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Monopoles and Geodesics

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Abstract. Using the holomorphic geometry of the space of straight lines in Euclidean 3-space, it is shown that every static monopole of charge k may be constructed canonically from an algebraic curve by means of the Atiyah-Ward Ansatz $\mathscr{A}_k$.

1. Introduction

It has been known for some time that the Bogomolny equations, describing static Yang-Mills-Higgs monopoles in the Prasad-Sommerfield limit, may be solved by twistor methods. Indeed, they can be reinterpreted as the self-duality equations in Euclidean four-space which are in addition time-translation invariant, and the methods of Penrose, Ward and Atiyah may be applied directly. During the past year significant progress has been made using this line of attack by Ward [15, 16], Prasad and Rossi [12], and Corrigan and Goddard [7]. They all use a variant of the Atiyah-Ward $\mathscr{A}_k$-Ansatz [3] to construct an SU(2) monopole of charge k. The main purpose of this paper is to show that *every* solution of the Bogomolny equations satisfying the appropriate boundary conditions can be constructed in a canonical manner by this method.

Our approach is again twistorial, but instead of passing from a problem in 3-space to one in 4-space, we use complex methods intrinsically associated to the Euclidean geometry of $\mathbb{R}^3$. We replace the set of points of $\mathbb{R}^3$ by the space of oriented geodesics (straight lines). This has the structure of a complex surface (in fact, the holomorphic tangent bundle $\mathbf{T}$ to the projective line) and a solution to the Bogomolny equations gives rise in a natural manner to a holomorphic vector bundle over this surface. Actually, this approach to problems in Euclidean space is by no means new – it was used by Weierstrass in 1866 to solve the minimal surface equations.

Briefly, our method consists of defining a vector bundle $\tilde{E}$ over the surface $\mathbf{T}$ of geodesics by associating to each straight line the null space of the differential operator

$$D = \nabla_u - i\Phi\,,$$

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where ∇_u is the covariant derivative in the unit direction of the line and Φ is the Higgs field. The bundle is holomorphic if (∇, Φ) satisfies the Bogomolny equations $\nabla\Phi = *F$. The Atiyah-Ward Ansatz entails a description of the holomorphic bundle $\tilde{E}$ as an extension of line bundles. This means finding a distinguished subbundle of $\tilde{E}$. We show that the bundle L^+ of solutions to $Ds=0$ which decay as $t \to +\infty$ is a holomorphic subbundle isomorphic to a certain standard bundle $L(-k)$ on $\mathbf{T}$. It follows that $\tilde{E}$ is an extension of $L(-k)$ by its dual and hence that the initial solution to the Bogomolny equations may be constructed by a canonical application of the Ansatz $\mathscr{A}_k$.

The space of lines for which $Ds=0$ admits an $\mathscr{L}^2$ solution forms an algebraic curve of genus $(k-1)^2$ in $\mathbf{T}$. We call it the spectral curve, for it is a sort of non-linear spectrum for the family of linear differential operators parametrized by $\mathbf{T}$. We use the global properties of this curve to give a geometrical description of the approach of Corrigan and Goddard, realizing a $4k-1$ parameter family of solutions. Finally we show how, with sheaf cohomology, the spectral curve determines the holomorphic bundle $\tilde{E}$ and hence the solution to the Bogomolny equations.

2. The Space of Geodesics

If M^n is a Riemannian manifold, then the geodesics in a suitable open set (for example, geodesically convex) are parametrized by a manifold of dimension $(2n-2)$. The tangent space to this manifold at the geodesic γ may be described as follows. Take a curve of geodesics $\gamma(t,s)$ with $\gamma(t,0)=\gamma(t)$ and consider

$$V = \left.\frac{\partial\gamma}{\partial s}\right|_{s=0}.$$

This is a vector field along the geodesic γ – a *Jacobi field* – which satisfies the equation

$$(\nabla_U)^2 V + R(V,U)U = 0, \tag{2.1}$$

where $U = \dfrac{d\gamma}{dt}$ is the unit tangent vector of the curve and $R(X,Y)Z$ the curvature tensor of the metric.

The $2n$-dimensional space of solutions to this equation contains the 2-dimensional subspace of Jacobi fields tangential to γ. These are not tangent to a deformation of the curve, but correspond to an affine reparametrization. The $(2n-2)$-dimensional space of Jacobi fields *orthogonal* to U does consist of genuine deformations of γ and the map

$$V \mapsto V - (V,U)U$$

defines an isomorphism from the tangent space at γ of the space of geodesics to the space of Jacobi fields orthogonal to the direction of γ.

Suppose M is now 3-dimensional and we consider the space G of *oriented* geodesics (fix a direction for U). Now if V is orthogonal to U, so is the vector cross product $U \times V$. Since U is constant along the geodesic,

$$(\nabla_U)^2 U \times V = U \times (\nabla_U^2 V),$$

and so if

$$R(U \times V, U)U = U \times R(V, U)U, \tag{2.2}$$

then we can define a linear map

$$J(V) = U \times V \tag{2.3}$$

which satisfies

$$J^2(V) = U \times (U \times V) = (U, V)U - (U, U)V = -V$$

since V is orthogonal to U.

In other words, we have an almost complex structure on the 4-dimensional real manifold G. Note that we needed the orientation to fix U and define J. The curvature condition (2.2) is only satisfied for a Riemannian 3-manifold if it has vanishing traceless Ricci tensor. In such a case the metric has constant curvature and the almost complex structure is integrable. Then G is a complex surface. Before we consider in detail the case $M = \mathbb{R}^3$ which is relevant here, let us note some properties of G.

Firstly it possesses a map $\tau : G \to G$ with no fixed points such that $\tau^2 = \mathrm{id}$, obtained by simply reversing the orientation on each geodesic. By the definition (2.3), τ takes the complex structure J to $-J$ and is thus an antiholomorphic involution, or *real structure* on G.

Secondly, consider a point $x \in M$. The oriented geodesics through x are parametrized by the unit 2-sphere in the tangent space T_x, via the exponential map. Since U is the unit normal to this sphere, it is clear from (2.3) that J preserves its tangent space and moreover defines the standard complex structure on the Riemann sphere. Thus a point x corresponds to a holomorphic projective line $P_x \subset G$. Two sufficiently close points x and y are joined by a unique geodesic, hence (taking both directions) by two *oriented* geodesics. This means that P_x and P_y intersect in two points and so the self-intersection number of P_x (the degree of its normal bundle) is 2. The line P_x is also clearly preserved by τ and hence is real.

We thus have a real complex surface G with a family of real lines of self-intersection number 2. It can be shown that any such surface may be obtained by the above geodesic construction, but using a Weyl structure rather than a Riemannian structure. The integrability condition (2.2) is then the analogue of Einstein's equations $(R_{(ij)} = \Lambda g_{ij})$ for the Weyl structure (see [10]). This is the general context of the description in this paper, but from now on we shall restrict ourselves to the simplest case $M = \mathbb{R}^3$.

3. Straight Lines in 3-Space

In $\mathbb{R}^3$ the geodesics are straight lines. We may parametrize them by assigning first the direction, a unit vector $\mathbf{u}$ and secondly by choosing an origin and taking the position vector $\mathbf{v}$ of the point nearest the origin. Thus

$$G \cong \{(\mathbf{u}, \mathbf{v}) \in S^2 \times \mathbb{R}^3 \mid \mathbf{u} \cdot \mathbf{v} = 0\},$$

and this is just the *tangent bundle* TS^2 of S^2. The line $\mathbf{x} = \mathbf{v} + t\mathbf{u}$ corresponds to $(\mathbf{u}, \mathbf{v})$.

The complex structure constructed on G in Sect. 2 is, we claim, the natural structure on the holomorphic tangent bundle of the projective line $\mathbb{P}_1 \cong S^2$.

To see this, we may consider the standard Riemannian connection on S^2 and split the tangent space of TS^2 into vertical and horizontal spaces, so that

$$T(TS^2) \cong \pi^* T \oplus \pi^* T,$$

where $\pi: TS^2 \to S^2$ is the projection. The natural complex structure of TS^2 is obtained by taking the standard almost complex structure on S^2 in both factors.

