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Kähler-Einstein Metrics on Complex Surfaces with $C_1 > 0$

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Dedicated to Walter Thirring on his 60[th] birthday

Abstract. Various estimates of the lower bound of the holomorphic invariant $\alpha(M)$, defined in [T], are given here by using branched coverings, potential estimates and Lelong numbers of positive, d-closed $(1,1)$ currents of certain type, etc. These estimates are then applied to produce Kähler-Einstein metrics on complex surfaces with $C_1 > 0$, in particular, we prove that there are Kähler-Einstein structures with $C_1 > 0$ on any manifold of differential type $CP^2 \# n\overline{CP^2}$ $(3 \leqq n \leqq 8)$.

The question of finding gravitational instantons has been important in mathematical physics. In this paper, we restrict ourselves to Kähler-Einstein metrics. In 1976, the second author solved Calabi's conjecture on the Kähler-Einstein metric. However, an important related question has not been solved yet. When a compact complex manifold has positive first Chern class, does it admit any Kähler-Einstein metric?

The theorem of Matsushima says that if such a metric exists, the automorphism group must be reductive. More recently, Futaki introduced more invariants related to the automorphism group and he demonstrated that these invariants are zero if the Kähler-Einstein metric exists. Some authors expressed the hope that if the automorphism group is discrete, then the Kähler-Einstein metric exists. However, there is another integrability condition, the tangent bundle of a Kähler-Einstein manifold has to be stable unless reducible. (The work of Bogomolov, Kobayashi, Lübke leads to such a conclusion.) Since the stability of the tangent bundle is more related to the linearized version of the equation, it is likely that a

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more nonlinear concept of stability will be involved. Inspired by work in the study of Yamabe's problem and harmonic mappings [Tr, SU, S], the first author introduces a holomorphic invariant $\alpha_G(M)$ on a compact Kähler manifold M with $C_1(M)>0$, where G is the maximal compact subgroup of the automorphism group $\mathrm{Aut}(M)$. Such a $\alpha_G(M)$ is an analogy of the best constant in the study of Yamabe's equation. In [T], it is proved that $\alpha_G(M)>\frac{m}{m+1}$ implies that M admits a Kähler-Einstein metric, where m is the dimension of M. In the case that M is a fermat hypersurface of degree $\geqq m$ in CP^{m+1}, $\alpha_G(M)$ is indeed greater than $\frac{m}{m+1}$ [T].

In the first part of this paper, we study the existence of the Kähler-Einstein metric on a complex surface with $C_1>0$. By the classification theory of surfaces [GH], if $C_1(M)>0$, the surface M must be of the following form, i.e. either $CP^1\times CP^1$, or $CP^2\#nCP^2$, the surface obtained by blowing up CP^2 at n generic points, where $0\leqq n\leqq 8$, and "generic" means that no three points are colinear, and no six points are in one quadratic curve in CP^2. As symmetric spaces, $CP^1\times CP^1$ and CP^2 have standard Kähler-Einstein metrics. For $n=1$ or 2, $CP^2\#nCP^2$ has no Kähler-Einstein metric, since its automorphism group is not reductive. For $n\geqq 3$, define $\mathfrak{M}_n=\{$all complex structures with $C_1>0$ on $CP^2\#nCP^2\}$, then it is known that $\mathfrak{M}_n$ is an analytic variety, and $\mathfrak{M}_3$, $\mathfrak{M}_4$ contain only one point, $\dim_{\mathbb{C}}\mathfrak{M}_n\geqq(n-4)$ for $n\geqq 5$. By exploiting various methods to estimate $\alpha_G(M)$ from below, we prove the following

Theorem. *For any $M\in U_n\subset\mathfrak{M}_n$ for $3\leqq n\leqq 8$, there is a Kähler-Einstein metric on M, where U_n are non-empty open subsets.*

In particular, the tangent bundle of these surfaces are stable. Note that in [Bu], Burns proves that any complex surface $CP^2\#\overline{nCP^2}$ has a stable tangent bundle for $2\leqq n\leqq 6$. Also note that we actually prove that any complex surface $M\cong CP^2\#8CP^2$ with $C_1(M)>0$ and nontrivial $\mathrm{Aut}(M)$ admits a Kähler-Einstein metric.

Another theorem in this paper is the following

Theorem. *When $C_1(M)>0$, M admits a Kähler-metric with its Ricci curvature representing $C_1(M)$ and bounded from below by a positive constant depending only on the dimension m and $C_1(M)^m$.*

In the course of proving the above theorems, we also show various estimates of $\alpha(M)$ in terms of different constructions of M. For example, if M is a branch cover of another manifold N, we can relate $\alpha(M)$ and $\alpha(N)$ under certain conditions.

Let us outline the contents of this paper. In Sect. 1, we first review Lelong numbers and positive, d-closed currents. For latter use, we will confine ourselves to positive currents of type $(1,1)$. Then we give a potential estimate for plurisubharmonic functions, whose proof is essentially due to Skoda [Sk]. We also recall the definition of the invariant $\alpha_G(M)$ defined in [T] and the relation with existence of Kähler-Einstein metrics (Theorem A). A lower bound of $\alpha_G(M)$ is given in terms of the upper bound of Lelong numbers of G-invariant, positive, d-closed $(1,1)$ currents, representing the cohomological class $C_1(M)$. As applications, we provide certain explicit lower bounds for $\alpha_G(M)$ in the case when $M = CP^m$, $G \subset \mathrm{Aut}(CP^m)$. In Sect. 2, we consider algebraic manifolds M that are branched covers of CP^m. Imposing certain symmetry conditions on M, we can give an estimate of $\alpha_G(M)$ in terms of an estimate of $\alpha_{G'}(CP^m)$, where G' is a group on CP^m induced from G by the projection. The latter quantity is computed when M is a certain complex surface with $C_1 > 0$ and diffeomorphic to $CP^2 \# n\overline{CP^2}$, $n = 5, 6, 7$. In particular, a Kähler-Einstein metric exists on such an M. In Sect. 3, by studying the automorphism group of M and the curves of a certain type with low degrees in M, we prove that $\alpha_G(M) \geqq 1$ if M is $CP^2 \# 3\overline{CP^2}$; $\alpha_G(M) \geqq \frac{3}{4}$ if M is $CP^2 \# 4\overline{CP^2}$. Hence, such M also admits a Kähler-Einstein metric. In Sect. 4, we study the existence of a Kähler-Einstein metric on complex surfaces with $C_1 > 0$ and differential type $CP^2 \# 8\overline{CP^2}$. We prove that any such surface M has $\alpha_G(M) \geqq 1$ and then admits a Kähler-Einstein metric if its automorphism group is nontrivial. We also produce a family of complex surfaces which fit our requirements. Such a family is parametrized by an open set in CP^2. Combining results proved in Sects. 2–4, the first main theorem is proved. In Sect. 5, we apply the generalized Jensen formula for plurisubharmonic functions, once used by Demailly [De], or [Sk 2], to our algebraic manifolds. We obtain an inequality between the Lelong number of a positive, d-closed current and its intersection number with hyperphase sections. We also explain briefly how this theorem relates to a conjecture in algebraic geometry to the problem of existence of certain Kähler metrics with Ricci curvatures bounded uniformly from below on Kähler manifolds with positive first Chern class. The conjecture is that there is a uniform bound for $(-K_M)^m$ for any algebraic manifold M with dimension m and ample anti-canonical line bundle.

In this paper, unless specified, M is always a compact Kähler manifold with positive first Chern class and g is the Kähler metric on M, locally, $g = g_{\alpha\bar\beta} dz^\alpha d\bar z^\beta$, $(g_{\alpha\bar\beta})$ is a positive hermitian matrix-valued function. Define the Kähler class $\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta} g_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta$ in local coordinates. It is globally defined. We also suppose that ω_g be in the class $C_1(M) \in H^2(M, R)$, usually, we use the symbol $g \sim C_1(M)$ to mean this.

There are two hopeful ways to improve our theorem. The first is to sharpen the lower bound of the holomorphic invariant $\alpha_G(M)$. In Sect. 1, by using a potential estimate for plurisubharmonic functions, we give a lower bound of $\alpha_G(M)$ in terms of the upper bound of Lelong numbers of G-invariant, positive, d-closed $(1,1)$ currents representing $C_1(M)$ (Theorem 1.5). This bound is not optimal. The best one should be $\alpha_G(M) = \frac{1}{L'_G(M)}$, where $L'_G(M) = \sup\left\{\frac{1}{m-p} L_g(u,z) \mid z \in M,\ u\right.$ is a positive, d-closed $(1,1)$-current, G-invariant, coholomogical to $C_1(M)$ and the set $(z \in M \mid L_g(u,z') \geqq L_g(u,z))$ has complex dimension $\left.p\right\}$, g is a Kähler metric on M, $m = \dim_{\mathbb{C}} M$. The reason for our belief in such an equality is a prior estimate of almost plurisubharmonic functions on a Kähler manifold (M, g), namely, for $\beta < 1$ there exists constants C, r, depending only on (M, g) and β, such that for any C^2 function φ on M, satisfying $\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar\partial\varphi \geqq 0$, $\sup_M \varphi = 0$ and

$$\frac{1}{r^{2m-2}}\int_{B_r(x)}\left(\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)\wedge\omega_g^{m-1}\leqq m\beta,$$

where $x\in M$. Then

$$\int_{B_{r/4}(x)} e^{-\varphi}dV_g\leqq C\int_{B_r(x)\backslash B_{r/2}(x)} e^{-\varphi}dV_g.$$

We are not far away from verifying this estimate. Note that the condition

$$\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\geqq 0$$

on M is crucial, since a local version of such an estimate is never true. On a Kähler surface (M, g), diffeomorphic to one of $CP^2\#n\overline{CP^2}$, where $n=5$, 6, 7, 8, for any positive, d-closed, G-invariant $(1,1)$ current u representing $C_1(M)$, define $E_\lambda=\{x\in M|L_g(u,x)\geqq\lambda\}$, then E_λ is analytic [Si 1]. It can be shown that $E_{1+\varepsilon}$ is a 0-dimensional variety and $E_{3-\varepsilon}$ is empty for ε small. Hence, in this case, $L_g'(M)\leqq 1$. Another improvement will follow from an interior C^0-estimate of certain complex Monge-Ampère equations. Namely, consider the equation $\det(u_{i\bar{j}})=F$ in the unit ball $B_1(0)$ of C^m, with $F>0$ bounded from above and u plurisubharmonic. Is there a positive number $p>0$ such that for any solution u of the above equation,

$$|u(0)|\leqq C\left(\left|\sup_{B_1(0)}u\right|+\int_{B_1(0)} e^{-pu}dV\right),$$

where C depends only on F, p? Note that in [T], by a standard L^2-estimate, it is proved that for any sequence of C^2-functions $\{u_i\}$ on (M, g) satisfying that $\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_i\geqq 0$ and $\sup_M u_i=0$, there are a subsequence $\{u_{i_k}\}$ and a subvariety $S\subset M$ such that in any compact set K of $M\backslash S$, the integrals $\int_K e^{-pu_{i_k}}dV_g$ are uniformly bounded. Combining this with the proposed interior C^0-estimate, one should be able to prove that either there is a Kähler-Einstein metric on (M, g), or the degenerate Monge-Ampére equation $\det(g_{i\bar{j}}+\varphi_{i\bar{j}})=0$ has a solution u. The solution u will be smooth outside an algebraic subvariety S of M and has logarithmic growth near S. Such a u certainly imposes some constraints on the manifold M. We expect that the understanding of those constraints will result in the solution for the problem of existence of Kähler-Einstein metrics on Kähler manifolds with $C_1>0$. Also, the role of u here should resemble that of the Green function of the conformal operator in the study of Yamabi's equation.

Finally, we would like to mention that Calabi claimed that he could show the existence of a Kähler-Einstein metric over $CP^2\#3\overline{CP^2}$ by an almost explicit construction. Siu [Si 2] also proves that $CP^2\#3\overline{CP^2}$ and the Fermat surface admit Kähler-Einstein metrics by estimating the lower bound of bisectional curvatures of some Kähler metrics constructed on these surfaces and studying the Green's functions of holomorphic curves. A more effective estimate enables us to prove that $\alpha(M)\geqq\frac{2}{3}$ for any Kähler surface $M\cong CP^2\#8\overline{CP^2}$ with $C_1(M)>0$. Since the estimate is being applied to the general case and being studied further, we would like to present it elsewhere.

