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Bernoulli Property for a One-Dimensional System with Localized Interaction

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Abstract. We consider a one-dimensional system of particles on the half line $\mathbb{R}_+ = [0, \infty)$ interacting through elastic collisions among themselves and with a "wall" at the origin. On the first particle a constant force E is acting, no external forces act on the other particles. All particles are identical except the first one which has a larger mass. We prove that if E is such that the Gibbs equilibrium state exists, the corresponding equilibrium dynamical system is a Bernoulli flow.

1. Introduction

Consider the semi-infinite mechanical system consisting of a gas of infinitely many particles on the half line $\mathbb{R}_+ = [0, +\infty)$, interacting through elastic collisions with each other and with a "wall" at the origin. The mass of the first particle (i.e. the one closest to the origin) M is assumed to be larger than the common mass m of the other particles. A constant force $E > 0$ is acting only on the first particle (or "heavy particle," henceforth h.p.). The Gibbs equilibrium measure for all values of the temperature and the particle density such that $E < P$, where P is the thermodynamic pressure that the gas exerts on the wall, is stationary in time. In [2], using techniques introduced in [1], it was proved that the corresponding dynamical system is a Bernoulli flow for $E < P/2$.

In this paper we extend the result to all values of $E < P$, by giving a simpler and more general proof. The main point is that instead of proving loss of memory by explicit probabilistic estimates, as in [1] and in [2], we make use of the following general features of the system: i) the fact that the invariant measure is locally absolutely continuous and the interacting subsystem (i.e. the h.p.) is confined, and ii) the local smoothness of the dynamics, i.e. the phase point of a finite particle system at time t is "almost always" a smooth function of the initial data. Our methods of proof in their present form could be applied to a large class of systems

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of classical statistical mechanics consisting of a confined interacting subsystem in a free gas bath.

We give now a brief description of the main ideas in the proof.

It is easy to see that a point in the infinite particle phase space $\mathscr{X}$ can be determined almost surely by knowing the history of the h.p. $\{q_0(t)\}_{t\in\mathbb{R}^1}$. We prove a condition of asymptotic independence between past and future for the process $\{q_0(t)\}$, which is stronger than α-mixing (or "strong mixing"), and implies that our dynamical system is Bernoulli.

This particular type of "loss of memory" for the process is brought about by the following circumstances, which have been given a mathematical expression already in [1, 2].

Since the interacting subsystem (the h.p.) is confined near the origin, all the incoming (i.e. negative velocity) particles which are far enough did not collide in the past. Therefore the conditional distribution for them, for the condition that the past history of the h.p. is fixed, is essentially given by the equilibrium one. This led in [1] to the proof of the almost sure existence of infinitely many "cluster times" (c.t.'s), i.e. times for which the sets of the particles colliding with the h.p. at earlier and later times are disjoint. C.t.'s are produced by particular configurations of incoming particles, which last long enough to allow the particles that collided at earlier times to escape so far that they cannot be reached by the h.p. The history of the h.p. after a c.t. can be reproduced inside "most" of the atoms of partition of phase space generated by the past history of the h.p., and this implies the K property for the dynamical system [1].

In the paper [2] the Bernoulli property could be proved by relying on a "local" mechanism of loss of memory provided by the dynamics. Namely, if we consider the trajectory of the h.p. $q_0(t)$ for some initial particle configuration x, and the corresponding trajectory $\bar{q}_0(t)$ for the configuration $\bar{x}$ obtained from x by moving the initial position and velocity of the h.p. of a small amount, then the difference $q_0(t)-\bar{q}_0(t)$ becomes smaller in a way that is roughly speaking exponential in the number of collisions up to time t, until maybe a discontinuity occurs (because the order of collisions changes), and the trajectories diverge. This important fact is basic in the construction of an isomorphism with the free gas via the scattering formalism (see [3]).

In the present paper we use this fact, together with the "almost deterministic behavior" of the h.p. when it gets a very high speed, to show that the incoming configuration which produces the c.t. can be chosen in such a way that, for "almost all" initial situations in a finite region near the origin, it drives the h.p. towards an attracting periodic cycle, independent of the initial situation. The property of the process we need can be deduced directly from that (see Sect. 4), thereby simplifying the proof of [2].

It is an unsatisfactory point of both papers [1] and [2] that the final step in the proof of the existence of c.t.'s, which amounts to proving that the h.p. does not catch up with the particles of its past, had to be made by using an explicit probability estimate, for the particular distribution which we get after the c.t. This estimate is based on a momentum balance, which involves the h.p. and the incoming particles (the distribution of which is known to be close to the equilibrium one), and neglects the outgoing particles, since we ignore their distribution. But the average momentum of the incoming particles makes up only a half of the total pressure P, so the argument works only for $E<P/2$.

In this paper we are able to introduce, by a general argument, configurations which, after producing the c.t. situation, approximately restore a typical situation near the origin. This, together with the properties of the measure and of the dynamics described above, allow us to make use of the equilibrium estimate for the confinement of the h.p., and to extend the result to all $E<P$.

The case of a nonconstant force E, subject to some natural conditions, would require only minor changes in the proof.

The plan of the paper is the following. In Sect. 2 we give some definitions, state the main results, and give the main facts about confinement of the h.p. and quasi-deterministic behavior. In Sect. 3 we give our modified construction of cluster situations, and in Sect. 4 we prove and the B-property.

2. Definitions, Statement of the Results and Some Preliminary Facts

The one-particle phase space is $\mathbb{R}_+^2=\{(q,v)\in\mathbb{R}^2: q\geqq 0\}$, where q and v denote particle position and particle velocity, respectively. $\mathscr{X}$ is the space of the locally finite particle configurations in $\mathbb{R}_+^2$, with the usual topology of pointwise convergence. $\mathscr{X}$ is a polish space and the σ-algebra of the Borel subsets of $\mathscr{X}$ will be denoted by $\mathfrak{M}$.

If $x\in\mathscr{X}$ and $A\subset\mathbb{R}_+^2$ is a measurable set, we denote by x_A the configuration $x\cap A$, and by $\mathfrak{M}_A$ the σ-algebra generated by x_A. $\mathscr{X}_A$ will denote the phase space of a particle system in A, which we shall sometimes identify with $\tilde{\mathscr{X}}_A=\{x\in\mathscr{X}: x_{\mathbb{R}_+^2\setminus A}=\emptyset\}$. μ_A will denote the measure induced by the measure μ on $\mathscr{X}_A$. We shall sometimes identify sets of $\mathfrak{M}_A$ and the corresponding subsets of $\mathscr{X}_A$. Since the space is Lebesgue, for any σ-algebra $\mathfrak{M}'\subset\mathfrak{M}$ there is a measurable partition π' associated to it such that the regular conditional probability $\mu(\cdot\,|\mathfrak{M}')$ is μ-a.e. equal to a measure on an atom of π'. In what follows we shall denote the σ-algebra and the partition associated to it by the same symbol and we write $\mu(\cdot\,|\mathfrak{M}'(x))$ for the measure on the atom containing x.

A point $x\in\mathscr{X}$ can be identified with a sequence: $x=\{q_k(x), v_k(x)\}_{k=0}^{\infty}$ in which the particles are labeled in order of increasing position, and, for equal position, in order of increasing velocity. The mass of the h.p. is M, and its coordinates are of course (q_0, v_0). $m<M$ is the common mass of the other particles.

For a fixed value of the chemical potential λ and of the inverse temperature β, consider the Gibbs measure μ^0 for a gas of free particles with common mass m. μ^0 is a Poisson stochastic field on $\mathbb{R}_+^2$. The thermodynamic pressure P and the particle density ϱ are given by

$$\beta P=\sqrt{\frac{2\pi}{\beta m}}\,e^{\beta\lambda},\qquad \varrho=\beta P. \tag{2.1}$$

If we now fix $E<P$, the Gibbs measure for the mechanical system we consider in this paper can be defined as the measure μ on $\mathscr{X}$ such that: i) the position q_0 of the h.p. is exponentially distributed with parameter $\varrho-\beta E=\beta(P-E)$, i.e.

$$\mu(q_0<x)=[\beta(P-E)]^{-1}\int_0^x \exp(-\beta(P-E)y)\,dy; \tag{2.2}$$

and ii) the distribution of the other particles $x\backslash(q_0, v_0)$ for a fixed position q_0 coincides with the corresponding distribution for μ^0; and iii) momenta are distributed independently of positions with a maxwellian law with average 0 and dispersion β^{-1}.