On the other hand consider the geodesic

$$\gamma = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{v} + t\mathbf{u} \quad \text{with} \quad \mathbf{u} \cdot \mathbf{v} = 0\}.$$

A tangent vector to γ is a pair of vectors $(\dot{\mathbf{u}}, \dot{\mathbf{v}})$ with

$$\mathbf{u} \cdot \dot{\mathbf{u}} = 0,$$

$$\dot{\mathbf{u}} \cdot \mathbf{v} + \mathbf{u} \cdot \dot{\mathbf{v}} = 0.$$

This defines the Jacobi field

$$V = t\dot{\mathbf{u}} + \dot{\mathbf{v}} - (\dot{\mathbf{v}} \cdot \mathbf{u})\mathbf{u}$$

orthogonal to the geodesic, i.e. a pair of tangent vectors $(\dot{\mathbf{u}}, \dot{\mathbf{v}} - (\dot{\mathbf{v}} \cdot \mathbf{u})\mathbf{u})$. However this splitting coincides with the one above, since the Riemannian connection on S^2 is obtained by projecting onto the tangent space using the flat connection in $\mathbb{R}^3$.

Now the complex structure at $(\mathbf{u}, \mathbf{v})$ is given by taking the cross product with the normal direction $\mathbf{u}$ in each factor. This is clearly the standard complex structure on S^2, so we have established our claim. From now on we shall denote the space of oriented lines in $\mathbb{R}^3$ with this complex structure by $\mathbf{T}$.

Since the straight lines through a fixed point $x \in \mathbb{R}^3$ are determined by their direction $\mathbf{u}$, the line P_x is a holomorphic *section* of $\pi: \mathbf{T} \to \mathbb{P}_1$. Every such section is a holomorphic vector field on $\mathbb{P}_1$ which (since the tangent bundle is of degree 2) may be written in terms of a quadratic polynomial:

$$s(\zeta) = (a\zeta^2 + b\zeta + c)\frac{d}{d\zeta}. \tag{3.1}$$

The real structure is defined by

$$\tau(\mathbf{u}, \mathbf{v}) = (-\mathbf{u}, \mathbf{v}),$$

and this is minus the natural action of the antipodal map α on the tangent bundle TS^2. In holomorphic coordinates $\alpha(\zeta) = -\bar{\zeta}^{-1}$, and it follows that the section $s(\zeta)$ is real iff

$$b = \bar{b}; a = -\bar{c}.$$

Thus a point $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ may be represented by the real section

$$s(\zeta) = ((x_1 + ix_2) - 2x_3\zeta - (x_1 - ix_2)\zeta^2)\frac{d}{d\zeta}, \tag{3.2}$$

and in these coordinates the natural conformal structure induced on $\mathbb{R}^3$ by the discriminant $b^2 - 4ac$ is the standard Euclidean one given by the metric $dx_1^2 + dx_2^2 + dx_3^2$.

We have just described $\mathbb{R}^3$ as the space of real holomorphic vector fields on $\mathbb{P}_1$, i.e. as the Lie algebra of SO(3). The surface **T** should of course be thought of as an affine vector bundle over $\mathbb{P}_1$ with no distinguished zero section. Indeed a choice of real section determines an origin in $\mathbb{R}^3$.

The relationship between **T** and the twistor space $\mathbb{P}_3 \backslash \mathbb{P}_1$ of $\mathbb{R}^4$ (see [1]) may be described in terms of the action of time translation. Translation is an isometry of $\mathbb{R}^4$ and induces a free holomorphic action of the additive group $\mathbb{R}$ on $\mathbb{P}_3 \backslash \mathbb{P}_1$, which is the real part of a holomorphic action of the complex numbers $\mathbb{C}$. The quotient space of $\mathbb{P}_3 \backslash \mathbb{P}_1$ by this action is the surface **T**.

4. The Bogomolny Equations

We shall see now how solutions to the Bogomolny equations, with no boundary conditions imposed, may be represented in terms of the complex geometry of **T**.

Consider a principal SU(2) bundle with connection ∇ on $\mathbb{R}^3$, and a section of the adjoint bundle Φ, the Higgs field. Let E be the associated rank 2 complex vector bundle on $\mathbb{R}^3$, and F the curvature. Now define a rank 2 vector bundle $\tilde{E}$ on **T** by

$$\tilde{E}_z = \{s \in \Gamma(\gamma_z, E) | (\nabla_U - i\Phi)s = 0\}. \tag{4.1}$$

Here U is the unit tangent vector along the oriented geodesic γ_z corresponding to a point $z \in \mathbf{T}$. Thus for each line γ_z we have a system of ordinary differential equations along that line. The finite-dimensional null space is the fibre of the vector bundle $\tilde{E}$ over the point $z \in \mathbf{T}$.

Theorem (4.2). *If* (∇, Φ) *satisfy the* SU(2) *Bogomolny equations* $\nabla\Phi = *F$, *then* $\tilde{E}$ *is in a natural way a holomorphic vector bundle on the space of geodesics* **T** *such that*

(i) $\tilde{E}$ *is trivial on every real section.*

(ii) $\tilde{E}$ *has a symplectic structure.*

(iii) $\tilde{E}$ *has a quaternionic structure, that is an anti-holomorphic linear map*

$$\sigma : \tilde{E}_z \to \tilde{E}_{\tau z},$$

such that $\sigma^2 = -1$.

Conversely, every such holomorphic vector bundle on **T** *defines a solution of the Bogomolny equations.*

Proof. We shall construct on $\tilde{E}$ a $\bar{\partial}$-operator, that is a linear differential operator

$$\bar{\partial} : \Gamma(\tilde{E}) \to \Gamma(\tilde{E} \otimes \Lambda^{0,1}),$$

such that

(a) $\bar{\partial}(fs) = f\bar{\partial}s + s \otimes \bar{\partial}f, f \in C^\infty(\mathbf{T})$,

(b) $\bar{\partial}^2 = 0$,

where $\bar{\partial}^2$ is defined in terms of the natural extension of $\bar{\partial}$ to an operator

$$\bar{\partial} : \Gamma(\tilde{E} \otimes \Lambda^{0,p}) \to \Gamma(\tilde{E} \otimes \Lambda^{0,p+1}).$$

It is a corollary of the Newlander-Nirenberg theorem (see [1], for example) that $\tilde{E}$ then has a holomorphic structure for which this operator is the natural $\bar{\partial}$-operator.

First, we paraphrase the construction of the complex structure on the space of geodesics, by considering the unit sphere bundle $S(T\mathbb{R}^3)$. The flat Riemannian connection splits this into a product $S^2 \times \mathbb{R}^3$. The geodesic flow X is a vector field which is horizontal relative to this splitting and the space of geodesics $\mathbf{T}$ is the quotient space of $S^2 \times \mathbb{R}^3$ by the flow:

$$p : S^2 \times \mathbb{R}^3 \to \mathbf{T},$$

$$p(\mathbf{u}, \mathbf{x}) = (\mathbf{u}, \mathbf{x} - (\mathbf{x} \cdot \mathbf{u})\mathbf{u}).$$

The orthogonal complement of X in the tangent space at $(\mathbf{u}, \mathbf{x}) \in S^2 \times \mathbb{R}^3$ projects isomorphically to the tangent space of $p(\mathbf{u}, \mathbf{x})$ in $\mathbf{T}$. The almost complex structure we defined consists of taking the standard complex structure of S^2 in the fibre directions and the standard one (cross-product with X) in the horizontal direction orthogonal to X. The integrability condition on the curvature is equivalent to the invariance of this structure along the flow.

This interpretation makes it easier to describe a section of $\tilde{E}$ over $\mathbf{T}$. If we denote by p_1 and p_2 the projections onto the two factors of $S^2 \times \mathbb{R}^3$, then a section $s \in \Gamma(\mathbf{T}, \tilde{E})$ consists of a section $\hat{s} \in \Gamma(S^2 \times \mathbb{R}^3, p_2^* E)$ which satisfies the equation $\nabla_X \hat{s} - i\Phi \hat{s} = 0$, using the connection pulled back from $\mathbb{R}^3$ by p_2.

