Phi)\|_3^2\,d\mu_{C_{l,\beta}}(\Phi)\}^{1/2}\\&\quad\times\sup_{0\leqq s\leqq 1}\{\int\|\tilde{K}(s)\|_3^8\,d\mu_{C_{l,\beta}}(\Phi)\}^{1/4}\sup_{0\leqq s\leqq 1}\|\tilde{R}(s)\|_{L_4}=o(1),\end{aligned}$$

as a consequence of Lemma II.3 and (43). This completes the proof of the lemma and the theorem.

## IV. Continuity of the Heat Kernel

In this subsection we prove strong continuity at $\beta=0$ of the semigroup

$$T(\beta)=\lim_{\kappa\to\infty}\exp(-\beta H(\kappa)).$$

This the content of Theorem III.3 of [2]. Since $T(\beta)$ is a semigroup, it is sufficient to establish weak continuity. Furthermore, since $H(\kappa)=Q(\kappa)^2\geqq 0$, it follows that $T(\beta)\leqq I$ and it is sufficient to show

$$\lim_{\beta\searrow 0}\langle\Omega,(T(\beta)-I)\Omega'\rangle=0, \tag{44}$$

for $\Omega,\Omega'$ vectors in a dense subspace of $\mathscr{H}$. In fact, it is sufficient to choose $\Omega$ of the form generated by a polynomial in fields applied to the Fock ground state, since linear combinations of such vectors are dense in $\mathscr{H}$.

We use $K_l(\Phi)$ defined in Sect. II for the $\tau$-independent case $\zeta=0$, and with the index $\beta$ suppressed (no periodicity in time). Let $F_l(\Phi)$ be the corresponding heat kernel density. We study a series of estimates whose aim is to reduce (44) to simple Feynman graph estimates and Proposition II.6. We express inner products of the form $\langle\Omega,T(\beta)\Omega'\rangle$ as a bosonic function-space integral

$$\langle\Omega,T(\beta)\Omega'\rangle=\int F_l(\Phi,\mathbf{f},\mathbf{g},\mathbf{h})\,d\mu_{C_l}(\Phi), \tag{45}$$

where

$$F_l(\Phi,\mathbf{f},\mathbf{g},\mathbf{h})=F_l(\Phi)\left(\bigwedge_{j=1}^{k} g_j,\bigwedge_{j=1}^{k}(I-K_l(\Phi))^{-1}S_lh_j\right)_{\bigwedge^k\mathscr{K}_0}\prod_{j=1}^{q}\Phi^{\#}(f_j), \tag{46}$$

and where $\#$ denotes possible complex conjugation. The identity (45) follows from the finite cutoff Feynman-Kac formula (Proposition VI.8 of [2]), Lemma II.7 and the following:

**Proposition IV.1.** *Let $f_j\in\mathscr{H}_{-1}(\mathbb{R}\times T^1)$, $g_j,h_j\in\mathscr{K}_{-1/2}(\mathbb{R}\times T^1)$. Then for all $1\leqq p<\infty$,*

$$F_l(\Phi,\mathbf{f},\mathbf{g},\mathbf{h})\in L_p(\mathscr{S}'(\mathbb{R}\times T^1),d\mu_{C_l}). \tag{47}$$

*Remark.* We already know that $F_l(\Phi)\in L_p$, $1\leqq p<\infty$, as a consequence of Proposition II.8 (ii). However, we cannot use Hölder's inequality. The $(I-K)^{-1}$ factors in (46), which arise from the fermion integration, are not $L_p(d\mu_{C_l})$ by themselves. These factors must be combined with $\det_3(I-K_l)$ to produce an $L_p$, Fredholm minor, as in the proof of Lemma III.2. In order to simplify notation in the remainder of the paper, we sometimes write $\det_3$ or $\det_4$ separately from factors of $(I-K)^{-1}$ or $(I-L)^{-1}$ or other related operators which compose the corresponding minor. We do not believe this should cause confusion to the reader.

*Proof.* Since $\Phi(f_j)\in L_p(d\mu_{C_l})$ for all $1\leqq p<\infty$, we can use Hölder's inequality to reduce the proposition to the case

$$F_l(\Phi,\mathbf{g},\mathbf{h})=F_l(\Phi)\left(\bigwedge_{j=1}^{k} g_j,\bigwedge_{j=1}^{k}(I-K_l(\Phi))^{-1}S_lh_j\right)_{\bigwedge^k\mathscr{K}_0}\in L_p. \tag{48}$$

For $K \in I_p$, the mapping $z \to \det_p(I-zK) \bigwedge^k (I-zK)^{-1}$ is an entire, operator-valued function of $z$. In our case $K \in I_3(\mathscr{H}_{1/2})$, so (48) is well defined for $g_j, h_j \in \mathscr{H}_{1/2}$. Using (25) we have

$$|F_l(\Phi, \mathbf{g}, \mathbf{h})| \leqq R_l(\Phi) \{\det_4((1-L_-)\}^{1/2} \left| \left( \bigwedge_{j=1}^{k} C_l^{1/2} g_j, \bigwedge_{j=1}^{k} (I-K_l(\Phi))^{-1} S_l h_j \right)_{\bigwedge^k \mathscr{H}_{1/2}} \right|. \tag{49}$$