The time evolution can be described as follows: all particles of mass m move with constant velocity, and the h.p. with constant acceleration E/M, until the h.p. either collides with the wall at $q=0$, and inverts its velocity, or it collides with another particle, according to the collision laws,

$$V' = \alpha V + (1-\alpha)u, \qquad \alpha = \frac{M-m}{M+m}, \tag{2.3a}$$
$$u' = -\alpha u + (1+\alpha)V, \tag{2.3b}$$

where V and u are the ingoing velocities of the h.p. and of the light particle, and V' and u' are their outgoing velocities, respectively. We are using the standard device of treating the light particles as "pulses," i.e. we let them exchange "names," instead of exchanging velocities, when they collide among themselves. At collision, particles are supposed to be in the outgoing configuration.

The dynamics defines a measure preserving flow $\{T_t\}_{t\in\mathbb{R}}$ on an invariant set of full measure. By $\{T_t^0\}_{t\in\mathbb{R}}$ we denote the free flow, i.e. the flow for configurations of particles of common mass m. The measure μ^0 is clearly invariant under this flow.

The main result of this paper is the following.

Theorem 2.1. *If $E<P$ the dynamical system $(\mathscr{X}, \mu, T_t)$ is a Bernoulli flow.*

Proof (Preliminaries). Consider the process defined by the trajectory of the h.p. $q_0(T_t x)$, which we sometimes by abuse of notation will denote by $q_0(t)$.

We denote by $\zeta_\tau(\zeta^\tau)$ the σ-algebra generated by $q_0(t)$ for $t\leqq\tau$ $(t\geqq\tau)$. The assertion of the theorem will follow from the proof that for any $\varepsilon>0$ for the set

$$A_\varepsilon^\tau = \left\{x : \sup_{B\in\zeta^\tau} |\mu(B|)\zeta_0(x)) - \mu(B)| > \varepsilon\right\}, \tag{2.4}$$

we have

$$\lim_{\tau\to\infty} \mu(A_\varepsilon^\tau) = 0 \tag{2.5}$$

(see [4]). Equality (2.5) will be proved in Sect. 4.

The following results about the long time behavior of the h.p. (confinement) are an easy consequence of the properties of μ and are proved in [1].

Lemma 2.1. *There is a constant $c>0$ such that for μ-a.a. $x\in\mathscr{X}$,*

i) $$\limsup_{t\to\pm\infty} \frac{|v_0(T_t x)|}{\log^{1/2}|t|} < c, \tag{2.6a}$$

ii) $$\limsup_{t\to\pm\infty} \frac{q_0(T_t x)}{\log|t|} < c. \tag{2.6b}$$

For what follows we have to single out some regions of the one-particle phase space.

Definition 2.1. For any $L>0$, $s>t>0$, we introduce the following subsets of $\mathbb{R}^2_+$,

$$S_L=\{(q,v)\in\mathbb{R}^2_+ : q\in[0,L)\}\,, \tag{2.7a}$$

$$C_L(t,s)=\{(q,v)\in\mathbb{R}^2_+ : q\geqq L,\, q+v\tau=L \text{ for some } \tau\in[t,s)\}\,, \tag{2.7b}$$

$$R_L(t)=\mathbb{R}^2_+\backslash C_L(t,\infty)\,, \tag{2.7c}$$

$$\Gamma_L=\{(q,v)\in\mathbb{R}^2_+ : q\geqq L,\, 0\geqq v\geqq -\sqrt{L}\exp(-(q/\sqrt{L}+1))\,. \tag{2.7d}$$

We will set henceforth for brevity $C_L=C_L(0,\infty)$ and $R_L=R_L(0,\infty)$. Note that $C_L(t,s)$ contains the particles which cross L between times t and s for free motion, and $\Gamma_L\subset C_L$.

Proposition 2.1. *Consider the sets*

$$\mathscr{A}_L=\{x\in\mathscr{X} : x_{\Gamma_L}=\emptyset\}\,, \tag{2.8a}$$

$$\mathscr{E}_L=\{x\in\mathscr{X} : q_0(T_t x_{R_L})<\sqrt{L}\log_+|t|,\, t\leqq 0\}\,. \tag{2.8b}$$

Then: i) $\lim_{L\to\infty}\mu(\mathscr{A}_L\cap\mathscr{E}_L)=1$ *and* ii) *for* $x\in\mathscr{A}_L\cap\mathscr{E}_L$. $T_t x=T_t x_{R_L}\cup T_t^0 x_{C_L}$ *for* $t\leqq 0$ *and there are no negative (i.e. negative velocity) particles which collided with the* h.p. *in the past at the right of L.*

[Roughly speaking Γ_L is the region where such particles must be located if the bound (2.8b) holds.]

Remark 2.1. If we introduce the "time-reversed" sets $R_L^+=\{(q,v)\in\mathbb{R}^2_+ : (q,-v)\in R_L\}$, $\Gamma_L^+=\{(q,v)\in\mathbb{R}^2_+ : (q,-v)\in\Gamma_L\}$, we see that from Proposition 2.1 it follows for the sets

$$\mathscr{A}_L^+=\{x\in\mathscr{X} : x_{\Gamma_L^+}=\emptyset\}\,, \tag{2.8a'}$$

$$\mathscr{E}_L^+=\{x\in\mathscr{X} : q_0(T_t x_{R_L^+})<\sqrt{L}\log_+ t,\, t\geqq 0\}\,, \tag{2.8b'}$$

we have i) $\lim_{L\to\infty}\mu(\mathscr{A}_L^+\cap\mathscr{E}_L^+)=1$ and ii) for $x\in\mathscr{A}_L^+\cap\mathscr{E}_L^+$ no positive particle at the right of L will collide with the h.p. in the future.

As in [1], we make use of the fact that the motion of the h.p., when it gets a speed much above average, becomes "almost deterministic." For L large we divide the interval $[0,L)$ in $\kappa(L)=[L^{1/5}]$ pieces of length $\ell=L/\kappa(L)$: $I_k=[(k-1)\ell,\,k\ell)$, $k=1,2,\ldots,\kappa(L)$. Consider the sets $\mathscr{B}_L^{(k)}=\{x\in\mathscr{X} : |\mathrm{card}\, x_{I_k\times\mathbb{R}^1}-\varrho\ell|<s\}$ with $s=s(\ell)=\ell^{3/5}$, and the sets

$$\mathscr{B}'_L=\bigcap_{k=1}^{\kappa(L)}\mathscr{B}_L^{(k)}\,, \tag{2.9}$$

$$\mathscr{B}''_L=\left\{x\in\mathscr{X} : \max_{(q,v)\in x_{[0,L)\times\mathbb{R}^1}}|v|<c\log^{1/2}L\right\}\,, \tag{2.10}$$

$$\mathscr{B}_L=\mathscr{B}'_L\cap\mathscr{B}''_L\,. \tag{2.11}$$

For $x\in\mathscr{B}_L$ the particles are "uniformly distributed" in $[0,L)$, and velocities are not too high.

Lemma 2.2. *There is a constant* $c>0$ *(see Eq.* (2.10)*) such that*

$$\lim_{L\to\infty}\mu(\mathscr{B}_L)=1\,.$$

Proof. The result is proved in [1, Lemma A.1] for the case $E=0$, and can be extended to all values of $E\in[0,P]$ almost without change.

The following proposition makes precise the idea of "almost deterministic behavior" for a high speed h.p.

Proposition 2.2. *Let $x\in\mathscr{B}_L$ and consider the configuration $\hat{x}=x_{R_L}\cup(q,v)$ with $-v(1-\alpha)\in(e^{aL}, e^{aL}+1)$, $a>a_0=\varrho\log(1/\alpha)$, and $q+vt_0=L$ for some $t_0\in(0,1/2)$. Then, for L large enough, the evolution of $\hat{x}$ for positive times is such that:*

i) *after colliding with (q,v) the* h.p. *hits the wall at a time $t_1>t_0$, and for $t>t_1$ never inverts its velocity;*

ii) *(q,v) and all the other particles which collide with the* h.p. *after time t_1 and in $[0,L)$, get a positive velocity larger than $w_0=e^{\lambda L}$, for $0<\lambda<\min(a,a-a_0)$;*

iii) *after time t_1 the* h.p. *crosses the points $k\ell+c\log^{1/2}L$ with a velocity smaller than $V_M(k)$, and at a time larger than $t_m(k)$, and crosses the points $k\ell-c\log^{1/2}L$ with a velocity larger than $V_m(k)$, and at times not exceeding $t_M(k)$, for $k=2,3,\ldots,\kappa(L)$, where the quantities,*

$$V_M(k)=(1-\alpha)\,|v|\,\alpha^{k\ell\varrho(1-\sigma)}(1+\varepsilon_L)\,, \tag{2.11a}$$

$$V_m(k)=(1-\alpha)\,|v|\,\alpha^{k\ell\varrho(1+\sigma)}(1-\varepsilon_L)\,, \tag{2.11b}$$

$$t_m(k)=t_0+\frac{\ell}{V_M(k-1)}, \tag{2.12a}$$

$$t_M(k)=t_0+\frac{\ell}{V_m(k)}, \tag{2.12b}$$

$\varepsilon_L=e^{-\sqrt{L}}$ and $\sigma=s/\ell$, do not depend on $x\in\mathscr{B}_L$.