We now define the $\bar{\partial}$-operator by:

$$(\bar{\partial} s)\hat{} = \nabla^{0,1} \hat{s}. \tag{4.3}$$

This means we consider the $(0,1)$ component of $\nabla \hat{s}$ relative to the complex structure in the orthogonal space to X in the tangent space of $S^2 \times \mathbb{R}^3$. We must show that this is well-defined, i.e. that $\bar{\partial} s$ is a section over $\mathbf{T}$ rather than on $S^2 \times \mathbb{R}^3$. Now since the holomorphic tangents to the real sections span the tangent space to $\mathbf{T}$ at each point, it is enough to check this on each such section.

Thus, we pick an origin $0 \in \mathbb{R}^3$ and consider all the straight lines through it. The exponential map at this point is

$$\exp : S^2 \times \mathbb{R} \to \mathbb{R}^3$$

$$(\mathbf{u}, t) \to t\mathbf{u},$$

and is the projection p_2 applied to the translation of $S^2 \times 0$ by the geodesic flow. We have a section $\hat{s} \in \Gamma(S^2 \times \mathbb{R}, p_2^* E)$ which satisfies

$$\left(\nabla_{\frac{\partial}{\partial t}} - i\Phi\right)\hat{s} = 0,$$

and from (4.3)

$$(\bar{\partial} s)\hat{} = \left(\nabla_{\frac{\partial}{\partial x}} + i\nabla_{\frac{\partial}{\partial y}}\right)\hat{s}\, d\bar{z},$$

where $z = x + iy$ is a holomorphic coordinate on S^2. We need to show that

$$\left(\nabla_{\frac{\partial}{\partial t}} - i\Phi\right)\left(\nabla_{\frac{\partial}{\partial x}} + i\nabla_{\frac{\partial}{\partial y}}\right)\hat{s} = 0,$$

for then we shall have a genuine section over **T**. But

$$\left[\nabla_{\frac{\partial}{\partial t}} - i\Phi, \nabla_{\frac{\partial}{\partial x}} + i\nabla_{\frac{\partial}{\partial y}}\right] = F\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) - \nabla_{\frac{\partial}{\partial y}}\Phi + i\left(F\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right) + \nabla_{\frac{\partial}{\partial x}}\Phi\right),$$

and this vanishes because of the Bogomolny equations $\nabla\Phi = *F$ and the definition of the complex structure. Hence since

$$\left(\nabla_{\frac{\partial}{\partial t}} - i\Phi\right)\hat{s} = 0, \qquad \left(\nabla_{\frac{\partial}{\partial t}} - i\Phi\right)\left(\nabla_{\frac{\partial}{\partial x}} + i\nabla_{\frac{\partial}{\partial y}}\right)\hat{s} = 0$$

and we have a well-defined operator $\bar{\partial}$ which clearly satisfies condition (a). As for condition (b), note that

$$(\bar{\partial}^2 s)^\wedge = F^{0,2}\hat{s}$$

from the definition (4.3). But the fibre of $p_2 : S^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a holomorphic direction, so any (0, 2) form on T must be a multiple of $\bar{u}_1 \wedge \bar{u}_2$, where u_2 is in the fibre direction. On the other hand the connection ∇ on $S^2 \times \mathbb{R}^3$ is pulled back by p_2, so the $\bar{u}_1 \wedge \bar{u}_2$ component of F must be zero. Hence $\bar{\partial}^2 s = 0$, and the bundle $\tilde{E}$ is holomorphic.

We must now check properties (i)–(iii) of the theorem.

(i) A real section of $\mathbf{T} \to \mathbb{P}_1$ consists of the geodesics that pass through a fixed point $0 \in \mathbb{R}^3$. Take a fixed vector $e_0 \in E_0$, then there is a unique solution to $\left(\nabla_{\frac{\partial}{\partial t}} - i\Phi\right)\hat{s} = 0$ with initial condition $\hat{s}(0) = e_0$. This defines a section s of $\tilde{E}$ on the projective line P_0, and since e_0 is independent of the direction,

$$(\bar{\partial}s)^\wedge = \nabla^{0,1}\hat{s} = \nabla^{0,1}e_0 = 0.$$

Thus s is holomorphic, and letting e_0 vary in E_0 we have a trivialization of $\tilde{E}$ on P_0, and similarly on every real section.

(ii) The connection ∇ preserves the SU(2) structure on E and in particular a symplectic form. The Higgs field Φ also preserves the SU(2) structure but $i\Phi$ being self-adjoint does not. However since Trace$(i\Phi) = 0$, the symplectic form is preserved so the form on E is inherited by the solutions of $(\nabla_U - i\Phi)s = 0$. Thus $\tilde{E}$ has a symplectic form, which is clearly compatible with the $\bar{\partial}$-operator and hence holomorphic.

(iii) The bundle E on $\mathbb{R}^3$ has a quaternionic structure $\sigma : E_x \to E_x$. So, if

$$(\nabla_U - i\Phi)s = 0,$$

then applying σ,

$$(\nabla_U + i\Phi)\sigma s = 0,$$

and so

$$(\nabla_{-U} - i\Phi)\sigma s = 0.$$

Thus, recalling that changing the direction U of the geodesic is the real structure τ on **T**, we see that σ defines an antilinear map

$$\sigma : E_z \to E_{\tau z}$$

with $\sigma^2 = -1$, easily seen to be antiholomorphic.

To prove the converse part of Theorem (4.2) we must consider the geometry of the complexification $\mathbb{C}^3$ of $\mathbb{R}^3$. In our twistor representation a point $(a,b,c)\in\mathbb{C}^3$ is represented by a holomorphic section $(a\zeta^2+b\zeta+c)\frac{d}{d\zeta}$ of $\mathbf{T}$ which is not necessarily real. Take a local coordinate system (η,ζ) on $\mathbf{T}$ defined by

$$(\eta,\zeta)\to\eta\frac{d}{d\zeta},$$

where ζ is a local affine coordinate on $\mathbb{P}_1$, and consider all sections which pass through a fixed point $(\eta_0,\zeta_0)\in\mathbf{T}$. This is the set

$$\{(a,b,c)\in\mathbb{C}^3\,|\,\eta_0=a\zeta_0^2+b\zeta_0+c\},$$

and is thus a plane Π in $\mathbb{C}^3$. The conformal structure on $\mathbb{C}^3$ was defined in Sect. 3 by the discriminant, so the complexification of the Euclidean metric is

$$g=(db)^2-4(da)(dc).$$

Now if

$$c=\eta_0-b\zeta_0-a\zeta_0^2,$$
$$dc=-\zeta_0\,db-\zeta_0^2\,da,$$

and so

$$g=db^2+4da(\zeta_0\,db+\zeta_0^2\,da)=(db+2\zeta_0\,da)^2.$$

Thus the induced metric on the plane Π is degenerate: Π is a *null plane*. So from a *holomorphic point of view a point in* $\mathbf{T}$ *represents a null plane* Π *in* $\mathbb{C}^3$. *The plane* Π and its conjugate $\bar{\Pi}$ intersect in a straight line (the geodesic of our original construction) and the orientation is determined by the *ordered* pair $(\Pi,\bar{\Pi})$.

Now suppose $\tilde{E}$ is a holomorphic vector bundle which is trivial on every real section P_x of $\mathbf{T}$. It will be trivial on the complex sections sufficiently close to $x\in\mathbb{C}^3$ and we can define a vector bundle E on a neighbourhood of x by

$$E_y=H^0(P_y,\tilde{E}). \tag{4.4}$$

The fibre consists of the holomorphic sections of $\tilde{E}$ along the projective line P_y. Furthermore, for all lines which pass through the point $z\in\mathbf{T}$, we have a canonical trivialization of the bundle E defined by the restriction isomorphism

$$H^0(P_y,\tilde{E})\xrightarrow{\cong}\tilde{E}_z. \tag{4.5}$$

Thus on a neighbourhood in Π of the geodesic γ_z, E has a natural flat connection ∇_Π. If $\tilde{E}$ is endowed with a symplectic form, then clearly E inherits one and this is preserved by the connection.