We now use the identity for operator norms,

$$\|T\|_{\mathscr{H}_{1/2}} = \|C_l^{-1/4} T C_l^{1/4}\|_{\mathscr{H}_0},$$

and we compute the adjoint of $K$ on $\mathscr{H}_{1/2}$. Using the Schwarz inequality on $\mathscr{H}_{1/2}$, for $g, h \in \mathscr{H}_{1/2}$, and $\|C_l^{1/2} h\|_{\mathscr{H}_{1/2}} = \|h\|_{\mathscr{H}_{1/2}}$, we have

$$|(C_l^{1/2} g, TS_l h)_{\mathscr{H}_{1/2}}| \leqq \|g\|_{\mathscr{H}_{1/2}} \|TS_l h\|_{\mathscr{H}_{1/2}} \leqq \|g\|_{\mathscr{H}_{1/2}} \|TS_l C_l^{-1/2}\|_{\mathscr{H}_{1/2}} \|h\|_{\mathscr{H}_{1/2}}.$$

Note that $C_l$ and $S_l$ commute, so

$$\|S_l C_l^{-1/2}\|_{\mathscr{H}_{1/2}} = \|S_l C_l^{-1/2}\|_{\mathscr{H}_0} = 1,$$

and therefore

$$|(C_l^{1/2} g, TS_l h)_{\mathscr{H}_{1/2}}| \leqq \|T\|_{\mathscr{H}_{1/2}} \|g\|_{\mathscr{H}_{1/2}} \|h\|_{\mathscr{H}_{1/2}}. \tag{50}$$

For $T$ we choose $(I-K)^{-1}$, and we write $\|T\|_{\mathscr{H}_{1/2}}$ as $(\|T^*T\|_{\mathscr{H}_{1/2}})^{1/2} = \|(I+L)^{-1}\|_{\mathscr{H}_{1/2}}^{1/2}$. Applying the bound (50) on the wedge product space $\bigwedge^k \mathscr{H}_{1/2}$, we bound (49) by

$$\begin{aligned} & R_l(\Phi) \left\| \bigwedge_{j=1}^{k} (I-L_-)^{-1} \det_4(I-L_-) \right\|_{\bigwedge^k \mathscr{H}_{1/2}}^{1/2} \prod_{j=1}^{k} \|g_j\|_{\mathscr{H}_{-1/2}} \|h_j\|_{\mathscr{H}_{-1/2}} \\ & \leqq R_l(\Phi) e^k \prod_{j=1}^{k} \|g_j\|_{\mathscr{H}_{-1/2}} \|h_j\|_{\mathscr{H}_{-1/2}}. \end{aligned} \tag{51}$$

The final bound is a consequence of the following inequality, Lemma 4.2 of [6]:

$$\|\bigwedge^k (I-T)^{-1} \det_p(I-T)\| \leqq \exp\left\{ k \sum_{j=1}^{p-1} \frac{1}{j} \right\}, \tag{52}$$

valid for $T \in I_p$ with $0 \leqq T \leqq I$.

We now study the difference between the operator defining the right side of (46) and the identity, namely

$$Y = F_l \bigwedge^k (I-K)^{-1} - I, \tag{53}$$

where $F_l \equiv F_l(\Phi)$. This operator acts on the space $\bigwedge^k \mathscr{H}_{1/2}$ (for fixed $\Phi$). Write $Y = Y_1 + Y_2$, where

$$Y_1 = (F_l - 1) I \tag{54}$$

and

$$Y_2 = F_l (\bigwedge^k (I-K)^{-1} - I). \tag{55}$$

For $0 \leqq s \leqq 1$ we set

$$R_l(s) = \exp(-W(s))\{\det_4(I+L(s)_+)\}^{1/2}, \tag{56}$$

where

$$L(s) = -sK - sK^* + s^2 K^* K, \tag{57}$$

and

$$W(s) = s\mathscr{A} + \tfrac{1}{4}s^4 \|K\|_4^4 - s^3 \operatorname{Re}\operatorname{Tr}(K^2K^*) - \tfrac{1}{6}\operatorname{Tr} L(s)^3. \tag{58}$$

We now give separate estimates on $Y_1$ and $Y_2$.

**Lemma IV.3.** *With the above definitions,*

$$\|Y_1\| \leqq \{|\mathscr{A}| + e\|K\|_3^3\} \int_0^1 R_l(s)\,ds. \tag{59}$$

*Here $\|\cdot\|$ denotes the operator norm on $\bigwedge^k \mathscr{K}_{1/2}$.*

*Proof.* We use an interpolation argument. Let $s \in [0,1]$ and consider $\det_3(I-sK)\exp(-s\mathscr{A})$. We follow the proof of (27) to obtain the bound

$$|\det_3(I-sK)|\exp(-s\mathscr{A}) = [\det_4(I+L(s))]^{1/2} e^{-W(s)} \leqq R_l(s), \tag{60}$$

where $R_l(s)$ is defined (56). For each $s$, we remark below that $R_l(s)$ is $L_p(d\mu_{C_l})$ for $p<\infty$. Thus it is natural to write

$$\begin{aligned} Y_1 = F_l - 1 &= \int_0^1 \frac{d}{ds}\{\det_3(I-sK)\exp(-s\mathscr{A})\}\,ds \\ &= -\mathscr{A}\int_0^1 \det_3(I-sK)\exp(-s\mathscr{A})\,ds \\ &\quad - \int_0^1 \operatorname{Tr}(K^3(I-sK)^{-1}\det_3(I-sK))s^2\exp(-s\mathscr{A})\,ds \\ &= J_1 + J_2 . \end{aligned}$$

The bound on $J_1$ is

$$|J_1| \leqq |\mathscr{A}| \int_0^1 R_l(s)\,ds, \tag{61}$$

which is the first term in (59).

