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Hyper-Kähler Metrics and Harmonic Superspace

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Abstract. The most general unconstrained superfield action for self-interacting $N=2$ matter hypermultiplets in analytic $N=2$ superspace is argued to produce a most general $N=2$ hyper-Kähler σ-model after eliminating an infinite set of auxiliary fields. This suggests a new possibility of classifying hyper-Kähler metrics according to the $N=2$ analytic superfield self-interactions and provides an effective tool to compute these metrics explicitly. As the simplest example the U(2)-invariant quartic self-coupling of a single q-hypermultiplet is analyzed and is shown to yield the familiar Taub-NUT metric. To see the geometric pattern directly in terms of $N=2$ superfields we introduce a new on-shell representation of q-hypermultiplets in $N=2$ harmonic superspace similar to the τ-description of $N=2$ gauge theories. For the U(2)-example this formulation is checked to coincide with that by Sierra and Townsend.

1. Introduction

Supersymmetry severely restricts a form of matter self-couplings. The scalar fields of any supersymmetric matter theory[1] in four dimensions are described by nonlinear σ-models, Kählerian in the $N=1$ case [1], hyper-Kählerian in the rigid $N=2$ case [2] and quaternionic in the local one [3]. These remarkable geometric properties are to be revealed most transparently within manifestly supersymmetric formulations based on unconstrained off-shell superfields. Indeed, any admissible superfield self-interactions should necessarily lead to the above-mentioned σ-models.

There is an exhaustive description of the Kähler geometry of $N=1$ matter in superspace [1, 4]. The bosonic manifold metric was shown to be related in a simple way to the superfield Lagrangian. These results were successfully used in phenomenological applications in $N=1$ supersymmetric GUT's [5]. Now attempts of utilization of $N=2$ supersymmetry getting started (see e.g. [6]). Until the last year $N=2$ matter Lagrangians have been constructed either at the

1 We mean the supermultiplets with the propagating spins 0, 1/2

component level [2, 3, 7] or in terms of $N=1$ superfields [8] with at most one manifest supersymmetry. In the latter case an off-shell formulation was achieved for some hyper-Kählerian σ-models, and some new hyper-Kählerian metrics were found. However, $N=0$ and $N=1$ formulations give no recipes how to write down general $N=2$ supersymmetric Lagrangians so as to automatically get hyper-Kählerian metrics for scalar fields. Therefore it is highly desirable to have a complete $N=2$ superspace description both having in mind future phenomenological applications and purely mathematical reasons. Indeed, manifest $N=2$ supersymmetry opens a way to explicitly construct hyper-Kählerian metrics (even for the simplest, 4-dimensional manifolds metrics are not known in a number of important cases, including the famous K_3-manifold).

In [9] we have developed a manifestly $N=2$ supersymmetric off-shell description of self-interacting $N=2$ matter[2] (q and ω-hypermultiplets) in harmonic superspace in terms of unconstrained analytic $N=2$ superfields. Thus, listing all the possible hypermultiplet self-couplings we may, in principle, list all possible hyper-Kählerian metrics and find their explicit form.

In the present paper we do the first steps in this direction and compute the metric for the simplest U(2) invariant quartic self-interaction of a q-hypermultiplet. The problem of finding the metric amounts to eliminating an infinite number of auxiliary fields. In the case under consideration we obtain the known hyper-Kählerian Taub-NUT metric. Details of computation are given in Sects. 2, 3. To make closer contact with the hyper-Kähler geometry we pass in Sect. 4 to another equivalent representation of self-interacting q-hypermultiplet which reveals unexpected analogies with the τ-description of $N=2$ Yang-Mills theory [9, 10]. For the U(2)-example we recover in this way the on-shell constrained $N=2$ superfield formulation of supersymmetric hyper-Kählerian σ-models given by Sierra and Townsend [11] within which the hyper-Kahler properties are manifest (Sect. 5). Section 6 contains a discussion of the most general self-coupling of hypermultiplets based on the dimensionality and analyticity considerations. We conjecture that action is an analytic superspace integral of arbitrary analytic Lagrange density. Finally, the appendix treats a general harmonic conservation law which may be of use in future calculations of bosonic metrics for more complicated hypermultiplet self-couplings.

2

In the present section we compute the bosonic metric associated with the U(2)-invariant self-coupling of a single q-hypermultiplet. The harmonic superspace action and the corresponding equations of motion are [9]:

$$S=\int dz_A^{(-4)}du\left[\overset{*}{q}{}^{+}D^{++}q^{+}+\frac{\lambda}{2}(q^{+})^{2}(\overset{*}{q}{}^{+})^{2}\right], \tag{2.1}$$

$$D^{++}q^{+}+\lambda(q^{+}\overset{*}{q}{}^{+})q^{+}=0\,,\qquad D^{++}\overset{*}{q}{}^{+}-\lambda(q^{+}\overset{*}{q}{}^{+})\overset{*}{q}{}^{+}=0\,. \tag{2.2}$$

2 As well as $N=2$ Yang-Mills and supergravity theories

Here[3] q^+ is a complex unconstrained $N=2$ superfield defined on the analytic $N=2$ superspace $\{z_A, u_i^\pm\} = \{x_A^{\alpha\dot\alpha}, \theta_\alpha^+, \bar\theta_{\dot\alpha}^+, u_i^\pm\}$, $q^+ = q^+(z_A, u)$, $\overset{*}{}$ is the analyticity preserving conjugation ($(\overset{*}{q}{}^+)^{*} = -q^+$), and D^{++} is the harmonic derivative in the analytic basis:

$$D^{++} = \partial^{++} - 2i\theta^+\sigma^a\bar\theta^+\partial_a\,, \qquad \partial^{++} \equiv u^{+i}\frac{\partial}{\partial u^{-i}}\,. \tag{2.3}$$