Since every null line lies in a unique null plane, we can use this connection to define parallel translation along a segment of each complex null geodesic through x. This defines, by differentiation at x, a matrix valued function A of null directions which is homogeneous of degree 1 and holomorphic. However, the space of null directions is a conic C in $P(T_x)=\mathbb{P}_2$, the projective space of the tangent space to $\mathbb{C}^3$ at x, and every homogeneous function of degree 1 defines a section of the bundle

$O(1)$ on $\mathbb{P}_2$ restricted to C. It follows, using the Künneth theorem, the fact that $H^1(C, O(1))=0$ and that every *holomorphic* section of $O(1)$ on C is the restriction of a linear form on T_x, that A is in fact *linear* in the null direction. Thus the notion of parallel translation is defined by a *connection* ∇ in a neighbourhood of x. This is analogous to Ward's original construction [14].

Now restrict this connection to the null plane Π. It agrees by definition with the natural flat connection ∇_Π in the null direction, so

$$\nabla - \nabla_\Pi = i\Phi\, dy \tag{4.6}$$

for some endomorphism Φ of E, where the null lines in Π are defined by $y=$const. If we make y a linear coordinate and dy of unit length, then dy is uniquely determined (up to a sign fixed by the orientation of $\mathbb{R}^3$). If we now consider all the null planes through x, then $\Phi(x, \Pi)$ is a holomorphic matrix valued function of the null directions, homogeneous of degree 0. This means it must be constant as a function of Π and so gives a well-defined endomorphism of the bundle E in a neighbourhood of $\mathbb{R}^3 \subset \mathbb{C}^3$. Furthermore, since ∇ and ∇_Π preserve the symplectic form on E, so does Φ.

Now since ∇_Π is flat, we obtain from (4.6) the equation:

$$F - i\nabla\Phi \wedge dy = 0\,. \tag{4.7}$$

The real structure σ on $\tilde{E}$ defines a quaternionic structure on the bundle E over the real points $\mathbb{R}^3 \subset \mathbb{C}^3$ and makes ∇ and Φ compatible with it. With this real structure the Eq. (4.7) leads directly to the Bogomolny equations. Indeed let (e_1, e_2, e_3) be an orthonormal basis of tangent vectors in $\mathbb{R}^3$ such that $e_1 + ie_2$ is the null direction of Π. Then (4.7) implies that

$$F_{13} + iF_{23} = i\nabla_1\Phi - \nabla_2\Phi\,,$$

and hence that

$$\nabla_1\Phi = F_{23}\,,$$
$$\nabla_2\Phi = F_{31}\,,$$

which, for all null directions, yields the Bogomolny equations. Let $z\in T$, and γ_z be the corresponding geodesic, in the direction e_3. Then

$$\tilde{E}_z \cong \{s \in \Gamma(\Pi, E) | \nabla_\Pi s = 0\}$$

from (4.4) and (4.5). But this is isomorphic to the sections of E along the line γ_z in Π which satisfy $\nabla_\Pi s = 0$. This however, from (4.6) is the null space of $\nabla_3 - i\Phi$, and so we have inverted the initial construction.

The theorem is now proved. Although it was phrased in terms of SU(2) solutions, with minor modifications to the real structure, it is valid for any real form of a complex Lie group.

5. The Line Bundle L

To illustrate the preceding construction, let us take the simplest case of a solution to the Bogomolny equations. Here we take the group to be $U(1)$, the bundle E to be a trivial line bundle with flat connection, and the Higgs field Φ to be the constant i. We shall obtain a holomorphic line bundle L on $\mathbf{T}$, trivial on each real section and where in this case the real structure gives an antiholomorphic isomorphism

$$\sigma : L_z \cong L^*_{\tau z}.$$

Following theorem (4.2), we define

$$L_z = \left\{ s \in C^\infty(\gamma_z) \,\middle|\, \frac{ds}{dt} + s = 0 \right\}.$$

Thus $s = \text{const}\, e^{-t}$. Now define the function $\hat{l}$ on $S^2 \times \mathbb{R}^3$ by

$$\hat{l}(\mathbf{u}, \mathbf{x}) = e^{-\mathbf{u}\cdot\mathbf{x}}. \tag{5.1}$$

On each straight line this defines a multiple of e^{-t}, so $\hat{l}$ defines a global non-vanishing section l of the line bundle L over $\mathbf{T}$. Note that l depends on the choice of origin.

We shall compute $\bar{\partial} l$ next. Note firstly that if $\mathbf{u}\cdot\mathbf{x} = 0$, then $\hat{l}$ is constant. Hence $\nabla^{0,1}$ is zero in the horizontal direction of $S^2 \times \mathbb{R}^3$. This means, in terms of $\mathbf{T}$, that $\bar{\partial} l$ vanishes in the fibre directions of $\pi : \mathbf{T} \to \mathbb{P}_1$.

Now let $\zeta = x + iy$ be a standard local coordinate on the fibre S^2 of $S^2 \times \mathbb{R}^3$. By stereographic projection we may write

$$\mathbf{u} = \left(\frac{2x}{1+\zeta\bar{\zeta}}, \frac{2y}{1+\zeta\bar{\zeta}}, \frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}} \right).$$

Now $\mathbf{x} = (x_1, x_2, x_3)$, so putting $a = x_1 + ix_2$ and $b = x_3$ we get

$$\mathbf{u}\cdot\mathbf{x} = \frac{a\bar{\zeta} + \bar{a}\zeta + b(1-\zeta\bar{\zeta})}{(1+\zeta\bar{\zeta})}.$$

Hence

$$\nabla^{0,1}(\mathbf{u}\cdot\mathbf{x}) = (a - 2b\zeta - \bar{a}\zeta^2)\frac{d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}. \tag{5.2}$$

But taking coordinates $(\eta, \zeta) \to \eta \dfrac{d}{d\zeta}$ on $\mathbf{T}$ and comparing with (3.2) we see that the projective line P_x corresponding to x is $\eta = a - 2b\zeta - \bar{a}\zeta^2$. Thus the section l of L on $\mathbf{T}$ satisfies

$$\bar{\partial} l = \frac{-l\eta\, d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}. \tag{5.3}$$

Note that l is holomorphic in each fibre of $\mathbf{T}$ and also on the zero section $\eta = 0$, corresponding to the chosen origin of $\mathbb{R}^3$.

If fl is a local holomorphic section of L, then $\bar{\partial}(fl) = 0$, so from (5.3)

$$\frac{\partial f}{\partial \bar{\eta}} = 0$$

and

$$\frac{\partial f}{\partial \bar{\zeta}} = \frac{f\eta}{(1+\zeta\bar{\zeta})^2}.$$

It follows that $f = g(\zeta,\eta)\exp\left(\dfrac{-\eta}{\zeta(1+\zeta\bar{\zeta})}\right)$, where $g(\zeta,\eta)$ is holomorphic.

The function $f_0 = \exp\left(\dfrac{-\eta}{\zeta(1+\zeta\bar{\zeta})}\right)$ is regular at $\zeta=\infty$ but has a singularity at $\zeta=0$. Similarly $f_1 = \exp\left(\dfrac{-\eta}{\zeta(1+\zeta\bar{\zeta})} + \dfrac{\eta}{\zeta}\right)$ is regular at $\zeta=0$ but singular at $\zeta=\infty$.

Thus $f_0 l$ defines a trivialization of L on the open set $U_0 = \{(\eta,\zeta)\in\mathbf{T} \mid \zeta \neq 0\}$ and $f_1 l$ on $U_1 = \{(\eta,\zeta)\in\mathbf{T} \mid \zeta \neq \infty\}$. On the intersection $U_0 \cap U_1$ we have $f_0 l = e^{-\eta/\zeta} f_1 l$, so the transition function ϕ_{01} is given by

$$\phi_{01} = \exp(-\eta/\zeta). \tag{5.4}$$

Since the real structure in these coordinates is

$$\tau(\eta,\zeta) = (-\bar{\eta}/\bar{\zeta}^2, -\bar{\zeta}^{-1}),$$

it is clear, noting that τ interchanges U_0 and U_1, that

$$\phi_{01}\tau = \exp(\bar{\eta}/\bar{\zeta}) = \bar{\phi}_{01}^{-1},$$

so we have an antiholomorphic isomorphism

$$\sigma : L_z \xrightarrow{\cong} L^*_{\tau z}.$$

From its description as a solution of the Bogomolny equations, it is clear that L is a canonical object on $\mathbf{T}$. In algebraic geometric terms, we can describe it as follows:

Recall that any compact complex manifold X^n has a fundamental class $\omega \in H^n(X,K)$, where K is the canonical bundle. Furthermore on the total space of any vector bundle E there is a tautological section s of π^*E. Hence taking $E=K^*$, the anticanonical bundle, there is a natural element $s\pi^*\omega \in H^n(K^*,O)$.