Let us now turn to $J_2$. By Hölder's inequality on the Schatten spaces $I_p$,

$$|\operatorname{Tr}(K^3(I-sK)^{-1}\det_3(I-sK))| \leqq \|K\|_3^3 \, \|(I-sK)^{-1}\det_3(I-sK)\|, \tag{62}$$

and therefore using (60) and (26),

$$\begin{aligned} |J_2 &\leqq \|K_3\|_3^3 \int_0^1 \|(I-sK)^{-1}\det_3(I-sK)\|\, s^2 \exp(-s\mathscr{A})\,ds \\ &= \|K\|_3^3 \int_0^1 \|(I+L(s))^{-1}\det_4(I+L(s))\|^{1/2}\exp(-W(s))\,ds \\ &\leqq \|K\|_3^3 \int_0^1 \|(I-L(s)_-)^{-1}\det_4(I-L(s)_-)\|^{1/2} R_l(s)\,ds \\ &\leqq e^{11/12}\|K\|_3^3 \int_0^1 R_l(s)\,ds, \end{aligned} \tag{63}$$

where in the last line we have used (52) with $k=1$, $p=4$. The lemma follows from (61) and (63).

**Lemma IV.4.** *The following estimate holds on* $\bigwedge^k \mathscr{H}_{1/2}$:

$$\|Y_2\| \leqq e^k 2^k R_l \sum_{m=1}^{k} \|K\|_3^m . \tag{64}$$

*Proof.* We write

$$Y_2 = F_l \sum_{\substack{\varepsilon \neq 0 \\ \varepsilon_j \in \{0,1\}}} D^{\varepsilon_1} \wedge \ldots \wedge D^{\varepsilon_k}, \tag{65}$$

where $D=(I-K)^{-1}K$. Using (27),

$$\begin{aligned} \|Y_2\| &\leqq R_l \sum_{\varepsilon \neq 0} \|\{\det_4(I-L_-)\}^{1/2} D^{\varepsilon_1} \wedge \ldots \wedge D^{\varepsilon_k}\| \\ &\leqq 2^k R_l \sum_{m=1}^{k} \|\{\det_4(I-L_-)\}^{1/2} \textstyle\bigwedge^m D\|_3 . \end{aligned} \tag{66}$$

Note that $D^*D = K^*(I+L)^{-1}K \leqq K^*(I-L_-)^{-1}K$. We then have

$$\begin{aligned} \|\{\det_4(I-L_-)\}^{1/2} \textstyle\bigwedge^m D\|_3^3 &= \operatorname{Tr}\{\det_4(I-L_-)\}^{3/2} (\textstyle\bigwedge^m (D^*D)^{3/2}) \\ &\leqq \|K\|_3^{3m} \|\det_4(I-L_-)^{1/2} \textstyle\bigwedge^m (I-L_-)^{-1/2}\|^3 \leqq \|K\|_3^{3m} e^{3m} . \end{aligned} \tag{67}$$

The estimates (66) and (67) yield (64).

**Lemma IV.5.** *There exist* $\eta>0$ *and* $C<\infty$ *such that for* $\beta$ *small, the following two bounds holds*

$$\int \|K_l(\Phi)\|_3^3 \, d\mu_{C_l} \leqq C\beta^\eta , \tag{68}$$

*and*

$$\|\mathscr{A}_l(\Phi)\|_{L_2} \leqq C\beta^\eta . \tag{69}$$

*Proof.* We use the Schwarz inequality to relate the $I_3$ norm to $I_2$ and $I_4$ norms. Since $K \notin I_2$, we transfer a small power of $C$ from the $I_4$ to the $I_2$ norm. For $\varepsilon>0$,

$$\|K_l(\Phi)\|_3^3 \leqq \{\operatorname{Tr}(C_l^\varepsilon K_l(\Phi)^* K_l(\Phi))\}^{1/2} \{\operatorname{Tr}(C_l^{-\varepsilon}(K_l(\Phi)^* K_l(\Phi))^2)\}^{1/2} .$$

After integration over $d\mu_{C_l}$ we then obtain by another Schwarz inequality

$$\begin{aligned} \int \|K_l(\Phi)\|_3^3 \, d\mu_{C_l} \leqq &\{\int \operatorname{Tr}(C_l^\varepsilon K_l(\Phi)^* K_l(\Phi)) \, d\mu_{C_l}\}^{1/2} \\ &\times \{\int \operatorname{Tr}(C_l^{-\varepsilon}(K_l(\Phi)^* K_l(\Phi))^2) \, d\mu_{C_l}\}^{1/2} . \end{aligned} \tag{70}$$