Besides the standard SU(2) invariance realized in $N=2$ superspace [9], this model has U(1) invariance,

$$q^{+\prime} = e^{i\alpha}q^+\,, \qquad \overset{*}{q}{}^{+\prime} = e^{-i\alpha}\overset{*}{q}{}^+\,, \tag{2.4}$$

leading to the conserved Noether current j^{++},

$$D^{++}j^{++} = 0\,, \qquad j^{++} = iq^+\overset{*}{q}{}^+\,. \tag{2.5}$$

This U(1) invariance will substantially simplify the computation of metric.

Since we are interested in the pure bosonic part of the action, we may omit the fermions in the $\theta^+, \bar\theta^+$ expansion of q^+,

$$\begin{aligned} q^+(z_A, u) = {} & F^+(x_A, u) + i\theta^+\sigma^a\bar\theta^+A_a^-(x_A, u) + \theta^+\theta^+M^-(x_A, u) \\ & + \bar\theta^+\bar\theta^+N^-(x_A, u) + \theta^+\theta^+\bar\theta^+\bar\theta^+P^{(-3)}(x_A, u)\,. \end{aligned} \tag{2.6}$$

Substituting this into (2.2), one gets the equations of motion in (x_A, u) space:

$$\partial^{++}F^+ + \lambda(F^+\overset{*}{F}{}^+)F^+ = 0\,, \tag{2.7a}$$

$$\partial^{++}A_a^- - 2\partial_aF^+ + \lambda F^+\overset{*}{F}{}^+A_a^- + \lambda(F^+)^2\overset{*}{A}{}_a^- = 0\,, \tag{2.7b}$$

$$\partial^{++}M^- + \lambda(F^+)^2\overset{*}{N}{}^- + 2\lambda F^+\overset{*}{F}{}^+M^- = 0\,, \tag{2.7c}$$

$$\partial^{++}N^- + \lambda(F^+)^2\overset{*}{M}{}^- + 2\lambda F^+\overset{*}{F}{}^+N^- = 0\,, \tag{2.7d}$$

$$\begin{aligned} & \partial^{++}P^{(-3)} + \partial^aA_a^- + \lambda(F^+)^2\overset{*}{P}{}^{(-3)} + 2\lambda F^+\overset{*}{F}{}^+P^{(-3)} - \frac{\lambda}{2}A^{-a}A_a^-\overset{*}{F}{}^+ \\ & - \lambda A^{-a}\overset{*}{A}{}_a^-F^+ + 2\lambda\overset{*}{F}{}^+M^-N^- + 2\lambda F^+(M^-\overset{*}{M}{}^- + N^-\overset{*}{N}{}^-) = 0\,. \end{aligned} \tag{2.7e}$$

All these equations except (2.7e) are kinematical and serve to eliminate an infinite tail of auxiliary fields appearing in harmonic expansion with respect to $u_i^\pm$. The last equation contains dynamics and hence will not be used in what follows.

Now we integrate in (2.1) over $\theta^+, \bar\theta^+$ using Eqs. (2.6), (2.7a–d). Contributions proportional to M^-, N^-, and $P^{(-3)}$ drop out, and the bosonic action reduces to

$$S_B = \tfrac{1}{2}\int d^4x\, du(\overset{*}{A}{}_a^-\partial^aF^+ - A_a^-\partial^a\overset{*}{F}{}^+)\,, \tag{2.8}$$

where $F^+(x, u)$ and $A_a^-(x, u)$ obey Eqs. (2.7a, b). The latter are easily solved due to U(1) invariance (2.4). Indeed the conservation law (2.5) implies $\partial^{++}(F^+\overset{*}{F}{}^+) = 0$. Whence

$$\begin{aligned} & F^+(x, u)\overset{*}{F}{}^+(x, u) = C^{(ij)}(x)u_i^+u_j^+\,, \\ & (F^+\overset{*}{F}{}^+)^{*} = -F^+\overset{*}{F}{}^+ \Rightarrow \overline{C^{(ij)}} = -\varepsilon_{i\ell}\varepsilon_{jn}C^{(\ell n)}\,. \end{aligned} \tag{2.9}$$

This suggests the following change of variables

$$F^+(x, u) = f^+(x, u)e^{\lambda\varphi}\,, \qquad \varphi(x, u) = -C^{(ij)}(x)u_i^+u_j^- = -\overset{*}{\varphi}(x, u)\,, \tag{2.10}$$

3 For the notation and details concerning harmonic superspace, see [9, 10]

which reduces (2.7a) to the linear equation

$$\partial^{++} f^{+}(x,u)=0 \Rightarrow f^{+}(x,u)=f^{i}(x)u_i^{+}\,. \tag{2.11}$$

Taking into account that

$$F^{+}\overset{*}{\tilde{F}}{}^{+}=f^{+}\overset{*}{\tilde{f}}{}^{*} \Rightarrow C^{(ij)}(x)=-f^{(i}(x)\bar{f}^{j)}(x)\,, \tag{2.12}$$

where $\bar{f}^{i}=\varepsilon^{ij}\bar{f}_j$, $\bar{f}_j\equiv(\overline{f^{j}})$, we obtain the general solution of (2.7a) in the form

$$F^{+}(x,u)=f^{i}(x)u_i^{+}\cdot e^{\lambda\varphi}=f^{i}(x)u_i^{+}\exp(\lambda f^{(j}(x)\bar{f}^{k)}(x)u_j^{+}u_\kappa^{-})\,. \tag{2.13}$$

Thus, all the components in the $u_i^{\pm}$-expansion of $F^{+}(x,u)$ are expressed in terms of $f^{i}(x)$ which is the physical bosonic field.