Proof. The proof is done in [1] for the case $E=0$ (Proposition A.1). Its extension to the case $E>0$ is straightforward, since the correction to the motion of the h.p. due to the force E becomes negligible for large L.

We need for what follows an easy extension of the results above.

Remark 2.2. Consider the sets $\mathscr{B}_L^{+(k)}=\{x\in\mathscr{X}:|\mathrm{card}\,x_{I_k\times[0,\infty)}-\varrho\ell/2|<s/2\}$, $k=\kappa(L)+1,\ldots,2\kappa(L)$, and

$$\mathscr{B}_L^+=\bigcap_{k=\kappa(L)+1}^{2\kappa(L)}\mathscr{B}_L^{+(k)}\,. \tag{2.9'}$$

As for Lemma 2.2, it is not hard to prove, following the lines of Lemma A.1 of [1] that $\lim_{L\to\infty}\mu(\mathscr{B}_L^+)=1$. Moreover if $x\in\mathscr{B}_L^+$ and (q,v) is as in Proposition 2.2 with $a>3a_0/2$, assertion iii) of Proposition 2.2 can be extended to all k up to $2\kappa(L)$, by taking for $k=\kappa(L)+h$, $h\geqq 1$,

$$V_M(k)=(1-\alpha)\,|v|\,\alpha^{\left(L+\frac{h}{2}\right)\varrho(1-\sigma)}(1+\varepsilon_L)\,, \tag{2.11a'}$$

$$V_m(k)=(1-\alpha)\,|v|\,\alpha^{\left(L+\frac{h}{2}\right)\varrho(1+\sigma)}(1-\varepsilon_L)\,. \tag{2.11b'}$$

From Remark 2.2 we obtain the following result.

Corollary 2.1. *If* $x \in \mathscr{B}_L \cap \mathscr{B}_L^+$ *and* (q, v) *is as in Remark 2.2, then in the history of the* h.p. *in the configuration* $\hat{x} = x_{R_L} \cup (q, v)$, *there is a time* $\hat{t} \in (t_0, 1)$, *depending only on* (q, v), *such that* i) $q_0(T_{\hat{t}}\hat{x}) \in (L+\ell, L+5\ell)$, ii) $v_0(T_{\hat{t}}\hat{x}) \in (V_m, V_M)$ *with* $V_m = V_m(\kappa(L)+5)$, $V_M = V_M(\kappa(L)+1)$, *and* iii) *all the particles which at time* 0 *were in* $[0, L)$ *collide by time* $\hat{t}$ *with the* h.p. *and get a positive velocity larger than* $e^{\lambda_0 L}$ *for some* $\lambda_0 > 0$.

Proof. As for Corollary A.1 of [1] the proof reduces to checking that $t_m(\kappa(L)+4) > t_M(\kappa(L)+2)$, which is a consequence of the fact that $\sigma(L/\ell)\downarrow 0$ as $L \to \infty$.

3. Construction of Cluster Situations and Loss of Memory

We first illustrate the local mechanism of loss of memory for the process $\{q_0(t)\}$ in a particular case, which we use in our construction.

Suppose that $E=0$, and that the h.p. is at $L/2$, $L>0$, with a velocity $w>0$, and collides there with a particle of velocity $-V = -\frac{1+\alpha}{1-\alpha}w$, followed by an infinite sequence of particles $\bar{\xi}_k = (L/2 + k\tau V, -V)$, $k = 1, 2, \ldots$, with $\tau = L/w$. Then the h.p. undergoes a back and forth motion colliding at time $t_k = k\tau$ with $\bar{\xi}_k$ at $L/2$, and moving with constant speed w. We now change the initial position and velocity of the h.p. a little bit, and consider incoming particles ξ_k close to $\bar{\xi}_k$. We describe the motion in terms of the quantities $b_k = \hat{q}_k - L/2$ and $c_k = w_k - w$, where $\hat{q}_k$ is the position which the h.p. has at time t_k if we neglect the k^{th} collision, and w_k is the ingoing velocity of the h.p. We set $\sigma_k = (b_k, c_k)$, $\xi_k = \bar{\xi}_k + (s_k - k\tau u_k, u_k)$ and $\lambda_k = (s_k, u_k)$. We have

$$\sigma_{k+1} = (b_{k+1}, c_{k+1}) = \varphi(\sigma_k, \lambda_k) = A\sigma_k + B\lambda_k \tag{3.1a}$$

with

$$A = \begin{pmatrix} -\alpha & -\alpha\tau \\ 0 & -\alpha \end{pmatrix}, \quad B = \begin{pmatrix} -(1-\alpha) & -(1-\alpha)\tau \\ 0 & -(1-\alpha) \end{pmatrix}. \tag{3.1b}$$

A has a double eigenvalue $-\alpha \in (-1, 0)$. This implies that if we have an initial $\sigma_0 \neq 0$ small enough, the motion of the h.p. induced by the particles $\bar{\xi}_k$ (i.e. for $\lambda_k = 0$) loses memory of the initial perturbation and goes exponentially fast to the periodic cycle described above. It is not hard to see that such "focalization property" of the sequence $\bar{\xi}_k$ holds also for $E>0$, by suitably redefining the limiting cycle and the map φ (see [2]).

In constructing cluster situations we have to take care of recollisions after focalization. We now identify some local configurations, in the interval $[0, L]$), which are not likely to give recollisions.

By Definition 2.1 we have $\mathbb{R}_+^2 = S_L \cup C_L \cup C_L^+$, with $C_L^+ = \{(q, v) \in \mathbb{R}_+^2 : (q, -v) \in C_L\}$. Correspondingly we decompose $x \in \mathscr{X}$ as $x = y \cup z \cup z^+$, with $y = x_{S_L}$, $z = x_{C_L}$, $z^+ = x_{C_L^+}$. Let ν_L be the distribution of y induced by μ. It is a measure on $\mathscr{X}_{S_L} = \bigcup_{j=0}^{\infty} \mathscr{X}_{S_L}^{(j)}$, where $\mathscr{X}_{S_L}^{(j)}$ is the j-particle space, which can be interpreted as a subset of $(S_L)^j$. We can then identify the restriction $\nu_L|_{\mathscr{X}_{S_L}^{(j)}} = \nu_L^{(j)}$ with a measure on $\mathbb{R}^{2j}$ which is equivalent to the Lebesgue measure m^{2j} on its support.

Consider the set

$$\hat{\mathscr{E}}_L^+ = \mathscr{E}_L^+ \cap \mathscr{A}_L \tag{3.2}$$

[see Eqs. (2.8a), (2.8b′)]. $\hat{\mathscr{E}}_L^+ \in \mathfrak{M}_{R_L^+}$ and we can identify it with a subset of $\mathscr{X}_{R_L^+}$. By Proposition 2.1 and Remark 2.1, if $x \in \hat{\mathscr{E}}_L^+$ and $x \cap \Gamma_L^+ = \emptyset$ there are no recollisions in the future with positive particles which are outside $[0, L)$.

$$\mu(\hat{\mathscr{E}}_L^+ \,|\, \mathfrak{M}_{S_L}(x)) = \mu_{C_L}^0(z : y \cup z \in \hat{\mathscr{E}}_L^+) . \tag{3.3}$$

So, setting

$$Z_y = \{z \in \mathscr{X}_{C_L} : y \cup z \in \hat{\mathscr{E}}_L^+\} , \qquad y \in \mathscr{X}_{S_L} , \tag{3.4}$$

and

$$Y_L = \{y \in \mathscr{X}_{S_L} : \mu_{C_L}^0(Z_y) > 1 - \varepsilon\} , \tag{3.5}$$

we find by the Chebyshev inequality that for L large enough

$$\nu_L(Y_L) = \mu(\{x \in \mathscr{X} : \mu(\hat{\mathscr{E}}_L^+ \,|\, \mathfrak{M}_{S_L}(x)) > 1 - \varepsilon\}) > 1 - \varepsilon . \tag{3.6}$$

If $y \in \mathscr{X}_{S_L}$, then in its evolution for $t < 0$, there will be, for ν_L-a.a. y, a time $s_0(y)$ such that for $t < s_0$, $T_t y$ contains only negative velocity particles. Therefore for some time $s < s_0$ we will have that i) all particles except the h.p. are at the right of L, and ii) $q_0(T_s y) > L/2$. A time for which conditions i) and ii) hold will be called an "exit time" for y.