In our case $X = \mathbb{P}_1$, $\mathbf{T}=K^*$ and the line bundle $L \in H^1(\mathbf{T},O^*)$ is defined by

$$L = \exp(s\pi^*\omega).$$

6. Boundary Conditions

Suppose now that (∇,Φ) satisfies the SU(2) Bogomolny equations, subject to the following conditions:

$$\begin{aligned} &\text{(i)} && \|\Phi\| = 1 - \frac{m}{r} + O(r^{-2}), \\ &\text{(ii)} && \frac{\partial\|\Phi\|}{\partial\Omega} = O(r^{-2}), \\ &\text{(iii)} && \|\nabla\Phi\| = O(r^{-2}), \quad \text{as} \quad r\to\infty. \end{aligned} \tag{6.1}$$

Here we define $\|\Phi\|^2 = -\frac{1}{2}\operatorname{Tr}\Phi^2$ and $\dfrac{\partial\|\Phi\|}{\partial\Omega}$ denotes the angular derivative of $\|\Phi\|$,

$$\frac{\partial\|\Phi\|}{\partial\Omega} = \left(\left(\frac{\partial\|\Phi\|}{\partial\theta}\right)^2 + \sin^2\theta\left(\frac{\partial\|\Phi\|}{\partial\phi}\right)^2\right)^{1/2}.$$

These conditions are independent of the choice of origin. Note that condition (ii) implies that m is a constant. Also, since Φ is non-vanishing at sufficiently large distances from (i), the bundle E splits as a direct sum $E \cong M \oplus M^*$ of eigenspaces of Φ. Restricted to a large sphere the line bundle M has an integer Chern class $\pm k$. By integration over the sphere (see [15]) one may show that the constant m is the positive integer k. We shall call k the *charge* of the solution.

Ward's approach to constructing solutions of charge k is to use the Ansatz $\mathscr{A}_k$. This means in holomorphic terms constructing the bundle $\tilde{E}$ on $\mathbf{T}$ as an extension. More precisely we take on $\mathbf{T}$ the holomorphic line bundle $O(-k)$ (the pull-back under $\pi : \mathbf{T} \to \mathbb{P}_1$ of the unique line bundle of degree $-k$), tensor it with the bundle L we constructed in Sect. 5, and consider vector bundles $\tilde{E}$ for which there is an exact sequence:

$$0 \to L(-k) \xrightarrow{\alpha} \tilde{E} \xrightarrow{\beta} L^*(k) \to 0. \tag{6.2}$$

This means that $\tilde{E}$ has a distinguished holomorphic subbundle isomorphic to the bundle $L(-k)$. Since $\tilde{E}$ has a symplectic form, the quotient line bundle must be isomorphic to $L^*(k)$.

To express a bundle as an extension in this way means that one may find transition functions which are upper triangular. In our case since $O(k)$ has the transition function ζ^k and L has the function $e^{-\eta/\zeta}$, an application of the $\mathscr{A}_k$ Ansatz requires looking for a bundle E with transition functions

$$\begin{pmatrix} \zeta^{-k} e^{\eta/\zeta} & f(\eta, \zeta) \\ 0 & \zeta^k e^{-\eta/\zeta} \end{pmatrix}.$$

The off-diagonal term $f(\eta, \zeta)$ defines a Čech cocycle representing a class in the sheaf cohomology group $H^1(\mathbf{T}, L^2(-2k))$. This group classifies all extensions of the form (6.2).

We shall now prove the following:

Theorem (6.3). *Let (∇, Φ) be a solution of the* SU(2) *Bogomolny equations of charge k on $\mathbb{R}^3$, satisfying the boundary conditions* (6.1). *Let $\tilde{E}$ be the corresponding holomorphic bundle on* $\mathbf{T}$ *defined by* (4.1) *and let L^+ denote the subbundle of $\tilde{E}$ consisting of solutions to $(\nabla_U - i\Phi)s = 0$ which decay as $t \to +\infty$. Then L^+ is a holomorphic subbundle isomorphic to $L(-k)$ and thus $\tilde{E}$ may be represented as an extension*

$$0 \to L(-k) \to \tilde{E} \to L^*(k) \to 0.$$

Proof. In the neighbourhood of an oriented line in $\mathbb{R}^3$, choose a trivialization (e_0, e_1) of the vector bundle E which consists of the orthogonal eigenvectors of Φ at large distances in the positive direction. Then for large r,

$$\Phi e_j = (-1)^j i \|\Phi\| e_j, \qquad (j=0,1).$$

Hence taking the covariant derivative,

$$(\Phi - (-1)^j i \|\Phi\|) \nabla e_j = (i\nabla \|\Phi\| - (-1)^j \nabla \Phi) e_j .$$

Since $\|\nabla \Phi\| = O(r^{-2})$ from (6.1) this implies that the connection matrix in this gauge is of the form:

$$\begin{pmatrix} A & B \\ -B^* & -A \end{pmatrix}, \quad \text{where} \quad \|B\| = O(r^{-2}).$$

Now consider the diagonal part of this matrix. It defines a connection which preserves the eigenspaces of Φ. Choose a gauge which is covariant constant in the radial directions for this connection. Then the form of the matrix is unchanged, but in polar coordinates the 1-form A has no dr term.

The curvature F is given by:

$$\begin{aligned} F &= \begin{pmatrix} dA - B^*B & dB \\ -dB^* & -(dA\text{-}B^*B) \end{pmatrix} \\ &= \begin{pmatrix} i*d\|\Phi\| & 0 \\ 0 & -i*d\|\Phi\| \end{pmatrix} + O(r^{-2}) \end{aligned}$$

by the Bogomolny equations $\nabla \Phi = *F$. Hence, if (u_1, u_2) are orthogonal coordinates on the unit sphere and $A = A_1\, du_1 + A_2\, du_2$, we have

$$\frac{\partial A_1}{\partial r} = i \frac{\partial \|\Phi\|}{\partial u_2} + O(r^{-2}),$$

and from condition (ii) of (6.1)

$$\frac{\partial A_1}{\partial r} = O(r^{-2}),$$

and similarly for A_2. Thus in the radial direction A_i tends to a limit $\bar{A}_i$ as $r \to \infty$.

Instead of choosing a fixed set of polar coordinates we may consider the pulled back connection $p_2^* \nabla$ on $S^2 \times \mathbb{R}^3$ and choose an analogous gauge in a neighbourhood of an orbit of the geodesic flow. Then, as $t \to \infty$ along the flow, the connection approaches a pulled back connection $p_1^* \bar{\nabla}$ which moreover preserves the eigenspaces of Φ.

Now consider the corresponding holomorphic vector bundle $\tilde{E}$ on $\mathbf{T}$. From (4.1)

$$\tilde{E}_z = \{ s \in \Gamma(\gamma_z, E) \mid (\nabla_U - i\Phi) s = 0 \}.$$

In the present gauge we may write this as

$$\frac{dx}{dt} - \begin{pmatrix} -1+\frac{k}{t} & 0 \\ 0 & 1-\frac{k}{t} \end{pmatrix} x + C(t)x = 0, \tag{6.4}$$

where $\|C(t)\| = O(t^{-2})$. This follows from condition (i).