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1. Positive Currents and Related Estimates

In this section, we study d-closed, positive currents. We shall give an estimate for plurisubharmonic functions. The basic idea is due to Skoda [Sk 1]. While most of the arguments work for arbitrary (p,p)-currents, we will confine ourselves to $(1,1)$-current. At the end of this section, we will recall the definition of $\alpha_G(M)$ in [T] and the theorem on the existence of Kähler-Einstein metrics proved in [T]. The results on $(1,1)$-current will then be used to estimate $\alpha_G(CP^m)$ for $G \subset U(m+1)$.

Let M be a Kähler manifold with Kähler metric g. In local coordinates $(z_1, \ldots, z_m)$, g is represented by a positive hermitian matrix $(g_{i\bar{j}})$. The corresponding Kahler form is given by $\omega_g = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

For the definition of positive current, we refer the readers to Siu [Si 1] or Griffiths and Harris [GH]. Now we recall the definition of the Lelong number of a positive, d-closed, $(1,1)$-current u in an open set $\Omega \subset M$. We define the total variation $\|u\|$ of u to be the positive measure $u \wedge \omega_g^{m-1}$, where $m = \dim_{\mathbb{C}} M$. For all $a \in \Omega$, we define the Lelong number $L_g(u,a)$ of u at a with respect to the metric g to be the limit

$$\lim_{r \to 0} \frac{1}{r^{2n-2}} \int_{B_r(a)} u \wedge \omega_g^{m-1} = \lim_{r \to 0} \frac{1}{r^{2n-2}} \|u\| (B_r(a)), \tag{1.1}$$

where $B_r(a)$ is the geodesic ball in M with radius r and center a. Such a limit exists. Lelong [Le] shows this for $\Omega \subset C^n$ and g to be the standard metric. The general case follows from this special case, since a Kähler metric can be approximated by the standard Euclidean metric at one point up to second order.

Lemma 1.1. *Suppose that we have a sequence of closed, positive* $(1,1)$*-currents* $\{u_i\}$, *weakly converging to a d-closed, positive* $(1,1)$*-current* u *in* Ω *in the sense of convergence of corresponding total measures. Suppose also that for all* $a \in \Omega$, $L_g(u,a) < +\infty$. *Then for all* $\varepsilon > 0$, *and compact subset* $K \subset \Omega$, *there exist* $r = r(\varepsilon, K)$, $N = N(\varepsilon, K)$, *such that for* $i > N$, $a \in K$,

$$\frac{1}{r^{2n-2}} \int_{B_r(a)} u_i \wedge \omega_g^{m-1} \leqq L_g(u,a) + \varepsilon. \tag{1.2}$$

Proof. Choose $r_1 < \operatorname{dist}(K, \partial\Omega)$, such that for $a \in K$,

$$\frac{1}{r_1^{2n-2}} \int_{B_{r_1}(a)} u \wedge \omega_g^{m-1} \leqq L_g(u,a) + \frac{\varepsilon}{3}. \tag{1.3}$$

For each $a \in K$, let $\varrho_a(t)$ be a cut-off function satisfying:

$$\varrho_a(\mathrm{t}) = \begin{cases} 1 & 0 \leqq t \leqq r_1 - \delta, \\ 0 & t \geqq r_1. \end{cases} \tag{1.4}$$

Let $r_a(x)$ be the geodesic distance function on M from a, then near a, $r_a(x)$ is smooth. Thus

$$\lim_{i\to\infty} \int_{B_{r_1}(a)} \varrho_a(r_a(x))u_i \wedge \omega_g^{m-1} = \int_{B_{r_1}(a)} \varrho_a(r_a(x))u \wedge \omega_g^{m-1} \leqq \int_{B_{r_1}(a)} u \wedge \omega_g^{m-1}. \tag{1.5}$$

By the compactness of K, it is easy to see that there exists $N(\varepsilon, K)$ such that for $i>N$,

$$\int_{B_{r_1}(a)} \varrho_a(r_a(x))u_i \wedge \omega_g^{m-1} \leqq \int_{B_{r_1}(a)} u \wedge \omega_g^{m-1} + r_1^{2n-2}\frac{\varepsilon}{3}, \qquad \forall a \in K, \tag{1.6}$$

Hence

$$\frac{1}{r_1^{2n-2}} \int_{B_{r_1-\delta}(a)} u_i \wedge \omega_g^{m-1} \leqq L_g(u,a) + \frac{2\varepsilon}{3}. \tag{1.7}$$

Let $r = r(\varepsilon, K) = r_1 - \delta$ with δ small enough. Then we have

$$\frac{1}{r^{2n-2}} \int_{B_r(a)} u_i \wedge \omega_g^{m-1} \leqq L_g(u,a) + \varepsilon \quad \text{for} \quad i>N,\ a\in K. \tag{1.8}$$

This finishes the proof of Lemma 1.1.

Before we state the next lemma, we need the following definition of slicing of a d-closed positive $(1,1)$-current u. Suppose $\Omega = B$ is an open small ball in M, then by the closedness of u, there exists a plurisubharmonic function φ on B such that $u = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$. For an analytic curve L in M, $L \cap B \neq \emptyset$, we define the slice $u|L$ of u by L as

$$u|L = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\varphi|L) \tag{1.8}$$

when it is meaningful. It is easy to see that the definition is independent of the particular choice of φ and $\partial\bar{\partial}(\varphi|L)$ is meaningful whenever $\varphi|L$ is not identically equal to $-\infty$ and L^1-integrable. An important case is given by $u = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ on M. When u is defined on M, and $\varphi \leqq 0$, $u|L$ is well-defined iff $\varphi|L$ is not identically equal to $-\infty$. In this case, $e^{-\varepsilon\varphi}$ is automatically integrable for ε sufficiently small which follows from Proposition 2.1 in [T].

Lemma 1.2 (Siu [Si 1, Lemma 7.5]). *Suppose u is a positive, d-closed $(1,1)$-current defined on Ω. Let $a \in \Omega$ and L be a smooth curve segment in Ω passing through a such that $u|L$ is well-defined, then*

$$L_{g|L}(u|L, a) \geqq L_g(u,a). \tag{1.9}$$

Remark. The statement here is slightly different from that in Lemma 7.5 of Siu [Si 1], but his proof still works.

The following proposition is essentially due to Skoda [Sk].

Proposition 1.3. *Given (M,g), there exists a positive number R with the following property. For any $\beta, \gamma > 0$, and $\beta < 1$, there exist $r = r(\beta,\gamma,M)$, $C = C(\beta,\gamma,M)$ such that for any plurisubharmonic function $\varphi \in C^2(B_R(a))$, $a \in M$, satisfying:*

$$\left\|\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right\| \leqq \gamma, \qquad \int_{\partial B_R(a)} |\varphi| dV_g \leqq \gamma, \tag{1.10}$$

and

$$R^{-2m+2}\int_{B_R(a)}\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)\wedge\omega_g^{m-1}\leqq\beta\,, \tag{1.11}$$

then

$$\int_{B_r(a)} e^{-\varphi(z)}dV_g(z)\leqq C\,. \tag{1.12}$$

Proof. The constant R will be chosen so that within the given complex coordinate charts, the ratio of the geodesic distance to the Euclidean distance is close to one and ω_g is close to the Euclidean metric $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}|Z|^2$. In this case, (1.11) is valid with g replaced by the Euclidean metric with β replaced by a slightly larger constant. Hence we shall assume the metric is Euclidean. For convenience, we also assume the ball to have radius one.

By Green's formula (see Gilbarg and Trudinger [GT]), note that we write the formula in complex coordinates,

$$\begin{aligned}(m-1)\varphi(Z)=&-\int_{B_1(0)}\left(\frac{1}{|Z-\zeta|^{2m-2}}-\frac{1}{|1-Z\bar{\zeta}|^{2m-2}}\right)\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)\\&\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}|\zeta|^2\right)^{m-1}\\&+\int_{\partial B_1(0)}\varphi(\zeta)\frac{\sqrt{-1}}{2\pi}\partial\left(\frac{1}{|Z-\zeta|^{2m-2}}-\frac{1}{|1-Z\bar{\zeta}|^{2m-2}}\right)\\&\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}|\zeta|^2\right)^{m-1}.\end{aligned} \tag{1.13}$$

In this proof, C will always denote constants depending only on β, γ, M,

$$\begin{aligned}|Z-\zeta|^{2m-2}(\partial\bar{\partial}|\zeta|^2)^{m-1}=&(\partial\bar{\partial}\log|Z-\zeta|^2)^{m-1}\\&+(m-1)\partial\log|Z-\zeta|^2\wedge\bar{\partial}\log|Z-\zeta|^2\wedge(\partial\bar{\partial}\log|Z-\zeta|^2)^{m-2}.\end{aligned} \tag{1.14}$$

Plugging this in the above Green formula, we have for $|Z|\leqq\frac{1}{4}$,

$$\begin{aligned}-\varphi(Z)\leqq&\frac{1}{m-1}\int_{B_1(0)}\frac{1}{|Z-\zeta|^{2m-2}}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi(\zeta)\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}|\zeta|^2\right)^{m-1}+C\\=&\frac{\sqrt{-1}}{2\pi}\int_{B_1(0)}\left(\frac{1}{m-1}\partial\bar{\partial}\log|Z-\zeta|^2+\partial\log|Z-\zeta|^2\wedge\bar{\partial}\log|Z-\zeta|^2\right)\\&\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|Z-\zeta|^2\right)^{m-2}+C\\\leqq&\frac{\sqrt{-1}}{2\pi}\int_{B_1(0)}p(|\zeta|)\left(\frac{1}{m-1}\partial\bar{\partial}\log|Z-\zeta|^2+\partial\log|Z-\zeta|^2\wedge\bar{\partial}\log|Z-\zeta|^2\right)\\&\wedge\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|Z-\zeta|^2\right)^{m-2}+C,\end{aligned} \tag{1.15}$$

where p is a cut-off function satisfying: $p(t)=1$ for $t\leqq\frac{1}{2}$, $p(t)=0$ for $t\geqq\frac{3}{4}$, and $|p'(t)|\leqq 4$. Note also that in the last inequality, C has absorbed the integral cut by $1-p(|\zeta|)$.

Integrating the second term on the right by parts in the above inequality,

$$\begin{aligned}
-\varphi(Z)\leqq &\int_{B_1(0)} p(|\zeta|)\left(\frac{1}{m-1}-\log|Z-\zeta|^2\right)\wedge\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi \\
&\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|Z-\zeta|^2\right)^{m-1} \\
&-\int_{B_1(0)\backslash B_{1/2}(0)} \log|Z-\zeta|^2\partial p(|\zeta|)\wedge\frac{\sqrt{-1}}{2\pi}\bar\partial\log|Z-\zeta|^2\wedge\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi \\
&\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|Z-\zeta|^2\right)^{m-2}+C \\
\leqq &-\int_{B_{1/2}(0)}\left(\log e^{-\frac{1}{m-1}}|Z-\zeta|^2\right)\wedge\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi \\
&\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|Z-\zeta|^2\right)^{m-1}+C.
\end{aligned}\tag{1.16}$$

Put

$$\mu(Z)=\int_{B_{1/2}(0)}\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|Z-\zeta|^2\right)^{m-1}\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi\right).\tag{1.17}$$

We have the following monotonicity formula (see Lelong [Le]), for $0\leqq r<R<1$,

$$\begin{aligned}
&\int_{B_R(0)\backslash B_r(0)}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi(Z+\zeta)\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|\zeta|^2\right)^{m-1} \\
&=\frac{1}{R^{2m-2}}\int_{B_R(0)}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi(Z+\zeta)\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial|\zeta|^2\right)^{m-1} \\
&\quad-\frac{1}{r^{2m-2}}\int_{B_r(0)}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi(Z+\zeta)\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial|\zeta|^2\right)^{m-1}.
\end{aligned}\tag{1.18}$$