We take a $j > 0$ such that $\nu_L^{(j)}(Y_L \cap \mathscr{X}_{S_L}^{(j)}) > 0$ and a point $y^* \in Y_L$ such that there is a neighborhood of y^*, $\mathscr{U}$, and a time s^*, so that $\nu_L^{(j)}(Y_L \cap \mathscr{U}) > 0$, s^* is an exit time for all $y \in \mathscr{U}$, and the map T_{s^*}, as a map $\mathbb{R}^{2j} \to \mathbb{R}^{2j}$ is C^∞ on $\mathscr{U}$ with nonzero jacobian. Then clearly the image $T_{s^*}\mathscr{U} = \mathscr{U}^{(1)}$ is an open set, and T_{-s^*} is a C^∞ map which inverts T_{s^*} on $\mathscr{U}^{(1)}$.

If $y_1 \in \mathscr{U}^{(1)}$ we can represent it as $y_1 = (Q, \mathring{y}_1)$, where $Q = y_1 \cap S_L = (q_0, v_0)$ gives position and velocity of the h.p. and $\mathring{y}_1 = y_1 \cap C_L = y_1 \backslash Q$. Let $\nu_*^{(j)}$ denote the measure induced by $\nu_L^{(j)}$ on $\mathscr{U}^{(1)}$ via T_{s^*}, and $Y_L^{(1)} = T_{s^*} Y_L \cap \mathscr{U}^{(1)}$. Then clearly $\nu_*^{(j)}(Y_L^{(1)}) > 0$, and, since by the properties of $\nu_L^{(j)}$ and of T_{s^*}, $\nu_*^{(j)}$ is equivalent to the Lebesgue measure m^{2j} on $\mathscr{U}^{(1)}$, $Y_L^{(1)}$ has at least a density point $y_1^* = (Q_*, \mathring{y}_1^*)$.

Definition 3.1. In what follows for each L and ε the integer $j > 0$, the negative time s^*, the neighborhood $\mathscr{U}^{(1)}$ and the density point $y_1^* = (Q_*, \mathring{y}_1^*)$ of the set $Y_L^{(1)}$ will denote fixed entities constructed as above.

We will now show that there is a set $\mathscr{C}_L$, $\mu(\mathscr{C}_L) > 0$ such that for some time $r > 0$ $(T_r x)_{S_L}$ is in the neighborhood $\mathscr{U}$. $\mathscr{C}_L$ is written as an intersection

$$\mathscr{C}_L = \mathscr{E}_L' \cap \mathscr{F}_L \cap \mathscr{A}_L , \tag{3.7}$$

where $\mathscr{A}_L$ is given by Eq. (2.8a), $\mathscr{E}_L' \in \mathfrak{M}_{R_L}$ is given by

$$\mathscr{E}_L' = \mathscr{E}_L \cap \mathscr{B}_L \cap \mathscr{B}_L^+ \cap \mathscr{A}_L^+ \tag{3.8}$$

[see Eqs. (2.8b), (2.10), (2.9′), (2.8a′)] and $\mathscr{F}_L \in \mathfrak{M}_{C_L(0,r)}$ for $r = s^* + \tau^*$ (τ^* is given below) is defined by the condition that $x_{C_L(0,r)}$ belongs to a neighborhood $\mathscr{V}_2$, to be specified later, of the configuration $\bar{x}_2$, which is made of the following particles:

i) a particle $(\bar{q}, \bar{v})$ as in Remark 2.2. By that remark in the configuration $\hat{x}' = x_{R_L} \cup (q, v)$ the h.p. will be at some time $\hat{t}$ depending only on $(\bar{q}, \bar{v})$, in the interval $(L+\ell, L+5\ell)$, with a velocity less than V_M;

ii) a particle $(\bar{q}_1, \bar{v}_1)$, $\bar{q}_1 > 5\ell + L$, such that $\bar{q}_1 + \bar{v}_1 \hat{t} = L + 3\ell$, and $\bar{v}_1 = -(1-\alpha)w$, with $\alpha V_M / w < \eta/3$, where η is as small as will be required later;

iii) a sequence of particles $\bar{\xi}_k = (\bar{q}_k, \bar{v}_k)$, $k = 2, 3, \ldots, N$, such that $\bar{v}_k = -V = -((1+\alpha)/(1-\alpha))w$, and $\bar{q}_k + (\hat{t} + L/2w + kL/w)\bar{v}_k = L/2$, $\bar{q}_k > 5\ell + L$.

Clearly in the evolution of $\hat{x}'' = \hat{x}' \cup (\bar{q}_1, \bar{v}_1) \cup \left(\bigcup_{k=1}^{N} \bar{\xi}_k \right)$ the h.p. collides with $(\bar{q}_1, \bar{v}_1)$ at some point $\tilde{q} \in (L+\ell, L+5\ell)$ at some time $\bar{t} : |\bar{t} - \hat{t}| < 2(1-\alpha)(\ell/V)$, and gets a negative velocity $-V(1-\varepsilon_1)$, $\varepsilon_1 \in (0, \eta/3)$. If $\ell/L = \mathcal{O}(L^{-1/5})$ is small enough, the h.p. bounces off the wall and crosses $L/2$ at some time $\hat{t} + (3L/2V)(1+\varepsilon_2)$ with $|\varepsilon_2| < \eta$. (This estimate holds for all $E < P$ since the force E gives a negligible contribution to the trajectory for w large.) η should be small enough for the motion of the h.p. to be attracted by the periodic cycle described at the beginning of this section.

In addition we place in $C_L(0, r)$

iv) a particle $\bar{\xi}_{N+1} = (\bar{q}_{N+1}, \bar{v}_{N+1})$ with $\bar{q}_{N+1} > 5\ell + L$ and $\bar{q}_{N+1} + \bar{\tau} v_{N+1} = q'$, for some $q' \in (q_0^*, L)$, where q_0^* is the position of the h.p. corresponding to $Q_* = (q_0^*, v_0^*)$ (Definition 3.1), $\bar{\tau}$ is the time at which the h.p. would cross q' if it collided with $(\bar{q}_N, \bar{v}_N)$ exactly at $L/2$ with outgoing velocity $-w$, and $\bar{v}_{N+1}$ is such that in the configuration $\hat{x}''' = \hat{x}'' \cup \bar{\xi}_{N+1}$ the h.p. inverts its velocity and crosses at time τ^* the point q_0^* with velocity v_0^*;

v) the whole configuration $T^0_{-\tau^*} \mathring{y}_1^*$.

By construction, in the evolution of $\hat{x} = \hat{x}''' \cup \mathring{y}_1^*$ at time τ^* a configuration close to y_1^* appears, and $(T_r \hat{x})_{S_L} \in \mathscr{U}$. For the whole configuration x the situation can be different because when the h.p. gets out of $[0, L)$, as described in i), it may collide with particles which at time 0 are outside the region $R_L \cup C_L(0, r)$. But if τ^* is chosen large enough, which can be done by increasing N, the region $\{(q, v) : q \in [L, L+5\ell],\ q + v\hat{t} < 5\ell\} \cap C_L(r, \infty)$ is contained in Γ_L. So, if $x \in \mathscr{E}'_L$, $x \cap C_L(0, r) = \bar{x}_2$, $x \in \mathscr{A}_L$, there are no such particles and $(T_r x)_{S_L} = (T_r \hat{x})_{S_L} \in \mathscr{U}$. Also τ^* should be large so that $T^0_{-\tau^*} \mathring{y}_1^* \cap \Gamma_L = \emptyset$ (this is possible because all velocities in $\mathring{y}_1^*$ are nonzero) otherwise our conditions are incompatible.