We claim that there exists $\eta>0$ such that both factors on the right-hand side of (70) are $O(\beta^\eta)$ as $\beta \to 0$. In fact,

$$\begin{aligned} &\int \operatorname{Tr}(C_l^\varepsilon K_l(\Phi)^* K_l(\Phi)) \, d\mu_{C_l} \\ &= \int_{([0,\beta] \times T^1)^2} C_l^{1/2+\varepsilon}(x-y) \, C_l^{1/2}(x-y) \, B_2(x-y) \, dx \, dy , \end{aligned} \tag{71}$$

where

$$B_2(x-y) = \int \partial^2 P(\Phi(x))^* \, \partial^2 P(\Phi(y)) \, d\mu_{C_l} \tag{72}$$

has only logarithmicc singularities at $x=y$. Since

$$G_l^{1/2+\varepsilon}(x-y) \leqq O(|x|^{-1+2\varepsilon}) \leqq O(|x_0|^{-1/2+\varepsilon}|x_1|^{-1/2+\varepsilon}), \tag{73}$$

for $|x|<1$, it follows that (71) is $O(\beta^n)$ as $\beta \to 0$.

To bound

$$\int \mathrm{Tr}(C_l^{-\varepsilon}(K_l(\Phi)^* K_l(\Phi))^2) d\mu_{C_l},$$

we write it as a sum of terms of the form

$$\int_{([0,\beta]\times T^1)^4} \mathscr{F}(\mathbf{x}) B_4(\mathbf{x}) d^4\mathbf{x}, \tag{74}$$

where

$$\mathscr{F}(x) = C_l^{1/2-\varepsilon}(x_1-x_2) C_l^{1/2}(x_2-x_3) C_l^{1/2}(x_3-x_4) C_l^{1/2}(x_4-x_1),$$

where $x_j \in [0,\beta] \times T^1$ for $j=1,2,3,4$, and where $B_4(\mathbf{x})$ has logarithmic singularities at points of coincidence. Using Hölder's inequality we bound (74) by

$$C\{\int |\mathscr{F}(\mathbf{x})|^p d^4\mathbf{x}\}^{1/p} \{\int |B_4(\mathbf{x})|^q d^4\mathbf{x}\}^{1/q}, \tag{75}$$

where we choose $1<p<3/(2+\varepsilon)$, and correspondingly $q=p/(p-1) \in (3/(1-\varepsilon), \infty)$.

The integral $\|\mathscr{F}\|_{L_p}^p$ can be bounded as follows. We set $f(x) \equiv |(C_l^{1/2})(x)|^p$ and $g(x) = |(C_l^{1/2-\varepsilon})(x)|^p$. By Hölder's inequality,

$$\|\mathscr{F}\|_{L_p}^p \leqq \int |(f*f*f)(x) g(-x)| dx \leqq \|f*f*f\|_{L_4} \|g\|_{L_{4/3}}.$$

Using Young's inequality

$$\|f*f*f\|_{L_4} \leqq \|f\|_{L_{4/3}} \|f*f\|_{L_2} \leqq \|f\|_{L_{4/3}}^3$$

and so

$$\|\mathscr{F}\|_{L_p}^p \leqq \|f\|_{L_{4/3}}^3 \|g\|_{L_{4/3}} = \|(C_l^{1/2})(\cdot)\|_{L_{4p/3}}^{3p} \|(C_l^{1/2-\varepsilon})(\cdot)\|_{L_{4p/3}}^p.$$

Using (73) we find that

$$\|(C_l^{1/2-\varepsilon})(\cdot)\|_{L_{4p/3}} \leqq O(\beta^{3/4p-1/2-\varepsilon}).$$

Thus for $p<3/(2+\varepsilon)$

$$\|\mathscr{F}\|_{L_p} \leqq O(\beta^{3/p-2-\varepsilon}), \tag{76}$$

for $\beta$ small. (The constants depend on $l$.)

The second factor $\|\mathscr{B}_4\|_{L_q}$ in (74) is the integral of a product of integral kernels $C_l(x_i - x_{i'})$ with logarithmic singularities on the diagonal. Extracting the volume dependent factor we obtain thus

$$\|\mathscr{B}_4\|_{L_q}^q \leqq C\beta^4 (\log\beta)^N,$$

where $N$ is some fixed number (which grows with the degree $n$ of the superpotential), and since we fix $q$, we infer that for $\eta>0$ sufficiently small,

$$\|\mathscr{B}_4\|_{L_q}^q \leqq O(\beta^\eta),$$

for $\beta$ small.

These estimates establish (68). The proof of (69) is similar.

**Lemma IV.6.** *Let $\beta$ be sufficiently small. There is a constant $C$ independent of $\beta$ such that*

$$\sup_{0 \leqq s \leqq 1} \int R_l(s)^4 d\mu_{C_l} \leqq C. \tag{77}$$

The proof of this lemma is similar to the proof of Proposition II.6. The uniformity in $\beta$ follows easily from the fact that the Feynman graphs generated by the terms in (37) are $O(\beta^\eta)$. This can be established by the method used in the proof of Lemma IV.5.