The remaining Eq. (2.7b) is simplified by the substitution $A_a^{-}(x,u)=B_a^{-}(x,u)e^{\lambda\varphi}$. Equation (2.7b) implies that harmonic expansion of B_a^{-} contains only linear ($\sim u^{-}$) and trilinear ($\sim u^{-}u^{-}u^{+}$) terms. Finally,

$$A_a^{-}=e^{\lambda\varphi}\left\{2\lambda f^{i}u_i^{+}\partial_a(f^{(k}\bar{f}^{j)}u_k^{-}u_j^{-})+2\partial_a f^{i}u_i^{-}\right.$$
$$\left.+\frac{\lambda f^{i}u_i^{-}}{1+\lambda f\bar{f}}(f^{j}\partial_a\bar{f}_j-\bar{f}_j\partial_a f^{j})\right\}. \tag{2.14}$$

Let us emphasize once more that this simple form of the solution is due to U(2) invariance of the action (2.1). More general self-couplings lead to much more complicated equations (see Sect. 4).

To find the action in terms of $f^{i}(x)$, we integrate (2.8) over $u_i^{\pm}$ using (2.13), (2.14), the reduction identities [9]

$$u_i^{+}u_{(j_1}^{+}\dots u_{j_n}^{+}u_{k_1}^{-}\dots u_{k_m)}^{-}=u_{(i}^{+}u_{j_1}^{+}\dots u_{k_m)}^{-}+\frac{m}{m+n+1}\varepsilon_{i(k_1}u_{j_1}^{+}\dots u_{j_n}^{+}u_{k_2}^{-}\dots u_{k_m)}^{-}\,,$$

$$u_i^{-}u_{(j_1}^{+}\dots u_{j_n}^{+}u_{k_1}^{-}\dots u_{k_m)}^{-}=u_{(i}^{-}u_{j_1}^{+}\dots u_{k_m)}^{-}-\frac{n}{m+n+1}\varepsilon_{i(j_1}u_{j_2}^{+}\dots u_{k_m)}^{-}$$

and the $u_i^{\pm}$ integration rules [9]

$$\int du(u^{+})^{(m}(u^{-})^{n)}(u^{+})_{(k}(u^{-})_{\ell)}=\begin{cases}\dfrac{(-1)^n m!\,n!}{(m+n+1)!}\delta_{(j_1}^{(i_1}\dots\delta_{j_{k+l})}^{i_{m+n})} & \text{if } \begin{matrix} m=\ell \\ n=k\end{matrix}, \\ 0 & \text{otherwise}\end{cases}$$

$$(u^{+})^{(m}(u^{-})^{n)}\equiv u^{+(i_1}\dots u^{+i_m}u^{-j_1}\dots u^{-j_n)}\,.$$

As a result, we arrive at the following bosonic action

$$S_B=-\tfrac{1}{2}\int d^4x(g_{ij}\partial_a f^{i}\partial^{a}f^{j}+\bar{g}^{ij}\partial_a\bar{f}_i\partial^{a}\bar{f}_j+2h_j^{i}\partial_a f^{j}\partial^{a}\bar{f}_i)\,, \tag{2.15}$$

where

$$\begin{aligned} g_{ij}&=\frac{\lambda(2+\lambda f\bar{f})}{2(1+\lambda f\bar{f})}\bar{f}_i\bar{f}_j\,,\qquad \bar{g}^{ij}=\frac{\lambda(2+\lambda f\bar{f})}{2(1+\lambda f\bar{f})}f^{i}f^{j}\,,\\ h_j^{i}&=\delta_j^{i}(1+\lambda f\bar{f})-\frac{\lambda(2+\lambda f\bar{f})}{2(1+\lambda f\bar{f})}f^{i}\bar{f}_j\,;\qquad f\bar{f}\equiv f^{i}\bar{f}_i\,. \end{aligned} \tag{2.16}$$

It is remarkable that the extremely simple monomial $N=2$ superfield interaction (2.1) entails a complicated nonpolynomial Lagrangian for the physical bosons. Note the manifest U(2)-invariance of (2.15), (2.16), which reflects the U(2)-invariance of the original action.

3

It is not so easy to see that the metric (2.16) is hyper-Kählerian, especially, because it is not manifestly Kählerian in coordinates $f^i, \bar{f}_i$. By simple (though lengthy) calculations one can verify that it is Ricci-flat. However, it is the necessary condition, not the sufficient one. One should also pick up three linearly independent covariantly constant complex structures and this is less trivial. In Sect. 5 we shall visualize these geometric properties of metric (2.6) by passing to the new, τ-representation of Eqs. (2.2). Here we prefer to proceed in a different way. Namely, we demonstrate that (2.15) is reduced by a change of variables to the well-known Taub-NUT metric, which belongs to the class of four-dimensional Euclidean gravitational instantons and is known to be hyper-Kähler.