If we now consider some $x_2 \in \mathscr{X}_{C_L(0,r)}$ close to $\bar{x}_2$, it will have a particle $\xi_{N+1} = (q_{N+1}, v_{N+1})$ close to $\bar{\xi}_{N+1}$. Position and velocity of the h.p. at time τ^* will then again be close to Q_* and are a function of $\sigma_{N+1} = (\hat{q}_{N+1} - L/2,\ w_{N+1} - w)$ (here $\hat{q}_{N+1}$ is the position of the h.p. at time $\tau_{N+1} = (N+3/2)L/w$ and w_{N+1} is its incoming velocity) and of ξ_{N+1}. Denoting by Q position and velocity of the h.p. at time τ^*, and setting $\lambda_{N+1} = (q_{N+1} + \bar{\tau} v_{N+1} - q',\ v_{N+1} - \bar{v}_{N+1})$, we find

$$\sigma^* = Q - Q_* = A^* \sigma_{N+1} + B^* \lambda_{N+1}, \tag{3.9}$$

where A^* and B^* are obtained by the matrices A and B of Eq. (3.1) by replacing τ with $t^* = \tau^* - \tau_{N+1}$.

We can now define the neighborhood $\mathscr{V}_2$ by prescribing that a configuration $x_2 \in \mathscr{V}_2$ is made of:

i') two particles (q, v) and (q_1, v_1) such that

$$\max(|v - \bar{v}|, |(q-L)/v - (\bar{q}-L)/\bar{v}|, |v_1 - \bar{v}_1|, |(q_1 - L)/v_1 - (\bar{q}_1 - L)/\bar{v}_1|) < \eta_1,$$

ii′) a sequence of particles $\xi_k=(q_k, v_k)$ such that $\lambda_k=(q_k+v_k\tau_k-L/2, v_k-\bar{v}_k)\in K_{\eta_2}(0)$ for $k=2,3,\ldots,N$, and $\tau_k(k+1/2)L/w$;

iii′) a particle $\xi_{N+1}=(q_{N+1}, v_{N+1})$ such that $\lambda_{N+1}\in(B^*)^{-1}K_\delta(0)$;

iv′) a configuration $T^0_{-\tau^*}\mathring{y}_1$ for $\mathring{y}_1\in \mathring{W}_\delta(\mathring{y}_1^*)$;

where $K_\delta(x_0)=\{x\in\mathbb{R}^2: |x-x_0|<\delta\}$, for $\delta>0$, $x_0\in\mathbb{R}^2$, $\mathring{W}_\delta(y_0)=\{y\in\mathbb{R}^{2j-2}: |y-y_0|<\delta\}$, $y_0\in\mathbb{R}^{2j-2}$. For any given choice of ε [in Eq. (3.5)] and $\mathscr{U}^{(1)}$, δ should be so small that $K_\delta(Q_*)\times \mathring{W}_\delta(\mathring{y}_1^*)\subset\mathscr{U}^{(1)}$. Clearly if η is small enough we can choose η_1, η_2, N, and δ such that $(T_r x)_{S_L}\in\mathscr{U}$ for $x\in\mathscr{C}_L$. It is plain that $\mu(\mathscr{C}_L)>0$.

We first investigate the distribution of position and velocity of the h.p. at time τ^*. We use the notation $x_1=x_{R_L}$, $x_2=x_{C_L(0,r)}$. Moreover $m^*_{x_1}$ will denote the probability distribution induced on σ^* [Eq. (3.9)] by the restriction of $\mu(\cdot\,|\mathfrak{M}_{R_L}(x))$ to $\mathscr{C}_L$ and m^* the normalized Lebesgue measure on $K_\delta(0)$ extended to $\mathbb{R}^2$ by setting $m^*(D)=m^*(D\cap K_\delta(0))$ for any measurable set $D\subset\mathbb{R}^2$. $\|\cdot\|$ will denote the variation distance between measures.

Proposition 3.1. *There is a constant c_1, independent of x such that for all $x\in\mathscr{C}_L$,*

$$\|m^*_{x_1}-m^*\|<c_1\delta\,. \tag{3.10}$$

Proof. In addition to $\sigma_{N+1}, \lambda_{N+1}$ we introduce the variables $\sigma_k, \lambda_k, k=2,3,\ldots,N$ corresponding to the collision times $\tau_k=t+(k+\frac{1}{2})L/w$. They are related by Eq. (3.1) for $\tau=L/w$, and, because of the contracting property of A we have $\|\sigma_{N+1}\|<\delta^2$ for η, η_1, η_2 small enough. Note that, since for $x\in\mathscr{E}'_L$ $\mu(\cdot\,|\mathfrak{M}_{R_L}(x))$ coincides with μ^0 when restricted to $\mathfrak{M}_{C_L}$, the λ_k's are independent of the σ_k's and, in particular, λ_{N+1} has a distribution which is concentrated on $(B^*)^{-1}K_\delta(0)$ by condition iii′) above, and is a.c. with a smooth, positive density. For $\sigma_{N+1}=0$ we have, by Eq. (3.9) $\sigma^*=B^*\lambda_{N+1}$, and the distribution of σ^*, $\hat{m}^*$, is concentrated on $K_\delta(0)$. $\hat{m}^*$ is a.c. with density $(M_\delta)^{-1}g(\sigma)$, where g is a function independent of δ and such that: a) $g\in C^\infty$, b) $\inf\limits_{\sigma\in K_\delta(0)} g(\sigma)=g_0>0$, and c) $\lim\limits_{\delta\to 0}\pi\delta^2 g(0)/M_\delta=1$. Since $\|\sigma_{N+1}\|<\delta^2$, $m^*_{x_1}$ has support in $K_{\delta+\delta^2}(0)$. If $v_{N+1}(\cdot\,|x_1)$ denotes the distribution of σ_{N+1} induced by the restriction of $\mu(\cdot\,|\mathfrak{M}_{R_L}(x))$ to $\mathscr{C}_L$ and $D\subset K_{\delta+\delta^2}(0)$ is a measurable set, we have by Eq. (3.9),

$$|m^*_{x_1}(D)-\hat{m}^*(D)|=|\textstyle\int v_{N+1}(d\sigma|x_1)\,(\hat{m}^*(D-A^*\sigma)-\hat{m}^*(D))|\,,$$

and

$$|m^*_{x_1}(D-A^*\sigma)-\hat{m}^*(D)|<(M_\delta)^{-1}\int_D |g(\lambda+A^*\sigma)-g(\lambda)|\,d\lambda<C'\delta\,.$$

A similar estimate shows that

$$\|\hat{m}^*-m^*\|<C''\delta\,.$$

C' and C'' are constants independent of δ. Proposition 3.1 is proved.

Consider now the probability distribution $\ell^{(1)}_{x_1}$ of the whole configuration $y_1=Q\cup(T_{\tau^*}x)_{C_L(0,|s^*|)}$ induced by the restriction of $\mu(\cdot\,|\mathfrak{M}_{R_L}(x))$ to $\mathscr{C}_L$. $\ell^{(1)}_{x_1}$ is a product

$$\ell^{(1)}_{x_1}=\tilde{m}^*_{x_1}\times\tilde{\mathring{m}}\,, \tag{3.11}$$

with support in $\tilde{\mathring{W}}_\delta = K_{\delta+\delta^2}(Q_*) \times \mathring{W}_\delta(\mathring{y}_1^*) \subset \mathbb{R}^{2j}$, where $\tilde{m}_{x_1}^*$ is obtained by shifting $m_{x_1}^*$ to $K_{\delta+\delta^2}(Q_*)$, and $\tilde{\mathring{m}}$ is the measure on $\tilde{\mathring{W}}_\delta$ induced by μ^0. We denote by $\ell^{(1)}$ the normalized Lebesgue measure on $W_\delta = K_\delta(Q_*) \times \mathring{W}_\delta(\mathring{y}_1^*)$, extended to $\mathbb{R}^{2j}$ as above.

Proposition 3.2. *There is a constant $c>0$ independent of x such that for $x \in \mathscr{C}_L$,*

$$\|\ell_{x_1}^{(1)} - \ell^{(1)}\| < c\delta . \tag{3.12}$$

Proof. Let $\mathring{m}$ denote the normalized Lebesgue measure on $\mathring{W}_\delta$. Since $\tilde{\mathring{m}}$, like $\hat{m}^*$ of Proposition 3.1, is obtained by normalizing a measure with smooth positive density on $\mathring{W}_\delta$, one easily derives that $\|\tilde{\mathring{m}} - \mathring{m}\| < c'\delta$, and the result follows from Proposition 3.1.

Finally we have

Proposition 3.3. $\lim_{\delta\to 0} \ell^{(1)}(W_\delta \cap Y_L^{(1)}) = 1$.

