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(Higgs)$_{2,3}$ Quantum Fields in a Finite Volume

III. Renormalization

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Abstract. This is the third paper of a series, and contains a proof of the bounds on the effective actions needed in the two previous papers. The proof is based on perturbative analysis of renormalization.

1. Introduction. Formulation of a Basic Estimate

In this paper we will prove the basic properties of the perturbation expansions used in the two previous papers [1, 2]. These properties form a very important part of the proof of ultraviolet stability. We will prove them for the considered model, but the method extends in a natural way to more complicated models. The characteristic feature of the method is that it almost does not use a momentum representation for the expression in the perturbation expansion. It is based entirely on real space properties of propagators. These properties hold for propagators in more complicated theories also (e.g. non-abelian gauge theories, fermion field theories) and it is the reason for these natural extensions.

In this paper we use notation, results and formulas of the two previous papers [1, 2] and we will refer to them adding I or II before the numberings of the corresponding papers. Many properties of perturbative expressions were used in [1, 2] and it is difficult to formulate them in the form of separate theorems; the formulations would be very long. Instead we will describe and analyze a general expression and we will prove for it some basic estimate from which all the necessary properties will follow easily. Let us recall the formula for interaction terms of the action of the model after k renormalization transformations and let us write it in the form appearing in the proof of the upper bound. We will write this formula rescaled to the unit lattice with respect to "new" field variables after k steps. The torus T_η is replaced by the corresponding subset $B^k(\Lambda_2^{(k-1)\prime})$, which will be denoted by Ω. The set $\Lambda_2^{(k-1)\prime}$ is a sum of big blocks, so the assumptions of Propositions I.2.1–I.2.3 are satisfied for Ω. The "old" vector field can be represent-

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ed in the following way:

$$A = A' + A^{(k)} = A'^{(0),\eta} + \ldots + A'^{(k-1),\eta} + A^{(k)}, \qquad \eta = L^{-k}, \tag{1.1}$$

where

$$A'^{(j),\eta} = a_j (L^j \eta)^{-2} G_j^\eta Q_j^* A_j', \qquad j = 1, \ldots, k-1, \qquad A'^{(0),\eta} = A_0', \tag{1.2}$$

and the fields A_j' are defined on $L^j\eta$-lattice. The formula (1.2) extends to $A^{(k)}$ with A_k' replaced by the "new" field A_k on the unit lattice. Let us remind the reader also of the definition (I.3.34) of $A(\Gamma_{y,x}^{(k)})$, $x \in B^k(y)$:

$$A(\Gamma_{y,x}^{(k)}) = \sum_{j=0}^{k-1} A'^{(j),\eta}(\Gamma_{x_{j+1},x}^{(j+1)}) + A^{(k)}(\Gamma_{y,x}^{(k)}), \tag{1.3}$$

where x_j is defined by the condition $x \in B^j(x_j)$, $x_0 = x$, and the contours are defined in Chap. I.2. For an arbitrary vector field A defined on η-lattice and an arbitrary contour Γ of this lattice we put $A(\Gamma) = \sum_{b \subset \Gamma} \eta A_b$. Now we define an auxiliary function

$$\begin{aligned} E_k(e', \lambda', \Omega, A^{(k)}, \phi) = -\log \Bigg[\prod_{j=0}^{k-1} \int d\mu_{C^{(j)}, L^j\eta}(A_j') T^\eta_{a_k, L^k, e' g_k A' + A^{(k)}} \\ \cdot \Bigg[\exp\Big(-\tfrac{1}{2} \langle \phi', (-\Delta^\eta_{e' g_k A' + A^{(k)}} + m^2 (L^k \varepsilon)^2) \phi' \rangle \\ - \lambda' \lambda (L^k \varepsilon) \sum_{x \in \Omega_1} \eta^d |\phi'(x)|^4 - \tfrac{1}{2} \sum_{x \in \Omega_1} \eta^d \delta m^2 (e', g_k, \lambda', \Omega_1, x) \\ \cdot (L^k \varepsilon)^2 |\phi'(x)|^2 - E_1(e', g_k, \lambda', \Omega_1) \Big) \Bigg] \Bigg], \end{aligned} \tag{1.4}$$

where $\Omega_1 = B^k(\Lambda_7^{(k-1)\prime})$ and $\delta m^2(e', g_k, \lambda', \Omega_1, x)$, $E_1(e', g_k, \lambda', \Omega_1)$ will be defined later. The function g_k is a smooth function with $\operatorname{supp} g_k \subset \Omega_1$, $g_k(x) = 1$ if $\operatorname{dist}(x, \partial\Omega_1) \geqq M$ and bounded by 1 together with derivatives up to second order. The interaction after k steps is given by the formula

$$\mathscr{P}^{(k)}(\Omega_1, A^{(k)}, \phi) = \sum_{1 \leqq \alpha + \beta \leqq \bar{n}} \frac{1}{\alpha! \beta!} \left(\frac{\partial^{\beta+\beta}}{\partial e'^\alpha \partial \lambda'^\beta} E_k(e', \lambda', \Omega, A^{(k)}, \phi) \right) \Bigg|_{e' = \lambda' = 0}. \tag{1.5}$$

We will give a graphical description of this expression, giving vertices and propagators, but at first let us describe how this expression is changed when the operations in the $k+1$ step are being done. In the first operation we remove the interaction from the set $B^k(\Lambda_7^{(k-1)\prime} \cap \Lambda_7^{(k)c})$ estimating it as in Proposition II.2.1. This estimate will follow easily from our general result on perturbation expressions with vertices localized in unit cubes. The next operation is the translation (II.2.80) and an expansion with respect to the small field $\theta_{k+1} A'^{(k)} + \tilde{B}^{(k)}$. Let us write the corresponding expansion for the expressions in (1.5) with $A^{(k)}$ replaced by $\tilde{A} + \tilde{B}$ and the expansion taken with respect to $\tilde{A}$, where $\tilde{A}$ and $\tilde{B}$ are quite general fields satisfying some regularity conditions. More precisely we will assume $|\tilde{A}|$, $|\partial^n \tilde{A}|$, $|\partial^n \tilde{B}|$ and their Hölder norms with exponent α_0 are $\leqq O(1) p(L^k \varepsilon)$ and $\operatorname{dist}(\operatorname{supp} \tilde{A}, \partial\Omega) > 2r(L^k \varepsilon)$. (In [2] the Hölder norms were not estimated, but similar considerations as for derivatives lead to the desired bounds.) The expansion we get can be represented graphically. Let us begin this description by writing down all the vertices. The first two describe self-interaction of scalar fields:

$$-\lambda(L^k\varepsilon)\sum_{x\in\Omega_1}\eta^d|\phi'(x)|^4,\ \phi'(x)\text{ is a scalar field leg}, \tag{1.6}$$

$$-\tfrac{1}{2}\sum_{x\in\Omega_1}\eta^d\delta m_i^2(x)\,(L^k\varepsilon)^2|\phi'(x)|^2,$$
$$\delta m_i^2(x)\text{ is one of the renormalization mass counterterms}. \tag{1.7}$$

The next group of vertices describes an interaction of scalar and vector fields:

$$(e(L^k\varepsilon))^{n+n'}\frac{(-1)^{n+n'}\eta^{n+n'-1}}{n!n'!}\sum_b\eta^d[(D_B^\eta\phi')(b)\cdot q^{n+n'}\phi'(b_-)]\,(g_kA_b')^n(\tilde{A}_b)^{n'},$$
$$n,n'\leqq\bar{n},\qquad n+n'\geqq 1, \tag{1.8}$$

and the corresponding R-vertices

$$(e(L^k\varepsilon))^{n+\bar{n}+1}\frac{(-1)^{n+\bar{n}+1}\eta^{n+\bar{n}}}{n!(\bar{n}+1)!}$$
$$\cdot\sum_b\eta^d[(D_B^\eta\phi')(b)\cdot q^{n+\bar{n}+1}R_{\bar{n}+1}(-\eta qe(L^k\varepsilon)\,\tilde{A}_b)\phi'(b_-)]\,(g_kA_b')^n(\tilde{A}_b)^{\bar{n}+1},\quad n\leqq\bar{n}. \tag{1.9}$$

Furthermore

$$(e(L^k\varepsilon))^{n+n'}\frac{\eta^{n+n'-2}}{n!n'!}\sum_b\eta^d[\phi'(b_-)\cdot q^{n+n'}\phi'(b_-)]\,(g_kA_b')^n(\tilde{A}_b)^{n'},$$
$$n+n'\text{ even},\qquad n,n'\leqq\bar{n},\qquad n+n'\geqq 2, \tag{1.10}$$

and the R-vertices

$$(e(L^k\varepsilon))^{n+\bar{n}+1}\frac{(-1)^{n+\bar{n}+1}\eta^{n+\bar{n}-1}}{n!(\bar{n}+1)!}$$
$$\cdot\sum_b\eta^d[\phi'(b_-)\cdot q^{n+\bar{n}+1}R_{\bar{n}+1}(-\eta qe(L^k\varepsilon)\tilde{A}_b)\phi'(b_-)]\,(g_kA_b')^n(\tilde{A}_b)^{\bar{n}+1}. \tag{1.11}$$

These vertices come from an expansion of the original lattice action. If k is so large that $L^k\varepsilon\approx 1$, $\eta\approx\varepsilon$, and if we take $\varepsilon\to 0$, then only the vertices (1.8) with $n+n'=1$ and (1.10) with $n+n'=2$ will survive the limit and will give the vertices of the classical continuous action; the rest will tend to 0. Unfortunately for fixed ε and η, $\varepsilon<\eta\leqq 1$, we have to take into account all of them. In fact, the renormalization counterterms are defined by the non-classical vertices also. The remaining vertices are connected with the expansion of the renormalization transformation for scalar fields. The expressions we get have always two summations over the η-lattice, so it is convenient to represent them by two vertices. Thus we will have

$$\phi(y),\qquad y\in T_1^{(k)}\cap\Omega, \tag{1.12}$$

$$-(Q_k(\tilde{B})\phi')(y)=-\sum_{x\in B^k(y)}\eta^dU(\tilde{B}(\Gamma_{y,x}^{(k)}))\phi'(x), \tag{1.13}$$

$$-(e(L^k\varepsilon))^{n+n'}\frac{1}{n!n'!}\sum_{x\in B^k(y)}\eta^d(A'(\Gamma_{y,x}^{(k)}))^n(\tilde{A}(\Gamma_{y,x}^{(k)}))^{n'}q^{n+n'}U(\tilde{B}(\Gamma_{y,x}^{(k)}))\phi'(x),$$
$$n,n'\leqq\bar{n},\qquad n+n'\geqq 1, \tag{1.14}$$

and the R-vertices

$$-(e(L^k\varepsilon))^{n+\bar{n}+1}\frac{1}{n!(\bar{n}+1)!}\sum_{x\in B^k(y)}\eta^d(A'(\Gamma^{(k)}_{y,x}))^n(\tilde{A}(\Gamma^{(k)}_{y,x}))^{\bar{n}+1}$$
$$\cdot q^{n+\bar{n}+1}R_{\bar{n}+1}(qe(L^k\varepsilon)\tilde{A}(\Gamma^{(k)}_{y,x}))U(\tilde{B}(\Gamma^{(k)}_{y,x}))\phi'(x)\,,\qquad n\leqq\bar{n}\,. \tag{1.15}$$

Now the expressions of the expansion are formed in the following way: we take an arbitrary pair of two expressions from (1.12)–(1.15), with the restriction that at least one of them has to be of the form (1.14) or (1.15), and we take their scalar product. We multiply it by $-a_k$ if the expressions are different, or by $-\frac{1}{2}a_k$ if they are equal, and finally we sum over $y\in T^{(k)}_1\cap\Omega$. In the sequel we will treat (1.13)–(1.15) as the vertices, and (1.12) is an external field. Thus we have described all the vertices of the perturbation expansion. Let us notice that the vertices of (1.5) are given by (1.6)–(1.8), (1.10), (1.13), and (1.14) with $n'=0$ and $\tilde{B}=A^{(k)}$. Finally let us recall that $R_{\bar{n}+1}(qA)$ is an analytic function of $A\in R$ with values in linear operators on R^N satisfying the inequality $|R_{\bar{n}+1}(qA)|\leqq 1$.

Next we have to describe propagators. From the formula (1.4) it follows that A'_j are independent Gaussian fields with covariances $C^{(j),L^j\eta}$. The formulas (1.1) and (1.2) and the basic composition formula (I.2.43) imply that the field A' has the covariance G_k. The basic propagator for the scalar field ϕ' is $G_k(\Omega,\tilde{B})$, but there are other propagators also. One of the operations in our procedure is a change of the set Ω. We replace it by another set $\Omega_2\subset\Omega$ satisfying the same conditions as Ω, and we replace the propagator $G_k(\Omega,\tilde{B})$ by $G_k(\Omega_2,\tilde{B})$, so we will have also the propagators $\delta G_k(\Omega,\Omega_2,\tilde{B})=G_k(\Omega,\tilde{B})-G_k(\Omega_2,\tilde{B})$. Finally if we expand the propagator $G_k(\Omega,\tilde{A}+\tilde{B})$ using the formulas (I.3.44), (I.3.45), then in the last term of this expansion, equal to

$$[G_k(\Omega,\tilde{B})V_k(\tilde{A},\tilde{B})]^n G_k(\Omega,\tilde{A}+\tilde{B})\,[V_k(\tilde{A},\tilde{B})G_k(\Omega,\tilde{B})]^{n'}\,, \tag{1.16}$$

we have the propagator $G_k(\Omega,\tilde{A}+\tilde{B})$. There are several possible ways of treating the above expression. Perhaps the simplest way is to treat it as an external field, because for n,n' sufficiently large, a kernel of the operator (1.16) is a sufficiently regular function of both variables. More exactly the Hölder norms of the covariant derivatives of this kernel, the norms defined for example in the inequalities (I.2.24) and (I.2.25) of Proposition I.2.1, are exponentially decaying with the distance of the arguments and are uniformly bounded by $O(1)(e(L^k\varepsilon)^{1-\alpha})^{n+n'}$, where $\alpha>0$ but can be arbitrarily small. This estimate follows easily from the properties of the propagators $G_k(\Omega,A)$ proved in the next paper. Also the propagator $\delta G_k(\Omega,\Omega_2,\tilde{B})$ will be treated as an external field when the estimates of perturbative expressions will be considered.

Now we can describe an arbitrary expression in the expansion. It consists of a number of vertices. The maximal number of vertices depends in a simple way on $\bar{n}$, but it is unessential here. All the A'-legs are contracted, i.e. they are divided into pairs and each pair is replaced by the corresponding propagator. Some ϕ'-legs are replaced by external scalar fields and the remaining are again divided into pairs and each pair is replaced by a propagator, i.e. by $G_k(\Omega,\tilde{B})$, $G_k(\Omega_2,\tilde{B})$, $\delta G_k(\Omega,\Omega_2,\tilde{B})$ or the operator (1.16). Of course the whole expression is multiplied by a proper combinatorial factor connected with the number of ways given expression can be obtained from Gaussian integrals in (1.5). The external scalar field is $a_k G_k(\Omega, \tilde{B}) Q_k(\tilde{B})\phi$ or (1.12), but we can easily add the field $G_k(\Omega, \tilde{B})f$ coming from external sources in the generating functional for Schwinger functions. Later we will consider a much wider class of external fields.

Now let us introduce a graphical description for the perturbative expressions. Let us denote:

an external scalar field, or a leg in a vertex

an external vector field, or a leg in a vertex

the scalar field propagator $G_k(\Omega, \tilde{B})$

the vector field propagator G_k

the operator (1.13) acting on a scalar field leg

the operator Q_k acting on a vector field leg

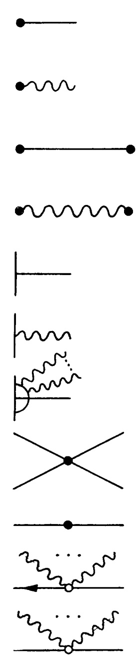
the vertices (1.14) or (1.15)

the vertex (1.6)

the vertex (1.7)

the vertices (1.8) or (1.9)

the vertices (1.10) or (1.11)



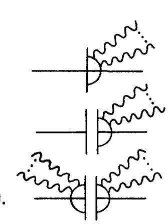
(1.17)

We introduce also some further notational conventions concerning the expressions obtained by the expansion of the kernel of the renormalization transformations. They are obtained by taking a scalar product of two expressions from (1.12)–(1.15) and multiplying it by $-a_k$, or $-\frac{1}{2}a_k$ if they are equal. We denote them as follows:

a product of (1.12) and (1.14) or (1.15)

a product of (1.13) and (1.14) or (1.15)

a product of two factors of the form (1.14) or (1.15).