To this end, let us first introduce "spherical" coordinates in the R^4-space $\{f^i, \bar{f}_i\}$:

$$
\begin{aligned}
f^1 &= \varrho \cos\frac{\theta}{2} \cdot \exp\frac{i}{2}(\psi+\varphi)\,,\\
f^2 &= \varrho \sin\frac{\theta}{2} \exp\frac{i}{2}(\psi-\psi)\,, \qquad f\bar{f} = \varrho^2\,.
\end{aligned}
\tag{3.1}
$$

Then

$$
\begin{aligned}
ds^2 &= g_{ij} df^i df^j + \bar{g}^{ij} d\bar{f}_i d\bar{f}_j + 2h^i_j df^j d\bar{f}_i\\
&= 2(1+\lambda\varrho^2) d\varrho^2 + \tfrac{1}{2}\varrho^2(1+\lambda\varrho^2)(d\theta^2 + \sin^2\theta d\varphi^2)\\
&\quad + \frac{\varrho^2}{2(1+\lambda\varrho^2)}(d\psi + \cos\theta d\varphi)^2\,.
\end{aligned}
\tag{3.2}
$$

We assume that (3.2) has no singularities in ϱ, so $\lambda > 0$. Then one makes a change of variables

$$
\varrho^2 = 2(r-m)m\,, \qquad r \geqq m = \frac{1}{2\sqrt{\lambda}}\,,
\tag{3.3}
$$

recasting ds^2 in the form

$$
\begin{aligned}
ds^2 = 2\Bigg\{ &\frac{1}{4}\frac{r+m}{r-m} dr^2 + \frac{1}{4}(r^2-m^2)(d\theta^2 + \sin^2\theta d\varphi^2)\\
&+ m^2 \frac{r-m}{r+m}(d\psi + \cos\theta d\varphi)^2 \Bigg\}\,,
\end{aligned}
\tag{3.4}
$$

which, up to a numerical coefficient, is the standard Taub-NUT metric (see, e.g. [12]).

4

We have seen above that the hyper-Kähler geometry in the $N=2$ analytic superspace description arises only upon eliminating an infinite tower of auxiliary fields, i.e. with partially putting the theory on-shell[4]. One may inquire how to expose the hyper-Kähler properties directly in terms of $N=2$ superfields. Clearly, it should essentially involve the use of superfield equations of motion. Here we derive another on-shell superfield representation of self-interacting q-hypermultiplets in which the geometry is expected to reveal itself more transparently and which bears an interesting analogy with the τ-description of $N=2$ gauge theory [9, 10].

For simplicity we restrict our study to a single q^+ self-interacting in a manifestly SU(2)-invariant manner (the general case will be treated elsewhere). The action is

$$S_q=\int dz_A^{(-4)}du(\overset{*}{\bar{q}}{}^+ D^{++}q^+ + \mathscr{L}_{\text{int}}^{(+4)})\,, \tag{4.1}$$

$$\mathscr{L}_{\text{int}}^{(+4)}=\frac{\lambda_1}{2}(q^+)^2(\overset{*}{\bar{q}}{}^+)^2+\lambda_2(q^+)^3\overset{*}{\bar{q}}{}^+-\bar{\lambda}_2(\overset{*}{\bar{q}}{}^+)^3q^+ +\lambda_3(q^+)^4+\bar{\lambda}_3(\overset{*}{\bar{q}}{}^+)^4\,. \tag{4.2}$$

Note that the kinetic term in (4.1) is invariant under some extra $\widetilde{\mathrm{SU}}(2)$ group [containing the U(1)-subgroup (2.4)] which is an analogue of the known Pauli-Gürsey group. With respect to this group q^+ and $\overset{*}{\bar{q}}{}^+$ form an isodoublet. If

$$q^{+a}\equiv(q^+,\,-\overset{*}{\bar{q}}{}^+)\,,\qquad (\overset{*}{\overline{q^{+a}}})=\overset{*}{\bar{q}}{}_a^+=-\varepsilon_{ab}q^{+b}\,, \tag{4.3}$$

then the kinetic term can be written in the form

$$\tfrac{1}{2}(\overset{*}{\bar{q}}{}^+D^{++}q^+-q^+D^{++}\overset{*}{\bar{q}}{}^+)=\tfrac{1}{2}q^{+a}D^{++}q_a^+\,. \tag{4.4}$$

Though self-couplings in (4.1) break this SU(2) symmetry, the $\widetilde{\mathrm{SU}}(2)$-notation is useful in that it allows one to write the equations of motion in a compact 2×2 matrix form[5]

$$(D^{++}+iV^{++})q^+=[D^{++}\delta_a^b+i(V^{++})_a^b]q_b^+=0\,, \tag{4.5}$$

$$V^{++}=\overset{*}{\overline{V^{++}}}=\frac{1}{i}\begin{pmatrix} -\lambda_1q^+\overset{*}{\bar{q}}{}^+-\lambda_2(q^+)^2+\bar{\lambda}_2(\overset{*}{\bar{q}}{}^+)^2 & -2\lambda_2q^+\overset{*}{\bar{q}}{}^+-4\lambda_3(q^+)^2 \\ -2\bar{\lambda}_2q^+\overset{*}{\bar{q}}{}^+ +4\bar{\lambda}_3(\overset{*}{\bar{q}}{}^+)^2 & \lambda_1q^+\overset{*}{\bar{q}}{}^+-\bar{\lambda}_2(\overset{*}{\bar{q}}{}^+)^2+\lambda_2(q^+)^2 \end{pmatrix}. \tag{4.5a}$$