These estimates now lead to the proof of continuity of $T(\beta)$ at $\beta = 0$. Returning to the definitions (45) and (50)–(52), we need only estimate

$$\int \Big(\prod_j \Phi^\#(f_j)\Big) \|Y\| d\mu_{C_l} \leqq \Big(\int \Big|\prod_j \Phi(f_j)\Big|^2 d\mu_{C_l} \int \|Y\|^2 d\mu_{C_l}\Big)^{1/2} \tag{78}$$

and to show it tends to zero with $\beta$. Write $Y = Y_1 + Y_2$, and let us first estimate the contribution from $Y_1$. Apply the Schwarz inequality to the $d\mu_{C_l}$ integral of (56) squared in order to split the $R$ factors from the remaining terms involving $K$ and $\mathscr{A}$. These terms can now be estimated by $O(\beta^{\eta'})$, $\eta' > 0$, as follows. We study

$$\int \{|\mathscr{A}| + e\|K\|_3^3\}^4 d\mu_{C_l}. \tag{79}$$

Each term in (79) can be estimated using Lemma IV.5 and is bounded by $O(\beta^\eta)$ as $\beta \to 0$. A similar estimate holds for $L_p$ norms of $\mathscr{A}$. Thus

$$\int \|Y_1\|^2 d\mu_{C_l} \leqq O(\beta^{\eta'}) \int_0^1 (\int R_l(s)^4 d\mu_{C_l})^{1/4} ds \leqq O(\beta^{\eta'}), \tag{80}$$

with the last inequality a consequence of Lemma IV.6.

In a similar fashion, we bound the $Y_2$ contribution by

$$O(1)\|R_l\|_{L_4}^2 \sum_{m=1}^{k} \{\int \|K_l\|_3^{2m} d\mu_{C_l}\}^{1/2}.$$

By Hölder's inequality (applied to the $m=1$ term) and the hypercontractivity bound (5) (applied to the terms with $m \geqq 2$) we can reduce the above sum to a form which can be estimated by means of (68). This yields

$$\int \|Y_2\|^2 d\mu_{C_l} \leqq O(\beta^{\eta'}) \|R_l\|_{L_4}^2 \leqq O(\beta^{\eta'}).$$

The continuity of $T(\beta)$ then follows since (78) vanishes as $\beta \to 0$. This completes the proof of (44).

## V. Convergence of $Q(\kappa)$ as a Form

In this section we establish norm convergence of $Q(\kappa)$ regularized by the heat kernel. The following theorem yields Theorem III.4 of [2]. The complete definition of $Q(\kappa)$ is given in (II.11) of [2]. We let

$$\delta Q = \delta Q(\kappa, \kappa') = (Q(\kappa) - Q(\kappa'))^-, \tag{81}$$

where $^-$ denotes the operator closure.

**Theorem V.I.** (i) *Let $\beta>0$ and $\kappa, \kappa'<\infty$. Then*

$$\operatorname{Range}(\exp(-\beta H(\kappa))) \subset \operatorname{Domain}(\delta Q).$$

(ii) *As $\kappa, \kappa' \to \infty$ and as $|\kappa-\kappa'| \to 0$,*

$$\|\exp(-\beta H(\kappa'))\delta Q \exp(-\beta H(\kappa))\|_2^2 = o(1). \tag{82}$$

The remainder of this section is devoted to the proof of this theorem. We establish convergence in Theorem V.1 as $\kappa, \kappa' \to \infty$. The proof of continuity as $\kappa' \to \kappa$ is similar, and we do not present details.

Let us introduce

$$w(x) = -\frac{i}{2}(\partial P(\varphi_\kappa(x)) - \partial P(\varphi_{\kappa'}(x))), \tag{83}$$

where $x \in T^1$. Then $\delta Q$ has the representation on the domain $\mathscr{D}_0$ of vectors in Fock space with $C^\infty$ wave functions and a finite number of particles:

$$\delta Q = \int_{T^1} [\psi_1(x)w(x) + \psi_2(x)w(x)^* + \bar{\psi}_2(x)w(x)^* + \bar{\psi}_1(x)w(x)]dx, \tag{84}$$

*Proof of Theorem V.1.* (i) Since $\kappa, \kappa'<\infty$, the difference operator $\delta Q$ is a polynomial in a finite number of bosonic and fermionic degrees of freedom. These are the degrees of freedom of the Hilbert space $\mathscr{H}_{\leqq}$ discussed in [2, following (VI.8)]. The Hamiltonian $H(\kappa)$ has the representation $H(\kappa) = H^{\leqq} \otimes I + I \otimes H_0^{>}$, and it follows that $\delta Q$ acts on $\mathscr{H}_{\leqq}$, namely $\delta Q = \delta Q^{\leqq} \otimes I$. In [7] it is shown that $\operatorname{Range}(\exp(-\beta H^{\leqq}))$ is contained in the Schwartz subspace of $\mathscr{H}$. Hence these vectors lie in the domain of $\delta Q$.

(ii) By part (i), the operator $\delta Q \exp(-\beta H(\kappa))$ is defined. We estimate the Hilbert-Schmidt norm of $\exp(-\beta H(\kappa'))\delta Q \exp(-\beta H(\kappa))$ uniformly in $\kappa, \kappa'$. Replacing $\beta$ by $\beta/2$, we can write

$$\begin{aligned}
&\|\exp(-\beta H(\kappa')/2)\delta Q \exp(-\beta H(\kappa)/2)\|_2^2 \\
&\quad = \operatorname{Tr}(\exp(-\beta H(\kappa)/2)\delta Q \exp(-\beta H(\kappa'))\delta Q \exp(-\beta H(\kappa)/2)) \\
&\quad = \operatorname{Tr} \int \exp(-\beta H(\kappa))\psi_1(x)w(x) \exp(-\beta H(\kappa'))\bar{\psi}_2(y)w(y)^* dx dy \\
&\qquad + 7 \text{ similar terms} \equiv \sum_{j=1}^{8} I_j,
\end{aligned} \tag{85}$$

where we have used the fact that only expectations with an equal number of $\psi$ and $\bar{\psi}$ factors are non-zero. Thus only eight of the sixteen pairs of factors from expanding the product $\delta Q \exp(-\beta H(\kappa'))\delta Q$ according to (84) will contribute. We give the details of how to bound $I_1$ above, the corresponding estimates on the other seven similar terms being similar.