Since $\|C(t)\|$ is integrable, we may apply a theorem of Levinson (see [5, 6]) on the asymptotic behaviour of ordinary differential equations to deduce that there are solutions x_0, x_1 such that

$$\begin{aligned} x_0(t)t^{-k}e^{t} &\to e_0, \\ x_1(t)t^{k}e^{-t} &\to e_1, \end{aligned} \quad \text{as} \quad t \to +\infty.$$

In particular there is a uniquely determined 1-dimensional subspace $L_z^+ \subset \tilde{E}_z$ of solutions (namely multiples of x_0) which decay as $t \to +\infty$.

The solution x_0 is obtained by using the fundamental solution of the unperturbed equation:

$$\begin{aligned} K(s,t) &= \begin{pmatrix} e^{s-t}\frac{t^k}{s^k} & 0 \\ 0 & 0 \end{pmatrix} \qquad s > t, \\ &= \begin{pmatrix} 0 & 0 \\ 0 & e^{t-s}\frac{s^k}{t^k} \end{pmatrix} \qquad s \leqq t. \end{aligned}$$

One then solves the equation

$$\begin{aligned} x(t) &= (t^k e^{-t})e_0 + \int_{t_1}^{\infty} K(s,t)\, C(s)x(s)\, ds \\ &= (t^k e^{-t})e_0 + (Tx)(t). \end{aligned}$$

Since $C(t)$ is integrable, T is a contraction mapping for large enough t_1, so there is a unique solution amongst functions bounded on $[t_1, \infty)$. Using this method, and the uniform bounds on $C(t)$ provided by the boundary conditions one may show that L^+ varies continuously with $z \in \mathbf{T}$, and moreover is differentiable by condition (ii). Furthermore, the derivative relative to some choice of initial value is given by:

$$\left.\frac{\partial x_0}{\partial z}\right|_{z=0} = \int_{t_1}^{\infty} K(s,t) \left[\frac{\partial C}{\partial z}(s,0)\, x_0(s,0) + C(s,0)\frac{\partial x_0}{\partial z}(s,0)\right] ds,$$

so since by condition (ii) $\frac{\partial C}{\partial z} = O(s^{-2})$ as $s \to \infty$, uniformly as z varies over a compact set, we have another contraction mapping and

$$\left\|\frac{\partial x_0}{\partial z}\right\| < \text{const}\, \|x_0\|.$$

Thus in particular $\frac{\partial x_0}{\partial z} \to 0$ as $t \to +\infty$.

We now wish to show that L^+ is a *holomorphic* subbundle of $\tilde{E}$. If s is a local differentiable section of L^+ then we may use the symplectic form of $\tilde{E}$ to express the obstruction to holomorphicity as $\langle \bar{\partial} s, s \rangle$.

In terms of the sphere bundle $S^2 \times \mathbb{R}^3$,

$$\langle \bar{\partial} s, s \rangle = \langle \nabla^{0,1} \hat{s}, \hat{s} \rangle,$$

which because of the Bogomolny equations is constant along the geodesic flow. But as we have seen $s(z,t)$ and $\nabla s(z,t)$ both tend to zero as $t \to +\infty$. Thus the constant is zero and L^+ is holomorphic. Note that we may define L^- as the subspace of solutions which decay as $t \to -\infty$ and the same argument shows that L^- is holomorphic. We shall consider the relationship between the two in the next section.

It remains to identify L^+ with $L(-k)$. The line bundle L corresponds to the trivial $U(1)$ solution of the Bogomolny equations. Consider the holomorphic bundle $\tilde{E} \otimes L^*$. This corresponds to a $U(2)$ solution where the Higgs field is replaced by $\Phi - i$. Hence the subbundle $L^+ \otimes L^*$ is isomorphic to the space of solutions to the equation

$$\frac{dx}{dt} - \begin{pmatrix} \frac{k}{t} & 0 \\ 0 & 2 - \frac{k}{t} \end{pmatrix} x + C(t)x = 0$$

such that $t^{-k} x(t) \to \lambda e_0$.

Now let s be a local section of $L^+ \otimes L^*$. It defines $\hat{s}(z,t)$ on $S^2 \times \mathbb{R}^3$. As $t \to +\infty$, the eigenspace of the Higgs field corresponding to the limiting eigenvalue i tends to a line bundle $p_1^* M$ of degree $\pm k$ (a bundle on the sphere of infinity) and the connection, as we have seen, tends to a pulled back connection $p_1^* \bar{\nabla}$.

Define $\alpha : L^+ \otimes L^* \to \pi^* M$ by

$$\alpha(s) = \lim_{t \to +\infty} t^{-k} \hat{s}(z,t).$$

Note that α is independent of the choice of origin $t=0$ on the line. By using the fundamental solution again we see that

$$\frac{\partial}{\partial z} \alpha(\hat{s}(z,t)) = \alpha \frac{\partial \hat{s}}{\partial z}(z,t),$$

and hence

$$(\partial^{0,1} + \bar{A}^{0,1}) \alpha(\hat{s}) = \alpha(\nabla^{0,1} \hat{s}(z,t)). \tag{6.5}$$

However a line bundle on $S^2 = \mathbb{P}_1$ with connection has a holomorphic structure defined by the $\bar{\partial}$ operator $\partial^{0,1} + \bar{A}^{0,1}$ and the holomorphic structure on $\tilde{E} \otimes L^*$ is defined by $\nabla^{0,1}$. Thus (6.5) implies that α is a holomorphic isomorphism. There is a unique complex structure on a line bundle over $\mathbb{P}_1$, hence

$L^+ \otimes L^* \cong \pi^* O(\pm k)$. In fact, since the bundle $\tilde{E}$ is holomorphically trivial on each real section of $\mathbf{T}$, L^+ must have negative degree, so

$$L^+ \cong L(-k),$$

and Theorem (6.3) is proved.

7. The Spectral Curve

We have established that the holomorphic vector bundle $\tilde{E}$ corresponding to a monopole is expressed as an extension

$$0 \to L^+ \to \tilde{E} \to (L^+)^* \to 0, \tag{7.1}$$

where $L^+ \cong L(-k)$. We also saw in the proof of Theorem (6.3) that it is an extension

$$0 \to L^- \to \tilde{E} \to (L^-)^* \to 0. \tag{7.2}$$

In fact the real structure $\sigma : \tilde{E} \to \tilde{E}$ takes L^+ to L^-, for since it changes orientation on each straight line, the solutions that decay at $+\infty$ go to those that decay at $-\infty$. Hence σ defines an antiholomorphic isomorphism

$$\sigma : L^+ \to L^-$$

and in particular $L^- \cong L^*(-k)$.

Now we may project L^- in $\tilde{E}$ onto $(L^+)^*$ in (7.1) and obtain a holomorphic section,

$$\psi \in H^0(\mathbf{T}, (L^+ \otimes L^-)^*) \cong H^0(\mathbf{T}, O(2k)).$$

We shall call the zero set of this section the *spectral curve* S. This is because it corresponds to those points $z \in \mathbf{T}$ for which $L_z^+ = L_z^-$. In other words, it describes those straight lines for which there are solutions to $(\nabla_U - i\Phi)s = 0$ which decay at both ends of the line. Since they decay exponentially they are also in $\mathscr{L}^2$. Thus we are considering a nonlinear family of linear differential operators on the line and we are seeking the particular parameter values for which there exist $\mathscr{L}^2$ solutions.

We shall prove the following properties of the curve S:

Proposition (7.3). *Let S be the spectral curve corresponding to an* SU(2) *monopole of change k. Then*

(i) *S is compact.*

(ii) *S is defined by an equation*

$$p(\eta, \zeta) = \eta^k + a_1(\zeta)\eta^{k-1} + \dots a_k(\zeta) = 0,$$

where $a_i(\zeta)$ is a polynomial of degree $2i$.