Let us also recall the analyticity conditions

$$D_\alpha^+q^{+a}=0\,,\qquad \bar{D}_{\dot\alpha}^+q^{+a}=0\,. \tag{4.6}$$

The quantity V^{++}, being a real analytic superfield in the adjoint representation of SU(2), can be regarded as a composite $N=2$ Yang-Mills prepotential [9]. Correspondingly, Eq. (4.5) is similar to the equation for the "bridge" between λ- and τ-representations of $N=2$ gauge theory [9], Eq. (IV.16b)). This suggests the

4 This has to be compared with the $N=1$ case where the Kähler properties are manifest already at the level of off-shell $N=1$ superfield action. Elimination of auxiliary fields there does not influence the form of the bosonic Lagrangian

5 Besides this, extra $\widetilde{\mathrm{SU}}(2)$ group effectively reduces also the number of independent coupling constants in (4.2) from 5 to 2

following substitution for q^+:

$$q^+ = e^{iv}\tilde{q}^+ , \tag{4.7}$$

$$(D^{++} + iV^{++})e^{iv} = 0 , \quad \text{or} \quad V^{++} = -ie^{iv}D^{++}e^{-iv} . \tag{4.8}$$

In terms of $\tilde{q}^+$, Eqs. (4.5), (4.6) reduce to

$$D^{++}\tilde{q}^+ = 0 \Rightarrow \tilde{q}^{+a} = \tilde{q}^{ia}(z)u_i^+ , \tag{4.9}$$

$$\mathscr{D}^+_{\alpha(\dot{\alpha})}\tilde{q}^+ \equiv (D^+_{\alpha(\dot{\alpha})} + iA^+_{\alpha(\dot{\alpha})})\tilde{q}^+ = 0 , \tag{4.10}$$

$$A^+_{\alpha(\dot{\alpha})} \equiv -ie^{-iv}D^+_{\alpha(\dot{\alpha})}e^{iv} = A^i_{\alpha(\dot{\alpha})}(z)u_i^+ . \tag{4.11}$$

Thus, we arrive at the on-shell description of the self-interacting hypermultiplet in terms of the ordinary $N=2$ superfield $\tilde{q}^i(z)$ constrained by the "covariantized" analyticity conditions (4.10). It directly generalizes the standard $N=2$ superfield formulation of a free hypermultiplet [9] and is related to the original analytic superspace description given by Eqs. (4.5), (4.6) like the τ-representation of $N=2$ Yang-Mills is related to the λ one [9]. In the q^+-language, analyticity is purely kinematic while the dynamics is concentrated in Eq. (4.5) which can be interpreted as the condition of "covariant" $u_i^\pm$-independence of q^+. On the contrary, in the $\tilde{q}^+$-language, the notion of $u_i^\pm$-independence is kinematic. The theory is put on-shell by the constrains (4.10) stating that $\tilde{q}^+$ is "covariantly" analytic.

Let us emphasize that Eq. (4.8) in different descriptions comes out as a definition of different objects. In the λ-description it defines the bridge e^{iv} while in the τ-description it defines the "prepotential" V^{++}. The expression of e^{iv} in forms of V^{++} can be obtained iteratively, by a general recipe given by us for the $N=2$ gauge theory [10]. This solution is nonlocal in harmonics and is independent of a specific form of V^{++}.

Note that again in a close analogy with the $N=2$ Yang-Mills theory [13, 9] we may define the τ-representation of q^+-hypermultiplet in more abstract terms, namely, by adding to Eq. (4.10) the constraints

$$\begin{aligned} &\{\mathscr{D}^+_\alpha, \mathscr{D}^+_\beta\} = \{\bar{\mathscr{D}}^+_{\dot{\alpha}}, \bar{\mathscr{D}}^+_{\dot{\beta}}\} = \{\mathscr{D}^+_\alpha, \bar{\mathscr{D}}^+_{\dot{\beta}}\} = 0 , \\ &[D^{++}, \mathscr{D}^+_\alpha] = [D^{++}, \bar{\mathscr{D}}^+_{\dot{\alpha}}] = 0 . \end{aligned} \tag{4.12}$$

Equations (4.10), with any $A^+_{\alpha(\dot{\alpha})}$ composed of $\tilde{q}^+$ and satisfying (4.12), are reduced after the redefinitions (4.7), (4.8) to the manifestly analytic Eqs. (4.5), (4.6), However, for (4.5) to be derivable from an action, V^{++} and, respectively, e^{iv} and $A^+_{\alpha(\dot{\alpha})}$ have to obey certain integrability conditions whose implications are not clear to us at the moment.

These considerations can be easily extended to the case of n hypermultiplets. Superfield q^{+a} (4.3) then acquires additional indices and so do V^{++} and $A^+_{\alpha(\dot{\alpha})}$, which become $2n \times 2n$ matrices.

In the next section the usefulness of the τ-representation will be illustrated by the $(q^+)^2(\overset{*}{\tilde{q}}{}^+)^2$-example.