Proof. Since W_δ is a neighborhood of fixed geometrical shape of y_1^*, which is a density point of $Y_L^{(1)}$, we have

$$\ell^{(1)}(W_\delta \cap Y_L^{(1)}) = \frac{m^{2j}(W_\delta \cap Y_L^{(1)})}{m^{2j}(W_\delta)} \xrightarrow[\delta\to 0]{} 1 .$$

We are now able to prove that for "most" $x \in \mathscr{C}_L$ τ^* is a cluster time, i.e. the particles that collide before τ^* and after it are two disjoint sets.

Proposition 3.4. *Let $\mathscr{C}'_L$ denote the subset of $\mathscr{C}_L$ for which τ^* is a* c.t. *Then for any $\varepsilon > 0$ one can find L_0 such that for $L > L_0$, and for a suitable choice of the parameters η_1, η_2, δ and N (in conditions* i′)–iv′) *above) we have*

$$\mu(\mathscr{C}_L) - \mu(\mathscr{C}'_L) < \varepsilon\mu(\mathscr{C}_L) .$$

Proof. If $x \in \mathscr{C}_L \subset \mathscr{A}_L \cap \mathscr{E}_L$ there are (by Proposition 2.1) no negative particles of the past with $q > L$. Moreover those that collide between times 0 and r do not recollide by construction until time r. So, by Remark 2.1, τ^* is a c.t. if $T_r x \in \mathscr{E}_L^+ \cap \mathscr{A}_L^+$. The condition $T_r x \in \mathscr{A}_L^+$ is satisfied for all $x \in \mathscr{C}_L$ because: i) $x \cap \Gamma_L^+ = \emptyset$ by definition of $\mathscr{E}'_L$ [Eq. (3.8)], ii) no positive particle in $[0, L)$ can be in Γ_L^+ after time $\hat{t}$ because they all collide with the h.p. and get a high velocity, and iii) all the other particles that collide before time r either go away with a velocity larger than that allowed by Γ_L^+ or are still in $[0, L)$ at time r (those specified by condition iv′)). The condition $T_r x \in \mathscr{E}_L^+$ is satisfied for $x \in \mathscr{C}_L$ if $y = y(x) = (T_r x)_{S_L} = T_{|s^*|} y_1 \in Y_L$ and $z = (T_r x)_{C_L} \in Z_y$ [see Eqs. (3.4), (3.5)]. So if we set

$$\mathscr{C}''_L = \{x \in \mathscr{C}_L : y_1(x) \in Y_L^{(1)}, z \in Z_y\} \tag{3.13}$$

(here and in the following $y = T_{|s^*|} y_1$) we have $\mathscr{C}''_L \subset \mathscr{C}'_L$. Consider the σ-algebra $\mathfrak{M}_{R'_L}$ with $R'_L = R_L \cup \Gamma_L \cup C_L(0, r)$. Clearly $\mathscr{C}_L \in \mathfrak{M}_{R'_L}$ and, since no negative particle outside $C_L(0, r)$ can collide before time r for $x \in \mathscr{C}_L$, y_1 is $\mathfrak{M}_{R'_L}$-measurable and $z \in Z_y$ is equivalent to $x \in Z'_y = \{x : x_{C_L(r,\infty)} \in T^0_{-r} Z_y\}$. So for $x \in \mathscr{C}_L$, we get

$$\mu(\mathscr{C}''_L | \mathfrak{M}_{R'_L}(x)) = \chi_{Y_L^{(1)}}(y_1)\mu^0(Z'_y | \mathscr{A}_L) .$$

Setting $R''_L = R_L \cup \Gamma_L \subset R'_L$ we have $\mathfrak{M}_{R''_L} \subset \mathfrak{M}_{R'_L}$ and, taking expectations, we find

$$\begin{aligned}\mu(\mathscr{C}''_L|\mathfrak{M}_{R''_L}(x)) &= \mathbb{E}(\chi_{\mathscr{C}_L}\cdot\chi_{Y_L^{(1)}}(y_1)\mu^0(Z'_y|\mathscr{A}_L)|\mathfrak{M}_{R''_L}(x))\\ &= \chi_{\mathscr{E}'_L\cap\mathscr{A}_L}(x)\mathbb{E}(\chi_{\mathscr{F}_L}\cdot\chi_{Y_L^{(1)}}(y_1)\mu^0(Z'_y|\mathscr{A}_L)|\mathfrak{M}_{R''_L}(x))\\ &= \mu(\mathscr{C}_L|\mathfrak{M}_{R''_L}(x))\int_{Y_L^{(1)}}\ell_{x_1}^{(1)}(dy_1)\mu^0(Z'_y|\mathscr{A}_L),\end{aligned} \tag{3.14}$$

where $x_1 = x_{R_L}$ and $\ell_{x_1}^{(1)}$ is the probability distribution of y_1 induced by the restriction of $\mu(\cdot|\mathfrak{M}_{R''_L}(x))$ to $\mathscr{C}_L$ (which coincides for $x\in\mathscr{E}'_L\cap\mathscr{A}_L$ with the one induced by the restriction of $\mu(\cdot|\mathfrak{M}_{R_L}(x))$ to $\mathscr{C}_L$). By Proposition 2.1 and the Definitions (3.5), (3.4) of Y_L and Z_y, we see that for L large enough $\mu^0(Z'_y|\mathscr{A}_L)$ is close to 1 for all $y\in Y_L$, so that by Propositions 3.2 and 3.3, if δ is small enough, we get for $x\in\mathscr{E}'_L\cap\mathscr{A}_L$,

$$\mu(\mathscr{C}''_L|\mathfrak{M}_{R''_L}(x)) \geqq \mu(\mathscr{C}_L|\mathfrak{M}_{R''_L}(x))\,(1-\varepsilon). \tag{3.15}$$

The proof is accomplished by taking expectations.

4. Proof of the Main Theorem

Throughout this section L, η, η_1, η_2, δ and N are supposed to satisfy all the requirements for which the previous results hold.

The proof is based on the fact that for almost all trajectories there is an infinite number of cluster times of the type described in the previous section.

Proposition 4.1. *For any $\varepsilon>0$ there is an L_0 such that for $L>L_0$ the measure of the subset of $\mathscr{X}$ for which the limit*

$$\lim_{K\to\infty}\frac{1}{2K+1}\sum_{j=-K}^{K}\chi_{\mathscr{C}''_L}(T_{kr}x) \tag{4.1}$$

(see Eq. (3.13)) exists and is positive is larger than $1-\varepsilon$.

Proof. Consider the σ-algebra $\hat{\zeta}_L = \mathfrak{M}_{R''_L}\vee\zeta_0$, where as above $R''_L = R_L\cup\Gamma_L$, and by $\mathfrak{M}'\vee\mathfrak{M}''$ we denote the smallest σ-algebra containing both $\mathfrak{M}'$ and $\mathfrak{M}''$. By Proposition 2.1 the atoms of $\hat{\zeta}_L$ which are contained in $\mathscr{E}_L\cap\mathscr{A}_L$ coincide with corresponding atoms of $\mathfrak{M}_{R''_L}$, so that Eq. (3.14) gives for μ-a.a. $x\in\mathscr{E}'_L\cap\mathscr{A}_L$,

$$\mu(\mathscr{C}''_L|\hat{\zeta}_L(x)) \geqq \mu(\mathscr{C}_L|\hat{\zeta}_L(x))\,(1-\varepsilon) > 0. \tag{4.2}$$

Since $\zeta_0\subset\hat{\zeta}_L$ we find

$$\mu(\{x : \mu(\mathscr{C}''_L|\zeta_0(x)) > 0\}) \geqq \mu(\mathscr{E}'_L\cap\mathscr{A}_L).$$

Consider now the discrete transformation $T_* = T_r$, and let ζ_ℓ^* be the σ-algebra of the ergodic components of T_*. Since the space is Lebesgue, ζ_ℓ^* is associated to a measurable partition and clearly $\zeta_\ell^* < \zeta_0 (\mathrm{mod}\,0)$. Therefore

$$\mu(\{x : \mu(\mathscr{C}''_L|\zeta_\ell^*(x)) > 0\}) \geqq \mu(\mathscr{E}'_L\cap\mathscr{A}_L),$$

and, since the right-hand side tends to 1 as $L\to\infty$, the result is proved. Note that since $\mathscr{C}''_L\subset\mathscr{C}'_L$, $\mu(\mathscr{C}''_L|\zeta^*_\ell(x))>0$ implies an infinite number of c.t.'s for x.