(iii) *The line bundle L^2 is holomorphically trivial on S.*

(iv) *S is preserved by the real structure τ.*

Proof. (i) Let s be a non-zero solution of $(\nabla_U - i\Phi)s = 0$ on some line, and consider the function $f = \|s\|^2$ on the line. Since s is the solution of a linear first order equation, $f > 0$. Now,

$$\frac{df}{dt} = (\nabla_U s, s) + (s, \nabla_U s) = 2(i\Phi s, s)$$

since ∇ preserves the inner product and $i\Phi$ is self-adjoint. Differentiating again, we obtain

$$\frac{d^2 f}{dt^2} = 2\{((i\nabla_U \Phi)s, s) + 2\|\Phi\|^2 f\}.$$

But from the boundary conditions (i) and (iii) of (6.1), there exists $R > 0$ such that

$$\|\Phi\|^2 > \tfrac{1}{2} \quad \text{and} \quad |((\nabla_U \Phi)s, s)| < \tfrac{1}{2}\|s\|^2$$

if $r > R$.

Hence if the whole line is a distance greater than R from the origin,

$$\frac{d^2 f}{dt^2} > f > 0$$

for *all* solutions s. Thus since f is convex and non-constant, it cannot be bounded and there are no non-zero solutions which decay at both ends of the line.

Hence the spectral curve S lies in the disc bundle of radius R of $\mathbf{T} = TS^2$ and is therefore compact.

(ii) The curve is defined by $\psi = 0$, where ψ is a holomorphic section of $O(2k)$ on $\mathbf{T}$. Let $\mathscr{I}$ be the ideal sheaf of a section P of $\mathbf{T}$, then for any vector bundle E there is an exact sequence of sheaves:

$$0 \to \mathscr{I}^{n+1} \otimes O(E) \to \mathscr{I}^n \otimes O(E) \to O_P(E \otimes N^{-n}) \to 0,$$

where N is the normal bundle. Since in our case $N \cong O_P(2)$, if we take $E = O(2k)$, we have

$$0 \to \mathscr{I}^{n+1} \otimes O(2k) \to \mathscr{I}^n \otimes O(2k) \to O_P(2k - 2n) \to 0. \tag{7.4}$$

Now if $n > k$, $H^0(\mathbb{P}_1, O(2k - 2n)) = 0$, so from the exact cohomology sequence of (7.4), if a section ψ of $O(2k)$ vanishes to order $(k+1)$ on P, it vanishes identically on $\mathbf{T}$. Thus a section is determined by its restriction to the k^{th} order neighbourhood of P. From the exact sequence of higher order neighbourhoods:

$$0 \to O_P(2k - 2n) \to O_P^n(2k) \to O_P^{n-1}(2k) \to 0,$$

we see that

$$\begin{aligned} \dim H^0(P, O_P^k(2k)) &= \sum_{n=0}^{k} \dim H^0(P, O(2k - 2n)) \\ &= \sum_{n=0}^{k} (2n+1) = (k+1)^2. \end{aligned}$$

Hence

$$\dim H^0(\mathbf{T}, O(2k)) \leqq (k+1)^2 . \tag{7.5}$$

On the other hand $O(2)$ is the pull-back of the tangent bundle of $\mathbb{P}_1$ and this on $\mathbf{T}$ has a canonical section $\eta \frac{d}{d\zeta}$ in our standard coordinate system. Hence if $p(\eta, \zeta)$ is any polynomial as in (ii), $\psi = p(\eta, \zeta)\left(\frac{d}{d\zeta}\right)^k$ is a holomorphic section of $O(2k)$. Since the dimension of the space of such sections is $(k+1)^2$, it follows from (7.5) that every such section is of this form, in particular ψ. In order for S to be compact, it is clear that the coefficient of η^k must be non-zero, hence S is defined by an equation $\eta^k + a_1(\zeta)\eta^{k-1} + \ldots + a_k(\zeta) = 0$.

(iii) Since $L^+ = L^-$ on S, $L^+ \cong L(-k)$ and $L^- \cong L^*(-k)$, it is clear that $L \cong L^*$ and hence L^2 is trivial.

(iv) Since $\sigma(L^+) = L^-$ it is obvious that S is real.

The curve S is thus determined by $(k+1)^2 - 1$ real parameters, but because of (ii) satisfies further constraints. Since L^2 has Chern class zero, its restriction to S corresponds to a point in the g-dimensional complex torus

$$H^1(S, O)/H^1(S, \mathbb{Z}),$$

where g is the genus of S. Since furthermore L^2 is real ($L^+ = L^-$ on S) there are g real constraints on the curve. If S is non-singular, we may compute the genus from the adjunction formula:

$$K \cdot S + S^2 = 2g - 2 .$$

Since S is the zero set of a section of $O(2k)$, its self-intersection number is $c_1(O(2k))[S]$. But because $p(\eta, \zeta)$ is of degree k in η, S is a k-fold covering of the base space $\mathbb{P}_1$. Hence

$$S^2 = 2k^2 ,$$

and since the canonical bundle $K \cong O(-4)$,

$$K \cdot S = -4k .$$

Thus

$$2g - 2 = 2k^2 - 4k$$

and

$$g = (k-1)^2 .$$

Hence the conditions we have described (a geometric version of those in [7]) amount to the choice of

$$(k+1)^2 - 1 - (k-1)^2 = 4k - 1$$

real parameters.

This suggests that the spectral curve determines the solution. This is in fact the case, as we shall see now.

We have described the bundle $\tilde{E}$ as an extension

$$0 \to L(-k) \to \tilde{E} \to L^*(k) \to 0\,,$$

in other words, defined by an element a in the sheaf cohomology group $H^1(\mathbf{T}, L^2(-2k))$. We may take the section $\psi \in H^0(\mathbf{T}, O(2k))$ and multiply to obtain $\psi(a) \in H^1(\mathbf{T}, L^2)$. This class is in fact trivial for general reasons.

We have the extension

$$0 \to L^+ \xrightarrow{\alpha} \tilde{E} \xrightarrow{\beta} (L^+)^* \to 0$$

and a homomorphism $\psi : L^- \to (L^+)^*$. The product defines an extension

$$0 \to L^+ \to F \to L^- \to 0\,.$$

This bundle may be defined by

$$F = \{(x, y) \in \tilde{E} \oplus L^- \mid \beta(x) = \psi(y)\}\,.$$

But in our case $\psi = \beta | L^-$, so the diagonal map

$$\begin{aligned} \Delta : L^- &\to F \\ y &\to (y, y) \end{aligned}$$

defines a splitting of the extension. Hence the class $\psi(a) = 0$.

Now consider the exact cohomology sequence of the sequence of sheaves

$$0 \to O(L^2(-2k)) \to O(L^2) \to O_S(L^2) \to 0\,.$$

We obtain

$$0 \to H^0(S, L^2) \xrightarrow{\delta} H^1(\mathbf{T}, L^2(-2k)) \xrightarrow{\psi} H^1(\mathbf{T}, L^2)\,.$$

Since $\psi(a) = 0$, $a = \delta(\alpha)$ for some $\alpha \in H^0(S, L^2)$. But L^2 is trivial on the compact connected curve S, so up to a factor (which does not change the bundle $\tilde{E}$), α is unique. Thus

Theorem (7.6). *If S is the spectral curve, the bundle $\tilde{E}$ is obtained as the extension $\delta(\alpha)$, where α is a trivialization of L^2 on S and*

$$\delta : H^0(S, L^2) \to H^1(\mathbf{T}, L^2(-2k))$$

the coboundary map.

In particular the curve S determines the bundle $\tilde{E}$.

Example (7.7). In the case $k=1$, the spectral curve S is defined by $\eta + a_1(\zeta) = 0$, where $a_1(\zeta)$ is of degree 2. This is just a section of $\pi : \mathbf{T} \to \mathbb{P}_1$ and since S is real it corresponds to a real section P_x, for some distinguished point $x \in \mathbb{R}^3$.

If we take x to be the origin, then the BPS monopole gives a solution to the charge 1 SU(2) Bogomolny equations [4, 13]. The Higgs field Φ for this solution is given by

$$\Phi = \hat{\mathbf{r}}\left(\coth r - \frac{1}{r}\right).$$

Hence if we consider the lines through the origin and use a gauge which is *covariant constant in the radial directions, the differential equation on each line is,*

$$\frac{dx}{dr} - \begin{pmatrix} \coth r - \frac{1}{r} & 0 \\ 0 & -\coth r + \frac{1}{r} \end{pmatrix} x = 0.$$

Clearly $x = \begin{pmatrix} 0 \\ r/\sinh r \end{pmatrix}$ is the unique solution which decays as $r \to +\infty$. Since it also decays as $r \to -\infty$, the spectral curve has as a component all the lines through x, i.e. P_x. Since it must be a section, then $S = P_x$. Finally since by (7.6) the spectral curve determines the solution, we see as a consequence that the BPS monopole is the *unique* solution for charge $k = 1$.