Then,

$$\begin{aligned}
\mu(Z)&\leqq\int_{|Z-\zeta|\leqq\frac{1}{2}+|Z|}\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|Z-\zeta|^2\right)^{m-1}\wedge\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi \\
&=\int_{|\zeta|\leqq\frac{1}{2}+|Z|}\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|\zeta|^2\right)^{m-1}\wedge\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi(Z+\zeta) \\
&\leqq\left(\tfrac{1}{2}+|Z|\right)^{-2m+2}\int_{|\zeta|\leqq\frac{1}{2}+|Z|}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi(Z+\zeta)\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial|Z+\zeta|^2\right)^{m-1} \\
&\leqq\left(\frac{\frac{1}{2}+2|Z|}{\frac{1}{2}+|Z|}\right)^{2m-2}\frac{1}{\left(\frac{1}{2}+2|Z|\right)^{2m-2}}\int_{|\zeta|\leqq\frac{1}{2}+2|Z|}\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi(\zeta) \\
&\quad\wedge\left(\frac{\sqrt{-1}}{2\pi}\partial\bar\partial|\zeta|^2\right)^{m-1}.
\end{aligned}\tag{1.19}$$

Since φ is plurisubharmonic, (1.18) implies

$$\mu(Z) \leqq \left(\frac{1+4|Z|}{1+2|Z|}\right)^{2m-2} \int_{|\zeta| \leqq 1} \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi(\zeta) \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}|\zeta|^2\right)^{m-1} \leqq \left(\frac{1+4|Z|}{1+2|Z|}\right)^{2m-2} \beta. \tag{1.20}$$

Let $\beta_1 = \beta + 1/2$, then $\beta < \beta_1 < 1$. Take r small enough, s.t.

$$\left(\frac{1+4r}{1+2r}\right)^{2m-2} \beta \leqq \beta_1, \tag{1.21}$$

then $\mu(Z) \leqq \beta_1$ for $|Z| \leqq r$, by the concavity of log,

$$\begin{aligned} -\varphi(Z) \leqq & \frac{\mu(Z)}{\beta_1} \log \left(\int_{B_{1/2}(0)} \left(e^{-\frac{1}{m-1}} |Z-\zeta|^2\right)^{-\beta_1} \right. \\ & \left. \times \frac{\left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|Z-\zeta|^2\right)^{m-1} \wedge \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi}{\mu(Z)} \right) + C \\ \leqq & \log \left(\int_{B_{1/2}(0)} \left(e^{-\frac{1}{m-1}} |Z-\zeta|^2\right)^{-\beta_1} \right. \\ & \left. \times \frac{\left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|Z-\zeta|^2\right)^{m-1} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi\right)}{\beta_1} \right) + \frac{\mu(Z)}{\beta_1} \log \frac{\beta_1}{\mu(Z)} + C. \end{aligned} \tag{1.22}$$

As $\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|Z-\zeta|^2$ is dominated by $C \frac{1}{|Z-\zeta|^2} \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}|\zeta|^2\right)$,

$$\begin{aligned} \int_{|Z| \leqq r} e^{-\varphi(Z)} dZ \leqq C & \int_{|Z| \leqq r} dZ \\ & \times \int_{|\zeta| \leqq \frac{1}{2}} |Z-\zeta|^{-2\beta_1 - 2m + 2} \frac{\left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}|\zeta|^2\right)^{m-1} \wedge \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi}{\beta_1} \leqq C. \end{aligned}$$

This completes the proof of Proposition 1.3.

Consider a Kähler manifold M with $C_1(M) > 0$, let us recall the definition of $\alpha_G(M)$ introduced in [T], where G is a compact subgroup in $\mathrm{Aut}(M)$. To define $\alpha_G(M)$, we first pick up a G-invariant metric $g \sim C_1(M)$.

Put $P_G(M, g) = \left\{ \varphi \in C^2(M, R) \mid \varphi \text{ is } G\text{-invariant and } \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi \geqq 0 \right\}$,

$$\alpha_G(M) := \sup \left\{ \alpha \mid \text{there exists } C \text{ such that } \int_M e^{-\alpha\left(\varphi(Z) - \sup_M \varphi\right)} dV_g \leqq C,\ \forall \varphi \in P_G(M, g) \right\}.$$

It is easy to show that $\alpha_G(M)$ is independent of the particular choice of g. $\alpha(M) = \alpha_G(M)$ in case that G is trivial. In [T], the following theorem is proved. We will use this theorem often.

Theorem 1.4. *For a compact Kähler manifold M with $C_1(M)>0$, if $\alpha_G(M)>\frac{m}{m+1}$, where $m=\dim_{\mathbb{C}} M$, G is the maximal compact subgroup of* $\mathrm{Aut}(M)$, *then M admits a Kähler-Einstein metric.*

On the given manifold M, we also have the following natural invariant:

$$L_G(M)=\sup\{L_g(u,z)|z\in M, u \text{ is a } d\text{-closed, positive } (1,1)\text{-current, cohomological to } C_1(M), \text{ and } u \text{ is } G\text{-invariant}\}, \tag{1.24}$$

where g is a Kähler metric on M, $g\sim C_1(M)$. As above, $L_G(M)$ is independent of the choice of g. By Lemma 1.2, and the fact that M is actually algebraic, it can be proven that $L_G(M)$ is bounded from above.

Theorem 1.5. $\alpha_G(M)\geqq\frac{1}{L_G(M)}$, *whenever M is Kähler and $C_1(M)>0$.*

Proof. Fix an arbitrary $\lambda<\frac{1}{L_G(M)}$, we have to prove that $\alpha_G(M)\geqq\lambda$. For that, it suffices to show that for any sequence $\{\varphi_i\}\in P_G(M,g)$, we can find a subsequence $\{\varphi_{i_k}\}$ and a constant C, such that

$$\int_M e^{-\lambda(\varphi_{i_k}(z)-\sup_M \varphi_{i_k})}dV_g\leqq C. \tag{$**$}$$

By taking the subsequence, one may assume that, as measures, $u_i=\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi_i$ converges weakly to a d-closed, positive $(1,1)$-current u, u is cohomological to $C_1(M)$. By Lemma 1.1, there exists $N>0$, $r>0$ such that for $i>N$, and $z\in M$,

$$\frac{1}{r^{2m-2}}\int_{B_r(z)} u_i\wedge\omega_g^{m-1}\leqq\frac{1}{\lambda}-\varepsilon,\qquad \varepsilon \text{ small}.$$

Note that $\frac{1}{\lambda}>L_g(u,z)$ for any $z\in M$. Locally, $\omega_g=\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\psi$ for a certain plurisubharmonic function, then $\psi+\varphi_i$ is plurisubharmonic for each i. Then Proposition 1.3 implies $(**)$ for $i>N$, $i=i_k$. Hence, the theorem is proved.

Remark. As we mentioned in the introduction, $\alpha_G(M)\cdot L'_G(M)=1$, where $L'_G(M)=\sup\left\{\frac{1}{m-p}L_g(u,z)|z\in M,\ u \text{ is a positive, } d\text{-closed } (1,1)\text{-current, } G\text{-invariant, cohomological to } C_1(M) \text{ and the set } (z'\in M|L_g(u,z')\geqq L_g(u,z)) \text{ has complex dimension } p\right\}$. Note the set $(z'\in M|L_g(u,z')\geqq L_g(u,z))$ is analytic, see Siu [Su 1].

Corollary 1.6. $\alpha(CP^m)\geqq\frac{1}{m+1}$.

Proof. Just take g as the $(m+1)$-multiple of Fubini-Study metric, i.e.

$$\omega_g=(m+1)\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|Z|^2, \text{ where } |Z|^2=|Z_0|^2+\ldots+|Z_m|^2,$$

$[Z_0,\ldots,Z_m]$ is the homogeneous coordinate of CP^m.

For any positive, d-closed $(1,1)$-current u, cohomological to $G(M)$, it can be easily proved that the slice of u by l_z is well-defined for each $z\in CP^m$ and generic line l_z through z in CP^m, so $L_g(u,z)$

$$L_g(u,z)\leqq L_{g|_{l_z}}(u_{|l_z},z)\leqq \int_{l_z} u=\int_{l_z} C_1(M)=m+1,$$

since u is cohomological to $C_1(M)$. Theorem 1.5 says that $\alpha(CP^m)\geqq\frac{1}{m+1}$. □

Let $G(p)\subset U(m+1)$ be the finite group generated by σ_i and permutations τ_{ij} $(0\leqq i\leqq m,\ 0\leqq i<j\leqq m)$, where, in the homogeneous coordinates $[Z_0,\ldots,Z_m]$ of CP^m.

$$\sigma_i:[Z_0,Z_1,\ldots,Z_i,\ldots,Z_m]\to[Z_0,\ldots,e_pZ_i\ldots Z_m],$$

$$e_p=\exp\left(\frac{2\pi\sqrt{-1}}{p}\right),\qquad \tau_{ij}:[Z_0,\ldots,Z_i,\ldots,Z_j,\ldots,Z_m]\to[Z_0,\ldots,Z_j,\ldots,Z_i,\ldots,Z_m].$$

By a sophisticated argument, one can prove that $\alpha_{G(p)}(CP^m)\geqq\frac{2}{m+1}$ for $p\geqq 2$. Actually, one should be able to demonstrate the sharp estimate $\alpha_{G(p)}(CP^m)\geqq\frac{p}{m+1}$. However, for simplicity, we only prove the following special estimate, which is sufficient for this paper.

Corollary 1.7. *$\alpha_{G(p)}(CP^2)\geqq\frac{1}{2}$ for $p\geqq 2$.*

Proof. It suffices to show that for any positive, d-closed, $G(p)$-invariant, $(1,1)$-current $u\sim C_1(M)$, $L_g(u,z)\leqq 2$, $z\in M$, where $g\sim C_1(M)$, and

$$\omega_g=3\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|Z_0|^2+|Z_1|^2+|Z_2|^2),$$

$[Z_0,Z_1,Z_2]=Z$ are the homogeneous coordinates of CP^2.

For each $z\in M$, the orbit $G(p)\cdot z$ has at least three distinct points. It is easy to check that three of $G(p)\cdot z$ are not colinear, suppose they are $z^1=z$, z^2, $z^3\in CP^2$. There is a family

$$\{C_{[\alpha,\beta,r]}\}_{[\alpha,\beta,r]\in CP^2}$$

of quadrics in CP^2 passing through z^1,z^2,z^3, and generic quadric is smooth. So one can find a smooth quadric C on which the slice of u is well-defined, thus by

Lemma 1.1 and G-invariance of u,

$$\begin{aligned}3L_g(u,z)&=L_g(u,z^1)+L_g(u,z^2)+L_g(u,z^3)\\&\leqq L_{g|_C}(u|_C,z^1)+L_{g|_C}(u|_C,z^2)+L_{g|_C}(u|_C,z^3)\\&\leqq \int_C (u|_C)=\int_C C_1(M)=6,\end{aligned}$$

i.e. $L_g(u,z)\leqq 2$. $\square$

2. Kähler-Einstein Metrics on $CP^2\#n\overline{CP^2}$ for $5\leqq n\leqq 7$

In this section, we assume that the manifold M has been embedded into CP^N, $m=\dim_{\mathbb{C}} M$, $N>m$, and the metric g is a multiple of the restriction of the Fubini-Study metric of CP^N to M. We also assume that the maximal compact subgroup G of $\mathrm{Aut}(M)$ is a subgroup in $U(N+1)$, the maximal compact subgroup of $\mathrm{Aut}(CP^N)$ $=PSL(N+1)$. In the homogeneous coordinates $[Z_0,\dots,Z_N]=Z$ of CP^N,

$$\omega_g=\frac{\mu\sqrt{-1}}{2\pi}\partial\bar\partial\log(|Z_0|^2+\dots+|Z_N|^2)|_M,$$

μ is a positive integer. Obviously, g is G-invariant.

Choose a $(N-m-1)$-dimensional projective subspace $F\cong CP^{N-m-1}$ in CP^N, such that $F\cap M=\emptyset$. We project M onto a m-dimensional subspace $F^\perp\cong CP^m$ $\subset CP^N$ from F. Denote the projection by π_F, then $\pi_F: M\to F^\perp$ is a branched covering, its covering degree is the same as the degree of M in CP^N. Take a function $\varphi\in C^2(M,R)$, define φ_F on $F^\perp$ as follows: $\forall x\in F^\perp$,

$$\varphi_F(x)=\frac{1}{d}\sum_{g\in\pi_F^{-1}(x)}\varphi(y),\tag{2.1}$$

where $d=\deg(\pi_F)$.