We use the functional integral representation for $I_1$ to obtain

$$\begin{aligned}
I_1 = \Xi_{l,2\beta} \int &((I - \tilde{K}_{l,2\beta}^{(\kappa',\kappa)}(\Phi))^{-1} \tilde{S}_{l,2\beta})_{12}(x,y) \det{}_3(I - \tilde{K}_{l,2\beta}^{(\kappa',\kappa)}(\Phi)) \\
&\times \exp(-\tilde{\mathscr{A}}_{l,2\beta}^{(\kappa',\kappa)}(\Phi)) W(x) W(y)^* d\mu_{C_{l,2\beta}}(\Phi) dx_1 dy_1,
\end{aligned} \tag{86}$$

where the notation follows Sect. III. For convenience, here we let $x = (\beta, x_1)$ and $y = (0, y_1)$. Also, $W(x) = -\frac{i}{2}(\partial P(\Phi_\kappa(x)) - \partial P(\Phi_{\kappa'}(x)))$ is the Euclidean counterpart to (83).

We now prove that $I_1$ converges to zero as $\kappa, \kappa' \to \infty$. We use the smoothing operator

$$\mu^{-\varepsilon} \equiv \mu_x^{-\varepsilon} = (-d^2/dx_1^2 + m^2)^{-\varepsilon/2}, \tag{87}$$

and set $C \equiv C_{l,2\beta}$. Let $\|\cdot\|_{L_p}$ denote the $L_p$ norm on path space with respect to $d\mu_C$.

**Lemma V.2.** *For any* $\varepsilon > 0$,

$$\sigma \equiv \|\mu^{-\varepsilon} W\|_{L_2}^2 = o(1), \tag{88}$$

*as* $\kappa, \kappa' \to \infty$.

*Proof.* Since the only singularities in $\|W\|_{L_2}$ are logarithmic, smoothing by $\mu^{-\varepsilon}$ removes them. The convergence of $\sigma$ to zero as $\kappa, \kappa' \to \infty$ then follows by standard constructive field theory estimates. Explicitly, let

$$H(x) = \int W(x)^* W(0) d\mu_C. \tag{89}$$

Then if $\mu^{-\varepsilon}$ has kernel $\mu^{-\varepsilon}(u, v)$, for $u, v \in T^1$,

$$f(u-v) = |\mu^{-\varepsilon}(u,v)| \leqq \text{const}(1 + |u-v|^{-1+\varepsilon}).$$

Thus for $0 < (p-1)$ sufficiently small, $\mu^{-\varepsilon}(u, 0) \in L_p(T^1)$. Since the integral is translation invariant,

$$\|\mu^{-\varepsilon} W\|_{L_2}^2 \leqq \int f(u) f(v) H(0, u-v) du dv.$$

Using Hölder's inequality, with $p^{-1} + q^{-1} = 1$, $p$ as above yields

$$\|\mu^{-\varepsilon} W\|_{L_2}^2 \leqq \|f\|_p^{2/p} (\int H(0,u)^q du)^{1/q} |T^1|^{1/q}, \tag{90}$$

where $|T^1|$ denotes the space volume. The function $H(0, u)$ is a sum of terms of the form

$$O(1) C^{(\kappa)}(u)^{k_1} C^{(\kappa\kappa')}(u)^{k_2}$$

with $k_1 + k_2 = n-1$, where $k_2 \geqq 1$, and where

$$C^{(\kappa\kappa')}(u) = (\chi_\kappa - \chi_{\kappa'}) * C * (\chi_\kappa - \chi_{\kappa'})(0, u).$$

Here $C^{(\kappa)} \in L_q$ for all $q < \infty$, with $\|C^{(\kappa)}\|_{L_q(T^1)} \leqq c(q)$, independent of $\kappa$. Also, given $q < \infty$, for $\kappa' \geqq \kappa$, there exists $\delta > 0$ such that

$$\|C^{(\kappa\kappa')}(\cdot)\|_{L_q(T^1)} \leqq O(\kappa^{-\delta}).$$

It follows that (88) can be bounded by $O(\kappa^{-\delta})$, and the proof of the lemma is complete.

We study explicitly the integrand $G(x, y)$ in the integral (86) over $x_1, y_1$, namely

$$G(x,y) = \Xi \int ((I-K)^{-1} S)_{12}(x,y) \det{}_3(I-K) e^{-\mathscr{A}} W(x) W(y)^* d\mu_C. \tag{91}$$

Here we suppress $l, \beta, \kappa, \kappa', \tilde{}$, etc. We isolate the singularity of $G(x, y)$ in the diagonal, and we show that $G(x, y)$ is integrable over $x_1, y_1$ as long as $\beta \neq 0$. In fact, we show that if $\beta = 0$, then $\int G(x, y) dy_1$ is singular.