5

In the τ-description of q^+ proposed above the basic geometric object is the composite spinor connection $A^+_{\alpha(\dot{\alpha})}(\tilde{q})$ restricted by the constraints (4.12), (4.10). On the other hand, Sierra and Townsend [11] have given a different on-shell superfield formulation of q^+-hypermultiplet, also in terms of the ordinary constrained $N=2$ superfields. For one hypermultiplet their constraint is as follows [11]:

$$E_{jb}^{(ia}(\tilde{q})D_{\alpha(\dot{\alpha})}^{k)}\tilde{q}^{jb}(z)=0\,, \qquad \bar{\tilde{q}}_{jb}=\varepsilon_{j\ell}\,\varepsilon_{bc}q^{\ell c}$$
$$(i,j=1,2;\ a,b=1,2)\,, \tag{5.1}$$

where the real superfields q^{jb} are assumed to parametrize a four-dimensional real Riemann space, $E_{jb}^{iq}(\tilde{q})$ is the corresponding inverse vielbein with the world indices jb and the tangent space indices ia. In terms of E_{jb}^{ia} the hyper-Kählerian geometry of self-interacting hypermultiplet manifests itself most clearly [11]. So it would be desirable to put our constraints (4.9)–(4.11) in the form (5.1). For the time being, we do not know whether it is always possible [the σ-models associated with the constraint (5.1) seem to require the SU(2) automorphism group to be unbroken while Eqs. (4.9)–(4.11) do not imply such a restriction]. Our aim here is to explicitly demonstrate that for the U(2)-case treated above this equivalence really takes place.

The relevant V^{++} is diagonal

$$V^{++}=\frac{1}{i}\begin{pmatrix}\lambda q^+\overset{*}{\tilde{q}}{}^+ & 0\\ 0 & -\lambda q^+\overset{*}{\tilde{q}}{}^+\end{pmatrix}, \tag{5.2}$$

and there arises an analogy with the Abelian $N=2$ gauge theory. Such a simplification allows us to obtain the bridge and spinor connections in a closed form

$$v=\frac{i\lambda}{2}(\tilde{q}^+\overset{*}{\tilde{q}}{}^-+\tilde{q}^-\overset{*}{\tilde{q}}{}^+)\,, \qquad \tilde{q}^{\pm}=\tilde{q}^i(z)u_i^{\pm}\,, \qquad \overset{*}{\tilde{q}}{}^{\pm}=-\bar{\tilde{q}}^i(z)u_i^{\pm}\,, \tag{5.3}$$

$$A_{\alpha(\dot{\alpha})}^{+}=\begin{pmatrix}D_{\alpha(\dot{\alpha})}^{+}v & 0\\ 0 & -D_{\alpha(\dot{\alpha})}^{+}v\end{pmatrix}. \tag{5.4}$$

Note that

$$\tilde{q}^i(z)|_{\theta=0}=f^i(x)\,, \qquad iv|_{\theta=0}=\lambda\varphi(x,u)\,,$$

where f^i and φ are the same as in Eqs. (2.9), (2.10).

By means of some easy algebra the constraints (4.10) with $A_{\alpha(\dot{\alpha})}^{+}$ (5.4) can be reduced to

$$E_{kb}^{+a}D_{\alpha(\dot{\alpha})}^{+}\tilde{q}^{kb}(z)=0\,, \qquad E_{kb}^{+a}=E_{kb}^{ia}(z)u_i^{+}\,, \tag{5.5}$$

$$E_{kb}^{ia}=\begin{pmatrix}E_{k1}^{i1} & E_{k1}^{i2}\\ E_{k2}^{i1} & E_{k2}^{i2}\end{pmatrix}=\begin{pmatrix}\delta_k^i\left(1+\frac{\lambda}{2}\tilde{q}\bar{\tilde{q}}\right)-\frac{\lambda}{2}\tilde{q}_k\bar{\tilde{q}}^i & \frac{\lambda}{2}\bar{\tilde{q}}_k\bar{\tilde{q}}^i\\ -\frac{\lambda}{2}\tilde{q}_k\tilde{q}^i & \delta_k^i\left(1+\frac{\lambda}{2}\tilde{q}\bar{\tilde{q}}\right)+\frac{\lambda}{2}\bar{\tilde{q}}_k\tilde{q}^i\end{pmatrix}. \tag{5.6}$$

Taking off the zweibeins u_i^+ from the right-hand side of (5.5) we may cast the latter equation just into the form (5.1). To achieve a complete agreement with [11], one should also take into account a freedom of rescaling (5.5) by a scalar function of $\tilde{q}$. It turns out that the hyper-Kähler properties become manifest in terms of the vielbeins

$$\tilde{E}_{kb}^{ia}=(\det E)^{-1/6}\cdot E_{kb}^{ia}=\frac{1}{(1+\lambda\tilde{q}\bar{\tilde{q}})^{1/2}}E_{kb}^{ia}\,. \tag{5.7}$$

One may explicitly check that the two-forms:

$$\Omega^{ij}=\tilde{E}^{(ia}_{kb}\tilde{E}^{j)c}_{\ell d}\varepsilon_{ac}d\tilde{q}^{kb}d\tilde{q}^{\ell d} \tag{5.8}$$

are closed, constitute a SU(2)-triplet, and are covariantly constant with respect to the connection constructed by the metric

$$q_{ib,ka}=\tilde{E}^{jc}_{ib}\tilde{E}^{ed}_{ka}\varepsilon_{jl}\varepsilon_{cd}\,. \tag{5.9}$$