Definition 4.1. For $t>0$ we set

$$R''_L(t)=\{(q,v)\in\mathbb{R}^2_+ : q+vt\in R_L\cup\Gamma_L\},$$
$$C''_L(t)=\mathbb{R}^2_+\backslash\mathbb{R}''_L(t),$$
$$\hat{\zeta}_{L,t}=\mathfrak{M}_{R''_L(t)}\vee\zeta_t.$$

Remark 4.1. i) Any function of $(T_tx)_{R''_L(t)}$, $t\in[0,\tau]$ is $\hat{\zeta}_{L,t}$-measurable, since all particles in $R''_L(t)$ either are in $R''_L(\tau)$ at time 0, or collide for some $t'\in(0,t]$, or both. ii) If $x\in\mathscr{E}_L\cap\mathscr{A}_L$, since $q_0(T_{-t}x)<\sqrt{L}\log_+ t=\inf\{q:(q,v)\in C''_L(t)\}$ (see Proposition 2.1) for all $t\geqq 0$, we have $T_{-\tau}\mathscr{C}_L=T_{-\tau}(\mathscr{E}'_L\cap\mathscr{A}_L)\cap\{x: T^0_\tau x_{C''_L(t)}\in\mathscr{F}_L\}$.

Definition 4.2. We set for brevity

$$\hat{\mathscr{E}}^{(k)}_L=T_{-kr}(\mathscr{E}'_L\cap\mathscr{A}_L)\,,\qquad \mathscr{F}^{(k)}_L=\{x: T^0_\tau x_{C''_L(\tau)}\in\mathscr{F}_L\}\,,\qquad \hat{\zeta}^{(k)}_L=\hat{\zeta}_{L,kr}\,,\tag{4.3a}$$

and

$$\mathscr{G}_k=\bigcap_{j=0}^{k}(\mathscr{F}^{(j)}_L)^c\,,\qquad \hat{\mathscr{C}}^{(k)}_L=T_{-kr}\mathscr{C}_L\cap\mathscr{G}_{k-1}\,,\qquad (\mathscr{G}_{-1}=\mathscr{X})\,,\tag{4.3b}$$

where $(\cdot)^c$ denotes the complement. By Remark 4.1, ii) the sets $\hat{\mathscr{C}}^{(k)}_L$ are disjoint.

Proposition 4.2. *For any $\varepsilon>0$, if L is large enough, we have*

$$\mu\left(\bigcup_{k=0}^{\infty}\hat{\mathscr{C}}^{(k)}_L\right)>1-\varepsilon\,.$$

Proof. We set $\mathscr{J}_k=\bigcap_{j=0}^{\infty}(T_{-jr}\mathscr{C}_L)^c\ k\geqq 0$, $\mathscr{J}_{-1}=\mathscr{X}$, $\tilde{\mathscr{C}}^{(k)}_L=T_{-kr}\mathscr{C}_L\cap\mathscr{J}_{k-1}$. By Proposition 4.1, if L is large enough $\mu\left(\bigcup_{k=0}^{\infty}\tilde{\mathscr{C}}^{(k)}_L\right)=\mu\left(\bigcup_{k=0}^{\infty}T_{-kr}\mathscr{C}_L\right)>1-\varepsilon/2$. Since $\mathscr{J}^c_{k-1}=\bigcup_{j=0}^{k-1}T_{-jr}\mathscr{C}_L\subset\mathscr{G}^c_{k-1}$, we have $\tilde{\mathscr{C}}^{(k)}_L\supset\hat{\mathscr{C}}^{(k)}_L$ and $\tilde{\mathscr{C}}^{(k)}_L\backslash\hat{\mathscr{C}}^{(k)}_L=T_{-kr}\mathscr{C}_L\cap\mathscr{J}_{k-1}\cap\mathscr{G}^c_{k-1}\supset\mathscr{J}_{k-1}\cap\mathscr{G}^c_{k-1}$. It is easy to see that $\bigcup_{k=0}^{\infty}(\mathscr{J}_k\cap\mathscr{G}^c_k)\subset\bigcup_{k=0}^{\infty}\Delta_k$ where $\Delta_k=(\hat{\mathscr{E}}^{(k)}_L)^c\cap\mathscr{F}^{(k)}_L\cap\mathscr{G}_{k-1}$. Setting $\mathscr{N}_L=\{x:q_0(x)<L\}$ we have $\mu\left(\bigcup_{k=0}^{\infty}\Delta_k\right)\leqq\mu(\mathscr{N}^c_L)+\sum_{k=0}^{\infty}\mu(\Delta_k\cap\mathscr{N}_L)$ and

$$\begin{aligned}\mu(\Delta_k\cap\mathscr{N}_L)&=\mathbb{E}(\chi_{\mathscr{G}_{k-1}}(x)\chi_{\mathscr{N}_L}(x)\,(1-\chi_{\hat{\mathscr{E}}(k)}(x))\mu(\mathscr{F}^{(k)}_L|\hat{\zeta}^{(k)}_L))\\&\leqq\mu^0(\mathscr{F}_L|\mathscr{A}_L)\,(\mu((\hat{\mathscr{E}}^{(k)}_L)^c)\mu(\mathscr{G}_{k-1}\cap\mathscr{N}_L))^{1/2}\end{aligned}\tag{4.4}$$

where we have used the Schwartz inequality, and the fact that $\mathscr{N}_L$, $\hat{\mathscr{E}}^{(k)}_L\in\hat{\zeta}^{(k)}_L$ and that for $x\in\mathscr{N}_L$, $\mu(\mathscr{F}^{(k)}_L|\hat{\zeta}^{(k)}_L(x))=\mu^0(\mathscr{F}^{(k)}_L|T^0_{-kr}\mathscr{A}_L)=\mu^0(\mathscr{F}_L|\mathscr{A}_L)$. Since $\mu(\mathscr{G}_{k-1}\cap\mathscr{N}_L)\leqq\mu(\mathscr{G}_{k-1}|\mathscr{N}_L)=(1-\mu^0(\mathscr{F}_L))^{k-1}$, we get, for L large $\mu(\Delta_k\cap\mathscr{N}_L)\leqq(\varepsilon/8)\,\mu^0(\mathscr{F}_L)\,(1-\mu^0(\mathscr{F}_L))^{(k-1)/2}$, and hence

$$\begin{aligned}\mu\left(\bigcup_{k=0}^{\infty}\Delta_k\right)&\leqq\mu(\mathscr{N}^c_L)+(\varepsilon/8)\mu^0(\mathscr{F}_L)\sum_{k=0}^{\infty}(1-\mu^0(\mathscr{F}_L))^{k/2}\\&<\mu(\mathscr{N}^c_L)+(\varepsilon/4)\end{aligned}\tag{4.5}$$

since $a\sum_{k=0}^{\infty}(1-a)^{k/2}<2$ for all $a\in(0,1)$. Since $\mu(\mathscr{N}^c_L)\xrightarrow[L\to\infty]{}0$ the result follows.

Proposition 4.3. *Let $B \in \zeta^\tau$, $\tau > (k+1)r$. Then there are numbers b_k such that*

$$\mu(B \cap \hat{\mathscr{C}}_L^{(k)} | \zeta_0(x)) = b_k \mu(\hat{\mathscr{C}}_L^{(k)} | \zeta_0(x)) (1 + \hat{\eta}_1(x)) \tag{4.6}$$

and for any $\varepsilon > 0$ if L is large enough we can assume $|\hat{\eta}_1(x)| < \varepsilon$, for all $k < (\tau/r) - 1$.