Lemma 2.1. *$F, F^\perp, \pi_F$ as above, then for each $\varphi\in C^2(M,R)$, and an open set $U\subset F^\perp$,*

$$\int_{\pi^{-1}(U)} e^{-\pi_F^*\varphi_F(y)}\left(\mathrm{Jac}(\pi_F(y))\cdot\frac{dV_{g_F}(\pi(y))}{dV_g(y)}\right)dV_g(y)=d\int_U e^{-\varphi_F(x)}dV_{g_F}(x),\tag{2.2}$$

where $g_F=$ the Fubini-Study metric of $F^\perp\cong CP^m$, i.e. the corresponding Kähler form is

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log(|Z_0|^2+\dots+|Z_N|^2)|_{F^\perp}.$$

$\mathrm{Jac}(\pi_F)$ is the Jacobian of π_F.

Proof. It follows from the transformation formula for integrals and the fact that π_F is a covering mapping outside the branched locus of complex codimension one.

Proposition 2.2. *M, g, G as above, suppose that $F_1,\dots,F_l$ be $(N-m-1)$-subspaces in CP^N, such that $F_1^\perp\cap\dots\cap F_l^\perp=\emptyset$, $F_i\cap M=\emptyset$ for $i=1,2,\dots,l$, then we have projections*

$\pi_i=\pi_{F_i}: M \to F_i^{\perp}$, $i=1,2,\ldots,l$. Furthermore, suppose that the group G contains all deck transformations of the projections π_i ($i=1,2,\ldots,l$), then

$$\alpha_G(M) \geqq \frac{m+1}{\mu} \min_{1 \leqq i \leqq 1} \alpha_{G_i}(F_i^{\perp}), \tag{2.3}$$

where G_i is the compact subgroup of $\mathrm{Aut}(F_i^{\perp})$ *induced by π_i from G, i.e. the group generated by elements of G preserving the fibres of π_i.*

Proof. First, we prove that for each i, there exists a constant C_i, depending only in i, λ, such that

$$\int_M e^{-\lambda(\varphi(x)-\sup_M \varphi)} \cdot \left(\mathrm{Jac}(\pi_i)\frac{dV_{g_{F_i}}(\pi_i(x))}{dV_g(x)}\right) dV_g(x) \leqq C_i \tag{2.4}$$

for each $\varphi \in P_G(M,g)$, where $\lambda < \frac{m+1}{\mu} \min_{1 \leqq i \leqq l} \alpha_{G_i}(F_i^{\perp})$ and g_{F_i} is the Fubini-Study metric on $F_i^{\perp}$. Clearly we can assume $\sup_M \varphi = 0$.

For simplicity, we assume that $i=1$,

$$F_1^{\perp}=\{[Z_0,\ldots,Z_m,0,\ldots,0] \in CP^N\}, \qquad F=\{[0,\ldots,0,Z_{m+1},\ldots,Z_N] \in CP^N\}$$

and $\pi_i: [Z_0,\ldots,Z_N] \to [Z_0,\ldots,Z_m,0,\ldots,0]$. Then

$$\omega_{g_{F_1}} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|Z_0|^2+\ldots+|Z_m|^2). \tag{2.5}$$

Put

$$\psi = \mu \log\left(\frac{|Z_0|^2+\ldots+|Z_N|^2}{|Z_0|^2+\ldots+|Z_m|^2}\right)\Bigg|_M, \tag{2.6}$$

since $M \cap F_1 = \emptyset$, such ψ is a smooth function on M.

Now for $\varphi \in P_G(M,g)$,

$$\mu\partial\bar{\partial}\log(|Z_0|^2+\ldots+|Z_m|^2)+\partial\bar{\partial}(\psi+\varphi)=\mu\partial\bar{\partial}\log(|Z_0|^2+\ldots+|Z_N|^2)+\partial\bar{\partial}\varphi \geqq 0, \tag{2.7}$$

it follows that

$$\mu\partial\bar{\partial}\log(|Z_0|^2+\ldots+|Z_m|^2)+\partial\bar{\partial}(\psi_{F_1}+\varphi_{F_1}) \geqq 0. \tag{2.8}$$

Obviously, $\psi_{F_1}+\varphi_{F_1}$ is G_1-invariant, moreover, since G contains all deck transformations of π_i and φ is G-invariant,

$$\pi_1^*(\psi_{F_1}+\varphi_{F_1}) = \pi_1^*\psi_{F_1}+\pi_1^*\varphi_{F_1} = \psi+\varphi. \tag{2.9}$$

Lemma 1.1 says:

$$\begin{aligned} &\int_M e^{-\lambda(\psi+\varphi)(x)}\left(\mathrm{Jac}(\pi_1(x))\frac{dV_{g_{F_1}}(\pi(x))}{dV_g(x)}\right) dV_g(x) \\ &= d_1 \int_{F_1^{\perp}} e^{-\lambda(\psi_{F_1}+\varphi_{F_1})(y)} dV_{g_{F_1}}(y), \quad d_1=\deg(\pi_1), \end{aligned} \tag{2.10}$$

we have proved that $\psi_{F_1}+\varphi_{F_1}\in P_{G_1}(F_1^\perp,\mu g_{F_1})=P_{G_1}(CP^m,\mu\cdot g_{F_1})$. The first Chern class of $F_1^\perp\cong CP^m$ is represented by $(m+1)\omega_{g_F}$,

$$\frac{m+1}{\mu}(\psi_{F_1}+\varphi_{F_1})\in P_{G_1}(CP^m, C_1(CP^m)), \tag{2.11}$$

now

$$\lambda<\frac{m+1}{\mu}\min_{1\leqq i\leqq l}\alpha_{G_i}(F_i^\perp)\leqq\frac{m+1}{\mu}\alpha_{G_1}(F_1^\perp), \tag{2.12}$$

i.e. $\frac{\mu\lambda}{m+1}<\alpha_{G_1}(F_1^\perp)$, hence, there exists a constant C_1', depending only on λ, such that

$$\int_{F_1^\perp} e^{-\lambda(\psi_{F_1}+\varphi_{F_1})(y)}dV_{g_{F_1}}=\int_{F_1^\perp} e^{-\frac{\mu\lambda}{m+1}\cdot\frac{m+1}{\mu}(\psi_{F_1}+\varphi_{F_1})(y)}dV_{g_{F_1}}\leqq C_1'. \tag{2.13}$$

ψ is a smooth function independent of φ, there exists a constant C_1 such that

$$\int_M e^{-\lambda\varphi(x)}\left(\operatorname{Jac}(\pi_1(x))\frac{dV_{g_{F_1}}(\pi(x))}{dV_g(x)}\right)dV_g(x)\leqq C_1. \tag{2.14}$$

Once we have (*), we sum them up

$$\int_M e^{-\lambda\varphi(x)}\left(\frac{\sum_{i=1}^{l}\operatorname{Jac}(\pi_i(x))dV_{g_{F_i}}(\pi(x))}{dV_g(x)}\right)dV_g(x)\leqq C_1+\ldots+C_l=C'. \tag{2.15}$$

Because $F_1^\perp\cap\ldots\cap F_l^\perp=\emptyset$, there exists a constant $C''>0$, such that

$$\sum_{i=1}^{l}\operatorname{Jac}(\pi_i(x))dV_{g_{F_i}}(\pi(x))\geqq C''dV_g(x). \tag{2.16}$$

Put $C=C'/C''$, then

$$\int_M e^{-\lambda(\varphi(x)-\sup_M\varphi)}dV_g(x)\leqq C \tag{2.17}$$

for each $\varphi\in P_G(M,g)$. It implies that

$$\alpha_G(M)\geqq\frac{m+1}{\mu}\min_{1\leqq i\leqq l}\alpha_{G_i}(F_i^\perp). \tag{2.18}$$

Now we consider the existence of Kähler-Einstein metrics on $CP^2\#5\overline{CP^2}$. It is known that the generic intersection of two quadrics in CP^4 is CP^2 blown up at five generic points (see Griffiths and Harris [GH, p. 550]), consider smooth surfaces

$$M=\{[Z_0,Z_1,Z_2,Z_3,Z_4]\in CP^4|Z_0^2+\ldots+Z_4^2=0,\ a_0Z_0^2+a_1Z_1^2+\ldots+a_4Z_4^2=0\},$$

where $a_i\neq a_j$ for $i\neq j$, $0\leqq i,j\leqq 4$. $M\cong CP^2\#5CP^2$. We take g as the restriction of Fubini-Study metric on CP^4 to M, then

$$\omega_g=\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log(|Z_0|^2+\ldots+|Z_4|^2)|_M,$$

which is just $C_1(M)$, so $g\sim C_1(M)$. For $i\neq j$, $1\leqq i,j\leqq 4$,

$$\begin{aligned} F_{ij} &= \{[0,\ldots,Z_i,\ldots,Z_j,0,\ldots,0]\in CP^4\}, \\ F_{ij}^{\perp} &= \left\{\left[Z_0,\ldots,\overset{(i)}{0},\ldots,\overset{(j)}{0},\ldots,Z_4\right]\in CP^4\right\}, \end{aligned} \tag{2.19}$$

the corresponding projections $\pi_{ij}: M\to F_{ij}^{\perp}$, $[Z_0,\ldots,Z_4]$ to $\left[Z_0,\ldots,\overset{(i)}{0},\ldots,\overset{(j)}{0},\ldots,Z_4\right]$,

$$F_{ij}\cap M=\{[0,\ldots,Z_i,\ldots,Z_j,\ldots,0]|Z_i^2+Z_j^2=0,\ a_iZ_i^2+a_jZ_j^2=0\}=\emptyset, \tag{2.20}$$

and

$$\bigcap_{\substack{i,j=0\\ i<j}}^{4} F_{ij}^{\perp}=\emptyset.$$

Let G be the maximal compact group in $\mathrm{Aut}(M)$, then G contains transformations

$$\tau_i:[Z_0,\ldots,Z_4]\to[Z_0,\ldots,-Z_i,\ldots,Z_4].$$

Since the deck transformations of π_{ij} are τ_i, τ_j, $\tau_i\tau_j=\tau_j\tau_i$, by Proposition 2.2,

$$\alpha_G(M)\geqq 3\min_{0\leqq i<j\leqq 4}\alpha_{G_{ij}}(F_{ij}^{\perp}),$$

where G_{ij} is the group on $F_{ij}^{\perp}$ induced from G by π_{ij}. By Corollary 1.6, we obtain $\alpha_G(M)\geqq 1$. Hence

Theorem 2.3. *Non-singular intersections*

$$\{Z_0^2+Z_1^2+\ldots+Z_4^2=0,\ a_0Z_0^2+a_1Z_2^2+\ldots+a_4Z_4^2=0\}$$

admit Kähler-Einstein metrics, where $a_i\neq a_j$ *for* $0\leqq i<j\leqq 4$.

Remark. Based on the same arguments, one can show the existence of Kähler-Einstein metrics on certain non-singular complete intersections, such as

$$\{Z_0^3+Z_1^3+\ldots+Z_6^3=0,\ a_0Z_0^3+a_1Z_1^3+\ldots+a_6Z_6^3=0\}\subset CP^6;$$

$$\{Z_0^2+\ldots+Z_6^2=0,\ a_0Z_0^2+\ldots+a_6Z_6^2=0,\ b_0Z_0^2+\ldots+b_6Z_6^2=0\}\subset CP^6.$$

Next, we suppose that M is diffeomorphic to $CP^2\#7\overline{CP^2}$, i.e. CP^2 blown up at seven generic points. By Riemann-Roch theorem and Kodaira vanishing theorem, $\dim_{\mathbb{C}}H^0(M,-K_M)=3$. It is known that this group gives a holomorphic branched covering $\pi: M\to CP^2$, with degree equal to two. It has deck transformation σ_M which exchanges two sheets over CP^2. Let $E\subset CP^2$ be the branch locus, then E is smooth, as M is smooth. From $(-K_M)^2=2$, it is easy to see that $\deg E=4$. Moreover, by applying the covering lemma to $M-E\to CP^2-E$, one can easily deduce that any transformation of CP^2 preserving E can be lifted to be an automorphism of M. Let $\mathrm{Aut}(E)$ be $\{\tau\in PSL(3)|\tau(E)\subset E\}$. Then $\mathrm{Aut}(E)$ is finite and we can assume $\mathrm{Aut}(E)\subset U(3)$. Also, $\mathrm{Aut}(E)\subset G\subset\mathrm{Aut}(M)$, where G is maximal compact. Let $g\sim C_1(M)$ be a G-invariant metric on M and

$$\omega_{g_0}=\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log(|Z_0|^2+|Z_1|^2+|Z_2|^2)$$

be the Chern form of the hyperplane line bundle H on CP^2. As $C_1(M)=\pi^*(C_1(H))$ $=\pi^*(\omega_{g_0})$, we have $\omega_g-\pi^*\omega_{g_0}=\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi$ for a smooth function $\psi\in C^2(M,R)$. Obviously, we can assume that ψ is σ_M-invariant and $\mathrm{Aut}(E)$-invariant. Therefore $\pi^*(\psi_F+\varphi_F)=\psi+\varphi$ in Lemma 2.1, where $\varphi\in P_G(M,g)$. By Lemma 2.1,

$$\int_M e^{-\lambda(\psi+\varphi-\sup_M(\psi+\psi))}\left(\mathrm{Jac}(\pi)\cdot\frac{dV_{g_0}\pi}{dV_g}\right)dV_g=2\int_{CP^2} e^{-\lambda(\psi_F+\varphi_F-\sup_{CP^2}(\varphi_F+\psi_F))}dV_{g_0}. \tag{2.21}$$