These properties are just characteristic of a hyper-Kähler manifold. The purely bosonic metric defined as the θ-independent part of (5.9) exactly coincides with (2.16),

$$g_{ik}(f)=\begin{pmatrix} \dfrac{\lambda(2+\lambda\varrho^2)}{2(1+\lambda\varrho^2)}\bar{f}_i\bar{f}_k & \varepsilon_{ik}(1+\lambda\varrho^2)+\dfrac{\lambda(2+\lambda\varrho^2)}{2(1+\lambda\varrho^2)}\bar{f}_i f_k \\ -\varepsilon_{ik}(1+\lambda\varrho^2)+\dfrac{\lambda(2+\lambda\varrho^2)}{2(1+\lambda\varrho^2)}f_i\bar{f}_k & \dfrac{\lambda(2+\lambda\varrho^2)}{2(1+\lambda\varrho^2)}f_i f_k \end{pmatrix}. \tag{5.10}$$

Thus, there exists a possibility to expose the structure of metrics associated with the q-self-couplings also in the τ-representation by passing to the constraints of the form (5.1). One may derive a general formula relating the vielbein E^{ia}_{jb} to the bridge e^{iv}. However, to restore e^{iv} by V^{++} is in general not easier than to compute the metric in the λ-representation. Perhaps, it would be more fruitful to deal at once with the $N=2$ Yang-Mills-like constraints (4.10), without transforming them to the form (5.1) or (and it would be most desirable) to learn how to reveal the geometric structures directly in the λ-representation which provides the natural framework for handling hypermultiplets.

In any case, there remains an actual and interesting task of computing the metrics for other self-couplings of q and ω-hypermultiplets by applying the straightforward method of Sect. 2. In particular, it is an intriguing question which self-coupling corresponds to the more familiar hyper-Kähler metric, that of Equchi and Hanson [14]. It appeared in the early investigations on supersymmetric hyper-Kähler σ-models and, like the Taub-NUT metric, exhibits U(2)-invariance (see Note added in proof).

6

Finally, we discuss the most general self-interactions of hypermultiplets. The dimensionality and analyticity arguments seem to completely determine their form. Indeed let us start with the case $N=0$. The standard $N=0$ σ-model action is

$$S=\frac{1}{\kappa^2}\int d^4x g_{ij}(f)\partial_a f^i\partial^a f^j\,, \tag{6.1}$$

where κ is a coupling constant (dimension mass^{-1}), f^i are dimensionless and are considered as coordinates of some manifold. To pass from f^i to the physical scalar field one has to rescale it as $f^i=\kappa f^i_{\text{phys}}$. The metric $g_{ij}(f)$ is dimensionless and does not explicitly depend on κ.

Correspondingly in the $N=1$ case matter is described by dimensionless chiral superfields ϕ^i ($\phi^i(x,\theta)=f^i+\theta\psi^i+\ldots$) which again play the rôle of coordinates of some (Kähler) manifold. The most general $N=1$ σ-model action is

$$S=\frac{1}{\kappa^2}\int d^4x d^4\theta K(\phi,\bar{\phi})\,. \tag{6.2}$$

The dimensional parameter κ enters again via the factor κ^{-2} and the structure of Lagrange density is controlled by dimension of measure $d^4xd^4\theta$.

As we know $N=2$ matter is represented by $N=2$ analytic superfields $q^+(z_A,u)$ and $\omega(z_A,u)$. Their θ-expansion again begins with geometric fields $f^i(x)$, so q^+ and ω are dimensionless as well. Under the natural assumption that the most general $N=2$ σ-model is formulated via q^+ or ω superfields, the only possible superspace action which results (after elimination of auxiliary fields) in (6.1) is

$$S=\frac{1}{\kappa^2}\int dz_A^{(-4)}du\mathscr{L}^{(+4)}(q^+,\omega,u^\pm,D^{++}q^+,D^{++}\omega,\ldots)\,, \tag{6.3}$$

where $\mathscr{L}^{(+4)}$ is dimensionless function of q^+, ω, harmonics $u^\pm$ and analyticity preserving derivatives $D^{++}q^+$, $D^{++}\omega$, etc. Note that $\mathscr{L}^{(+4)}$ cannot contain harmonic nonlocalitics[6], because for analyticity such terms would inevitably include spinor derivatives $(D^+)^4$. The latter is forbidden by the above dimensionality arguments.

Thus, we conjecture that any hyper-Kählerian σ-model is supersymmetrized to some $\mathscr{L}^{(+4)}$ (6.3). This provides a technique to explicitly compute hyper-Kählerian metrics by choosing a Lagrangian and eliminating auxiliary bosonic fields[7].

Concluding the paper we wish to emphasize the importance of establishing a classification of the hyper-Kähler metrics according to $N=2$ superfield Lagrange densities. The simplest case considered above is an example.

Appendix

We derive here a general conservation law for self-interacting q-hypermultiplets which may be useful in practical calculations.