Proof. By Remark 4.1, observing that an atom of $\hat{\zeta}_L^{(k)}$ contained in $T_{-kr}(\mathscr{E}'_L \cap \mathscr{A}_L)$ evolves into an atom of $\hat{\zeta}_L = \hat{\zeta}_L^{(0)}$ (see Proposition 4.1) contained in $\mathscr{E}'_L \cap \mathscr{A}_L$, we have

$$\begin{aligned} \mu(B \cap \hat{\mathscr{C}}_L^{(k)} | \hat{\zeta}_L^{(k)}(x)) &= \chi_{\mathscr{G}_{k-1}}(x) \mu(\mathscr{C}_L \cap T_{-kr} B | \hat{\zeta}_L(T_{kr}x)) \\ &= \chi_{\mathscr{G}_{k-1}}(x) \mu(\mathscr{C}''_L \cap T_{-kr} B | \hat{\zeta}_L(T_{kr}x)) (1 - \hat{\eta}_2) \end{aligned} \tag{4.7}$$

with $\hat{\eta}_2 \in (0, \varepsilon/2)$ [we have used Definition 4.2 and Ineq. (4.2)]. Since, for $x \in \mathscr{C}''_L$ $y_1(x) \in Y_L^{(1)}$ and τ^* is a c.t., the history of the h.p. for $t > \tau^*$ depends only on y_1 and $z'(x) = x_{C_L(r, \infty)} \in T_{-r}^0 Z_y$. Therefore, reasoning as in Proposition 3.4 [Eq. (3.14)], and observing again that an atom of $\hat{\zeta}_L$ contained in $\mathscr{E}_L \cap \mathscr{A}_L$ coincides with a corresponding atom of $\mathfrak{M}_{R'_L}$, we find, setting $x_1 = (T_{kr}x)_{R_L}$,

$$\mu(\mathscr{C}''_L \cap T_{-kr} B | \hat{\zeta}_L(T_{kr}x)) = \mu(\mathscr{C}_L | \hat{\zeta}_L(T_{kr}x)) \int_{Y_L^{(1)}} \ell_{x_1}^{(1)}(dy_1) \mu^0(B_{y_1}^{(k)} | \mathscr{A}_L) \tag{4.8}$$

with

$$B_{y_1}^{(k)} = \{x : (y_1(x), z'(x)) \in B^{(k)}\},$$

where

$$B^{(k)} = \{(y_1(x), z'(x)) : x \in \mathscr{C}''_L \cap T_{-kr} B\}.$$

By Proposition 3.2 we find, putting together Eqs. (4.7) and (4.8), and setting $b_k = \int_{Y_L^{(1)}} \ell^{(1)}(dy_1) \mu^0(B_{y_1}^{(k)} | \mathscr{A}_L)$

$$\begin{aligned} \mu(B \cap \hat{\mathscr{C}}_L^{(k)} | \hat{\zeta}_L^{(k)}(x)) &= \chi_{\mathscr{G}_{k-1}}(x) \mu(\mathscr{C}_L | \hat{\zeta}_L(T_{kr}x)) b_k (1 + \hat{\eta}'_1) \\ &= \mu(\hat{\mathscr{C}}_L^{(k)} | \hat{\zeta}_L^{(k)}(x)) b_k (1 + \hat{\eta}'_1). \end{aligned}$$

with $|\hat{\eta}'_1| < \varepsilon$ if L is large enough and δ small enough. Equation (4.6) follows by taking conditional expectations.

To accomplish the proof of Eq. (4.1) we need the following result.

Proposition 4.4. *For any $\varepsilon > 0$, if L is large enough*

$$\mu\left(\left\{x : \sum_{k=0}^{\infty} |\mu(\hat{\mathscr{C}}_L^{(k)} | \zeta_0(x)) - \mu(\hat{\mathscr{C}}_L^{(k)})| > \varepsilon\right\}\right) < \varepsilon, \tag{4.9}$$

Proof. With the notation of Proposition 4.2, and by Definition 4.2, we have $\hat{\mathscr{C}}_L^{(k)} \cap \Delta_k = \emptyset$ and $\hat{\mathscr{C}}_L^{(k)} \cup \Delta_k = \mathscr{F}_L^{(k)} \cap \mathscr{G}_{k-1} \equiv \mathscr{H}_k$, which implies

$$\mu(\hat{\mathscr{C}}_L^{(k)} | \hat{\zeta}_L) - \mu(\hat{\mathscr{C}}_L^{(k)}) = \mu(\mathscr{H}_k | \hat{\zeta}_L) - \mu(\mathscr{H}_k) - \mu(\Delta_k | \hat{\zeta}_L) + \mu(\Delta_k).$$

Moreover for $x \in \mathscr{N}_L$ $\mu(\mathscr{H}_k | \hat{\zeta}_L(x)) = \mu(\mathscr{H}_k | \mathscr{N}_L) = \mu^0(\mathscr{H}_k)$, so that for $x \in \mathscr{N}_L$ $\mu(\mathscr{H}_k | \hat{\zeta}_L(x)) - \mu(\mathscr{H}_k) = \mu(\mathscr{H}_k | \mathscr{N}_L) \mu(\mathscr{N}_L^c)$. Taking expectations, since $\zeta_0 \subset \hat{\zeta}_L$, we see that

$$\mu(\hat{\mathscr{C}}_L^{(k)}|\zeta_0)-\mu(\hat{\mathscr{C}}_L^{(k)})=\mu(\mathscr{H}_k\cap\mathscr{N}_L^c|\zeta_0)-\mu(\mathscr{H}_k)\mu(\mathscr{N}_L^c)$$
$$+(\mu(\mathscr{N}_L^c))^2\mu(\mathscr{H}_k|\mathscr{N}_L)-\mu(\Delta_k|\zeta_0)+\mu(\Delta_k).$$

Summing up, and taking into account the fact that the Δ_k's and the $\mathscr{H}_k$'s are disjoint, we find

$$\sum_{k=0}^{\infty}|\mu(\hat{\mathscr{C}}_L^{(k)}|\zeta_0)-\mu(\hat{\mathscr{C}}_L^{(k)})|\leqq\mu(\mathscr{N}_L^c|\zeta_0)+\mu(\mathscr{N}_L^c)+\mu\left(\bigcup_{k=0}^{\infty}\Delta_k\right)$$
$$-\mu\left(\bigcup_{k=0}^{\infty}\Delta_k|\zeta_0\right),$$

and the result follows, as in Proposition 4.2, from Ineq. (4.5) and the fact that $\lim_{L\to\infty}\mu(\mathscr{N}_L^c)=0.$

Proof of Theorem 2.1. Let $\eta>0$ be a small number. By Propositions 4.1 and 4.4 we can choose the parameters in such a way that $\mu\left(\bigcup_{k=0}^{\infty}\mathscr{C}_L^{(k)}\right)>1-\eta$, and the measure of the set $\mathscr{M}_1$, for which $\sum_{k=0}^{\infty}|\mu(\mathscr{C}_L^{(k)}|\zeta_0(x))-\mu(\mathscr{C}_L^{(k)})|<\eta$ and $\mu\left(\bigcup_{k=0}^{\infty}\hat{\mathscr{C}}_L^{(k)}|\zeta_0(x)\right)>1-\eta$, is larger than $1-\eta$. Moreover we can assume that $|\hat{\eta}_1|<\eta$ (Proposition 4.3). For $x\in\mathscr{M}_1$ we have

$$|\mu(B|\zeta_0(x))-\mu(B)|\leqq\sum_{k=0}^{\infty}|\mu(B\cap\hat{\mathscr{C}}_L^{(k)}|\zeta_0(x))-\mu(B\cap\mathscr{C}_L^{(k)})|+2\eta.$$

Moreover, by Proposition 4.3, since $b_k\in[0,1]$, for $x\in\mathscr{M}_1$,

$$\sum_{k=0}^{[\frac{\tau}{r}]-1}|\mu(B\cap\hat{\mathscr{C}}_L^{(k)}|\zeta_0(x))-\mu(B\cap\hat{\mathscr{C}}_L^{(k)})|$$
$$\leqq\sum_{k=0}^{[\frac{\tau}{r}]-1}b_k|\mu(\hat{\mathscr{C}}_L^{(k)}|\zeta_0(x))-\mu(\hat{\mathscr{C}}_L^{(k)})|+2\eta<3\eta.$$

Hence

$$|\mu(B|\zeta_0(x))-\mu(B)|<5\eta+\sum_{k=[\frac{\tau}{r}]}^{\infty}(\mu(\hat{\mathscr{C}}_L^{(k)}|\zeta_0(x))+\mu(\hat{\mathscr{C}}_L^{(k)})).$$

Since $\lim_{k_0\to\infty}\sum_{k=k_0}^{\infty}\mu(\mathscr{C}_L^{(k)})=0$, if τ is large enough $|\mu(B|\zeta_0(x))-\mu(B)|<6\eta$ on a set $\mathscr{M}_2$, $\mu(\mathscr{M}_2)>1-2\eta$. Taking $\eta=\varepsilon/2$, if $3\varepsilon<a$, we get

$$\limsup_{\tau\to\infty}\sup_{B\in\zeta^\tau}\mu(\{x:|\mu(B|\zeta_0(x))-\mu(B)|>a\})<\mu(\mathscr{M}_2^c)<\varepsilon.$$

Since ε is arbitrarily small, Eq. (2.5), and hence Theorem 2.1, are proved.

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