Let h be the global section of line bundle $[\pi^{-1}(E)]$, defining $\pi^{-1}(E)$, then

$$\mathrm{Jac}(\pi(x))\frac{dV_{g_0}(\pi(x))}{dV_g(x)}\geqq C_1|h|^2(x), \tag{2.22}$$

where C_1 is a constant and $|h|^2$ is the norm of h with respect to certain fixed hermitian metric for $[\pi^{-1}(E)]$. Since ψ is smooth, there exists a constant C_2 such that

$$\int_M e^{-\lambda(\varphi(x)-\sup_M\varphi)}|h|^2dV_g\leqq C_2\int_{CP^2} e^{-\lambda(\varphi_F(y)+\psi_F(y)-\sup_M(\varphi_F+\psi_F))}dV_{g_0} \tag{2.23}$$

for every $\varphi\in P_G(M,g)$.

$$\pi^*\left(\omega_{g_0}+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\psi_F+\varphi_F)\right)=\pi^*\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$$
$$=\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\geqq 0,$$

on the other hand, $3\omega_{g_0}\sim C_1(CP^2)$, so $3(\psi_F+\varphi_F)\in P_{G_0}(CP^2,3g_0)$, where $G_0=\mathrm{Aut}(E)$. Thus the right-handed side of (2.23) is uniformly bounded independent of φ, whenever $\lambda<3\alpha_{G_0}(CP^2)$.

By Hölder inequality and the fact that $\int_M |h|^{-\frac{2}{\alpha-1}}dV_g<+\infty$ for $\alpha>2$, we have

$$\int_M e^{-\frac{\lambda}{\alpha}(\varphi(x)-\sup_M\varphi)}dV_g=\int_M\left(e^{-\frac{\lambda}{\alpha}(\varphi(x)-\sup_M\varphi)}\cdot|h|^{2/\alpha}\right)|h|^{-2/\alpha}dV_g$$
$$\leqq\left(\int_M e^{-\lambda(\varphi(x)-\sup_M\varphi)}|h|^2dV_g\right)^{1/\alpha}\left(\int_M |h|^{\frac{-2}{\alpha-1}}dV_g\right)^{\frac{\alpha-1}{\alpha}}$$
$$\leqq C_3\int_{CP^2} e^{-\lambda(\varphi_F(y)+\psi_F(y)-\sup_M(\varphi_F+\psi_F))}dV_{g_0},$$

where C_3 is a constant depending only on λ, M, α. Thus we prove that

$$\alpha_G(M)\geqq\tfrac{3}{2}\alpha_{G_0}(CP^2)\quad\text{and}\quad\alpha_G(M)>\tfrac{2}{3}$$

whenever $\alpha_{G_0}(CP^2)>\frac{4}{9}$.

Now we take E such that the corresponding $G_0=\mathrm{Aut}(E)$ contains a $G(p)$ defined in Sect. 1 for certain p, for example, we can take E to be the Fermat quartic curve $\{Z_0^4+Z_1^4+Z_2^4=0\}$ in CP^2 and in this case $\mathrm{Aut}(E)=G(4)$. For such a curve E, Corollary 1.7 says that $\alpha_{G_0}(CP^2)\geqq\frac{1}{2}$, i.e. $\alpha_G(M)\geqq\frac{3}{4}$, where M is the corresponding double covering of CP^2 branched along E. By Theorem 1.4, such an M admits a Kähler-Einstein metric.

Theorem 2.4. *For $E\subset CP^2$, a quartic curve such that* $\mathrm{Aut}(E)$ *contains $G(p)$, $p\geqq 2$, the double covering M of CP^2 branched along E admits a Kähler-Einstein metric. Note such a $M\cong CP^2\#7CP^2$.*

Remark. If one can prove $\alpha_G(M)\cdot L'_G(M)=1$ as stated in the remark after Theorem 1.5, then $\alpha(M)\geqq 1$ for $M\cong CP^2\#7CP^2$. This concludes that any such a M admits a Kähler-Einstein metric.

Combining a result in [T], we have proved that there exists $M\cong CP^2\#n\overline{CP^2}$ admitting a Kähler-Einstein metric for each n among 5, 6, 7. A standard argument by using implicit function theorem shows the following

Theorem 2.5. *There exist non-empty open sets $U_n\subset\mathfrak{M}_n$ for $n=5,6,7$, such that each M in U_n admits a Kähler-Einstein metric.*

It is well-known that any algebraic manifold is a branched covering of the projective space of same dimension. In many cases, the branched locus is smooth and the pull-back of the anticanonical line bundle of the base manifold N is proportional to that of the covering manifold M. Then one can estimate $\alpha(M)$ in terms of $\alpha(N)$, the covering degree and the ratio of two anticanonical line bundles, precisely, one can prove the following proposition. The proof is based on the same argument in the estimate of $\alpha_G(M)$ above when M is diffeomorphic to $CP^2\#7CP^2$.

Proposition 2.6. *Suppose that M, N are two compact Kähler manifolds with positive first Chern classes, and suppose there exist ad-branched covering map $\pi: M\to N$ such that the branch locus B is simple; i.e. $B=\{x\in N\,|\,\pi^{-1}(x)$ consists of a single point$\}$ and B is smooth. Then if $\pi^*C_1(N)=\mu C_1(M)$, μ is a rational number, $\alpha(M)\geqq\dfrac{1}{d^2\mu}\alpha(N)$. If the maximal compact group G of* $\mathrm{Aut}(M)$ *contains all deck transformations of π, then $\alpha_G(M)\geqq\dfrac{1}{d\mu}\alpha_{G_0}(N)$, where G_0 is the subgroup of* $\mathrm{Aut}(N)$ *induced by G and π.*

Remark. The condition on the branch locus can be weakened.

3. Kähler-Einstein Metrics s on $CP^2\#3\overline{CP^2}$ and $CP^2\#4\overline{CP^2}$

In this section, we prove that $\alpha_G(M)>\frac{2}{3}$ if M is diffeomorphic to either $CP^2\#3\overline{CP^2}$ or $CP^2\#4\overline{CP^2}$, and thus conclude the existence of Kähler-Einstein metric on such M. The case that $M\cong CP^2\#3\overline{CP^2}$ is also considered by Calabi and Siu [Si 2].

We start with a lemma taken from [T].

Lemma 3.1 ([T], Lemma 3.2). *Let $B_{R_1}^{m-1}(0)\times B_{R_2}(0)\subset C^{m-1}\times C^1$ be the product of balls,*

$$S_\beta=\Big\{\varphi\in C^2(B_{R_1}^{m-1}(0)\times B_{R_2}(0))\,|\,\forall z\in B_{R_1}^{m-1},\ \varphi_z=\varphi(z,\cdot)\ \text{is subharmonic},$$

$$\varphi\leqq 0,\ \int_{B_{R_2}(0)}\Delta_w\varphi_z(w)dw\leqq\beta\Big\}.$$

Then for each ε, $\delta>0$, *there exist* $r_2=r_2(\varepsilon,R_2)>0$, $C=C(\delta,\beta)$, *such that* $\forall\varphi\in S_\beta$,

$$\iint_{\substack{|z|\leqq R_1\\|w|\leqq r_2}} e^{-\left(\frac{4\pi}{\beta}-\delta\right)\varphi(z,w)}dz\,dw\leqq\frac{CR_2^2}{r_2^2}\iint_{\substack{|z|\leqq R_1\\r_2\leqq|w|\leqq 2r_2}} e^{-(1+\varepsilon)\left(\frac{4\pi}{\beta}-\delta\right)\varphi(z,w)}dz\,dw\,, \tag{3.1}$$

where Δ_w *is the real Laplacian of* w.

Lemma 3.2. *Let* φ *be a radically symmetric subharmonic function in* $B_1(0)\subset C^1$, *then*

$$\varphi(Z)-\varphi\left(\frac{Z}{|Z|}\right)=\varphi(Z)-\varphi(1)\geqq\frac{1}{2\pi}\left(\int_{B_1(0)}\Delta\varphi(w)dw\right)\log|Z|\,. \tag{3.2}$$

Proof. Let $r=|Z|$, $rf(r)=r\Delta\varphi\geqq 0$, then $rf=\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)$, note that φ can be considered as a function of r. Integrating on both sides of the above equality,

$$r\frac{d\varphi}{dr}(r)=\int_0^r sf(s)ds\,. \tag{3.3}$$

So

$$\begin{aligned}\varphi(r)-\varphi(1)&=\int_1^r\frac{d\varphi}{ds}ds=\int_1^r\frac{ds}{s}\int_0^s tf(t)dt\\&\geqq\frac{1}{2\pi}\int_0^1 sf(s)ds\cdot\int_1^r\frac{ds}{s}=\frac{1}{2\pi}\left(\int_{B_1(0)}\Delta\varphi(Z)dZ\right)\log|Z|\,.\end{aligned} \tag{3.4}$$

Theorem 3.3. (i) *If* M *is diffeomorphic to* $CP^2\#3\overline{CP}^2$, *then* $\alpha_G(M)\geqq 1$,

(ii) *If* M *is diffeomorphic to* $CP^2\#4\overline{CP}^2$, *then* $\alpha_G(M)\geqq\frac{3}{4}$, *where* G *is the maximal compact subgroup of* Aut(M). *In particular, both manifolds in* (i), (ii) *admit Kähler-Einstein metrics.*

The rest of this section is devoted to the proof of this theorem. First, let us assume $M\cong CP^2\#3\overline{CP^2}$, i.e. M is CP^2 blown up at three generic points. After an automorphism of CP^2, we may assume that the blown-up points are $[1,0,0]$, $[0,1,0]$, $[0,0,1]$. Aut(M) consists of all those projective transformations on CP^2, permutating the blown-up points, so G is generated by

$$\begin{pmatrix}e_1&0&0\\0&e_2&0\\0&0&e_3\end{pmatrix}\in PSL(3)\,,\qquad |e_1|=|e_2|=|e_3|=1$$

and

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\ \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\in PSL(3)\,.$$

Now the metric g is G-invariant. In order to estimate $\alpha_G(M)$, we take a sequence $\{\varphi_i\}\subset P_G(M,g)$. Fix a $\lambda<1$. By taking a subsequence if necessary, we may assume that there is an analytic subvariety S_λ, such that $\dim_C S_\lambda\leqq 1$ and for each

$z \in M - S_\lambda$, $\exists r > 0$, $C > 0$, such that

$$\int_{B_r(z)} e^{-\lambda\left(\varphi_i(\zeta) - \sup_M \varphi_i\right)} dV_g(\zeta) \leqq C \text{ for all } i; \tag{3.5}$$

for $z \in S_\lambda$,

$$\lim_{i \to +\infty} \int_{B_r(z)} e^{-\lambda\left(\varphi_i(\zeta) - \sup_M \varphi_i\right)} dV_g(\zeta) = +\infty \quad \text{for any } r > 0 \text{ [T]}. \tag{3.6}$$

For our purpose, it suffices to show that $S_\lambda = \emptyset$. On the other hand, by Proposition 2.1, $e^{-\varepsilon\left(\varphi_i(\zeta) - \sup_M \varphi_i\right)}$ have uniform L^1-integral bound for ε small and all i, so we may assume that $\varphi_i - \sup_M \varphi_i$ converge to φ in L^2, then $e^{-\varepsilon\varphi}$ is L^1-integrable and $u_i = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_i$ converge weakly to the positive, d-closed $(1,1)$-current $u = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi$, where $\partial\bar{\partial}\varphi$ is in the sense of distribution. Define $E_\lambda = \left\{ z \in M \,|\, L_g(u, z) \geqq \frac{1}{\lambda} \right\}$, then E_λ is an analytic subvariety (see Siu [Si 1]). The arguments in the proof of Theorem 1.5 shows that $E_\lambda \supseteqq S_\lambda$. Note that they are not equal in general.