(1.18)

The above notations in (1.17) and (1.18) are not precise, but they can be made quite precise if we specify a number and nature of legs. Now a graph for us is a collection of internal lines, external legs, and vertices connected in the usual sense. There is at least one internal line, and every internal line has a vertex at each endpoint. The construction of graphs is otherwise arbitrary. We will never describe more precisely the external fields however. In the sequel we will apply the same graphical description to slightly changed expressions.

To complete the definition of the expressions (1.4) and (1.5) we have to define the mass renormalization and vacuum energy counterterms. Let us begin with the mass renormalization counterterm δm^2. It is determined by the two-point Schwinger function

$$G^{\varepsilon}_{ab}(x,x')=\langle \phi_a(x)\phi_b(x')\rangle^{\varepsilon}=(Z^{\varepsilon})^{-1}\int dA\int d\phi e^{-S^{\varepsilon}(A,\phi)}\phi_a(x)\phi_b(x'), \qquad x,x'\in T_{\varepsilon}, \tag{1.19}$$

where $S^{\varepsilon}(A,\phi)$ is the lattice action of the model given by

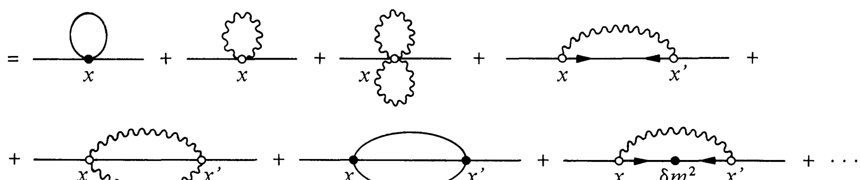
$$\begin{aligned}S^{\varepsilon}(A,\phi)=\tfrac{1}{2}\langle\phi,(-\Delta^{\varepsilon}_A+m^2)\phi\rangle+\sum_{x\in T_{\varepsilon}}\varepsilon^d(\lambda|\phi(x)|^4+\tfrac{1}{2}\delta m^2|\phi(x)|^2)\\ +\tfrac{1}{2}\langle A,(-\Delta^{\varepsilon}+\mu_0^2)A\rangle,\end{aligned} \tag{1.20}$$

and Z^{ε} is the partition function. The function G^{ε} has a perturbative expansion of the following structure

$$G^{\varepsilon}=\sum_{n=0}^{\infty}C^{\varepsilon}_0[(-\delta m^2+\Sigma^{\varepsilon}+\partial^{\varepsilon *}\Sigma^{\varepsilon}_1+\Sigma^{\varepsilon *}_1\partial^{\varepsilon}+\partial^{\varepsilon *}\Sigma^{\varepsilon}_2\partial^{\varepsilon})C^{\varepsilon}_0]^n, \tag{1.21}$$

where $C^{\varepsilon}_0=(-\Delta^{\varepsilon}_0+m^2)^{-1}$ and $\Sigma^{\varepsilon},\Sigma^{\varepsilon}_1,\Sigma^{\varepsilon}_2$ are given by amputated, one-particle-irreducible graphs of the expansion of G^{ε}. Here we have a graphical description of the same type as in (1.17), but with some simplifications. We have $\eta=\varepsilon$ (hence $L^k\varepsilon=1$) and the only vertices are (1.6), (1.7) [with δm^2 instead of $\delta m_i^2(x)$], (1.8), and (1.10) with $n'=0$, $\tilde{B}=0$, $g_k=1$ (but without any restrictions on n). The propagators are C^{ε}_0 for the scalar field and $C^{\varepsilon}=(-\Delta^{\varepsilon}+\mu_0^2)^{-1}$ for the vector field (of course internal indices and vector indices are understood here). Let us write a few terms of the expansion of Σ^{ε}:

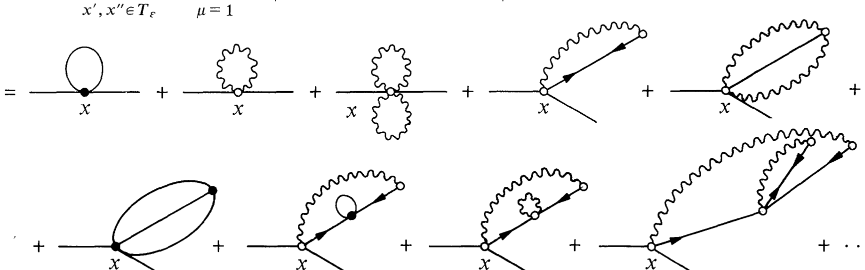
$$\begin{aligned}\Sigma^{\varepsilon}(x-x')&=-4(N+2)\lambda C^{\varepsilon}_0(0)\delta^{\varepsilon}(x-x')+e^2dC^{\varepsilon}(0)q^2\delta^{\varepsilon}(x-x')\\ &+\frac{2\cdot 3}{4!}de^4\varepsilon^2(C^{\varepsilon}(0))^2q^4\delta^{\varepsilon}(x-x')-e^2\sum_{\mu=1}^{d}q(\partial^{\varepsilon}_{\mu}C^{\varepsilon}_0\partial^{\varepsilon *}_{\mu})(x-x')qC^{\varepsilon}(x-x')\\ &+2de^4q^2C^{\varepsilon}_0(x-x')q^2(C^{\varepsilon}(x-x'))^2+4^2(2N+4)\lambda^2(C^{\varepsilon}_0(x-x'))^3\\ &+e^2\sum_{\mu=1}^{d}\sum_{x''\in T_{\varepsilon}}\varepsilon^d q(\partial^{\varepsilon}_{\mu}C^{\varepsilon}_0)(x-x'')\delta m^2(C^{\varepsilon}_0\partial^{\varepsilon *}_{\mu})(x''-x')qC^{\varepsilon}(x-x')+\dots\end{aligned}$$



(1.22)

Here we did not write, and we will not write in the future, combinatoric factors before the graphs, understanding that they are a part of the graphical description. It is easily seen that the expressions in (1.22) are divergent as $\varepsilon\to 0$ (except the third). It will follow from our future considerations that $\Sigma^{\varepsilon}_1,\Sigma^{\varepsilon}_2$ are convergent and the terms of order higher than 4 (in coupling constants) in the expansion of Σ^ε are convergent also. The mass renormalization counterterm δm^2 is chosen in such a way that $-\delta m^2+\Sigma^\varepsilon$ is convergent. Of course this condition does not determine δm^2 uniquely, and usually it is defined as a solution of the equation $-\delta m^2+\sum\limits_{x\in T_\varepsilon}\varepsilon^d\Sigma^\varepsilon(x)=0$. This equation can be solved recursively if δm^2 and Σ^ε are expanded into power series in e,λ. In our case δm^2 will be defined by the terms of order $\leqq 4$. More exactly we write $\delta m^2=\sum\limits_{2\leqq\alpha+2\beta\leqq 4} e^\alpha\lambda^\beta\delta m^2_{(\alpha,\beta)}$ and we insert this into Σ^ε. This gives us an expansion of Σ^ε in coupling constants and we take a sum of terms of order $\leqq 4$: $\sum\limits_{2\leqq\alpha+2\beta\leqq 4} e^\alpha\lambda^\beta\Sigma^\varepsilon_{(\alpha,\beta)}$. The counterterms $\delta m^2_{(\alpha,\beta)}$ are defined by the equations $-\delta m^2_{(\alpha,\beta)}+\sum\limits_{x\in T_\varepsilon}\varepsilon^d\Sigma^\varepsilon_{(\alpha,\beta)}(x)=0$. It is convenient to write them in a different way. A term $e^\alpha\lambda^\beta\Sigma^\varepsilon_{(\alpha,\beta)}$ can be written also as a sum of terms Σ^ε_G, the summation over a family of one-particle-irreducible graphs with two external legs of scalar fields. The same for the term $e^\alpha\lambda^\beta\delta m^2_{(\alpha,\beta)}$, and we define δm^2_G by the equation $-\delta m^2_G+\sum\limits_{x\in T_\varepsilon}\varepsilon^d\Sigma^\varepsilon_G(x)=0$. This implies some obvious graphical representation of the terms of δm^2. If a graph G representing $\Sigma^\varepsilon_G(x-x')$ has the external legs localized in x,x', then $\delta m^2_G=\sum\limits_{x'\in T_\varepsilon}\varepsilon^d\Sigma^\varepsilon_G(x-x')$ will be represented by the same graph G but with both external legs localized in x and with the summation over x'. For example the expressions in (1.22) define the following counterterms:

$$\begin{aligned}\delta m^2=&-4(N+2)\lambda C^\varepsilon_0(0)+e^2dC^\varepsilon(0)q^2+\frac{2\cdot 3}{4!}de^4\varepsilon^2(C^\varepsilon(0))^2q^4\\&-e^2\sum_{x'\in T_\varepsilon}\varepsilon^d\sum_{\mu=1}^{d}q(\partial^\varepsilon_\mu C^\varepsilon_0\partial^{\varepsilon *}_\mu)(x-x')qC^\varepsilon(x-x')\\&+2de^4\sum_{x'\in T_\varepsilon}\varepsilon^dq^2C^\varepsilon_0(x-x')q^2(C^\varepsilon(x-x'))^2+4^2(2N+4)\lambda^2\sum_{x'\in T_\varepsilon}\varepsilon^d(C^\varepsilon_0(x-x'))^3\\&+e^2\sum_{x',x''\in T_\varepsilon}\varepsilon^{2d}\sum_{\mu=1}^{d}q(\partial^\varepsilon_\mu C^\varepsilon_0)(x-x'')\delta m^2_1(C^\varepsilon_0\partial^{\varepsilon *}_\mu)(x''-x')qC^\varepsilon(x-x')+\dots\end{aligned}$$



(1.23)

where δm^2_1 denote a sum of terms $\delta m^2_{(\alpha,\beta)}$ of the order $\alpha+2\beta=2$. Thus we have determined the counterterm δm^2.

Now it is easy to define the vacuum energy counterterm E_1. It is defined by the following perturbation expansion:

$$E_1=\sum_{1\leqq\alpha+\beta\leqq\bar n}\frac{1}{\alpha!\,\beta!}e^\alpha\lambda^\beta\left(\frac{\partial^{\alpha+\beta}}{\partial e^\alpha\partial\lambda^\beta}\log\int dA\int d\phi\, e^{-S^\varepsilon(A,\phi)}\right)\Bigg|_{e=\lambda=0}\tag{1.24}$$

with $\bar{n}>12$. In S^ε, defined in (I.1.11), we of course have dropped the term E. Terms of this expansion are described by connected graphs without external legs (vacuum graphs). To renormalize the theory it is sufficient to take the terms in the expansion (1.24) restricted by the condition $2\leqq\alpha+2\beta\leqq 6$; the other terms are convergent as $\varepsilon\to 0$.

Let us define now the expressions $\delta m^2(e', g_k, \lambda', \Omega_1, x)$, $E_1(e', g_k, \lambda', \Omega_1)$ appearing in (1.4). Let us recall that we finish our procedure for $k=K$, K is defined by the conditions $L^K\varepsilon\leqq\varepsilon_0$, $L^{K+1}\varepsilon>\varepsilon_0$, where $\varepsilon_0>0$ is some fixed sufficiently small number, independent of $\varepsilon, T_\varepsilon$. At first we replace the propagators $C^\varepsilon, C_0^\varepsilon$ in the perturbative expansions of δm^2, E_1 by $G_K^\varepsilon, G_K^\varepsilon(0)$ correspondingly, where $G_K^\varepsilon(0) =(-\Delta_0^\varepsilon+m^2+a_K(L^K\varepsilon)^{-2}P_K)^{-1}$ and a similar formula for G_K. We have

$$C_0^\varepsilon = G_K^\varepsilon(0)+a_K(L^K\varepsilon)^{-2}G_K^\varepsilon(0)P_K C_0^\varepsilon, \tag{1.25}$$

similarly for C^ε and the second term on the right side above is an operator with a regular kernel, as it follows from Proposition I.2.1 and the properties of C_0^ε. "Regular" means here that the localized Hölder norms of the derivatives for both variables are uniformly bounded and exponentially decaying, i.e. as in the formulation of Proposition I.2.1. It will follow from our future analysis of perturbative expansions that only a finite error is made by considering propagators G_K^ε instead of C^ε. More exactly we have

$$\delta m^2 = \delta m_{\mathrm{fin}}^2(x)+\delta m_K^2(x), \qquad E_1 = E_{\mathrm{fin}}+E_K, \tag{1.26}$$

where $\delta m_K^2(x)$, E_K are given by the same expressions (same graphs) as δm^2, E_1, but with propagators G_K^ε, $G_K^\varepsilon(0)$ instead of C^ε, C_0^ε, and $\delta m_{\mathrm{fin}}^2(x)$, E_{fin} are convergent as $\varepsilon\to 0$. The propagators G_K^ε, $G_K^\varepsilon(0)$ are not translation invariant, so we have the dependence on x. The terms $\delta m_K^2(x)$, E_K are transformed further. We have the following equality [the normalization group equation (I.2.43)]

$$\begin{aligned} G_K^\varepsilon(0) &= \sum_{j=1}^{K-1} a_j^2(L^j\varepsilon)^{-4}G_j^\varepsilon(0)Q_j^* C^{(j), L^j\varepsilon}(0)Q_j G_j^\varepsilon(0)+C^{(0),\varepsilon}(0) \\ &= G_k^\varepsilon(0)+\sum_{j=k}^{K-1} a_j^2(L^j\varepsilon)^{-4}G_j^\varepsilon(0)Q_j^* C^{(j), L^j\varepsilon}(0)Q_j G_j^\varepsilon(0) \\ &= G_k^\varepsilon(0)+G_{K,k}^\varepsilon(0), \end{aligned} \tag{1.27}$$

and the same for G_K^ε. We substitute it in $\delta m_K^2(x)$, E_K and we get the following decomposition

$$\delta m_K^2(x) = \delta m_{K,k}^2(x)+\delta m_k^2(x), \qquad E_K = E_{K,k}+E_k, \tag{1.28}$$

where $\delta m_k^2(x)$, E_k have the same form as $\delta m_K^2(x)$, E_K, only the propagators are replaced by $G_k^\varepsilon(0)$, G_k^ε correspondingly, and $\delta m_{K,k}^2(x)$, $E_{K,k}$ are formed by the remaining terms. Future considerations will imply that $\delta m_{K,k}^2(x)=O((L^k\varepsilon)^{-1})$, hence $\delta m_{K,k}^2(x)(L^k\varepsilon)^2=O(L^k\varepsilon)$. Now it is convenient to rescale the expressions defining the counterterms $\delta m_k^2(x)$, E_k from ε-lattice to η-lattice. The counterterms $\delta m_k^2(x)(L^k\varepsilon)^2$, E_k are then transformed into counterterms $\delta m_k^2(e(L^k\varepsilon), \lambda(L^k\varepsilon), x)$, $E_k(e(L^k\varepsilon), \lambda(L^k\varepsilon))$ by rescaling from the ε-lattice to the η-lattice. Again they have the same form as $\delta m_k^2(x)$, E_k, only with ε replaced by η and the coupling constants e, λ replaced by $e(L^k\varepsilon)$, $\lambda(L^k\varepsilon)$. The final transformation is the localization. We localize the vertices (1.6) and (1.7) restricting summations to the set Ω_1. The other vertices are localized by functions g_k multiplying each leg of the vector field (or each coupling constant e). We get counterterms $\delta m_k^2(e(L^k\varepsilon)g_k, \lambda(L^k\varepsilon), \Omega_1, x)$, $E_k(e(L^k\varepsilon)g_k, \lambda(L^k\varepsilon), \Omega_1)$, and we define

$$\begin{aligned}\delta m^2(e', g_k, \lambda', \Omega_1, x)(L^k\varepsilon)^2 &= \delta m_{\mathrm{fin}}^2(e', \lambda', x)(L^k\varepsilon)^2 + \delta m_{K,k}^2(e', \lambda', x)(L^k\varepsilon)^2 \\ &\quad + \delta m_k^2(e'e(L^k\varepsilon)g_k, \lambda'\lambda(L^k\varepsilon), \Omega_1, x), \end{aligned} \tag{1.29}$$

$$E_1(e', g_k, \lambda', \Omega_1) = E_{\mathrm{fin}}(e', \lambda') + E_{K,k}(e', \lambda') + E_k(e'e(L^k\varepsilon)g_k, \lambda'\lambda(L^k\varepsilon), \Omega_1), \tag{1.30}$$

the dependence on e', λ' is obtained by replacing e, λ by $e'e, \lambda'\lambda$. Let us illustrate the definition (1.29) for one of the basic counterterms in (1.23):

$$-e^2 \sum_{x' \in T_\varepsilon} \varepsilon^d \sum_{\mu=1}^{d} q(\partial_\mu^\varepsilon C_0^\varepsilon \partial_\mu^{\varepsilon *})(x-x')qC^\varepsilon(x-x')(L^k\varepsilon)^2, \qquad x \in T_\varepsilon,$$

is replaced by

$$\begin{aligned}&\Big[-e^2 a_k (L^K\varepsilon)^{-2} \sum_{x' \in T_\varepsilon} \varepsilon^d \sum_{\mu=1}^{d} q(\partial_\mu^\varepsilon G_K^\varepsilon(0) P_K C_0^\varepsilon \partial_\mu^{\varepsilon *})(x, x')qC^\varepsilon(x-x')(L^k\varepsilon)^2 \\ &\quad - e^2 a_k (L^K\varepsilon)^{-2} \sum_{x' \in T_\varepsilon} \varepsilon^d \sum_{\mu=1}^{d} q(\partial_\mu^\varepsilon G_K^\varepsilon(0) \partial_\mu^{\varepsilon *})(x, x')q(G_K^\varepsilon P_K C^\varepsilon)(x, x')(L^k\varepsilon)^2\Big] \\ &\quad + \Big[-e^2 \sum_{x' \in T_\varepsilon} \varepsilon^d \sum_{\mu=1}^{d} q(\partial_\mu^\varepsilon G_{K,k}^\varepsilon(0) \partial_\mu^{\varepsilon *})(x, x')qG_K^\varepsilon(x, x')(L^k\varepsilon)^2 \\ &\quad - e^2 \sum_{x' \in T_\varepsilon} \varepsilon^d \sum_{\mu=1}^{d} q(\partial_\mu^\varepsilon G_k^\varepsilon(0) \partial_\mu^{\varepsilon *})(x, x')qG_{K,k}^\varepsilon(x, x')(L^k\varepsilon)^2\Big] \\ &\quad + \Big[-(e(L^k\varepsilon))^2 \sum_{x' \in T_\eta} \eta^d \sum_{\mu=1}^{d} g_k((L^k\varepsilon)^{-1}x)q(\partial_\mu^\eta G_k(0) \partial_\mu^{\eta *})((L^k\varepsilon)^{-1}x, x') \\ &\quad \cdot qg_k(x')G_k((L^k\varepsilon)^{-1}x, x')\Big]. \end{aligned} \tag{1.31}$$