The isolation of the singular part of $G(x, y)$ can be seen from perturbation theory. In other words, we expand

$$(I-K)^{-1} S = S + KS + K(I-K)^{-1} KS, \tag{92}$$

and insert this into (91). The first two terms in (92) yield contributions to $G(x, y)$ which are singular on the diagonal, but which are integrable in $x_1 - y_1$ as long as $\beta \neq 0$. The final term in (92) yields a contribution to $G(x, y)$ which exists on the diagonal, and hence which is integrable also for $\beta = 0$. Thus the first terms in (92) yield the precise singularity as $\beta \to 0$ of $\|\delta Q \exp(-\beta H(\kappa))\|_2$ for $\kappa, \kappa' \to \infty$, as opposed to $\|\exp(-\beta H(\kappa'))\delta Q \exp(-\beta H(\kappa))\|_2$, which is bounded as $\beta \to 0$, uniformly in $\kappa, \kappa'$.

Let $\mathscr{I}_k$ denote the contribution of the $k$th term in (92) to $I_1$, multiplied by $\Xi^{-1}$. Since $1 \leqq \Xi \leqq 1 + O(e^{-\beta})$, it is sufficient to bound $\mathscr{I}_k$. First, we study $\mathscr{I}_1$,

$$\mathscr{I}_1 = \int \left(\int \det{}_3(I-K) e^{-\mathscr{A}} (\mu^{-\varepsilon/2} W)(x) (\mu^{-\varepsilon/2} W)(y)^* d\mu_C\right) (\mu^\varepsilon S)(x-y) dx_1 dy_1 \,. \quad (93)$$

We bound $\mathscr{I}_1$ using Hölder's inequality,

$$|\mathscr{I}_1| \leqq |T^1| \, \|\mu^\varepsilon S(\beta, \cdot)\|_{L_1(T^1)} \|\mu^{\varepsilon/2} W\|_{L_4}^2 \|\det{}_3(I-K) e^{-\mathscr{A}}\|_{L_2} \,. \quad (94)$$

We bound this product as follows. The singularity of $(\mu^\varepsilon S)(x)$ on the diagonal is $|x|^{-(1+\varepsilon)}$, which is not $L_1$. However, for $\beta \neq 0$, $(\mu^\varepsilon S)(\beta, \cdot)$ is pointwise bounded and continuous, hence $L_1(T^1)$. Secondly, using hypercontractivity (5), and Lemma V.2,

$$\|\mu^{\varepsilon/2} W\|_{L_4} \leqq 3^{(n-1)/2} \|\mu^{\varepsilon/2} W\|_{L_2} = o(1) \,.$$

Finally, by Lemma III.1 (ii) the last factor in (94) is $O(1)$ as $\kappa, \kappa' \to \infty$. Combining these estimates we obtain

$$|\mathscr{I}_1| \leqq o(1) \,, \quad (95)$$

provided $\beta > 0$.

Next we study $\mathscr{I}_2$. The integral $\mathscr{I}_2$ is a sum of terms similar to (93) but with the factor $(\mu^\varepsilon S)(x-y)$ in the integrand replaced by a factor of the form

$$v_+(x, y) = \int (\mu^{\varepsilon/2} S)(x-z) \Lambda_+ \partial^2 P(\Phi(z)) (\mu^{\varepsilon/2} S)(z-y) dz$$

or $v_-$ with $\Lambda_- \partial^2 P(\Phi(z))^*$ replacing $\Lambda_+ \partial^2 P(\Phi(z))$. We suppress the regularization functions in $v_\pm$, as they complicate the notation and do not change the uniform character of our estimates. As in the bound (94) we then have by Hölder's inequality

$$|\mathscr{I}_2| \leqq |T^1| \, |||v||| \, \|\mu^{\varepsilon/2} W\|_{L_4}^2 \|\det{}_3(I-K) e^{-\mathscr{A}}\|_{L_4} \,,$$

where

$$|||v||| \equiv \int (\|v_+(x, y)\|_{L_4}) dx_1 dy_1 \,.$$

By hypercontractivity, we can bound the $L_4$ norms by a constant times the $L_2$ norm of $v_+$ and $v_-$. Thus

$$|||v||| \leqq \text{const} \left(1 + \int_{T^1 \times T^1} \left(\int_{T^2 \times T^2} |\log|z-z'||^{n-1} |z-x|^{-1-\varepsilon} |z-y|^{-1-\varepsilon} \right.\right.$$
$$\left.\left. \times |z'-x|^{-1-\varepsilon} |z'-y|^{-1-\varepsilon} d^2z d^2z' \right)^{1/2} dx_1 dy_1 \right). \quad (96)$$

This integral over $d^2z d^2z'$ is bounded if $x \neq y$ (i.e. for $\beta > 0$). Hence $|||v||| \leqq \text{const}$ and as in the bound of $\mathscr{I}_1$ we conclude from (96) that for $\beta > 0$,

$$|\mathscr{I}_2| \leqq o(1) \,, \quad \text{as} \quad \kappa, \kappa' \to \infty \,. \quad (97)$$