Let $\pi: M \to CP^2$ be the natural projection, C_0, C_1, C_2 be the exceptional curves of M over $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, F_0, F_1, F_2 are quadratic transformations of lines

$$\{[0, Z_1, Z_2] \in CP^2\}, \quad \{[Z_0, 0, Z_2] \in CP^2\}, \quad \{[Z_0, Z_1, 0] \in CP^2\}.$$

Claim. For $\lambda < 1$, $E_\lambda \subset (C_0 \cup C_1 \cup C_2) \cap (F_0 \cup F_1 \cup F_2)$.

Since E_λ is G-invariant and analytic, it is easy to see that E_λ is contained in

$$F_0 \cup F_1 \cup F_2 \cup C_0 \cup C_1 \cup C_2.$$

If $\dim_C E_\lambda = 1$, then either $F_0 \cup F_1 \cup F_2 \subset E_\lambda$, or $C_0 \cup C_1 \cup C_2 \subset E_\lambda$. In the former case, since $e^{-\varepsilon\varphi}$ is L^1-integrable for ε small, we find a generic line $l \subset CP^2$, avoiding $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, such that φ is not identically equal to $-\infty$ on $\pi^{-1}(l) \cong l$, so the slice $u|_{\pi^{-1}(l)}$ is well-defined. This l intersects F_i at one point P_i, $i = 0, 1, 2$, $P_0, P_1, P_2 \in E_\lambda$, so by Lemma 1.2,

$$\begin{aligned} \int_{\pi^{-1}(l)} C_1(M) = \int_{\pi^{-1}(l)} \left(\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi \right) &\geqq \sum_{i=0}^{2} L_{g|_{\pi^{-1}(l)}}(u|_{\pi^{-1}(l)}, P_i) \\ &\geqq \sum_{i=0}^{2} L_g(u, P_i) \geqq \frac{3}{\lambda} > 3, \end{aligned} \tag{3.7}$$

but $C_1(M) = \pi^*(3H) - [C_0] - [C_1] - [C_2]$, where H is the hyperplane line bundle on CP^2, so

$$\begin{aligned} \int_{\pi^{-1}(l)} C_1(M) &= (\pi^*(3H) - [C_0] - [C_1] - [C_2]) \cdot [\pi^{-1}(l)] \quad \text{([GH])} \\ &= 3. \end{aligned} \tag{3.8}$$

A Contradiction. In the second case, by taking a generic quadric Q in CP^2 passing through $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, we can also get a contradiction by the same argument as above.

Hence, $\dim_C E_\lambda = 0$, i.e. E_λ consists of finite points. For any point $p \in M$ other than

$$\left(\bigcup_{i=0}^{2} C_i\right) \cap \left(\bigcup_{i=0}^{2} F_i\right),$$

the orbit $G \cdot p$ has at least real dimension one, so $p \notin E_\lambda$. Thus the claim is proved.

Quadrics $C_\delta = \{Z_0 Z_2 = \delta Z_1^2\}$ of CP^2 $(\delta \in CP^1)$ pass through $[1,0,0]$, $[0,0,1]$. Their quadratic transformations $\pi^* C_\delta$ pass through points $C_0 \cap F_2$, $C_2 \cap F_0$. $\pi^* C_\delta$ is smooth except $\delta = \infty \in CP^1$.

At point $C_0 \cap F_2$, the local coordinates are $(x, \eta) \to [1, x, x\eta] \times [1, \eta] \in M$, the automorphism

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$$

acts on M near $(0,0) = C_0 \cap F_2$ by sending (x, η) to $(e_2 e_1^{-1} x,\ e_3 e_2^{-1} \eta)$, so

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{pmatrix}$$

acts on M by $(x, \eta) \to (ex, e\eta)$, since φ_i is G-invariant, $\varphi_i(x, \eta) = \varphi_i(ex, e\eta)$ for any $e \in C$ with $|e| = 1$, so $\varphi_i(x, \eta) = \varphi_i(|x|, |\eta|)$.

For R small enough, $B_R(C_0 \cap F_2)$ is contained in the chart of local coordinates (x, η). Since the metric g is G-invariant, $\pi^* C_\delta \cap B_R(C_0 \cap F_2)$ is spherically symmetric in the usual sense of local coordinates (x, η), so

$$\pi^* C_\delta \cap B_R(C_0 \cap F_2) = \{(x, \eta) \,|\, \eta = \delta x, |x|^2 + |\eta|^2 \leqq R_\delta\}$$

for certain $R_\delta > 0$. On each $\pi^* C_\delta \cap B_R(C_0 \cap F_2)$, each φ_i is radically symmetric.

Automorphism $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in G$ maps each $\pi^* C_\delta$ into itself and maps $C_0 \cap F_2$ to $C_2 \cap F_0$.

$$4 = \pi^* C_\delta \cdot C_1(M) = \int_{\pi^* C_\delta} C_1(M) = \int_{\pi^* C_\delta} \left(\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_i\right)$$

for any φ_i, but φ_i is G-invariant, so

$$\int_{B_R(C_0 \cap F_2) \cap \pi^* C_\delta} \left(\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_i\right) \leqq \tfrac{1}{2} \int_{\pi^* C_\delta} \left(\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_i\right) = 2.$$

Locally, $\omega_g = \dfrac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi$, ψ is also G-invariant, then

$$\theta_i(x, \eta) = \psi(x, \eta) + \varphi_i(x, \eta) - \sup_M \varphi_i = \theta_i(|x|, |\eta|)$$

is radically symmetric on each $\pi^*C_\delta \cap B_R(C_0 \cap F_2)$. Each θ_i is plurisubharmonic and

$$\int_{\pi^*C_\delta \cap B_R(C_0 \cap F_2)} \Delta\theta_i(\zeta)d\zeta = 2\pi \int_{\pi^*C_\delta \cap B_R(C_0 \cap F_2)} \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_i\right) \leqq 4\pi,$$

so by Lemma 3.2,

$$\theta_i(x,\eta) \geqq 2\log(|x|^2 + |\eta|^2) + \theta_i(z(x,\eta)),$$

where $z(x,\eta) \in \pi^*C_\delta \cap \partial B_R(C_0 \cap F_2)$. Thus, since $\lambda < 1$, by using the polar coordinates, there exists $C > 0$, such that

$$\int_{B_R(C_0 \cap F_2)} e^{-\lambda\theta_i(x,\eta)} dV_g \leqq C \int_{\partial B_R(x,\eta)} e^{-\lambda\theta_i(x,\eta)} dV_g,$$

then

$$\int_{B_R(C_0 \cap F_2)} e^{-\lambda(\varphi_i - \sup_M \varphi_i)} dV_g \leqq C \int_{\partial B_R(C_0 \cap F_2)} e^{-\lambda(\varphi_i - \sup_M \varphi_i)} dV_g.$$

Since we have proved that $S_\lambda \subseteqq E_\lambda$ contains at most $\left(\bigcup_{i=0}^{2} C_i\right) \cap \left(\bigcup_{i=0}^{2} F_i\right)$, integrating on R from R_1 to $2R_1$, R_1 small, it follows that there exists C such that

$$\int_{B_R(C_0 \cap F_2)} e^{-\lambda\left(\varphi_i - \sup_M \varphi_i\right)} dV_g \leqq C \quad \text{for all } i,$$

i.e. $C_0 \cap F_2 \notin S_\lambda$. Similarly, $C_2 \cap F_0 \notin S_\lambda$, and $C_i \cap F_j \notin S_\lambda$ $(i \neq j)$, so $S_\lambda = \emptyset$. Therefore, we have proved that $\alpha_G(M) \geqq 1$.

Next, we turn to the proof of (ii). In this case, we may assume that M is CP^2 blown up at four points $[1,0,0], [0,1,0], [0,0,1], [1,1,1]$. There is a fibration of M over CP^1 by conics, precisely, if $\pi: M \to CP^2$ is the projection, the fibration $f: M \to CP^1$ is given by mapping $\pi^*C_{[\alpha,\beta]}$ to $[\alpha,\beta] \in CP^1$, where

$$C_{[\alpha,\beta]} = \{\alpha Z_0(Z_1 - Z_2) + \beta Z_2(Z_1 - Z_0) = 0\} \subset CP^2.$$

Let $G_0 \subset G$ be the subgroup preserving the fibration, then G_0 is generated by

$$\sigma_0 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

The fixed points of σ_i are D_i,

$$D_0 = \pi^*\{[Z_0, Z_1, Z_2] \in CP^2 | Z_1 + Z_2 = Z_0\} \cup \{[0,1,1]\},$$

$$D_1 = \pi^*\{[Z_0, Z_1, Z_2] \in CP^2 | Z_0 + Z_2 = Z_1\} \cup \{[1,0,1]\},$$

$$D_2 = \pi^*\{[Z_0, Z_1, Z_2] \in CP^2 | Z_0 + Z_1 = Z_2\} \cup \{[1,1,0]\},$$

$$D_0 \cap D_1 = D_1 \cap D_2 = D_2 \cap D_0 = \{[1,0,1], [0,1,1], [1,1,0]\}.$$

As before, we fix a $\lambda < \frac{3}{4}$ and take a sequence $\{\varphi_i\} \subset P_G(M,g)$, we also have $S_\lambda, E_\lambda, S_\lambda \subseteqq E_\lambda$. It is sufficient to show that $S_\lambda = \emptyset$.

Claim. $\dim_{\mathbb{C}} S_\lambda = 0$.

First of all, E_λ cannot contain a curve in fibres of f. In fact, if not, E_λ contains at least one fibre $C_{[\alpha,\beta]}$. As G acts on M without fixed point, E_λ contains at least two fibres $C_{[\alpha,\beta]}$, $C_{[\alpha',\beta']}$. Choose a generic line $l \subset CP^2$, away from four blown-up points, such that $\varphi \not\equiv -\infty$ on π^*L, where φ is the limit of $\varphi_i - \sup_M \varphi_i$ as before. Then, by Bertini theorem [GH], l will intersect $C_{[\alpha,\beta]}$, $C_{[\alpha',\beta']}$ at two points, respectively. The argument used in Lemma 1.2 then gives a contradiction. So $S_\lambda \subset E_\lambda$ does not contain a curve in the fibre. If $\dim_C S_\lambda = 1$, then S_λ is generically transversal to the fibres of f. $\pi^*C_{[\alpha,\beta]} \cdot C_1(M) = 2$, and G_0 acts on each smooth $\pi^*C_{[\alpha,\beta]}$ without fixed point, since $[1,1,0]$, $[0,1,1]$, $[1,0,1]$ are only fixed points of G_0, and they are in three singular fibres $C_{[1,0]}$, $C_{[0,1]}$, $C_{[1,1]}$. Hence, at each point p where S_λ intersects a smooth $\pi^*C_{[\alpha,\beta]}$ transversally, we may find a neighborhood U so that in proper local coordinates, $U = B_{R_1}^{m-1}(0) \times B_{R_2}(0)$, $p = (0,0)$, and $U \cap \pi^*C_{[\alpha',\beta']}$ is one of $z \times B_{R_2}(0)$, $z \in B_{R_1}^{m-1}(0)$, moreover,

$$\int_{U \cap \pi^*C_{[\alpha',\beta']}} \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_i\right) \leqq \tfrac{1}{2}\pi^*C_{[\alpha',\beta']} \cdot C_1(M) = 1\,.$$

By Lemma 3.1, one sees that $p \notin S_\lambda$, a contradiction, and we have proved $\dim_{\mathbb{C}} S_\lambda = 0$.

Furthermore, the above argument actually shows that S_λ contains at most $[1,1,0]$, $[1,0,1]$, $[0,1,1]$. They are equivalent under the action of G. Now we estimate Lelong numbers of $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ at those points.