Thus we have finished the description of the perturbative expressions and now we will make some final preparations in order to formulate the basic theorem of this paper. At first let us notice that we have to estimate also some expressions obtained by an integration and a cumulant expansion in k^{th} step, where some external legs are contracted by propagators $C^{(k)}_{\Lambda_5^{(k)}}(B^k(\Lambda_2^{(k)}), \tilde{B})$, $C^{(k)}_{\Lambda_5^{(k)}}$, the others have the field ϕ replaced by $aL^{-2}C^{(k)}_{\Lambda_4^{(k)}}(B^k(\Lambda_2^{(k)}), \tilde{B})Q^*(\tilde{B})\psi$. In these expressions we have to change some domains of summation, remove some boundary conditions, again change domains of summation, and so on. Each time we have to estimate the differences. Also let us recall that in the estimates we treat $\delta G_k(\Omega, \Omega_2, \tilde{B})$ and the operators (1.16) as external fields. In order to have a formulation of the basic theorem covering all necessary cases we will consider external fields in the form of functions $\Phi_{\mathrm{ext}}(x, x', x'', \dots)$, $A_{\mathrm{ext}}(y, y', y'', \dots)$ of many variables instead of a product. We will need some norms for these functions. The basic norm is the Hölder norm $\|\cdot\|_{1,\alpha}$ appearing in the formulation of Proposition I.2.1. Let us recall the definition. For a scalar field f of one variable we define

$$\|f\|_{1,\alpha}=\sup_x|f(x)|+\sup_{x,\mu}|(D^{\eta}_{\tilde{B},\mu}f)(x)| + \sup_{x,x',\mu}\frac{1}{|x'-x|^{\alpha}}|U(\tilde{B}(\Gamma_{x,x'}))(D^{\eta}_{\tilde{B},\mu}f)(x')-(D^{\eta}_{\tilde{B},\mu}f)(x)|, \tag{1.32}$$

where $\Gamma_{x,x'}$ is a shortest contour connecting x and x'. This definition extends in a natural way to functions of many variables. For external vector fields we have the same definition, but with $\tilde{B}=0$. The next thing we need is a further localization in the vertices. For the vertices (1.6) and (1.7) we localize simply by representing Ω_1 as a sum of unit cubes of the η-lattice. For the remaining vertices we localize, taking for each leg of the vector field a smooth partition of unity satisfying the condition that a support of each function is contained in a cube with sides of length 2. We perform this localization for the terms of $\delta m_k^2(e(L^k\varepsilon)g_k,\lambda(L^k\varepsilon),\Omega_1,x)$, $E_k(e(L^k\varepsilon)g_k,\lambda(L^k\varepsilon),\Omega_1)$ also. The expressions (1.12)–(1.15) are localized by fixing a point $y\in T_1^{(k)}\cap\Omega$, or the unit cube $B^k(y)$. Thus with each expression there is connected some localization $\{\square(v)\}_{v\in G}$ (v is a vertex of a graph G corresponding to this expression).

Our last preparatory remark is most strictly connected with renormalization. We gather some graphs into families denoted G_{ren}. In the next chapter we will describe precisely how it is done; let us mention only that an ultimate aim is to cancel divergencies. For example if a graph G contains a subgraph G_0 with two external scalar legs and the expression corresponding to G_0 is divergent, then we add another graph G' which is obtained from G by replacing G_0 by the vertex (1.7) with the counterterm $\delta m^2_{G_0}(x)$. Let us denote by $E(G,\{\square(v)\}_{v\in G},\Phi_{\mathrm{ext}},A_{\mathrm{ext}})$ the expression corresponding to graph G with localizations $\{\square(v)\}_{v\in G}$ and external fields $\Phi_{\mathrm{ext}},A_{\mathrm{ext}}$. The same symbol with G_{ren} instead of G denotes a sum of these expressions for $G\in G_{\mathrm{ren}}$. Let us denote by $d_s(v)$ an order of the coupling constant λ for the vertex v, and by $d_v(v)$ an order of the coupling constant e. Finally let $d_s(G)=\sum_{v\in G}d_s(v)$, $d_v(G)=\sum_{v\in G}d_v(v)$. The numbers $d_s(G)$, $d_v(G)$ are well-defined for G_{ren} because all graphs in the family G_{ren} have the same orders. We have the following basic estimate

Proposition 1. *There exist positive constants δ_0, $O(1)$ such that*

$$\begin{aligned}&|E(G_{\mathrm{ren}},\{\square(v)\}_{v\in G_{\mathrm{ren}}},\Phi_{\mathrm{ext}},A_{\mathrm{ext}})|\\ &\quad\leqq O(1)(e(L^k\varepsilon))^{d_v(G_{\mathrm{ren}})}(\lambda(L^k\varepsilon))^{d_s(G_{\mathrm{ren}})}\exp[-\delta_0 d(\{\square(v)\}_{v\in G_{\mathrm{ren}}})]\\ &\quad\cdot\|h\Phi_{\mathrm{ext}}\|_{1,\alpha_0}\|h'A_{\mathrm{ext}}\|_{1,\alpha_0},\end{aligned}\tag{1.33}$$

where $0<\alpha_0<1$, $d(\{\square(v)\}_{v\in G})$ denotes a length of a shortest tree graph connecting the vertices v localized in $\square(v)$, $v\in G$, and the functions h,h' describes localizations of external fields. More exactly if in a vertex v there is a leg of external field, then we multiply it by a smooth function h such that $h=1$ on $\square(v)$ and $h=0$ outside some neighborhood of $\square(v)$. The constant δ_0 depends on the dimension d only (in estimates we always assume that $a=1$ in the definition of the renormalization transformation and M is fixed in an optimal way, i.e. the smallest possible). The constant $O(1)$

depends on α_0, $\bar{n}$ *(in fact on* $\bar{n}$ *only if* α_0 *is chosen not too close to* 0, *e.g. if we take* $\alpha_0=\frac{1}{2}$*), and is independent of* ε, k, *the domains* $\Omega, \Omega_1, \Omega_2$, *the vector field* $\tilde{B}$ *(if they satisfy the conditions mentioned previously).* □

The proof of this theorem is given in the rest of the paper. Now let us make a few comments about the applications of this theorem. All the statements about perturbative expansions used in the previous papers [1, 2] are rather easy consequences of it. Let us make a short survey of them. Proposition I.3.1 is completely obvious if we sum over localizations, use the restrictions on the fields and take $\bar{n}$ large enough, e.g. $\bar{n}>6$. Proposition I.3.2 is also obvious if we take the external fields of the form $(a_k G_k(B^{(k+1)})Q_k^*(B^{(k+1)}))(\cdot, x)\phi'(x)$, $(a_k G_k Q_k^*)(\cdot, x')A'(x')$, $x, x' \in T_1^{(k)}$, i.e. with fields localized in one point. The same applies to the corresponding theorems in paper [2], and for example in Proposition II.2.1 we need really a localized form of the estimate, but there are some difficulties now. They are connected with the weaker restrictions on new scalar fields. These fields are of the order $O\left(\frac{p(L^k\varepsilon)}{\lambda(L^k\varepsilon)^{1/4}}\right)$ and we have to count carefully powers of $L^k\varepsilon$. Let us consider all vertices. The worst situation is when all legs of the scalar field are replaced by external fields with the above restrictions. For the vertex (1.6) there is only one graph for which this situation can hold, the vertex itself with all external legs, and the corresponding expression is treated separately, the positivity property is used in estimates. For all other graphs the vertices (1.6) can have at most 3 external legs and we get a factor $O(\lambda(L^k\varepsilon)^{1/4}p(L^k\varepsilon)^3)$. The vertices (1.7) give also a positive power of $L^k\varepsilon$ depending on a term in (1.29). The situation is similar for the vertices (1.8)–(1.11), for (1.8) because the covariant derivative of the external scalar field $a_k G_k(\Omega, \tilde{B})Q_k^*(\tilde{B})\phi$ is of the order $O(p(L^k\varepsilon))$ by Lemma II.2.4. Among the vertices (1.18) only the first two can be dangerous for $n+n'=1$. For example, let us consider the first, given by the expression

$$a_k e(L^k\varepsilon)\phi(y)\cdot \sum_{x\in B^k(y)} \eta^d A'(\Gamma_{y,x}^{(k)})qU(\tilde{B}(\Gamma_{y,x}^{(k)}))(a_k G_k(\Omega, \tilde{B})Q_k^*(\tilde{B})\phi)(x).$$

Again by Lemma II.2.4 it is equal to

$$\begin{aligned} &a_k e(L^k\varepsilon)\phi(y)\cdot \sum_{x\in B^k(y)} \eta^d A'(\Gamma_{y,x}^{(k)})qU(\tilde{B}(\Gamma_{y,x}^{(k)}))U(\tilde{B}(\Gamma_{x,y}^{(k)}))\phi(y) \\ &\quad + O(e(L^k\varepsilon)\lambda(L^k\varepsilon)^{-1/4}p(L^k\varepsilon)^2) \\ &= a_k e(L^k\varepsilon)(\phi(y)\cdot q\phi(y)) \sum_{x\in B^k(y)} \eta^d A'(\Gamma_{y,x}^{(k)}) + O\left((L^k\varepsilon)^{\frac{4-d}{4}} p(L^k\varepsilon)^2\right) \\ &= O\left((L^k\varepsilon)^{\frac{4-d}{4}} p(L^k\varepsilon)^2\right). \end{aligned}$$

Similarly situation holds for the second vertex, thus in all cases we get some positive power of $L^k\varepsilon$, in fact we get $O\left((L^k\varepsilon)^{\frac{4-d}{4}-\beta}\right)$ for positive but arbitrarily small β. This implies Proposition II.2.1 and the analogs of Propositions I.2.1 and I.3.2 in the case of the restrictions of paper [2] also, with $\bar{n}$ sufficiently large. The above considerations give us $\bar{n}>12$. The other estimates in [2] are connected with changes of boundary conditions, changes of domains of summation, and are again the obvious consequences of Proposition 1 because very small factors $O(e^{-\delta_1 r(L^k\varepsilon)})$ are produced each time.

Finally there are some estimates which were not mentioned explicitly in [2], except in the formula (II.2.14), but which are clearly needed, as can be seen in the definitions (1.29) and (1.30). Passing from k to $k+1$ we have to change some terms in these expressions, more exactly the terms in $\delta m^2_{K,k}(x)(L^k\varepsilon)^2$ and $E_{K,k}$ for which the operators in the decomposition (1.27) have the indices $j \leqq k$ and at least one of them is equal to k. We have to rescale these terms from the ε-lattice to the η-lattice and then localize them properly, which means that we have to estimate the terms with bad localizations. Let us discuss it briefly for mass renormalization counterterms. These terms are given by graphs with two external legs, one localized in a point x. If a line of any such graph corresponds to the operator with the index k, then the kernel of this operator is a regular function of both variables and we can treat it as an external field. This means that the line of the graph is replaced by two external legs and we get a graph with four external legs. Now if we add all other graphs of this type forming a renormalized graph, then we get a convergent expression satisfying (1.33). Summing properly over localizations we get an estimate of the terms with bad localizations by a constant proportional to a power of rescaled coupling constants for x in a neighbourhood of a boundary of $B^k(\Lambda_7^{(k)})$, e.g. for $x \in B^k(\Lambda_7^{(k)} \cap \Lambda_8^{(k)c})$, and by this constant times $O(e^{-\delta_1 r(L^k\varepsilon)})$ for $x \in B^k(\Lambda_8^{(k)})$ (the notations are the same as in [2]). These estimates are sufficient to estimate the expressions containing the mass renormalization vertex with these counterterms with bad localizations. The corresponding considerations for vacuum energy counterterms are even simpler and we omit them.

2. Classification of Divergent Graphs and Their Renormalization

We will introduce now the fundamental notions needed in the renormalization. At first we will define a notion of degree of a vertex in a given graph G. It is a sum of dimensions of elements forming the vertex. We assume that the dimension of a leg of scalar or vector field is equal to $-\frac{d-2}{2}$, with the exception of legs of vector fields in the vertices (1.4) and (1.15) for which we assume that the dimension is $-\frac{d-2}{2}+1$. A dimension of differentiation is -1, each factor η has a dimension $+1$ and external fields have a dimension equal to 0. We define a degree $D_G(v)$ of a vertex v in a graph G in the following way:

$$\begin{aligned} D_G(v) = {} & (\text{the number of factors } \eta \text{ in } v) + (\text{the number of legs of scalar or vector fields in } v \text{ belonging to internal lines} \\ & \text{of the graph } G)\frac{-d+2}{2} - (\text{the number of differentiations in } v \text{ acting on internal lines of the graph } G) \\ & + [\text{the number of legs of vector fields in } v \text{ belonging to internal lines of the graph } G \text{ in the case when } v \text{ is a} \\ & \text{vertex of the form (1.14) and (1.15)}]. \end{aligned} \tag{2.1}$$

Also we define a degree $D(v)$ of a vertex v in the same way as above assuming that G contains all the elements of the vertex v.

Let us write these degrees for all vertices:

(i) if v_1 is the vertex (1.6), then $D(v_1)=d+4\frac{-d+2}{2}=4-d$,

(ii) if v_2 is the vertex (1.7), then $D(v)=d+2\frac{-d+2}{2}=2$, but generally we have to take into account a degree of mass renormalization counterterm (this will be explained later),

(iii) if v_3 is one of the vertices (1.8) and (1.9), then

$$D(v_3)=d+n+n'-1+(2+n)\frac{-d+2}{2}-1=n\frac{4-d}{2}+n',$$
$$n+n'\geqq 1,\quad n\leqq \bar{n},\quad n'\leqq \bar{n}+1,$$

(iv) if v_4 is one of the vertices (1.10) and (1.11), then

$$D(v_4)=d+n+n'-2+(2+n)\frac{-d+2}{2}=n\frac{4-d}{2}+n',$$
$$n+n'\geqq 2,\quad n\leqq \bar{n},\quad n'\leqq \bar{n}+1,$$

(v) (1.12) has only an external scalar field and for v_5 of the form (1.13)–(1.15) we have $D(v_5)=d+(n+1)\frac{-d+2}{2}+n=\frac{d+2}{2}+n\frac{4-d}{2}$, $0\leqq n\leqq \bar{n}$.