We start with the most general q-hypermultiplet action containing no more than one harmonic derivative.

$$S=S_{\text{free}}+S_{\text{int}}=\int dz_A^{(-4)}du[\overset{*}{q}{}^{+a}D^{++}q^{+a}+\mathscr{L}_{\text{int}}^{(+4)}(q^+,\overset{*}{q}{}^+,u^\pm,D^{++}q^+)]\,. \tag{A.1}$$

The relevant equations of motion are

$$\begin{aligned}
&D^{++}q^{+a}+\frac{\partial\mathscr{L}_{\text{int}}^{(+4)}}{\partial\overset{*}{q}{}^{+a}}-D^{++}\frac{\partial\mathscr{L}_{\text{int}}^{(+4)}}{\partial(D^{++}\overset{*}{q}{}^{+a})}=0\,,\\
&D^{++}\overset{*}{q}{}^{+a}-\frac{\partial\mathscr{L}_{\text{int}}^{(+4)}}{\partial q^{+a}}+D^{++}\frac{\partial\mathscr{L}_{\text{int}}^{(+4)}}{\partial(D^{++}q^{+a})}=0\,.
\end{aligned} \tag{A.2}$$

6 E.g. like those occurring in the $N=2$ Yang-Mills action [10]

7 Recently Rosly and Schwarz [15] have suggested a geometric action for hyper-Kähler supersymmetric σ-models in the analytic $N=2$ superspace starting with the Sierra-Townsend approach [11], where hyper-Kähler metrics are assumed to be given in advance

Let us compute

$$
\begin{aligned}
D^{++}\mathcal{L}_{\text{int}}^{(+4)} =& \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial q+a} D^{++}q^{+a} + \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial\overset{*}{q}{}^{+a}} D^{++}\overset{*}{q}{}^{+a} + \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial(D^{++}q^{+a})}(D^{++})^2 q^{+a} \\
&+ \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial(D^{++}\overset{*}{q}{}^{+a})}(D^{++})^2\overset{*}{q}{}^{+a} + \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial u^{-i}}\cdot u^{+i} \\
=& D^{++}\left\{\frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial(D^{++}\overset{*}{q}{}^{+a})}\frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial q^{+a}} - \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial(D^{++}q^{+a})}\cdot\frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial\overset{*}{q}{}^{+a}}\right. \\
&\left.+ \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial(D^{++}q^{+a})}\cdot D^{++}\frac{\partial\mathcal{L}^{(+4)}}{\partial(D^{++}\overset{*}{q}{}^{+a})} - \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial(D^{++}\overset{*}{q}{}^{+a})}\cdot D^{++}\frac{\partial\mathcal{L}^{(+4)}}{\partial(D^{++}q^{+a})}\right\} \\
&+ \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial u^{-i}} u^{+i}\,.
\end{aligned}
\tag{A.3}
$$

Using once more Eqs. (A.2), we observe that the quantity

$$
T^{(+4)} = \mathcal{L}_{\text{int}}^{(+4)} - \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial(D^{++}q^{+a})}\cdot D^{++}q^{+a} - \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial(D^{++}\overset{*}{q}{}^{+a})} D^{++}\overset{*}{q}{}^{+a}
\tag{A.4}
$$

obeys the conservation-like identity,

$$
D^{++}T^{(+4)} = \frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial u^{-i}} u^{+i}\,,
\tag{A.5}
$$

which becomes exact if $\mathcal{L}_{\text{int}}^{(+4)}$ contains no explicit u^--dependence.

$$
u^{+i}\frac{\partial\mathcal{L}_{\text{int}}^{(+4)}}{\partial u^{-i}} = 0 \Rightarrow D^{++}T^{(+4)} = 0\,.
\tag{A.6}
$$

Equation (A.6) implies that in coordinates of the central basis,

$$
T^{(+4)} = T^{(ijke)}(z) u_i^+ u_j^+ u_k^+ u_e^+\,.
\tag{A.7}
$$

In the case when $\mathcal{L}_{\text{int}}^{(+4)}$ does not contain derivatives, $T^{(+4)}$ coincides with $\mathcal{L}_{\text{int}}^{(+4)}$. This conservation law is especially simple for U(2)-invariant coupling (2.1):

$$
D^{++}[(q^+\overset{*}{q}{}^+)]^2 = 0 \Rightarrow D^{++}(q^+\overset{*}{q}{}^+) = 0\,.
\tag{A.8}
$$

An interesting point about the conservation law (A.6) is that it can be related by the standard Noether procedure to the invariance of action (A.1) with respect to the following transformations

$$
\begin{aligned}
&\delta u_i^- = c^{--}u_i^+\,, \qquad \delta u_i^+ = 0\,, \\
&\delta x_a^m = -2ic^{--}\theta^+\sigma^m\bar\theta^+\,, \qquad \delta\theta^+ = \delta\bar\theta^+ = 0\,,
\end{aligned}
\tag{A.9}
$$

$$
\delta^* q^+ = -c^{--}D^{++}q^+\,,
\tag{A.10}
$$

provided c^{--} is a double U(1) charged constant independent of u $(D^{++}c^{--} = 0, c^{--} \neq 0)$. Such a constant looks rather unusual. However, one may recall the familiar isospin transformations. Here, e.g. in the transformation of proton via neutron $\delta p^{(+)} = i\alpha^{(+)} n^{(0)}$, parameter $\alpha^{(+)}$ also has an electric charge $+1$. We prefer to postpone a discussion of the exact meaning of transformations (A.9), (A.10) to the future.

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Note added in proof. Recently [15] we have found harmonic superspace actions corresponding to a wide class of hyper-Kähler metrics including multi-Eguchi-Hanson and Calabi ones. In particular, the familiar Eguchi-Hanson metric [14] is described by the following ω-hypermultiplet action

$$S_{EH} = -\frac{1}{4\varkappa^2}\int dz_A^{(-4)} du[(D^{++}\omega)^2 - (\xi^{++})^2\omega^{-2}], \qquad \xi^{++} = \xi^{ij}u_i^+ u_j^+$$

with ξ^{ij} being constants.