Lemma 3.4. *The generic curve in the family*

$$\{\alpha Z_0(Z_0 - Z_1 - Z_2)^2(Z_1 - Z_2) + Z_2(Z_2 - Z_0 - Z_1)^2(Z_0 - Z_1) = 0\}_{\alpha \in CP^1}$$

in CP^2 is smooth except at $[1,1,0]$, $[1,0,1]$, $[0,1,1]$, *where the curve has ordinary double points.*

Proof. Let $C_\alpha = \{\alpha Z_0(Z_0 - Z_1 - Z_2)^2(Z_1 - Z_2) + Z_2(Z_2 - Z_1 - Z_0)^2(Z_0 - Z_1) = 0\}$. It is trivial to see that C_0 and C_∞ have no common component. By Bezout's theorem (see Griffiths and Harris [GH, p. 670], also [H, p. 54]).

$$16 = \sum_{p \in C_0 \cap C_\infty} \operatorname{int}(C_0, C_\infty, p)\,,$$

where $\operatorname{int}(C_0, C_\infty, p)$ denotes the intersection multiplicity of C_0 and C_∞ at p. C_0, C_∞ pass through points $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, $[1,1,0]$, $[0,1,1]$, $[1,0,1]$, $[1,1,1]$ and have multiplicities $=2$ at $[1,1,0]$, $[0,1,1]$, $[1,0,1]$.

It is well known that $\operatorname{int}(C_0, C_\infty, p) \geqq \operatorname{mult}(C_0, p)\operatorname{mult}(C_\infty, p)$ [H, Exercise 5.4], thus at $p =$ one of $[1,1,0]$, $[0,1,1]$, $[1,0,1]$, $\operatorname{int}(C_0, C_\infty, p) \geqq 4$.

Hence,

$$\operatorname{int}(C_0, C_\infty, p) = 4 \text{ for } p = [1,1,0], \text{ or } [0,1,1], \text{ or } [1,0,1]\,,$$

$$\operatorname{int}(C_0, C_\infty, p) = 1 \text{ for } p = \text{one of } [1,0,0], [0,1,0], [0,0,1], [1,1,1]\,.$$

By Bertini theorem [GH], the generic C_α is smooth outside $[1,1,0]$, $[0,1,1]$, $[1,0,1]$.

A direct computation shows that for $\alpha \neq -1, 0, \infty$, C_α has ordinary double points at $[1,1,0]$, $[1,0,1]$, $[0,1,1]$.

As before, by Lemma 3.4, we can find a generic quartic curve C_α, smooth outside $[1,1,0]$, $[1,0,1]$, $[0,1,1]$, where C_α has ordinary double points, such that the slice of $u=\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ is well-defined on π^*C_α, then

$$\begin{aligned} 8=\pi^*C_\alpha\cdot C_1(M)=\int_{\pi^*C_\alpha} C_1(M)=\int_{\pi^*C_\alpha} u &\geqq L_{g|_{\pi^*C_\alpha}}(u|_{\pi^*C_\alpha},[1,1,0]) \\ &\quad + L_{g|_{\pi^*C_\alpha}}(u|_{\pi^*C_\alpha},[1,0,1]) + L_g(u|_{\pi^*C_\alpha},[0,1,1]) \\ \text{(Lemma 1.2)} \qquad &\geqq 2L_g(u,[1,1,0])+2L_g(u,[1,0,1])+2L_g(u,[0,1,1]) \\ &=6L_g(u,[1,1,0]), \end{aligned}$$

i.e.

$$L_g(u,[1,0,1])=L_g(u,[0,1,1])=L_g(u,[1,1,0])\leqq \tfrac{4}{3},$$

so $[1,1,0]$, $[1,0,1]$, $[0,1,1]\notin E_\lambda$, it follows that $S_\lambda=\emptyset$. Therefore, $\alpha_G(M)\geqq\frac{3}{4}$. We complete the proof of Theorem 3.3.

4. Kähler-Einstein Metrics on $CP^2 \# 8\overline{CP^2}$

In this section, we investigate the existence of Kähler-Einstein metrics on complex surfaces with $C_1>0$ and diffeomorphic to $CP^2 \# 8\overline{CP^2}$. As before, it suffices to estimate the lower bound of $\alpha_G(M)$ for $M\cong CP^2 \# 8\overline{CP^2}$. Such a surface is obtained by blowing up CP^2 at generic eight points as explained in the introduction. Now, $C_1(M)^2=1$ and $h^0(M,\theta_M(-K_M))=2$, i.e. the anti-canonical bundle $-K_M$ has a pencil of elliptic curves as its complete linear system. Such a pencil corresponds to the pencil of cubic curves $\{C\}_{\delta\in CP^1}$ in CP^2 passing through the blown-up points. Because of the general positions of those blown-up points, one easily checks that each C_δ is irreducible, so the singular C_δ is the rational curve with either an ordinary double point or a cusp. $\mathrm{Aut}(M)$ consists of all automorphisms in $\mathrm{Aut}(CP^2)$ preserving the set of blown-up points. Clearly, $\mathrm{Aut}(M)$ is finite.

Lemma 4.1. *Any non-trivial $\sigma\in\mathrm{Aut}(M)$ doesn't preserve the singular curve in the pencil $\{C_\delta\}_{\delta\in CP^1}$.*

Proof. Suppose that σ preserves the singular curve C_{δ_0}. Then σ fixes the singular point P_1 of C_{δ_0}. σ must interchange the cubic curves in the given pencil. Since all these cubic curves intersect at one point P_2, which is not one of the blown-up points. Thus, σ fixes P_2. Now, σ preserves the tangent line $T_{P_2}C_{\delta_0}=l$. If $l\cap C_{\delta_0}$ contains more than one point, then σ fixed at least three points of C_{δ_0}. Since C_{δ_0} is rational, $\sigma|_{C_{\delta_0}}=\text{identity}$. It follows that σ is the identity, a contraction. Hence, $l\cap C_{\delta_0}=\{P_2\}$. $CP^2\backslash l=C^2$, choose coordinates $[x,y,z]$ of CP^2 such that $l=\{z=0\}$, $P_1=[0,0,1]$, and $P_2=[0,1,0]$, then for certain $a,b\in C$, $C_{\delta_0}=\{y^2=ax^3+bx^2\}$, where $a\neq 0$.

Now, σ is a linear transformation of C^2. It follows that either $\sigma=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, or $\sigma^2=\begin{pmatrix}\omega&0\\0&1\end{pmatrix}$, $\omega=\exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$, and $b=0$. In the case, since σ^2 preserves the set of eight blown-up points, it must fix at least two of them, thus σ^2 fixes at least four points of C_{δ_0}, as before, it follows that $\sigma^2 = \text{identity}$, which is impossible! Hence, $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, but a direct computation shows that if $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then either three of eight blown-up points are colinear, or six of them are on a quadratic curve. Both are against the assumption of the generic position of blown-up points. The lemma is proved.

Example. We construct a family of complex surfaces $M_{[\alpha,\beta,\gamma]}$ parametrized by an open subset in CP^2 such that $M_{[\alpha,\beta,\gamma]} \cong CP^2 \# 8CP^2$, $C_1(M_{[\alpha,\beta,\gamma]}) > 0$, and

$$\operatorn{Aut}(M_{[\alpha,\beta,\gamma]}) \neq \{\text{identity}\},$$

i.e. non-trivial.

Fix an element

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad \omega = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$$

and

$$P_1 = [1,0,0],\ P_2 = [0,1,0],\ P_3 = [1,1,1],\ P_4 = [1,\omega,\omega^2],$$
$$P_5 = [1,\omega^2,\omega],\ P_6 = [\alpha,\beta,\gamma],\ P_7 = \sigma(P_6),\ P_8 = \sigma^2(P_6).$$

Take $V_1 = \{xyz = 0\} \cup \Big(\bigcup_{1 \leqq i \leqq j \leqq 5} l_{ij}\Big)$, where l_{ij} is the line through P_i, P_j,

$$\begin{aligned} V_2 = &\{\omega yz - xy - (1+\omega)xz = 0\} \cup \{yz - xy - \omega(1+\omega)xz = 0\} \\ &\cup \{yz - \omega xy - xz(1+\omega) = 0\} \cup \{z^2 = xy\} \cup \{x = y\} \\ &\cup \{x = \omega y\} \cup \{x = \omega^2 y\}. \end{aligned}$$

Then it is straightforward to check that for any $P_6 \in CP^2 \setminus V_1 \cup V_2$, $P_1, P_2, \ldots, P_8$ are in general position. Blowing up CP^2 at these P_i, we obtain the required $M_{[\alpha,\beta,\gamma]}$ with $\sigma \in \operatorname{Aut}(M_{[\alpha,\beta,\gamma]})$. Moreover, one can directly verify Lemma 4.1 for σ.

Theorem 4.2. *For Kähler surface* $M \cong CP^2 \# 8\overline{CP^2}$, $C_1(M) > 0$, *if* $\operatorname{Aut}(M)$ *is nontrivial, then* $\alpha_G(M) \geqq 1$. *In particular, such a Kähler surface M admits a Kähler-Einstein metric.*

Proof. Note that $G = \operatorname{Aut}(M)$ in this case. Fix a $\lambda < 1$, in order to prove that $\alpha_G(M) \geqq \lambda$, as before, we take a sequence $\{\varphi_i\}$ from $P_G(M,g)$, where g is a Kahler metric on M, invariant under G. It suffices to prove that $S_\lambda = \emptyset$, where S_λ is defined as in the proof of Theorem 3.3. Let $\{C_\delta\}_{\delta \in CP^1}$ be the pencil of cubic curves on M, which generate $H^0(M, \theta_M(-K_M))$. Then by $C_1(M)^2 = 1$, i.e. $C_1(M) \cdot C_\delta = 1$ and Lemma 3.1, one concludes that S_λ consists of those singular points of certain C_δ's, which are finite. Let $\sigma \in \operatorname{Aut}(M)$, $\sigma \neq \text{id}$, and a singular point $P_1, P_2 = \sigma(P_1)$. By Lemma 4.1 and nontriviality of $\operatorname{Aut}(M)$, $P_1 \neq P_2$. Moreover, suppose that $C_{\delta_1}, C_{\delta_2}$ pass through P_1, P_2, respectively.

By the Riemann-Roch theorem [GH], $h^0(M, \theta_M(-2K_M)) = 4$. Thus there is a pencil of divisors of $-2K_M$ passing through P_1 and P_2. Clearly, $C_{\delta_1} + C_{\delta_2}$ is one of them and the other one has no common component with this one. Hence, one can pick up an irreducible divisor D of $-2K_M$ in that pencil such that $\varphi \not\equiv -\infty$ on D, where φ is the limit of the sequence $\{\varphi_i\}$. It follows that the slicing $u|_D$ of the d-closed, positive $(1,1)$-current $u=\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ is well-defined. Since $2 \geqq \mathrm{int}(C_{\delta_1}, D, P_1) \geqq \mathrm{mult}(C_{\delta_1}, P_1)\,\mathrm{mult}(D, P_1) \geqq 2\,\mathrm{mult}(D, P_1)$ ([H], Exercise 5.4), D is smooth at P_1. Similarly, D is smooth at P_2. Then

$$\begin{aligned} 2 = D \cdot C_1(M) = \int_D u &\geqq L_{g|_D}(u|_D, P_1) + L_{g|_D}(u|_D, P_2) \\ &= L_g(u, P_1) + L_g(u, P_2) = 2L_g(u, P_1), \end{aligned}$$

thus, $L_g(u, P_1) \leqq 1$. By Lemma 1.1 and Proposition 1.3, it follows that $P_1 \notin S_\lambda$. Therefore, $S_\lambda = \emptyset$.

Corollary. $M_{[\alpha,\beta,\delta]}$, *constructed in the previous example, admits a Kähler-Einstein metric.*

A standard argument using the implicit function theorem shows the following

Theorem 4.3. *There is an open, non-empty set $U_8 \subset \mathfrak{M}_8$ such that each $M \in U_8$ admits a Kähler-Einstein metric.*

Combining this with Theorem 2.5 and Theorem 3.3, we finish the proof of the main theorem of this paper.

5. A Lower Bound of $\alpha(M)$ in Terms of $(-K_M)^m$

In this section, we apply the Jensen formula to obtain a useful inequality.