Thus for all vertices we have $D(v)\geqq \frac{4-d}{2}$. Of course we have also $D_G(v)\geqq D(v)$, except for the vertices (1.14) and (1.15). Now we will define the most important notion, a degree of a connected graph. At first let us define it in a special case: if a graph G does not have the vertices of the form (1.7) (mass renormalization vertices), then we define

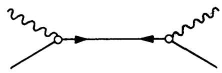
$$D(G)=\sum_{v\in G} D_G(v)-d\,. \tag{2.2}$$

We take the same definition in the case when mass renormalization counterterms are given by the terms of the first two expressions on the right side of (1.29), i.e. $\delta m_{\mathrm{fin}}^2$ or $\delta m_{K,k}^2$. We will say also that the degrees of these counterterms are equal to 0. If we have a counterterm from δm_k^2, then it corresponds to some graph G_0 and we define a degree of this counterterm as the degree of the graph G_0. This is given by (2.2) if the assumption stated before this definition is satisfied. In the general case we take the following definition, which in fact is an inductive definition:

$$D(G)=\sum_{v\in G} D_G(v)-d+(\text{a sum of degrees of mass renormalization counterterms connected with the vertices of the graph } G). \tag{2.3}$$

Thus the degree of G is the same as the degree of a graph G' obtained from G by attaching the corresponding graphs to the mass renormalization vertices.

Graphs with a nonpositive degree play a special role. Usually they correspond to divergent expressions, as it will be clarified later, so they have to be analyzed carefully. Now we will prove a theorem inverse in a certain sense to the above statement. We will prove Proposition 1 in a special case. To have not too much restricted formulation of this theorem, let us introduce the following special graph with a degree equal to 0:



(2.4)

We have the following theorem which partly clarifies the meaning of the notion of degree.

Proposition 2.1. *Let G be a connected graph such that its each connected subgraph, with the possible exception of the subgraphs* (2.4), *has a positive degree. Then we define $G_{\mathrm{ren}}=\{G\}$ and Proposition 1 holds in this case.*

In the proof of this theorem we will present our basic strategy underlying all the considerations of this paper. At first let us recall that some lines in the graph correspond to the operators $\delta G_k(\Omega,\Omega_2,\tilde{B})$ or (1.16). In the estimates we treat them as external fields and we use the inequalities:

$$\begin{aligned}&\|h(\text{an operator } \delta G_k(\Omega,\Omega_2,\tilde{B}) \text{ or } (1.16))h'\|_{1,\alpha}\\ &\quad \leqq O(e^{-\delta_0 \operatorname{dist}(\Omega_2,\partial\Omega)} \text{ or } (e(L^k\varepsilon)p(L^k\varepsilon))^{n+n'})e^{-\delta_0 \operatorname{dist}(\operatorname{supp} h,\operatorname{supp} h')}, \qquad (2.5)\end{aligned}$$

where h,h' are functions giving the localizations of the vertices. Replacing these lines by pairs of external fields we get a new graph G'. Of course connected subgraphs of G' have positive degrees, with the exception of subgraphs (2.4).

Our first step in the proof of the theorem is to write the expression

$$E(G,\{\square(v)\}_{v\in G},\Phi_{\mathrm{ext}},A_{\mathrm{ext}})=E(G',\{\square(v)\}_{v\in G},\Phi'_{\mathrm{ext}},A_{\mathrm{ext}})$$

as a sum obtained by decomposing all the propagators corresponding to the lines of G' according to the equality

$$\begin{aligned}G_k(\Omega,\tilde{B})&=C^{(0),\eta}(\Omega,\tilde{B})+\sum_{j=1}^{k-1} a_j^2(L^j\eta)^{-4}G_j^\eta(\Omega,\tilde{B})Q_j^*(\tilde{B})C^{(j),L^j\eta}(\Omega,\tilde{B})\\ &\quad\cdot Q_j(\tilde{B})G_j^\eta(\Omega,\tilde{B})=\sum_{j=0}^{k-1} G^\eta_{(j)}(\Omega,\tilde{B}), \qquad (2.6)\end{aligned}$$

and the similar equality for the vector field propagator. Thus we get a sum of new expressions obtained by replacing in each line l of the graph G' the corresponding propagator by a propagator $G^\eta_{(j_l)}$, $0\leqq j_l\leqq k-1$. Let us add index j_l to the line and let us denote by $G'(j)$ the graph G' with indices added, $j=\{j_l\}_{l\in G'}$. We can write $E(G')$ as a sum $\sum_j E(G'(j))$. The part of this sum with indices j different can be represented in a natural way as a sum over orderings of the lines and for fixed ordering $\tilde{l}=\{l(1),\dots,l(m)\}$ a sum over indices j satisfying the condition $j_{l(1)}<j_{l(2)}<\dots<j_{l(m)}$. This sum can be supplemented to the sum over all indices j if equalities are admitted in the above condition. Thus for each ordering $\tilde{l}$ we assign some set $J(\tilde{l})$ of the indices j satisfying the condition

$$j_{l(1)}\leqq j_{l(2)}\leqq \dots \leqq j_{l(m)}$$

in such a way that $\bigcup_{\tilde{l}} J(\tilde{l})$ is the set of all indices and the components of this union are disjoint sets. We get the equality

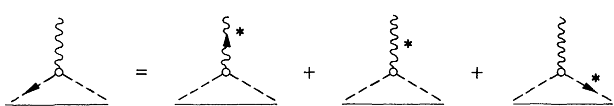
$$E(G', \{\square(v)\}_{v\in G}, \Phi'_{\text{ext}}, A_{\text{ext}}) = \sum_{\text{orderings}\,\tilde{l}} \sum_{j\in J(\tilde{l})} E(G'(j), \{\square(v)\}_{v\in G}, \Phi'_{\text{ext}}, A_{\text{ext}}). \tag{2.7}$$

To prove the theorem it is sufficient to prove the estimate (1.33) for the sum with fixed ordering $\tilde{l}$ on the right side above. This reduction is an important, although very simple, step in the proof because an order of summations over indices j is fixed now. For every such ordering we define an increasing sequence of subgraphs $G_1, G_2, \ldots, G_m = G'$ of the graph G' in the following way: G_1 is formed by the line $l(1)$ and two vertices at the endpoints of this line, ..., G_{i+1} is built by adding to the graph G_i the line $l(i+1)$ and two vertices at the endpoints of $l(i+1)$ (if they do not belong to G_i already), ...

In general the subgraphs G_i are not connected and we represent them as sums of disjoint connected subgraphs: $G_i = \bigcup_\alpha G_i^{(\alpha)}$. According to the assumption of the theorem, we have that either $D(G_i^{(\alpha)}) > 0$, or $G_i^{(\alpha)}$ is the graph (2.4). Thus there is a possibility that the first few graphs $G_1, G_2^{(2)}, G_3^{(3)}, \ldots$ are graphs (2.4) (let us notice that $G_2 = G_1 \cup G_2^{(2)}$, $G_3 = G_1 \cup G_2^{(2)} \cup G_3^{(3)}, \ldots$). Our next step is to transform the whole expression in such a way that these graphs are replaced by graphs with positive degrees. It can be done, e.g. by integration by parts. For a vertex in (2.4) we have

$$\begin{aligned}
&-e(L^k\varepsilon)\sum_{b\subset T_\eta}\eta^d[(D^\eta_B\phi')(b)\cdot q\phi'(b_-)]g(b_-)A'_b\\
&\quad= -e(L^k\varepsilon)\sum_{x\in T_\eta}\eta^d\phi'(x)\cdot q(D^{\eta*}_B\phi' gA')(x)\\
&\quad= -e(L^k\varepsilon)\sum_{x\in T_\eta}\eta^d[\phi'(x)\cdot q\phi'(x)]g(x)(\partial^{\eta*}A')(x)\\
&\qquad -e(L^k\varepsilon)\sum_{x\in T_\eta}\eta^d[\phi'(x)\cdot q\phi'(x)]\sum_{\mu=1}^{d}(\partial^{\eta*}_\mu g)(x)A'_\mu(x-\eta e_\mu)\\
&\qquad -e(L^k\varepsilon)\sum_{x\in T_\eta}\eta^d\sum_{\mu=1}^{d}[\phi'(x)\cdot q(D^\eta_{B,\mu}\phi')(x)]g(x-\eta e_\mu)A'_\mu(x-\eta e_\mu),
\end{aligned} \tag{2.8}$$

and it can be written graphically in the following way



(2.9)

(here we have defined the new graphical notations). The effect of this transformation is that the graphs (2.4) are replaced by the graphs with degree $+1$. We do it for all graphs $G_1, G_2^{(2)}, G_3^{(3)}, \ldots$ described previously and this way we represent G' as a sum of graphs $\{G'^*\}$. Further let us notice that if the graph (2.4) is a subgraph of some graph G_0, then after the transformation (2.9) G_0 is represented as a sum of three graphs and the degrees of these graphs are $\geqq D(G_0)$. It is so because the degrees of the vertices on the right side of (2.8) or (2.9) are bigger or equal to the degree of the left side. From this it follows that for each graph G'^* the subgraphs

$G_1, G_2, \ldots, G_m = G'^*$ defined as previously have positive degrees. Thus it is sufficient to prove the estimate (1.33) under this assumption. For simplicity let us denote G'^* by G again. In the next step we make a first estimate of the expression. We estimate it taking absolute values of all factors. The external fields are estimated further by the Hölder norms and for the operators δG_k and (1.16) we apply the inequality (2.5). For the propagators $G^{\eta}_{(j)}$ we apply the inequality

$$|G^{\eta}_{(j)}(\Omega, \tilde{B}; x, x')| \leqq O(1)(L^j\eta)^{-d+2} e^{-\delta_1 (L^j\eta)^{-1}|x-x'|}, \tag{2.10}$$

and if the propagator is differentiated, then for each differentiation, there is an additional factor $(L^j\eta)^{-1}$ on the right side. This applies also to Hölder norms, e.g. we have

$$\begin{aligned}&\frac{1}{|x_2-x_1|^{\alpha}} |U(\tilde{B}(\Gamma_{x_1,x_2}))(D^{\eta}_{\tilde{B},\mu} G^{\eta}_{(j)})(\Omega, \tilde{B}; x_2, x) - (D^{\eta}_{\tilde{B},\mu} G^{\eta}_{(j)})(\Omega, \tilde{B}; x_1, x)| \\ &\quad \leqq O(1)(L^j\eta)^{-d+1-\alpha} e^{-\delta_1 (L^j\eta)^{-1} \operatorname{dist}(\{x_1,x_2\},x)}, \qquad 0 \leqq \alpha < 1.\end{aligned} \tag{2.11}$$

These inequalities will be used in the next chapter. They all are obtained by rescaling from the η-lattice to the L^{-j}-lattice and application of Propositions I.2.1 and I.2.3. Of course the same inequalities hold for vector field propagators, but we have to estimate some additional expressions also. If a leg A' of the line is in one of the vertices (1.14) and (1.15), then we have the expression $A'^{(j),\eta}(\Gamma^{j+1}_{x_j+1,x})$ on the basis of (1.3). For each such expression we have an additional factor $L^j\eta$ on the right side, e.g. we have

$$|G^{\eta}_{(j)}(\Gamma^{(j+1)}_{x_j+1,x}, b)| \leqq O(1)(L^j\eta)^{-d+3} e^{-\delta_1 (L^j\eta)^{-1} \operatorname{dist}(B^j(x), b)}. \tag{2.12}$$

Finally in vertices we apply the inequalities $|q| \leqq 1 |R_{\bar{n}+1}(\cdot)| \leqq 1$. In the obtained expression we make some partial summations. For a vertex $v \in G(j)$ let $j(v)$ be a lowest index of the lines with an end in this vertex. We localize further the expression to cubes $\Delta(v)$ of the size $L^{j(v)}\eta$, i.e. we have

$$\sum_{x(\text{or } b) \in \square(v)} \eta^d \ldots = \sum_{\Delta(v) \subset \square(v)} (L^{j(v)}\eta)^d \ldots .$$

For each line we extract a part of the exponential factors on the right sides of (2.5), (2.10), and (2.12) and we estimate them by $\exp[-\frac{1}{2}\delta_1 \operatorname{dist}(\square(v), \square(v'))]$, where $\square(v)$, $\square(v')$ are localizations of endpoints of the line. After all these operations we get the following inequality:

$$\begin{aligned}&\left| \sum_{j \in J(\tilde{l})} E(G(j), \{\square(v)\}_{v\in G}, \Phi'_{\text{ext}}, A_{\text{ext}}) \right| \\ &\quad \leqq O(1)(e(L^k\varepsilon))^{d_v(G)} (\lambda(L^k\varepsilon))^{d_s(G)} \exp\left[-\frac{\delta_1}{2} d(\{\square(v)\}_{v\in G}) \right] \\ &\quad \cdot \|h\Phi_{\text{ext}}\|_1 \|h' A_{\text{ext}}\|_1 \sum_{j \in J(\tilde{l})} \tilde{E}(G(j), \{\square(v)\}_{v\in G}),\end{aligned} \tag{2.13}$$

where $O(1)$ is a constant depending on $\bar{n}$ only, and

$$\tilde{E}(G(j), \{\square(v)\}_{v\in G}) = \sum_{\{\Delta(v)\}} \prod_{\text{vertices } v\in G} \Bigg((L^{j(v)}\eta)^d \prod_{\substack{\text{legs of lines } l\\ \text{with ends in } v}} (L^{j_l}\eta)^{-\frac{d-2}{2}}$$

$$\prod_{\substack{\text{differentiations in } v\\ \text{acting on lines } l}} (L^{j_l}\eta)^{-1}$$

$\cdot$[a proper power of $L^{j(v)}\eta$ for vertices (1.8), (1.9), (1.10), (1.11)].

$$\prod_{\substack{\text{legs of lines } l \text{ of vector fields}\\ \text{in the case when } v \text{ is of the form (1.14), (1.15)}}} L^{j_l}\eta \Bigg) \prod_{\text{lines } l\in G} \exp[-\tfrac{1}{2}\delta_1 (L^{j_l}\eta)^{-1} \operatorname{dist}(\Delta(v), \Delta(v'))]\,. \tag{2.14}$$

To prove the theorem it is sufficient to prove that

$$\sum_{j\in J(\tilde{l})} \sum_{\{\Delta(v)\}} \tilde{E}(G(j), \{\Delta(v)\}_{v\in G}) \leqq O(1) \tag{2.15}$$

with $O(1)$ depending on $\bar{n}$ and δ_1 only.