Finally, we bound $\mathscr{I}_3$. In this case the condition $\beta>0$ is unnecessary. We suppress convolutions with the regularization functions and write

$$\mathscr{I}_3=\int(\int\exp(-\mathscr{A})(\int g(u;x,\Phi)\det_3(I-K)((I-K)^{-1}S)(u,z)h(z;y,\Phi)dudz)\times(\mu^{-\varepsilon}W)(x)(\mu^{-\varepsilon}W)(y)^* d\mu_C(\Phi))dx_1dy_1+3\text{ similar terms}, \tag{98}$$

where in this term

$$g(u;x,\Phi)=(\mu^{\varepsilon}S)(x-u)\Lambda_+\partial^2P(\Phi(u)), \tag{99}$$

and

$$h(z;y,\Phi)=\Lambda_+\partial^2P(\Phi(z))(\mu^{\varepsilon}S)(z-y). \tag{100}$$

In the other terms $\Lambda_-\partial^2P\Phi(z)^*$ may replace $\Lambda_+\partial^2P$. Using the technique explained in Sect. IV to bound the Fredholm minor, we obtain the bound on part of the integrand in (98),

$$|\exp(-\mathscr{A})\int g(u;x,\Phi)\det_3(I-K)((I-K)^{-1}S)(u,z)h(z;y,\Phi)dudz| \leqq O(1)R(\Phi)\|g(\cdot;x,\Phi)\|_{\mathscr{K}_{-1/2}}\|h(\cdot;y,\Phi)\|_{\mathscr{K}_{-1/2}}. \tag{101}$$

We use (101) to bound $\mathscr{I}_3$. Using Hölder's inequality and the hypercontractivity estimate we obtain

$$|\mathscr{I}_3|\leqq O(1)\|\mu^{-\varepsilon}W\|_{L_2}^2\|R(\Phi)\|_{L_2}\{\int\|g(\cdot;0,\Phi)\|_{\mathscr{K}_{-1/2}}^2d\mu_C(\Phi)\}^{1/2}\times\{\int\|h(\cdot;0,\Phi)\|_{\mathscr{K}_{-1/2}}^2d\mu_C(\Phi)\}^{1/2}, \tag{102}$$

which, because of Lemma V.2, is $o(1)$ as $\kappa,\kappa'\to\infty$, provided that

$$\begin{aligned}&\int\|g(\cdot;0,\Phi)\|_{\mathscr{K}_{-1/2}}^2d\mu_C(\Phi)\leqq O(1),\\&\int\|h(\cdot;0,\Phi)\|_{\mathscr{K}_{-1/2}}^2d\mu_C(\Phi)\leqq O(1).\end{aligned} \tag{103}$$

Let us prove the first of the inequalities (103) (the proof of the second one is identical). The left-hand side of (103) can be bounded by

$$O(1)\int C^{1/4}(u-u')C^{1/4}(u-u'')|\Lambda_+T(-u')^*T(-u'')\Lambda_+|\mathscr{H}(u'-u'')dudu'du'', \tag{104}$$

where $T\equiv\mu^{\varepsilon}S$ has kernel $T(u-v)$ and

$$\mathscr{H}(u'-u'')=\int\partial^2P(\Phi(u'))^*\partial^2P(\Phi(u''))d\mu_C(\Phi).$$

The smoothing kernels $C^{1/4}$ arise from the $\mathscr{K}_{-1/2}$ norm. Apply Hölder's inequality to (104), and use the fact that $\mathscr{H}(u'-u'')$ has only logarithmic singularities on the diagonal; hence $\mathscr{H}\in L_p$ for all $p<\infty$. Thus (104) is bounded by

$$O(1)\{\int(C^{1/4}(u+u')C^{1/4}(u+u''))^q|T(u')^*T(u'')|^q\}^{1/q}dudu'du''\|\mathscr{H}\|_p \tag{105}$$

with $0<q-1$ chosen sufficiently small and $p^{-1}+q^{-1}=1$. Let $k(u)=|T(u)|^q$ and $f(u)=|C^{1/4}(u)|^q$. Then (105) is bounded by a constant times

$$\|f*k\|_2^{2/q}\leqq(\|f\|_\alpha\|k\|_{\alpha'})^{2/q},$$

where $\alpha^{-1}+\alpha'^{-1}=3/2$. Here we use Young's inequality. Note that the singularity of $f$ is bounded by

$$|f(u)|\leqq O\left(\frac{1}{|u|}\right)^{3q/2}\in L_\alpha,\quad \frac{1}{\alpha}>\frac{3q}{4},$$

and that of $k$ is bounded by

$$|k(u)| \leqq O\left(\frac{1}{|u|}\right)^{(1+\varepsilon)q} \in L_{\alpha'}, \qquad \frac{1}{\alpha'} > \frac{(1+\varepsilon)q}{2}.$$

Thus choosing $\varepsilon > 0$ small, $q$ close to 1, we take $\alpha, \alpha'$ such that $\alpha^{-1} + \alpha'^{-1} = 3/2$. This is possible as

$$\frac{3q}{4} + \frac{(1+\varepsilon)q}{2} = \frac{5}{4} + \frac{5(q-1)+2\varepsilon q}{4}$$

can be chosen close to 5/4 and hence less than 3/2.

It follows that $|\mathscr{I}_3| \leqq o(1)$, and this completes the bound on (91) and the proof of Theorem V.1.

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