Lemma 5.1 [De, Sk 2]. *Let X be a stein manifold, u be a positive, d-closed $(1,1)$-current. Let ψ be an exhaustive function of X. Define $B(r) = \{z \in X \mid \psi(z) < r^2\}$, then for $0 < r_1 < r_2 < \sup_X \psi$,*

$$(\pi r_2^2)^{m-1} \int_{B(r_2)} u \wedge \beta^{m-1} - \frac{1}{(\pi r_1^2)^{m-1}} \int_{B(r_1)} u \wedge \beta^{m-1} = \int_{B(r_1, r_2)} u \wedge \alpha^{m-1},$$

where

$$m = \dim X, \qquad B(r_1, r_2) = B(r_2) \setminus B(r_1),$$

$$\beta = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi, \qquad \alpha = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\psi.$$

Theorem 5.2. *Let M be an algebraic manifold in CP^N and g be the restriction of the Fubini-Study metric on CP^N to M. Then for any d-closed, positive $(1,1)$-current u,*

$$\int_M u \wedge \omega_g^{m-1} \geqq L_g(u, z) \quad \textit{for each } z \in M.$$

Proof. Let $[Z_0, \ldots, Z_N]$ be homogeneous coordinate of CP^N such that the point $z \in M$ corresponds to $[1, 0, 0, \ldots, 0] \in CP^N$ and $T_z M = \{[1, Z_1, \ldots, Z_m, 0, \ldots, 0] \in CP^N$.

Let $H=$ the hyperplane $\{Z_0=0\}$ in CP^N, $X=M\backslash H\subset C^N$ and $\psi=\sum_{i=1}^{N}|Z_i|^2$. Then ψ is obviously a plurisubharmonic exhaustive function of X. Hence by the above lemma, for $R>r>0$,

$$\frac{1}{(\pi R^2)^{m-1}}\int_{B_R(0)} u\wedge\beta^{m-1}-\frac{1}{(\pi r^2)^{m-1}}\int_{B_r(0)} u\wedge\beta^{m-1}\geqq 0,$$

where $\beta=\frac{\sqrt{-1}}{2}\partial\bar\partial|Z|^2$.

First, we assume that u is smooth and consider

$$\lim_{R\to+\infty}\frac{1}{(\pi R^2)^{m-1}}\int_{B_R(0)} u\wedge\beta^{m-1}.$$

By Stoke's theorem,

$$\lim_{R\to+\infty}\frac{1}{(\pi R^2)^{m-1}}\int_{B_R(0)} u\wedge\beta^{m-1}=\lim_{R\to+\infty}\frac{\sqrt{-1}}{2\pi}\int_{\partial B_R(0)} u\wedge\bar\partial\log|Z|^2\wedge\alpha^{m-2},$$

where $\alpha=\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|Z|^2$.

Let $\widetilde{CP}^N$ be the manifold produced by blowing up CP^N at $[1,0,\dots,0]$, $\pi:\widetilde{CP}^N\to CP^N$ the natural projection. There is a natural fibration $p:\widetilde{CP}^N\to CP^{N-1}=H$ by CP^1's, under which

$$\pi^*\alpha=p^*\omega'|_{\pi^*M},$$

where $\omega'=\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log(|Z_1|^2+\dots+|Z_N|^2)$ is the Fubini-Study metric of CP^{N-1}, π^*M is the quadratic transformation of M.

$$\begin{aligned}
\lim_{R\to\infty}\frac{1}{(\pi R^2)^{m-1}}\int_{B_R(0)} u\wedge\beta^{m-1}&=\lim_{R\to\infty}\frac{\sqrt{-1}}{2\pi}\int_{\partial B_R(0)}\pi^*u\wedge\bar\partial\log|Z|^2\wedge(p^*\omega')^{m-2}\\
&=\lim_{R\to\infty}\frac{\sqrt{-1}}{2\pi}\int_{\partial B_R(0)}\pi^*u\wedge(p^*\omega')^{m-2}\wedge\left(\bar\partial\log\frac{|Z|^2}{1+|Z|^2}+\bar\partial\log(1+|Z|^2)\right)\\
&=\lim_{R\to\infty}\left(-\frac{\sqrt{-1}}{2\pi}\int_{\pi^{-1}(M\backslash B_R(0))}\pi^*u\wedge(p^*\omega')^{m-2}\wedge\pi^*\left(\partial\bar\partial\log\frac{|Z|^2}{1+|Z|^2}\right)\right.\\
&\quad\left.+\int_{\pi^*(B_R(0))}\pi^*u\wedge(p^*\omega')^{m-2}\wedge\pi^*\omega_g\right)\\
&=\lim_{R\to\infty}\int_{\pi^*(B_R(0))}\pi^*u\wedge\pi^*\omega_g\wedge(p^*\omega')^{m-2},
\end{aligned}$$

since $\mathrm{Vol}(\pi^*(M\backslash B_R(0)))\to 0$ as $R\to\infty$ and $\log\frac{|Z|^2}{1+|Z|^2}$ is smooth in $M\backslash\{Z\}$. Then

$$\begin{aligned}
\lim_{R\to+\infty}\frac{1}{(\pi R^2)^{m-1}}\int_{B_R(0)} u\wedge\beta^{m-1}&=\lim_{R\to\infty}\frac{\sqrt{-1}}{2\pi}\int_{\pi^*(\partial B_R(0))}\pi^*(u\wedge\omega_g)\wedge(p^*\omega')^{m-3}\wedge\bar\partial\log|Z|^2\\
&=\lim_{R\to\infty}\frac{\sqrt{-1}}{2\pi}\int_{\partial B_R(0)} u\wedge\omega_g\wedge\bar\partial\log\psi\wedge\alpha^{m-3}.
\end{aligned}$$

Hence, inductively, we obtain

$$\lim_{R\to\infty}\frac{1}{(\pi R^2)^{m-1}}\int_{B_R(0)} u\wedge\beta^{m-1}=\lim_{R\to\infty}\int_{B_R(0)} u\wedge\omega_g^{m-1}=\int_M u\wedge\omega_g^{m-1}.$$

Thus, for each $r>0$, and smooth u, we have

$$\int_M u\wedge\omega_g^{m-1}\geqq\frac{1}{(\pi r^2)^{m-1}}\int_{B_r(0)} u\wedge\beta^{m-1}. \tag{$*$}$$

By taking the smooth approximation, one can easily see that $(*)$ still holds for general u. Now we let r go to zero and show that the limit is exactly $L_g(u,Z)$.

By the choice of the homogeneous coordinate, there exist holomorphic functions $f_{m+1},\dots,f_N$ near $z=(0,\dots,0)\in C^m$, such that

$$X=\{(Z_1,\dots,Z_m,f_{m+1}(Z_1,\dots,Z_m),\dots,f_N(Z_1,\dots,Z_m))\}$$

locally at origin of C^N. Since $T_0X=\{(Z_1,\dots,Z_m,0,\dots,0)\in C^N\}$, $f_j=0$, $df_j=0$ at origin for $j=m+1,\dots,N$, so

$$\sum_{j=m+1}^{N}|f_j(Z_1,\dots,Z_m)|^2=O((|Z_1|^2+\dots+|Z_m|^2)^2).$$

On the other hand, $\left|\omega_g-\frac{1}{\pi}\beta\right|=O((|Z_1|^2+\dots+|Z_m|^2)^2)$ locally at origin

$$\left|\sum_{i=1}^{N}|Z_i|^2-\log\left(1+\sum_{i=1}^{N}|Z_i|^2\right)\right|=O((|Z_1|^2+\dots+|Z_N|^2)^2).$$

Hence, there exists a function $\varepsilon(r)$, such that $\varepsilon(r)>0$, $\varepsilon(r)\to 0$ as $r\to 0$ and

$$B_{r(1-\varepsilon(r))}(0)\subset B_g(0,r)\subset B_{r(1+\varepsilon(r))}(0),\qquad \frac{(1-\varepsilon(r))}{\pi}\beta\leqq\omega_g\leqq\frac{1+\varepsilon(r)}{\pi}\beta,$$

where $B_g(0,r)$ is the geodesic ball of (M,g) at 0.

$$\begin{aligned}
L_g(u,z)&=\lim_{r\to 0}\frac{1}{r^{2m-2}}\int_{B_g(0,r)} u\wedge\omega_g^{m-1}\leqq\lim_{r\to 0}\frac{(1+\varepsilon(r))^{m-1}}{(\pi r^2)^{m-1}}\int_{B_{r(1+\varepsilon(r))}(0)} u\wedge\beta^{m-1}\\
&=\lim_{r\to 0}\frac{1}{(\pi(1+\varepsilon(r))^2r^2)^{m-1}}\int_{B_{r(1+\varepsilon(r))}(0)} u\wedge\beta^{m-1}=\lim_{r\to 0}\frac{1}{(\pi r^2)^{m-1}}\int_{B_r(0)} u\wedge\beta^{m-1}\\
&=\lim_{r\to 0}\frac{\left(\frac{1}{1-\varepsilon(r)}\right)^{m-1}}{r^{2m-2}}\int_{B_g\left(0,\frac{r}{1-\varepsilon(r)}\right)} u\wedge\omega_g^{m-1}=\lim_{r\to 0}\frac{1}{r^{2m-2}}\int_{B_g(0,r)} u\wedge\omega_g^{m-1}.
\end{aligned}$$

Hence,

$$L_g(u,z)\leqq\int_M u\wedge\omega_g^{m-1}.\quad\square$$

Recall the Monge-Ampére equations $(*)_t$

$$\left.\begin{aligned}&\det(g_{i\bar j}+\varphi_{i\bar j})=\det(g_{i\bar j})e^{F-t\varphi}\\ &(g_{i\bar j}+\varphi_{i\bar j})>0\end{aligned}\right\}\quad\text{on } M, \tag{$*$}_t$$

where g is a Kähler metric on M with $C_1(M)>0$, $g\sim C_1(M)$, $\frac{\sqrt{-1}}{2\pi}\mathrm{Ric}(g)$ $=\omega_g+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}F$. For any solution φ_t of $(*)_t$, $g_{i\bar{j}}+\varphi_{ti\bar{j}}=g_{ti\bar{j}}$ define a new Kähler metric on M, the Ricci curvature $\mathrm{Ric}(g_t)=tg_t+(1-t)g>tg_g$. It is actually proved in [T] that $(*)_t$ is solvable for $t<\frac{m+1}{m}\alpha(M)$.

Theorem 5.3. *If M is a compact Kähler manifold with $C_1(M)>0$, then M admits a Kähler metric $g_M\sim C_1(M)$, with $\mathrm{Ric}(g_M)>C(m)$, where $C(m)$ depends only on the dimension m and $C_1(M)^m$.*

Proof. By the arguments in [M], one can prove that in case $C_1(M)>0$, there is an integer $N>0$, depending only on the upper bound of $C_1(M)^m$, such that $(-K_M)^N$ gives an embedding of M into certain projective space, let g_F be the restriction of the Fubini-Study metric of the projective space to M, then $\omega_{g_F}=NC_1(M)$, the metric $g\sim C_1(M)$ may be taken as $\frac{1}{N}g_F$. Then by Theorem 5.2,

$$L_g(u,z)=L_{g_F}(u,z)\leqq\int_M u\wedge(NC_1(M))^{m-1}=N^{m-1}C_1(M)^{m-1}$$

for any $z\in M$ and d-closed, positive $(1,1)$-current $u\sim C_1(M)$. It follows (Theorem 1.5) that $\alpha(M)\geqq\frac{1}{N^{m-1}C_1(M)^m}$. By the previous remark before the statement of the theorem, we finish the proof. □

In algebraic geometry, there is a famous conjecture that for each m there exists a $C(m)>0$ such that $C_1(M)^m\leqq C(m)$ for any algebraic manifold M with $C_1(M)>0$ and $\dim_{\mathbb{C}}M=m$. The conjecture is true trivially in case $m=1,2$. In case $m=3$, such a $C(3)$ exists, as proved by L'Vouskiĭ in [Lv]. Moreover, for $m\leqq 3$, $C(m)$ is just $C_1(CP^m)^m$. It is still unknown for the cases $m\geqq 4$. If the conjecture is true, Corollary 4.4 says that for each dimension m, there is a uniform constant $\frac{1}{C(m)}>0$, such that any Kähler manifold M with $C_1(M)>0$ admits a Kähler metric $g_M\sim C_1(M)$ with $\mathrm{Ric}(g_M)>\frac{1}{C(m)}$. Conversely, by the comparison theorem on volume, a uniform lower bound of $\mathrm{Ric}(g_M)$ will result in a uniform estimate of $\mathrm{Vol}(g_M)$, which is nothing but $C_1(M)^m$. Hence, we have built the equivalence of the conjecture in algebraic geometry and the existence of certain Kähler metrics with Ricci curvatures bounded uniformly from below.

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