Now we will prove this inequality. We begin with summations connected with the subgraph G_1, i.e. the line $l(1)$ and two vertices v, v', The index j of the line is the lowest index among $J(\tilde{l})$, so $|\Delta(v)| = |\Delta(v')| = (L^j\eta)^d$. Let us consider the case $v \neq v'$ and let us take all lines l' of G outgoing from v'. We represent the exponential factor for the line $l(1)$ as a product of as many equal factors as there are legs in v' (now by a leg we mean a leg of some line of the graph). These factors combined with the exponential factors of the lines l' give us the factors

$$\exp[-\delta_2 (L^{j_{l'}}\eta)^{-1} \operatorname{dist}(\Delta(v), \Delta(v''))]\,,$$

where v'' is a second vertex of the line l'. We have still one exponential

$$\exp[-\delta_2 (L^j\eta)^{-1} \operatorname{dist}(\Delta(v), \Delta(v'))]$$

for the line l and we use it to make the summation over $\Delta(v')$. We get some constant $O(1)$ depending on δ_1 and $\bar{n}$ only, because the linear sizes of $\Delta(v)$, $\Delta(v')$ are equal to $L^j\eta$ (δ_2 depends on δ_1 and $\bar{n}$). We represent graphically these operations as obtained by shrinking the line $l(1)$ to one point. After this shrinking the graph G_1 becomes a new vertex v_1 with the same external legs as the graph G_1 and with localization in $\Delta(v)$. We denote the new graph by G/G_1. Next we fix a localization $\Delta(v_1)$ of the vertex v_1, $|\Delta(v_1)| = (L^{j(v_1)}\eta)^d$, and we sum over $\Delta(v) \subset \Delta(v_1)$ using the factor $(L^{j(v)}\eta)^d$ connected with the vertex v. Now it is easy to see that the remaining powers of $L^j\eta$ connected with the graph G_1 give us $(L^j\eta)^{D(G_1)}$. Because $D(G_1) > 0$ we can make the summation over j and this gives us

$$\sum_{j=0}^{j_{l(2)}} (L^j\eta)^{D(G_1)} \leqq O(1)\, (L^{j_{l(2)}}\eta)^{D(G_1)}\,.$$

In the case $v=v'$ we make the same operations beginning with the summation over $\Delta(v)$. The effect of these operations can be written in the form of the inequality

$$\sum_{j\in J(\tilde{l})}\sum_{\{\Delta(v)\}_{v\in G}}\tilde{E}(G(j),\{\Delta(v)\}_{v\in G})$$
$$\leqq \sum_{j\in J(\tilde{l})_1}\sum_{\{\Delta(v)\}_{v\in G/G_1}} O(1)(L^{j_{l(2)}}\eta)^{D(G_1)}\tilde{E}(G/G_1(j),\{\Delta(v)\}_{v\in G/G_1}). \tag{2.15}$$

The exponential factors for the expression on the right side have the exponent δ_2 instead of $\frac{1}{2}\delta_1$; $J(\tilde{l})_1$ denotes a projection of $J(\tilde{l})$ obtained by omitting $j_{j(1)}$. Now we can formulate an inductive assumption. A graph G/G_i is defined as a graph obtained from G by shrinking all connected components $G_i^{(\alpha)}$ of G_i to points. This way we get some new vertices $v_i^{(\alpha)}$ having the same external legs as the graphs $G_i^{(\alpha)}$ and with the corresponding expressions defined as in (2.14). Here $J(\tilde{l})_i$ denotes a set of indices $\{j_{l(i+1)},\dots,j_{l(m)}\}$ obtained by projecting $J(\tilde{l})$. We assume that

$$\sum_{j\in J(\tilde{l})}\sum_{\{\Delta(v)\}_{v\in G}}\tilde{E}(G(j),\{\Delta(v)\}_{v\in G})$$
$$\leqq \sum_{j\in J(\tilde{l})_i}\sum_{\{\Delta(v)\}_{v\in G/G_i}} O(1)(L^{j_{l(i+1)}}\eta)^{\sum_{\alpha} D(G_i^{(\alpha)})}\tilde{E}(G/G_i(j),\{\Delta(v)\}_{v\in G/G_i}), \tag{2.16}$$

where $O(1)$ depends on $\delta_1,\bar{n}$ only and an exponent δ_{i+1} in the exponential factors on the right side is positive and depends on $\delta_1,\bar{n}$. We can prove (2.16) for $i+1$ repeating the same reasoning as in the proof of (2.15). Thus we have (2.16) for all $i\leqq m$. For $i=m$ we get the constant $O(1)$ only on the right side of (2.16), hence the proof of the theorem is finished. In this proof the summations over localizations and the use of exponential factors can be analyzed much more carefully and a better dependence on $\bar{n}$ can be obtained, but it is unimportant here. In this simple case we have illustrated the basic method of analyzing perturbation expressions. It is easily seen from the proof that the theorem can be generalized to a much wider class of graphs and expressions. We can have vertices with an arbitrary number of legs and arbitrary power of η. We can have lines with the same exponential factors but with arbitrary dimensions instead of $-d+2$. The only thing which matters is that propagators have representations corresponding to (2.6) with the estimates corresponding to (2.10)–(2.12), so that we have the inequality (2.13) with the proper generalization of (2.14). We can formulate these remarks as the theorem:

Proposition 2.2. *Proposition 2.1 holds for the described above generalized expressions and graphs.*

Now we will analyze and describe the graphs with nonpositive degrees. Let us consider at first a graph G with at least one vertex v of the form (1.13)–(1.15). There is also a line $l\in G$ outgoing from v. If it is a vector field line, then we always have $D_G(v)-d\geqq 0$. If both endpoints of l belong to v, then we have $D_G(v)-d>0$, hence $D(G)>0$. If not, then there is another vertex $v'\in G$ with positive degree, hence $D(G)>0$ also. In the case when l is a scalar field line we have at least one vertex

$v' \in G$ more and then

$$D(G) \geqq D_G(v) - d + D_G(v') \geqq \frac{-d+2}{2} + \frac{4-d}{2} \geqq 0,$$

but analyzing more carefully the possibilities we arrive at the conclusion $D(G) > 0$ again.

Also it is easy to notice that if a graph G has an R-vertex of the form (1.9) and (1.11), then it has positive degree. For the graphs G which do not contain vertices of the form (1.13)–(1.15) we can easily prove the following estimate

$$\begin{aligned} D(G) \geqq & -2\frac{d-2}{2} - 1 + ((\text{a number of vertices}) - 1)\frac{4-d}{2} \\ & + ((\text{a number of external legs}) - 1)\frac{d-2}{2}, \end{aligned} \tag{2.17}$$

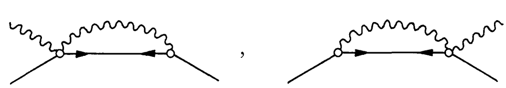
from which it follows that the graphs with more than four external legs have positive degree in $d=3$. In $d=2$ graphs are more convergent, so degrees are still positive. We formulate all these conclusions in

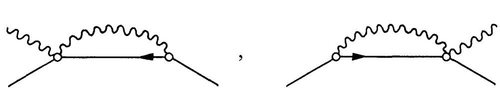
Corollary 2.3. *If a graph G has a vertex of the form* (1.13)–(1.15), *or an R-vertex, or a vertex* (1.7) *with finite counterterms, or it has more than four external legs, then* $D(G) > 0$.

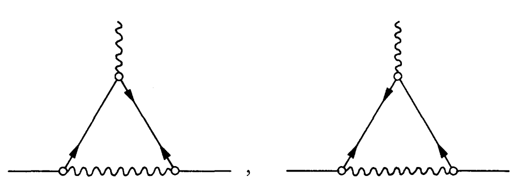
Now we can describe classes of divergent graphs i.e. graphs with nonpositive degree. They can be built of the vertices (1.6)–(1.8) and (1.10) only, on the basis of the above corollary.

There is only one graph with four external legs and it is the graph (2.4). In the proof of Proposition 2.1 it was shown how to treat such graphs. There are many graphs with three external legs and they fall into two classes: graphs with three external legs of vector fields, and graphs with two external legs of scalar fields and one external vector field leg. The first class will be considered later.

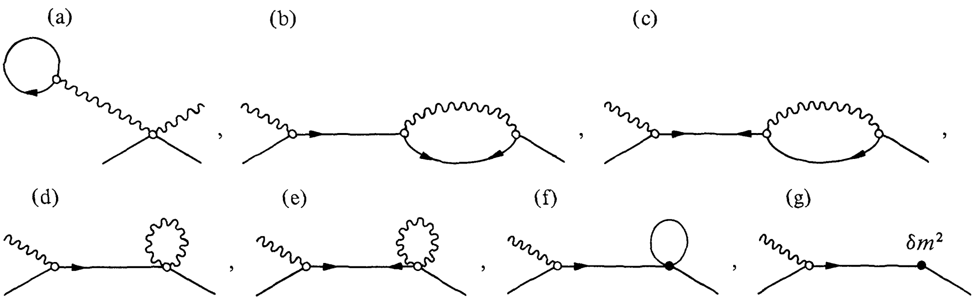
For the second class of graphs, the classes G_{ren} are defined as the smallest sets of graphs, symmetric with respect to permutation of external scalar field legs and with each divergent subgraph renormalized, e.g. each pair of graphs below form one class:

 (2.18)

 (2.19)

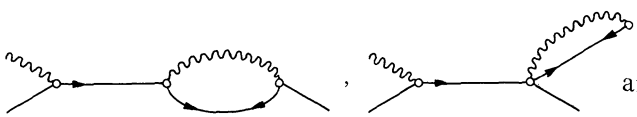
 (2.20)

In fact the only other divergent graphs of this class are:



(2.21)

and permuted graphs, and some of them have divergent subgraphs. We form classes collecting all graphs necessary to renormalize each divergent subgraph, e.g. we take as one class



and permuted graphs (2.22)

Thus our definition of the classes G_{ren} is in fact inductive and relies on definitions of these classes for lower order graphs of all kinds. The same applies to the remaining definitions.

The next class of graphs is formed by graphs with two external scalar field legs. The estimate (2.17) implies that we can have such graphs with at most four vertices. We consider one-particle-irreducible graphs only. We form classes G_{ren} for these graphs taking all graphs necessary to renormalize all divergent proper subgraphs, and adding one-vertex graphs of the form $\overset{\delta m_k^2(G)}{-\!\bullet\!-}$ with mass renormalization counterterms corresponding to the previous graphs, so we take one counterterm for each graph.

There is a similar class of graphs with two external vector field legs, and these graphs have at most four vertices again. To define the renormalized classes G_{ren} we have to consider at first the Ward-Takahashi identities. These identities express a gauge invariance of the scalar field part of the theory. The basic identity has the form:

$$\begin{aligned}&\int d\phi \exp[-\tfrac{1}{2}\langle\phi,(-\Delta^{\eta}_{A-e_k\varphi\partial^{\eta}\lambda}+M^2)\phi\rangle]F(\phi)\\&=\int d\phi\exp[-\tfrac{1}{2}\langle\phi,(-\Delta^{\eta}_{A}+M^2)\phi\rangle]F(\phi),\end{aligned} \tag{2.23}$$

where $e_k=e(L^k\varepsilon)$, $M^2>0$, $F(\phi)$ is an arbitrary gauge-invariant function of scalar fields. Taking a first order differential with respect to λ we get:

$$\int d\phi\exp[-\tfrac{1}{2}\langle\phi,(-\Delta^{\eta}_{A}+M^2)\phi\rangle]F(\phi)\langle D^{\eta}_{A}\phi,\partial^{\eta}\lambda q\phi\rangle=0\,. \tag{2.24}$$

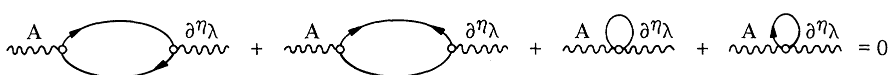
Now differentiating (2.24) with respect to A, connecting the vertices by properly localized propagators, and calculating the Gaussian integrals [$F(\phi)$ is chosen as a polynomial, e.g. we can take $F(\phi)=-\lambda_k:|\phi(x)|^4:$, or $F(\phi)=-\frac{1}{2}\delta m_i^2(x):|\phi(x)|^2:$, or $F(\phi)=-\lambda_k C^{\eta}_{M^2}(0):|\phi(x)|^2:$], we get a set of Ward-Takahashi identities. Let us give few examples. Taking $F=1$, $A=0$, we get

$$\int d\mu_{C^{\eta}_{M^2}}(\phi)\langle\partial^{\eta}\phi, \partial^{\eta}\lambda q\phi\rangle=0. \tag{2.25}$$

Taking $F=1$, differentiating with respect to A and next taking $A=0$ and using the identity (2.25), we get

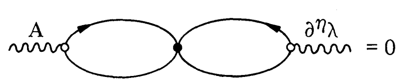
$$\begin{aligned}
&\int d\mu_{C^{\eta}_{M^2}}(\phi)\,[(-e_k\langle\partial^{\eta}\phi, Aq\phi\rangle)(:\langle\partial^{\eta}\phi, \partial^{\eta}\lambda q\phi\rangle:)\\
&\qquad -e_k\langle\phi, A\cdot\partial^{\eta}\lambda q^2\phi\rangle - e_k\langle\partial^{\eta}\phi, \eta A\partial^{\eta}\lambda q^2\phi\rangle]\\
&\quad=-e_k\sum_{b,b'}\eta^{2d}A_b\operatorname{tr} q(C^{\eta}_{M^2}\partial^{\eta *})(b_-,b')q(C^{\eta}_{M^2}\partial^{\eta *})(b'_-,b)(\partial^{\eta}\lambda)(b')\\
&\qquad+e_k\sum_{b,b'}\eta^{2d}A_b\operatorname{tr} qC^{\eta}_{M^2}(b_-,b'_-)q(\partial^{\eta}C^{\eta}_{M^2}\partial^{\eta *})(b',b)(\partial^{\eta}\lambda)(b')\\
&\qquad-e_k\sum_{b}\eta^{d}A_b(\partial^{\eta}\lambda)(b)\operatorname{tr} q^2C^{\eta}_{M^2}(0)\\
&\qquad-e_k\sum_{b}\eta^{d}\eta A_b(\partial^{\eta}\lambda)(b)\operatorname{tr} q^2(C^{\eta}_{M^2}\partial^{\eta})(b_-,b)=0,
\end{aligned} \tag{2.26}$$

or graphically



(2.27)

where now the propagators are $C^{\eta}_{M^2}$. Doing next the same operations as above but in the presence of $F(\phi)=-\lambda_k\sum_{x\in\Delta}\eta^d:|\phi(x)|^4:$, we get the identities represented graphically in the following way:



(2.28)

Taking other functions F, or differentiating (2.24) to higher order in A, we can get all necessary Ward-Takahashi identities. They hold for free boundary conditions also by taking a limit of the identities with periodic boundary conditions. Each such identity is connected with some set of graphs. Now we define classes G_{ren} as containing all graphs necessary to renormalize divergent proper subgraphs and next all graphs connected with these graphs by Ward-Takahashi identities. If we refrain from taking the smallest classes, then we can describe simply the classes G_{ren} as containing all possible connected graphs of the considered type (i.e. having two external vector field legs) of a fixed order in coupling constants.

There are no graphs with one external leg of scalar field and one external leg of vector field, and there are no graphs with one external leg of scalar field [more exactly every such graph necessarily has a vertex of the form (1.13), (1.14) or (1.15)]. There are graphs with one external vector field leg. These graphs, the graphs with three external vector field legs mentioned previously, and also the graphs with two external scalar field legs, but at least one differentiated, do not introduce any new divergences, and the classes G_{ren} are formed by graphs necessary to renormalize all divergent proper subgraphs.

Graphs with no external legs, vacuum energy graphs, are gathered together into classes G_{ren} by the same condition on divergent subgraphs, and then all vacuum energy counterterms corresponding to these graphs are added.

Finally for an arbitrary graph G we form a class G_{ren} containing this graph adding all graphs necessary to renormalize all divergent proper subgraphs. Of course localizations in these graphs should be in agreement.

3. A Proof of Proposition 1

Let us consider an arbitrary G_{ren} and the corresponding properly localized expression $E(G_{\mathrm{ren}}, \{\square(v)\}_{v\in G_{\mathrm{ren}}}, \Phi_{\mathrm{ext}}, A_{\mathrm{ext}})$. At first we will enlarge essentially the set of external fields. It was already stressed that the operators δG_k and (1.16) are considered as the external fields. Now if two vertices, v, v' have localizations satisfying $\mathrm{dist}(\square(v), \square(v')) \geqq 1$, then we consider every propagator corresponding to a line connecting these vertices as an external field also. Such a possibility is assured by the following estimates

$$\|hG_k(\Omega, \tilde{B})h'\|_{1,\alpha} \leqq O(1)e^{-\delta_0 \mathrm{dist}(\square(v), \square(v'))}, \tag{3.1}$$

and similarly for the vector field propagator, h, h' are localization functions. Thus we remove from every $G \in G_{\mathrm{ren}}$ all the lines corresponding to the operators of the types described above, more exactly we replace these lines by pairs of external legs. We get some new family of graphs $\{G'\}$, G' may be unconnected, and we have to know that it can be represented as a sum of classes G'_{ren}. Generally, if we replace a line in an arbitrary graph G by a pair of external fields, then the degree of the new graph is greater or equal to the degree of G, thus a convergent graph is transformed into a convergent one. From our analysis in previous chapters it follows that most of the divergent graphs are transformed into convergent ones, with the possible exception of graphs with one external vector field leg and vacuum graphs. These graphs occur together with all other graphs forming a renormalized class, and inspecting all possible cases it is easily seen that if we remove a line in any graph of the class, then for the obtained graph we can find in this class all graphs necessary to form a new renormalized class. Thus we can represent our expression as a sum of the expressions $E(G'_{\mathrm{ren}}, \{\square(v)\}_{v\in G'_{\mathrm{ren}}}, \Phi'_{\mathrm{ext}}, A'_{\mathrm{ext}})$, with the property that each line of each connected component of G'_{ren} is localized in cubes $\square(v)$, $\square(v')$ with $\mathrm{dist}(\square(v), \square(v')) = 0$ (by the definition of new graphs it should be <1, but by the construction of localizations it is then $=0$). From this it follows that the localizations of each connected component are contained in some cube of a linear size depending on a number of vertices, hence on $\bar{n}$ only. To prove (1.33) it is sufficient to prove the estimates

$$|E(G'_{\mathrm{ren}}, \{\square(v)\}_{v\in G'_{\mathrm{ren}}}, \Phi'_{\mathrm{ext}}, A'_{\mathrm{ext}})| \leqq O(1)(e(L^k\varepsilon))^{d_v(G'_{\mathrm{ren}})} \cdot (\lambda(L^k\varepsilon))^{d_s(G'_{\mathrm{ren}})} \|\Phi'_{\mathrm{ext}}\|_{1,\alpha_0} \|A'_{\mathrm{ext}}\|_{1,\alpha_0}, \tag{3.2}$$

because these estimates and (2.5), (3.1) imply (1.33), and the exponential factor in (1.33) is obtained from the estimates of external fields in (3.2).

To prove (3.2) we will transform further the expression using the remark about localizations of connected components of G'_{ren}. We can consider the expressions corresponding to these components separately. The localization cubes of each component are contained in some cube of a size $O(\bar{n})$. We take a cube $\square_1$ of size $r(L^k\varepsilon)$ containing the above cube in the center, and next we take a cube $\square$ of size $3r(L^k\varepsilon)$ and with $\square_1$ in the center. We assume that $\square_1, \square$ are sums of big blocks of the unit lattice. Now we replace each scalar field propagator $G_k(\Omega, \tilde{B})$ [or $G_k(\Omega_2, \tilde{B})$] in the expression by $G_k(\square, \tilde{B})$, more exactly we represent $G_k(\Omega, \tilde{B}) = G_k(\square, \tilde{B}) + \delta G_k(\Omega, \square, \tilde{B})$, and we treat $\delta G_k(\Omega, \square, \tilde{B})$ as an external field. Thus again we drop the lines corresponding to these operators and decompose the obtained set of graphs into renormalized classes. The field $\tilde{B}$ is regular in the sense that $\|\partial^\eta_\mu \tilde{B}\|_{0,\alpha} \leqq O(p(L^k\varepsilon))$. Hence there exists a constant field B_0 such that $|\tilde{B} - B_0| \leqq O(r(L^k\varepsilon)p(L^k\varepsilon)) \leqq O(p(L^k\varepsilon)^2)$ on the cube $\square$. We have $\tilde{B} = B_0 + \tilde{B}'$, and we expand in $\tilde{B}'$ the expressions connected with each renormalized class of graphs using the formulas (I.3.14), (I.3.44), and (I.3.45). After this expansion we get a sum of expressions corresponding to some new big set of graphs, which can be decomposed into a sum of renormalized classes. We repeat some of the operations done before, i.e. we localize new vertices in the way described in Chap. 1 and we include the operators (1.16) and propagators with localizations $\square(v)$, $\square(v')$ satisfying the condition $\mathrm{dist}(\square(v), \square(v')) \geqq 1$ into the external fields. These operations give us again a sum of renormalized classes. If a class contains an old vertex coming from the expression before the expansion, then all vertices of this class are localized in $\square_1$. If a class does not contain any old vertex, then it is created by an expansion of a propagator $G_k(\square, \tilde{B})$ and it consists of one graph having the form of a chain of propagators $G_k(\square, \tilde{B}_Q)$, with vertices of the form (1.8)–(1.11) and (1.13)–(1.15) with external vector field $\tilde{B}'$ only. For such chains, every subgraph has positive degree or contains (2.4), and the estimate (3.2) for them is a consequence of Proposition 2.1. Thus we have to consider the renormalized classes having some old vertices and localized in $\square_1$. Vertices are now of the same type (1.6)–(1.11) and (1.13)–(1.15), but with B_0 instead of $\tilde{B}$ and with the new external vector field $\tilde{B}'$ beside the old one. A scalar field propagator is $G_k(\square, B_0)$. It is easy to see that the expression corresponding to any such renormalized class is gauge-invariant with respect to gauge transformations of the field B_0, if the external scalar fields are simultaneously transformed. Our next operation is the gauge transformation which removes the field B_0. It was described more precisely in the proof of Lemma II.2.4. We get the same expressions as above with $B_0 = 0$ only (and external scalar fields gauge transformed). Finally we replace the scalar field propagator $G_k(\square, 0)$ by $G_k(0)$, i.e. we substitute $G_k(\square, 0) = G_k(0) + \delta G_k(\square, \eta Z^d, 0)$ and we treat δG_k as an external scalar field. After all these operations we get a sum of the expressions $E(G_{\mathrm{ren}}, \{\square(v)\}_{v \in G_{\mathrm{ren}}}, \Phi_{\mathrm{ext}}, A_{\mathrm{ext}})$ with localizations in some cube of size $O(\bar{n})$ and with the same graphical description as before, but with the scalar field propagator equal to $G_k(0)$. We use the same graphical notations for these new expressions. We have to prove the inequality (3.2) for these expressions. Considering each connected component of G_{ren} separately we can assume additionally that G_{ren} is connected. We can extract also all coupling constants, so finally we have to prove the estimate

$$|E(G_{\mathrm{ren}}, \{\square(v)\}_{v \in G_{\mathrm{ren}}}, \Phi_{\mathrm{ext}}, A_{\mathrm{ext}})| \leqq O(1) \|\Phi_{\mathrm{ext}}\|_{1,\alpha_0} \|A_{\mathrm{ext}}\|_{1,\alpha_0}. \tag{3.3}$$

Let us remark that the norms on the right side above are defined in the usual way, with the vector field $\tilde{B}=0$, but the external scalar fields appear with the gauge transformation, so these norms are equal to the norms defined by (1.32) with $\tilde{B}=B_0$. When an expression under the norm depends on $\tilde{B}$, then the norm can be estimated further by the norm (1.32) using the restrictions on the field $\tilde{B}'$. Thus we get the estimate (3.2) exactly.

The inequality (3.3) will be proven following the same strategy as applied in the proof of Proposition 2.1. We apply the decomposition (2.6) to the propagators $G_k(0)$, G_k and we get

$$E(G_{\mathrm{ren}}, \{\square(v)\}_{v\in G_{\mathrm{ren}}}, \Phi_{\mathrm{ext}}, A_{\mathrm{ext}}) = \sum_{G\in G_{\mathrm{ren}}} \sum_{\mathrm{orderings}\,\tilde{l}} \sum_{j\in J(\tilde{l})} E(G(j), \{\square(v)\}_{v\in G}, \Phi_{\mathrm{ext}}, A_{\mathrm{ext}}). \tag{3.4}$$

With each ordering $\tilde{l}$ of lines of the graph G a sequence of subgraphs $G_1, G_2, \dots, G_m = G$ of G is connected. In Proposition 2.1 we had the assumption that $D(G_i)>0$, with the possible exception of the graph (2.4). Now for a general graph G we can have subgraphs with $D(G_i)\leqq 0$ and our method will be to transform the corresponding expressions in such a way that we obtain the expressions to which Proposition 2.2 can be applied. More exactly we will prove the following statement: To each class G_{ren} there corresponds some family G'_{fin} of generalized graphs such that

$$\begin{aligned}&\sum_{G\in G_{\mathrm{ren}}} \sum_{\tilde{l}} \sum_{j\in J(\tilde{l})} E(G(j), \{\square(v)\}_{v\in G}, \Phi_{\mathrm{ext}}, A_{\mathrm{ext}})\\ &\quad= \sum_{G'\in G'_{\mathrm{fin}}} \sum_{\tilde{l}'} \sum_{j\in J(\tilde{l}')} \gamma(G', \tilde{l}') E'(G'(j), \{\square(v)\}_{v\in G'}, \Phi_{\mathrm{ext}}, A_{\mathrm{ext}}),\end{aligned} \tag{3.5}$$

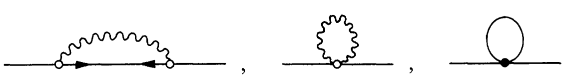
where $E'(G', \dots)$ is a generalized expression corresponding to the graph G' and $\gamma(G', \tilde{l}')$ is a characteristic function of some set of orderings $\tilde{l}'$. This set has the property that a sequence of subgraphs $G'_1, G'_2, \dots, G'_m = G'$ defined by an ordering $\tilde{l}'$ from the set consists of subgraphs with positive degrees $D(G'_i)>0$. In the rest of this chapter we will prove the above statement. In the proof we will describe the form of generalized graphs and the corresponding generalized expressions. Of course this statement and Proposition 2.2 imply the inequality (3.3).

We will prove (3.5) step by step, starting with the subgraphs of lowest order. At first let us remark that the expressions corresponding to graphs with an odd number of external vector field legs (and no other external legs) are equal to 0. This follows from the fact that every graph of this type has at least one loop of scalar field lines with an odd number of vector field legs, thus with an odd power of q, and we have $\mathrm{tr}\, q^{2n+1}=0$. Thus we have no divergent subgraphs with one or three external vector field legs.

Let us take an arbitrary $G\in G_{\mathrm{ren}}$, some ordering $\tilde{l}$ of lines of this graph and the corresponding sequence of subgraphs $G_1, G_2, \dots, G_m = G$. Some of them can have nonpositive degrees and we will construct a representation (3.5) constructing successively these representations for the subgraphs G_i.

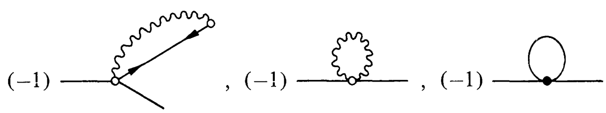
Let us take a first divergent subgraph G_0 in the sequence. If it is a graph of the form (2.4), and if every subgraph in the sequence containing it is convergent, then we can apply the same procedure as in the proof of Proposition 2.1, i.e. we apply the integration by parts formula (2.8). If it is a subgraph of some divergent subgraph in the sequence, then we temporarily postpone considering it until we will treat it with the larger subgraph. There are the following remaining possibilities for nonvacuum graphs: G_0 has two external scalar field legs, G_0 has two external vector field legs, G_0 has two external scalar field legs and one vector field leg. We will consider successively all these possibilities.

Let us start with self-energy graphs for scalar fields. The graphs of lowest order are



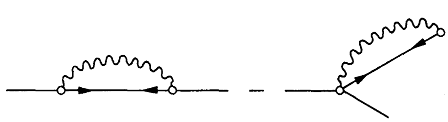
$(D=-d+2)$, (3.6)

and the renormalized class G_{ren} contains the corresponding mass renormalization counterterms also:



(3.7)

The two last terms in (3.7) cancel exactly the two last terms in (3.6), so the expressions containing these terms vanish (Wick ordering). We will consider in detail an expression corresponding to



(3.8)

The method described below, and even a simplified one, will be applied to all other primitively divergent graphs (graphs whose every proper subgraph is convergent). The expression corresponding to (3.8) is

$$\begin{aligned}
&-\sum_{x,x'}\eta^{2d}\phi(x)\cdot\left[\sum_{\mu=1}^{d} q(\partial_\mu^\eta G_{(j)}(0)\partial_\mu^{\eta *})(x,x')qg(x)G_{(j')}(x,x')g'(x')\right]\phi'(x')\\
&+\sum_{x,x'}\eta^{2d}\phi(x)\cdot\left[\sum_{\mu=1}^{d} q(\partial_\mu^\eta G_{(j)}(0)\partial_\mu^{\eta *})(x,x')qg(x)G_{(j')}(x,x')g'(x')\right]\phi'(x),
\end{aligned}\tag{3.9}$$

where g, g' are localization functions. To this expression we apply Taylor's formula in the form

$$f(y)=f(x)+\sum_{\mu=1}^{d}(y_\mu-x_\mu)(\partial_\mu^\eta f)(x)+\sum_{b\subset\Gamma_{x,y}}\eta|b_- - x|^\alpha\frac{(\partial^\eta f)(b)-(\partial^\eta f)((b)_x)}{|b_- - x|^\alpha},\tag{3.10}$$

where $(b)_x$ denotes a bond b parallel-transported to the point x. We apply it to a leg ϕ', but let us notice that the expression (3.9) is symmetric in ϕ, ϕ', so we could equally well apply it to the leg ϕ. We choose the leg with a smaller j-index and write it as ϕ'. We get

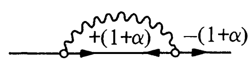
$$(\text{the expression }(3.9)) = -\sum_{\mu=1}^{d}\sum_{x}\eta^d\phi(x)\cdot\left[\sum_{x'}\eta^d\sum_{\nu=1}^{d} q(\partial_\nu^\eta G_{(j)}(0)\partial_\nu^{\eta *})(x,x')q\right.$$

$$\left.\cdot g(x)G_{(j')}(x,x')g'(x')(x'_\mu - x_\mu)\right](\partial_\mu^\eta\phi')(x)$$

$$-\sum_{x,x'}\eta^{2d}\phi(x)\cdot\left[q\sum_{\mu=1}^{d}(\partial_\mu^\eta G_{(j)}(0)\partial_\mu^{\eta *})(x,x')qg(x)G_{(j')}(x,x')g'(x')|x'-x|^{1+\alpha}\right]$$

$$\cdot\sum_{b\subset\Gamma_{x,x'}}\frac{\eta|b_- - x|^\alpha}{|x'-x|^{1+\alpha}}\,\frac{(\partial^\eta\phi')(b)-(\partial^\eta\phi')((b)_x)}{|b_- - x|^\alpha}. \tag{3.11}$$

The second expression above already has the right form because the factor $|x-x|^{1+\alpha}$ adds to the degree of the graph the number $1+\alpha$, thus the degree of the expression is equal to $-d+3+\alpha$ now. The operator acting on the leg ϕ' is a differentiation of the order $1+\alpha$, so we will represent graphically this expression by the generalized graph



(3.12)

To make the above statements about the degree quite clear let us show how this expression will be estimated. We localize additionally the vertices in cubes $\Delta(v)$, $\Delta(v')$, $|\Delta(v)| = (\Delta(v')| = (L^{j_1}\eta)^d$, $j_1 = \min\{j, j'\}$, and we have

$$(\text{the expression }(3.12)) \leqq O(1)\sum_{\Delta(v),\Delta(v')}\sup_{x\in\Delta(v)}|\phi(x)|(L^{j_1}\eta)^{2d}(L^j\eta)^{-d}$$

$$\cdot e^{-\delta_0(L^j\eta)^{-1}\operatorname{dist}(\Delta(v),\Delta(v'))}(L^{j'}\eta)^{-d+2}e^{-\delta_0(L^{j'}\eta)^{-1}\operatorname{dist}(\Delta(v),\Delta(v'))}$$

$$\cdot(L^{j_1}\eta)^{-1}\operatorname{dist}(\Delta(v),\Delta(v')))^{1+\alpha}$$

$$\cdot(L^{j_1}\eta)^{1+\alpha}\sup_{x\in\Delta(v),\,x'\in\Delta(v')}\ \sup_{y\in\Gamma_{x,x'},\mu}\frac{|(\partial_\mu^\eta\phi')(y)-(\partial_\mu^\eta\phi')(x)|}{|y-x|^\alpha}. \tag{3.13}$$

We can estimate the factor $((L^{j_1}\eta)^{-1}\operatorname{dist}(\Delta(v),\Delta(v')))^{1+\alpha}$ by $O(1)$ using half of the exponential factor with index j_1. If ϕ' is a leg of a propagator with an index j'', whose second leg is localized in $\Delta(v'')$, then the last supremum in (3.13) can be estimated by

$$O((1)(L^{j''}\eta)^{-d+1-\alpha}\sup_{x\in\Delta(v),\,x'\in\Delta(v')}\exp[-\delta_0(L^{j''}\eta)^{-1}\operatorname{dist}(\Gamma_{x,x'},\Delta(v''))],$$

and the exponential factor together with another exponential factor in (3.13) give us the estimate

$$(\text{the expression }(3.12)) \leqq O(1)\sum_{\Delta(v),\Delta(v')}\sup_{x\in\Delta(v)}|\phi(x)|(L^{j_1}\eta)^{2d}(L^j\eta)^{-d}$$

$$\cdot\exp[-\tfrac{1}{2}\delta_0(L^j\eta)^{-1}\operatorname{dist}(\Delta(v),\Delta(v'))](L^{j'}\eta)^{-d+2}\exp[-\tfrac{1}{2}\delta_0(L^{j'}\eta)^{-1}\operatorname{dist}(\Delta(v),\Delta(v'))]$$

$$\cdot(L^{j_1}\eta)^{1+\alpha}(L^{j''}\eta)^{-d+1-\alpha}\exp[-\tfrac{1}{2}\delta_0(L^{j''}\eta)^{-1}\operatorname{dist}(\Delta(v),\Delta(v''))]$$

(there may be an additional negative power of $L^{j''}\eta$ coming from differentiations in the vertex v''). (3.14)

This estimate shows that for (3.12) we have exactly the same factors as in the expression (2.14), but with different degrees; thus we have the generalized expressions and generalized graphs in the sense explained before Proposition 2.2. The same statements apply to all expressions appearing in the future and we will not repeat them.

Let us consider the first expression on the right side of (3.11). The same expression appears for all orderings of the lines of the graph G_0 with the only condition that they are earlier than the external lines. Making summations over these orderings and indices means that we sum with respect to j,j' from 0 to j'', where j'' is the lowest index of the external lines. After the summations we get

$$-\sum_{\mu=1}^{d}\sum_{x}\eta^d\phi(x)$$
$$\cdot\left[\sum_{x'}\eta^d\sum_{\nu=1}^{d} q(\partial_\nu^\eta G_{j''}^\eta(0)\partial_\nu^{\eta *})(x,x')qg(x)G_{j''}^\eta(x,x')g'(x')(x'_\mu-x_\mu)\right](\partial_\mu^\eta\phi')(x), \quad (3.15)$$

and this expression is represented graphically by . Let us estimate the coefficient in the vertex, i.e. the expression in the square brackets in (3.15). Rescaling it from the η-lattice to the $L^{-j''}$-lattice $(x, x', = L^{j''}\eta y, L^{j''}\eta y', y, y' \in T_{L^{-j''}})$, we get the same expression but with $L^{-j''}$ instead of η. For simplicity let us denote $L^{-j''}=\xi$. Next we replace $G_{j''}^\xi(0)$ by

$C^\xi = (-\Delta^\xi+1)^{-1}$: $G_{j''}^\xi(0) = C^\xi + G_{j''}^\xi(0)(1-m_{j''}^2-a_{j''}P_{j''})C^\xi$. We have $\sum_{\nu=1}^{d}\partial_\nu^\xi C^\xi \partial_\nu^{\xi *}$ $= -\Delta^\xi C^\xi = \delta^\xi - C^\xi$ and $\delta^\xi(y'-y)(y'_\mu-y_\mu)=0$, so the expression is equal to

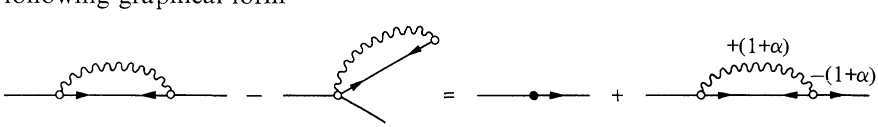
$$-q^2\sum_{y'}\xi^d C^\xi(y-y')g(y)G_{j''}^\xi(y,y')g(y')(y'_\mu-y_\mu)$$
$$+q^2\sum_{y'}\xi^d\left(\sum_{\nu=1}^{d}(\partial_\nu^\xi G_{j''}^\xi(0)(1-m_{j''}^2-a_{j''}P_{j''})C^\xi\partial_\nu^{\xi *})(y,y')\right)$$
$$\cdot g(y)G_{j''}^\xi(y,y')g'(y')(y'_\mu-y_\mu). \quad (3.16)$$

Using the inequalities

$$|C^\xi(y-y')| \leqq O(1)\frac{e^{-\frac{1}{2}|y-y'|}}{|y-y'|}, \qquad |G_{j''}^\xi(0;y,y')| \leqq O(1)\frac{e^{-\delta_0|y-y'|}}{|y-y'|},$$

and the corresponding inequalities for derivatives, we can estimate (3.16) by a constant.

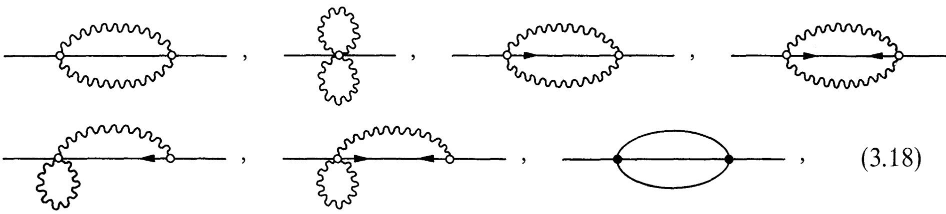
Thus we have finished the analysis of (3.8) and we can summarize it in the following graphical form



(3.17)

It is of the required form (3.5).

Let us consider the other cases of self-energy graphs for scalar fields. All the remaining divergent graphs of this type have degrees equal to 0. Primitively divergent graphs, i.e. the graphs

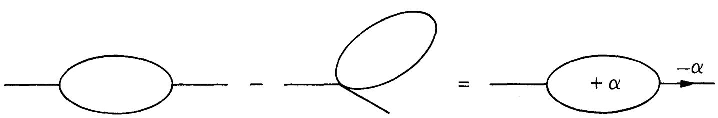


(3.18)

are treated in a simpler way. If $\Sigma(x,x')$ is an expression corresponding to any such graph, then we have a graph with mass renormalization counterterm of the form $-\sum_{x'} \eta^d \Sigma(x,x')$, and we write

$$\begin{aligned}&\sum_{x,x'} \eta^{2d}\phi(x)\cdot\Sigma(x,x')\phi'(x') - \sum_{x}\eta^d\phi(x)\cdot\left(\sum_{x'}\eta^d\Sigma(x,x')\right)\phi'(x)\\ &\quad= \sum_{x,x'} \eta^{2d}\phi(x)\cdot(\Sigma(x,x')|x'-x|^\alpha)\frac{\phi'(x')-\phi'(x)}{|x'-x|^\alpha}.\end{aligned} \tag{3.19}$$

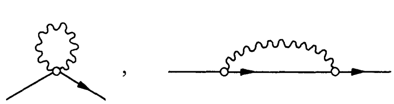
We get a generalized expression of degree $+\alpha$ represented by some generalized graph whose every subgraph has a positive degree also. Graphically we write it as follows



(3.20)

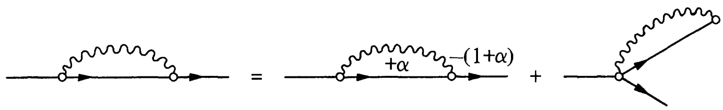
The same procedure is applied to all divergent self-energy graphs G_0 if no divergent subgraphs appear within the given ordering, with the possible exception of the graphs (2.4). When this graph appears, then we apply the integration by parts formula (2.8) again. For graphs of higher orders usually there is a chain of propagators connecting a vertex localized in x with a vertex localized in x', and then we estimate $|x'-x|^\alpha \leqq |x'-x_1|^\alpha + \ldots + |x_l - x|^\alpha$. It is necessary to mention here that G_0 can have divergent subgraphs and there is a whole renormalized class of graphs containing G_0, but we treat graphs separately. This applies also to graphs with renormalization mass counterterms. Within the considered ordering they do not give any divergences.

We have to consider another class of graphs with two external scalar field legs, the graphs with one leg differentiated. There are only two such graphs:



(3.21)

The expression corresponding to the first graph is in fact convergent, because $\eta G_k(x,x)$ is convergent to some finite constant as $\eta \to 0$. The second expression is transformed in a way similar to (3.17) and (3.20):



(3.22)

The first graph on the right side has positive degree, the second is treated in the same way as the expression (3.15): we sum over proper orderings and j-indices and we get

$$\sum_{\mu=1}^{d} \sum_{x} \eta^d \phi(x) \cdot \left[q^2 \sum_{x'} \eta^d (\partial_\mu^\eta G_{j''}^\eta(0))(x,x') g(x) G_{j''}^\eta(x,x') g'(x') \right] (\partial_\mu^\eta \phi')(x). \tag{3.23}$$

Further if we take $g'(x') = g'(x) + \dfrac{g'(x') - g'(x)}{|x'-x|}|x'-x|$, then the expression in the square bracket in (3.23) containing the second term will be convergent and the expression with the first term is equal to

$$q^2 g(x) g'(x) \sum_{x'} \eta^d (\partial_\mu^\eta G_{j''}^\eta(0))(x,x') G_{j''}^\eta(x,x').$$

Rescaling from the η-lattice to the $L^{-j''}$-lattice and using the same method as in (3.16) we get some convergent expressions plus

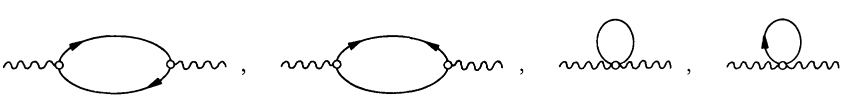
$$q^2 g(x) g'(x) \sum_{x'} \xi^d (\partial_\mu^\xi C^\xi)(x-x') C^\xi(x-x').$$

We have further

$$\begin{aligned} \sum_{x'} \xi^d (\partial_\mu^\xi C^\xi)(x-x') C^\xi(x-x') &= (2\pi)^{-d} \int_{|p| \leq \frac{\pi}{\xi}} dp \frac{\partial_\mu^\xi(p)}{(\Delta^\xi(p)+1)^2} \\ &= (2\pi)^{-d} \int_{|p'| \leq \pi} dp' \frac{\cos p'_\mu - 1}{(\Delta^1(p') + \xi^2)^2} \xi^{3-d}, \end{aligned} \tag{3.24}$$

and the expression on the right side has a finite limit as $\xi \to 0$. Hence the last graph in (3.22) defines a vertex with some convergent function.

The next class of graphs is the class of self-energy graphs for vector fields. The graphs of lowest order are



$(D = -d+2)$, (3.25)

and they form a renormalized class of graphs connected by Ward-Takahashi identities (2.26) and (2.27). Let us analyze in detail the expressions corresponding to these graphs. Applying the same transformations as for (3.9), we get

$$
\begin{aligned}
&-\sum_{x,x'} \eta^{2d} \sum_{\mu,\mu'=1}^{d} g(x)A_\mu(x)\,\mathrm{tr}\,q^2(G^\eta_{(j)}(0)\partial^{\eta*}_{\mu'})(x,x')(G^\eta_{(j')}(0)\partial^{\eta*}_{\mu})(x',x)g'(x')A'_{\mu'}(x')\\
&\quad+\sum_{x,x'} \eta^{2d} \sum_{\mu,\mu'=1}^{d} g(x)A_\mu(x)\,\mathrm{tr}\,q^2 G^\eta_{(j)}(0;x,x')(\partial^\eta_{\mu'}G^\eta_{(j')}(0)\partial^{\eta*}_{\mu})(x',x)g'(x')A'_{\mu'}(x')\\
&\quad-\sum_{x}\eta^{d}\sum_{\mu=1}^{d} g(x)A_\mu(x)g'(x)A'_\mu(x)\,\mathrm{tr}\,q^2G^\eta_{(j'')}(0,x,x)\\
&\quad-\sum_{x}\eta^{d}\sum_{\mu=1}^{d} g(x)A_\mu(x)g'(x)A'_\mu(x)\,\mathrm{tr}\,q^2\eta(G^\eta_{(j'')}(0)\partial^{\eta*}_\mu)(x,x)\\
&=-\Bigg[\sum_{x,x'} \eta^{2d} \sum_{\mu,\mu'=1}^{d} g(x)A_\mu(x)\,\mathrm{tr}\,q^2(G^\eta_{(j)}(0)\partial^{\eta*}_{\mu'})(x,x')(G^\eta_{(j')}(0)\partial^{\eta*}_{\mu})(x',x)g'(x)A_{\mu'}(x)\\
&\quad+\sum_{x,x'} \eta^{2d} \sum_{\mu,\mu'=1}^{d} g(x)A_\mu(x)\,\mathrm{tr}\,q^2 G^\eta_{(j)}(0;x,x')(\partial^\eta_{\mu'}G^\eta_{(j')}(0)\partial^{\eta*}_{\mu})(x',x)g'(x)A_{\mu'}(x)\\
&\quad-\sum_{x}\eta^{d}\sum_{\mu=1}^{d} g(x)A_\mu(x)g'(x)A'_\mu(x)\,\mathrm{tr}\,q^2G^\eta_{(j'')}(0;x,x)\\
&\quad-\sum_{x}\eta^{d}\sum_{\mu=1}^{d} g(x)A_\mu(x)g'(x)A'_\mu(x)\,\mathrm{tr}\,q^2\eta(G^\eta_{(j'')}(0)\partial^{\eta*}_\mu)(x,x)\Bigg]\\
&\quad+\Bigg\{\sum_{\nu=1}^{d}\sum_{x,x'}\eta^{2d}\sum_{\mu,\mu'=1}^{d} g(x)A_\mu(x)\\
&\quad\cdot\mathrm{tr}\,q^2[-(G^\eta_{(j)}(0)\partial^{\eta*}_{\mu'})(x,x')(G^\eta_{(j')}(0)\partial^{\eta*}_{\mu})(x',x)(x'_\nu-x_\nu)\\
&\quad+G^\eta_{(j)}(0;x,x')(\partial^\eta_{\mu'}G^\eta_{(j')}(0)\partial^{\eta*}_{\mu})(x',x)(x'_\nu-x_\nu)](\partial^\eta_\nu g'A'_{\mu'})(x)\Bigg\}\\
&\quad+\Bigg\{\sum_{x,x'}\eta^{2d}\sum_{\mu,\mu'=1}^{d} g(x)A_\mu(x)\,\mathrm{tr}\,q^2[-(G^\eta_{(j)}(0)\partial^{\eta*}_{\mu'})(x,x')(G^\eta_{(j')}(0)\partial^{\eta*}_{\mu})(x',x)|x'-x|^{1+\alpha}\\
&\quad+G^\eta_{(j)}(0;x,x')(\partial^\eta_{\mu'}G^\eta_{(j')}(0)\partial^{\eta*}_{\mu})(x',x)|x'-x|^{1+\alpha}]\\
&\quad\cdot\sum_{b\subset\Gamma_{x,x'}}\frac{\eta|b_--x|^\alpha}{|x'-x|^{1+\alpha}}\,\frac{(\partial^\eta g'A'_{\mu'})(b)-(\partial^\eta g'A'_{\mu'})((b)_x)}{|b_--x|^\alpha}\Bigg\},
\end{aligned}
\tag{3.26}
$$

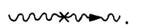
where A, A' are external vector field legs. The expressions in the last curly bracket above are the generalized expressions of the same form as in (3.11), they have positive degree $-d+3+\alpha$ and can be analyzed as in (3.13), (3.14), and Proposition 2.2 can be applied.

The remaining expressions are analyzed in the same way as the first term in (3.11). If j_0 denotes a smallest j-index of external legs, then we sum with respect to j, j', j'' from 0 to j_0 and we get the same expressions but with the propagator $G^\eta_{j_0}(0)$. The expression in the square bracket on the right side of (3.26) has the form

$$\sum_x \eta^d \sum_{\mu,\mu'=1}^{d} g(x)A_\mu(x)\Pi^{(\eta,j_0)}_{\mu\mu'}(x)g'(x)A'_{\mu'}(x),$$

graphically it is , and the expression in the first curly bracket has the form

$$\sum_{\nu=1}^{d} \sum_{x} \eta^{d} \sum_{\mu,\mu'=1}^{d} g(x) A_{\mu}(x) \Pi^{(\eta,j_0)}_{\mu\mu'\nu}(x) (\partial^{\eta}_{\nu} g' A'_{\mu'})(x),$$

graphically . Now let us rescale the functions $\Pi^{(\eta,j_0)}_{\mu\mu'}$, $\Pi^{(\eta,j_0)}_{\mu\mu'\nu}$ from the η-lattice to the L^{-j_0}-lattice. We get

$$\Pi^{(\eta,j_0)}_{\mu\mu'}(x) = (L^{j_0}\eta)^{-d+2} \Pi^{(L^{-j_0},j_0)}_{\mu\mu'}(y),$$
$$\Pi^{(\eta,j_0)}_{\mu\mu'\nu}(x) = (L^{j_0}\eta)^{-d+3} \Pi^{(L^{-j_0},j_0)}_{\mu\mu'\nu}(y), \qquad y = (L^{j_0}\eta)^{-1} x.$$

We consider the case $d=3$, so $-d+2=-1$, $-d+3=0$. Next we replace the propagator $G_{j_0}(0)$ by C^{ξ}, $\xi = L^{-j_0}$, using the same equation as in (3.16). If at least one propagator $G_{j_0}(0)$ is replaced by $G_{j_0}(0)(1-m^2_{j_0} - a_{j_0} P_{j_0}) C^{\xi}$, then we get a convergent expression. Hence it is enough to consider the expressions with the propagator C^{ξ} only. For the expression $\Pi^{(L^{-j_0},j_0)}_{\mu\mu'\nu}$, we have to consider the term

$$\begin{aligned}
&\operatorname{tr} q^2 \sum_{y'} \xi^d [-(C^{\xi}\partial^{\xi *}_{\mu'})(y-y')(C^{\xi}\partial^{\xi *}_{\mu})(y'-y)(y'_{\nu}-y_{\nu}) + C^{\xi}(y-y')(\partial^{\xi}_{\mu'} C^{\xi} \partial^{\xi *}_{\mu})(y'-y)(y'_{\nu}-y_{\nu})] \\
&= \tfrac{1}{2} \operatorname{tr} q^2 \sum_{y'} \xi^d [-(\partial^{\xi *}_{\mu} C^{\xi})(y')(\partial^{\xi}_{\mu'} C^{\xi})(y') y'_{\nu} + C^{\xi}(y')(\partial^{\xi *}_{\mu} \partial^{\xi}_{\mu'} C^{\xi})(y') y'_{\nu} \\
&\quad + (\partial^{\xi}_{\mu} C^{\xi})(y')(\partial^{\xi *}_{\mu'} C^{\xi})(y') y'_{\nu} - C^{\xi}(y')(\partial^{\xi *}_{\mu'} \partial^{\xi}_{\mu} C^{\xi})(y') y'_{\nu}] \\
&= \tfrac{1}{2} \operatorname{tr} q^2 \sum_{y'} \xi^d (C^{\xi}(y'+\xi e_{\mu}) - C^{\xi}(y'-\xi e_{\mu}))(\partial^{\xi}_{\mu'} C^{\xi})(y') \delta_{\mu\nu} - (\mu \leftrightarrow \mu').
\end{aligned} \tag{3.27}$$

We have used the identities $C^{\xi}(-y') = C^{\xi}(y')$, $(\partial^{\xi}_{\mu} C^{\xi})(-y') = (\partial^{\xi *}_{\mu} C^{\xi})(y')$ and the integration by parts formula. Furthermore, for the last expression we have

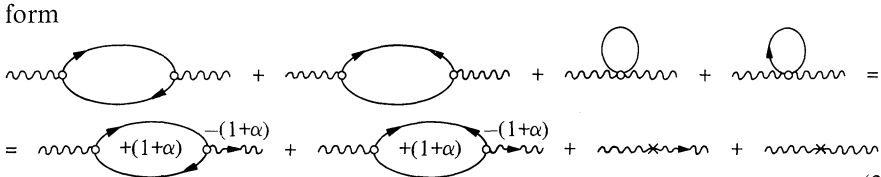
$$\begin{aligned}
&\sum_{y'} \xi^d (C^{\xi}(y'+\xi e_{\mu}) - C^{\xi}(y'-\xi e_{\mu}))(\partial^{\xi}_{\mu'} C^{\xi})(y') = 2(2\pi)^{-d} \int_{|p| \le \frac{\pi}{\xi}} dp \frac{\xi^{-1} \sin \xi p_{\mu} \sin \xi p_{\mu'}}{(\Delta^{\xi}(p)+1)^2} \\
&= \frac{2}{d} (2\pi)^{-d} \int_{|p'| \le \pi} dp' \frac{\sum_{\mu=1}^{d} \sin^2 p'_{\mu}}{(\Delta^1(p') + \xi^2)^2} \xi^{3-d} \delta_{\mu\mu'},
\end{aligned} \tag{3.28}$$

which is symmetric in μ, μ', hence finally the term (3.27) is equal to 0. Let us consider the corresponding term coming from $\Pi^{(L^{-j_0},j_0)}_{\mu\mu'}$, and let us rescale the external legs A, A' from the η-lattice to the ξ-lattice also. We get the expression

$$\begin{aligned}
&(L^{j_1}\eta)^{-\frac{d-2}{2}} (L^{j_0}\eta)^{d-\frac{d-2}{2}} \Bigg[-\sum_{y,y'} \xi^{2d} \sum_{\mu,\mu'=1}^{d} g(y) A_{\mu}(y) \operatorname{tr} q^2 (C^{\xi}\partial^{\xi *}_{\mu'})(y-y') \\
&\quad \cdot (C^{\xi}\partial^{\xi *}_{\mu})(y'-y) g'(y) A'_{\mu'}(y) \\
&\quad + \sum_{y,y'} \xi^{2d} \sum_{\mu,\mu'=1}^{d} g(y) A_{\mu}(y) \operatorname{tr} q^2 C^{\xi}(y-y')(\partial^{\xi}_{\mu'} C^{\xi} \partial^{\xi *}_{\mu})(y'-y) g'(y) A'_{\mu'}(y) \\
&\quad - \sum_{y} \xi^d \sum_{\mu=1}^{d} g(y) A_{\mu}(y) g'(y) A'_{\mu'}(y) \operatorname{tr} q^2 C^{\xi}(0) \\
&\quad - \sum_{y} \xi^d \sum_{\mu=1}^{d} g(y) A_{\mu}(y) g'(y) A'_{\mu'}(y) \operatorname{tr} q^2 (C^{\xi}\partial^{\xi *}_{\mu})(0) \Bigg].
\end{aligned} \tag{3.29}$$

Now if we introduce the function $\lambda(y')=\sum_{\mu'=1}^{d} g'(y)A'_{\mu'}(y)(y'_{\mu'}-y_{\mu'})$, then $(\partial^{\xi}_{\mu'}\lambda)(y')=g'(y)A'_{\mu'}(y)$, and the expression in the square bracket above has exactly the form appearing in the Ward-Takahashi identity (2.26) with $M^2=1$, hence it is equal to 0.

Let us write the effect of the above transformations in the following graphical form



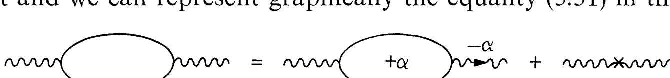
(3.30)

where the coefficient at the vertex is bounded, and the coefficient at the vertex is proportional to $(L^{j_0}\eta)^{-d+2}$, and j_0 is the lowest j-index of the external legs.

The other primitively divergent graphs are considered in a simpler way, because they have degree 0. We write

$$\begin{aligned}
&\sum_{x,x'}\eta^{2d}\sum_{\mu,\mu'=1}^{d} g(x)A_{\mu}(x)\Pi_{\mu\mu'}(x,x')g'(x')A'_{\mu'}(x')\\
&\quad=\sum_{x,x'}\eta^{2d}\sum_{\mu,\mu'=1}^{d} g(x)A_{\mu}(x)\Pi_{\mu\mu'}(x,x')|x'-x|^{\alpha}\frac{g'(x')A'_{\mu'}(x')-g'(x)A'_{\mu'}(x)}{|x'-x|^{\alpha}}\\
&\quad\quad+\sum_{x}\eta^{d}\sum_{\mu,\mu'=1}^{d} g(x)A_{\mu}(x)\Big(\sum_{x'}\eta^{d}\Pi_{\mu\mu'}(x,x')\Big)g'(x)A'_{\mu'}(x)\,, \qquad (3.31)
\end{aligned}$$

and the first term on the right side is convergent. To the second term we apply the same transformations as previously, i.e. we rescale it from the η-lattice to the L^{-j_0}-lattice, we resum over proper orderings and j-indices, we replace the propagators $G_{j_0}(0)$, G_{j_0} by C^{ξ} and finally to the expressions containing the propagators C^{ξ} only we apply a proper Ward-Takahashi identity. This gives us that the second term is convergent and we can represent graphically the equality (3.31) in the form



(3.32)

We have to remark that if the subgraph (2.4) appears in the first term on the right side above and at least one of the vertices of this subgraph is different from the vertices with external legs, then we apply the integration by parts formula (2.8) to this vertex. If both vertices have external legs, then the factor $|x'-x|^{\alpha}$ renders the integration by parts unnecessary.

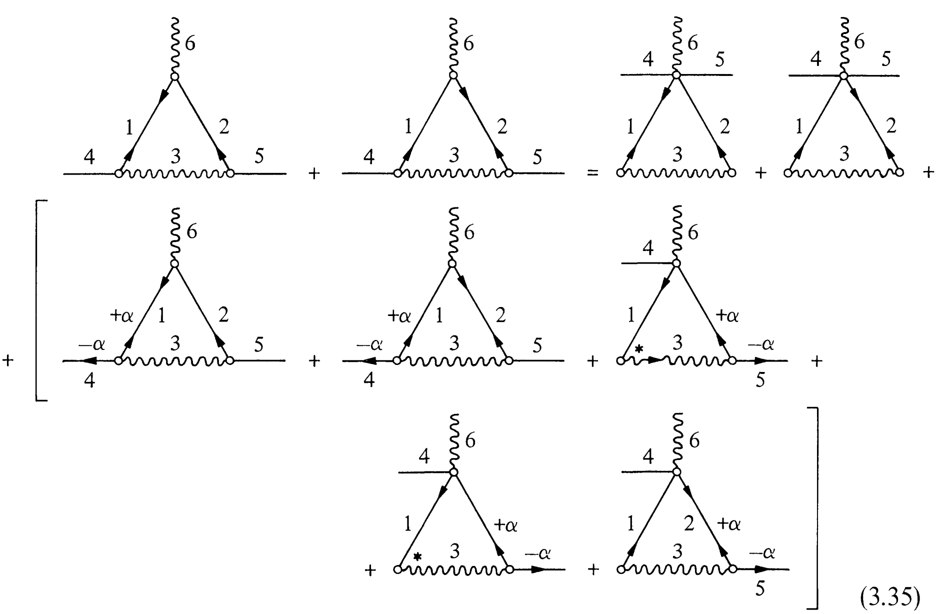
Our last class of graphs is the class of graphs with two external scalar field legs and one vector field leg. They were described graphically in (2.18)–(2.21). It is possible to give a general description of renormalization, but there are only a few graphs, so let us be more specific and consider them separately. At first let us recall that the expression corresponding to the graph (2.21a) vanishes. Graph (2.21e) has an additional power of η, so we can treat it like the graph (2.4) multiplied by a finite expression. The graphs (2.21d) and (2.21f) are cancelled by the corresponding counterterms in (2.21g). Thus we have to consider the graphs (2.18)–(2.20), (2.21b), and the counterterms (2.21g) and (2.21c). Only the first three classes are primitively divergent. Generally an expression corresponding to these graphs has the form

$$\sum_{x,x',x''} \eta^{3d} \sum_{\mu=1}^{d} g(x)A_\mu(x)\phi'(x')\cdot\Gamma_\mu(x,x',x'')\phi''(x'') \tag{3.33}$$

and the degree is 0. We transform this expression transporting all the legs to one vertex. More precisely we transport the legs with the lowest j-indices to a vertex having a leg with highest index. For example, if ϕ' is such a leg, then

$$\begin{aligned}(3.33)= \quad &\sum_{x,x',x''} \eta^{3d} \sum_{\mu=1}^{d} \Bigg[\frac{g(x)A_\mu(x)-g(x')A_\mu(x')}{|x-x'|^\alpha}\phi'(x')\\ &\cdot\Gamma_\mu(x,x',x'')|x-x'|^\alpha\phi''(x'')+g(x')A_\mu(x')\phi'(x')\\ &\cdot\Gamma_\mu(x,x',x'')|x''-x'|^\alpha\frac{\phi''(x'')-\phi''(x')}{|x''-x'|^\alpha}\Bigg]\\ &+\sum_{x'}\eta^d\sum_{\mu=1}^{d} g(x')A_\mu(x')\phi'(x')\cdot\Bigg(\sum_{x,x''}\eta^{2d}\Gamma_\mu(x,x',x'')\Bigg)\phi''(x').\end{aligned} \tag{3.34}$$

A more detailed inspection of the difergent graphs shows that we can transport the legs in such a way that the subgraphs (2.4) either are made convergent by a factor of the form $|x_1-x_2|^\alpha$, or the formula for integration by parts can be applied without differentiations acting on external legs. For example for the graphs (2.20) we have within the indicated ordering:



(3.35)

This way all the expressions on the right side of (3.34) are convergent, except the first two. We transform further this expression moving all localization functions to the corresponding vertex. This gives us a convergent expression plus the expression

$$\sum_{x'}\eta^d \sum_{\mu=1}^{d} g(x')A_\mu(x')\phi'(x')\cdot(\text{a product of the values of all the}$$
$$\text{localization functions at the point } x')\Big(\sum_{x,x''}\eta^{2d}\Gamma'_\mu(x,x',x'')\Big)\phi''(x'), \quad (3.36)$$

where Γ'_μ is given by the same formula as before, but with the summations unrestricted. Next we sum the expressions (3.36) over admissible orderings and indices and we get the expressions of the same type but with propagators $G^\eta_{j_0}(0)$, $G^\eta_{j_0}$, where j_0 is the lowest index of the external legs.

Now we can analyze the factor $\sum_{x,x''}\eta^{2d}\Gamma'_\mu(x,x',x'')$ for all the renormalized classes. It is easily seen that it vanishes for the graphs (2.21b) and (2.21g) with the proper renormalization mass counterterm. It vanishes also for the graphs (2.18) because they correspond to the same expressions but with the opposite signs. To analyze the remaining graphs we have to transform it further. The expression $\sum_{x,x''}\eta^{2d}\Gamma'_\mu(x,x',x'')$ is of degree 0, so it is equal to the same expression but on the scale ξ instead of η, $\xi=L^{-j_0}$. Now we replace the propagators $G^\eta_{j_0}(0)$, $G^\eta_{j_0}$ by C^ξ in the way described several times. We get a convergent expression plus $\sum_{y,y''}\xi^{2d}\Gamma''_\mu(y,y',y'')$ defined with the help of the propagator C^ξ. This expression for the graphs (2.19) equals

$$-q^3\sum_y \xi^d[(C^\xi\partial^{\xi *}_\mu)(y-y')C^\xi(y-y')-(\partial^\xi_\mu C^\xi)(y-y')C^\xi(y-y')]=0\,. \quad (3.37)$$

For the graph (2.21c) it equals 0 also because by translation invariance it can be written as a derivative of a constant. Finally for the graphs (2.20) it equals

$$\begin{aligned}
&-q^3\sum_{y,y''}\xi^{2d}\sum_{\nu=1}^{d}[(\partial^\xi_\nu C^\xi)(y'-y)(\partial^\xi_\mu C^\xi\partial^{\xi *}_\nu)(y-y'')C^\xi(y'-y'')\\
&\quad-(\partial^\xi_\nu C^\xi\partial^{\xi *}_\mu)(y'-y)(C^\xi\partial^{\xi *}_\nu)(y-y'')C^\xi(y'-y'')]\\
&=-q^3\sum_{y,y''}\xi^{2d}(-\Delta^\xi C^\xi)(y'')C^\xi(y''-y)((\partial^\xi_\mu C^\xi)(y)-(\partial^{\xi *}_\mu C^\xi)(y))\\
&=-q^3\sum_y \xi^d C^\xi(y)\frac{C^\xi(y+\xi e_\mu)-C^\xi(y-\xi e_\mu)}{\xi}\\
&\quad+q^3\sum_y \xi^d(C^\xi * C^\xi)(y)\,\frac{C^\xi(y+\xi e_\mu)-C^\xi(y-\xi e_\mu)}{\xi}=0\,,
\end{aligned} \quad (3.38)$$

because the functions which we are summing are odd.

We have finished the renormalization of all primitively divergent graphs, or primitively divergent within a given ordering, and we have proved the formula (3.5) for such graphs. In fact we have a precise description of the class of generalized graphs. Now we can prove the decomposition (3.5) for an arbitrary divergent graph G_0, assuming that it holds for all its subgraphs. We can assume that G_0 is not primitively divergent. We form a renormalized class $G_{0,\mathrm{ren}}$

containing G_0 and we consider all orderings of lines of the graphs in $G_{0,\mathrm{ren}}$, with j-indices smaller that the lowest j-index of the external legs. Each ordering defines a sequence of subgraphs and we take the last divergent subgraph in this sequence. We consider a subset of the set of all orderings consisting of the orderings for which the given subgraph appears and we apply (3.5). It is necessary to notice here that the differentiations acting on external legs of this subgraph and connected with the construction of the decomposition (3.5) do not act on external legs of the whole graph G_0. Thus the degree of G_0 is unchanged, and its situation inside some larger graph is unmodified. This way we have decomposed the expressions corresponding to $G_{0,\mathrm{ren}}$ into a sum of expressions corresponding to some set of generalized graphs. A graph of this set has the property that each of its subgraphs appearing within some ordering has positive degree. Thus the only divergent graphs can be the whole graphs of the set. Now we apply one of the formulas (3.19), (3.31), (3.34) [graphically (3.20) and (3.32)], and we finally renormalize the whole graphs, i.e. we have a representation (3.5) for them. Let us notice that in order to cancel the divergent expressions we have to perform the same transformations in mass renormalization counterterms, or in the expressions in Ward-Takahashi identities, as in the expressions corresponding to $G_{0,\mathrm{ren}}$. For divergent vacuum graphs we do not need to do all these transformations because they are canceled by vacuum energy counterterms. Thus we have finished the description of the renormalization of divergent graphs.

Now for arbitrary G_{ren} we get the decomposition (3.5) as above, taking these decompositions for all divergent subgraphs appearing in all possible orderings.

This ends the proof of Proposition 1.

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3. For more detailed exposition, see Harvard preprints Ultraviolet stability for a model of interacting scalar and vector fields. I. A lower bound. HUTMP 82/B116
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