Example (7.8). Consider the axially symmetric situation treated by Prasad and Rossi [12] and, using different methods, by Forgacs et al. [9]. In our twistorial approach a rotation of θ about the x_3-axis corresponds to the action of $U(1)$ on $\mathbf{T}$ given by

$$(\eta, \zeta) \to (e^{i\theta}\eta, e^{i\theta}\zeta).$$

Hence the spectral curve S given by

$$\eta^k + a_1(\zeta)\eta^{k-1} + \ldots + a_k(\zeta) = 0$$

is invariant iff $a_i(\zeta) = c_i \zeta^i$ for some constants c_i, $1 \leqq i \leqq k$. But then the equation factorizes as

$$\prod_{i=1}^{k} (\eta - \lambda_i \zeta) = 0,$$

where the λ_i are the roots of the equation $\xi^k + c_1 \xi^{k-1} + \ldots + c_k = 0$.

Geometrically this means that the curve S is reducible and is the sum of k sections of $\mathbf{T}$, each passing through the two points $(\eta, \zeta) = (0, 0)$, $(0, \infty)$.

On each individual section, the bundle L^2 is trivial, since it is of degree zero. We thus have a complex $\mathbb{C}^*$ solution to the Bogomolny equations on $\mathbb{C}^3$. From Sect. 5 it corresponds to a line bundle with trivial connection and Higgs field $2i$. We shall describe the condition that L^2 be trivial on S in terms of this connection.

Take two component curves P_u and P_v of S corresponding to the points $u = (0, 0, \lambda_i/2)$, $v = (0, 0, \lambda_j/2)$ in $\mathbb{C}^3$. By the holomorphic approach to the Bogomolny equations in the proof of Theorem (4.2), trivializations of L^2 on P_u and P_v agree at a point of intersection $z \in P_u \cap P_v$ if the corresponding vectors at u and v are related by parallel translation in the plane Π corresponding to z, with respect to the flat connection ∇_Π. Hence they are compatible at both points of intersection z, z' if parallel translation along the geodesic joining u to v is the same for both connections ∇_Π and $\nabla_{\Pi'}$.

Now from (4.6), $\nabla - \nabla_\Pi = i\Phi\, dy$ and $\nabla - \nabla_{\Pi'} = i\Phi\, dy'$. On the line of intersection $dy' = -dy$, so parallel translation is the same if

$$\exp\left(-\int_u^v 2dy\right) = \exp\left(\int_u^v 2dy\right),$$

i.e. if the distance between u and v is a multiple of $i\pi/2$.

Putting in the reality condition and choosing a suitable origin, the spectral curve for an axially symmetric solution is of the form

$$\eta \prod_{i=1}^{l} (\eta^2 + k_i^2 \pi^2 \zeta^2) = 0$$

for $k = 2l+1$, with $k_i \in \mathbb{Z}$ and

$$\prod_{i=1}^{l} (\eta^2 + (k_0/2 + k_i)^2 \pi^2 \zeta^2) = 0$$

for $k = 2l$.

In particular, as noted by Prasad and Rossi, apart from the choice of origin and axis of symmetry there are no continuous parameters.

8. Summary

We have shown that every SU(2) monopole satisfying the boundary condition (6.1) defines an algebraic curve in the surface $\mathbf{T} = T\mathbb{P}_1$ and that the curve determines the monopole. It is thus possible in principle to investigate properties of the monopole by studying algebraic curves. There are a number of questions we have not answered, however, by this construction:

(1) Which curves in $\mathbf{T}$ satisfy the necessary conditions of Proposition (7.3) in order to be a spectral curve?

The answer involves the periods of the curve, and hence is transcendental in nature. Any curve which does satisfy those conditions defines by the coboundary construction of Theorem (7.6) a vector bundle $\tilde{E}$ over $\mathbf{T}$ expressed as an extension $\delta(\alpha)$ in $H^1(\mathbf{T}, L^2(-2k))$. Moreover, by reversing the proof of that theorem one may show that the conjugate element $\sigma\delta(\alpha) \in H^1(\mathbf{T}, L^{-2}(-2k))$ defines an isomorphic bundle, so that $\tilde{E}$ has a quaternionic structure. In view of Theorem (4.2) it defines a solution of the Bogomolny equations on those points of $\mathbb{R}^3$ which correspond to sections of $\mathbf{T}$ over which $\tilde{E}$ is trivial. This provokes the next question:

(2) For which curves is $\tilde{E}$ trivial on every real section?

Since $\tilde{E} \cong O(m) \oplus O(-m)$ for some m on every projective line, we can characterize the sections P_x for which $\tilde{E}$ is trivial by the condition $H^0(P_x, \tilde{E}(-1)) = 0$. From the exact sequence

$$0 \to L(-k-1) \to \tilde{E}(-1) \to L^*(k-1) \to 0,$$

this holds iff the coboundary map

$$H^0(P_x, L^*(k-1)) \to H^1(P_x, L(-k-1)) \tag{8.1}$$

is an isomorphism. This map is obtained by multiplying by the extension class $\delta(\alpha) \in H^1(P_x, L^2(-2k))$. Now the trivialization α of L^2 on S is related to a fixed trivialization β of L^2 on P_x by

$$\alpha = \lambda_i \beta, \qquad 1 \leqq i \leqq 2k$$

at the points of intersection $\zeta_i \in P_x \cap S$. If the section P_x is defined by the equation $\eta = a(\zeta)$ and the spectral curve by $\psi(\eta, \zeta) = 0$ then we may define a quadratic form on polynomials of degree $(k-1)$ by

$$Q(f, g) = \sum \lambda_i \operatorname*{Res}_{\zeta = \zeta_i} \frac{f(\zeta)\, g(\zeta)}{\psi(a(\zeta), \zeta)}$$

which, identifying $H^1(P_x, L(-k-1)) \cong H^0(P_x, L^*(k-1))$ by Serre duality, is the map (8.1). Thus the non-degeneracy of Q is the required condition. In practice this is difficult to obtain, even in the axially symmetric case.

Having considered interior regularity, one may ask for the boundary conditions.

(3) Does a solution to the Bogomolny equations obtained this way satisfy the boundary conditions (6.1)?

This seems quite likely – indeed a partial answer is found in Corrigan and Goddard [7].

In some ways, the difficulties encountered in proving non-singularity parallel those in the algebraic curve approach to instantons [3]. The monad approach [2] gave a happier method of constructing solutions, though with some loss of explicitness. An analogous description for monopoles (cf. Nahm [11]) would get around some of the difficulties of the method presented here.

9. Appendix: Minimal Surfaces

As a postscript let us consider how Weierstrass used the complex geometry of **T** to solve the minimal surface equations in $\mathbb{R}^3$ (see [8, 17]). Recall that a minimal surface may be considered as a map $h: U \to \mathbb{R}^3$ from some open set $U \subseteqq \mathbb{R}^2$ which satisfies:

(i) h is harmonic,

(ii) h is conformal.

If we identify $\mathbb{R}^2$ with $\mathbb{C}$, then we may write

$$h(z) = \mathbf{x}(z) + \bar{\mathbf{x}}(z),$$

where $\mathbf{x}(z) = (x_1(z),\ x_2(z),\ x_3(z))$ is holomorphic, because of condition (i). Condition (ii) will be satisfied if the (2, 0) component of the pulled back metric vanishes, i.e.

$$\sum_{i=1}^{3} \frac{\partial x_i}{\partial z}^2 = 0.$$

Thus $\mathbf{x}=\mathbf{x}(z)$ describes a holomorphic null curve in $\mathbb{C}^3$. Through each point of the curve there is a unique tangential null plane, so this describes a curve in the space of null planes of $\mathbb{C}^3$, i.e. the twistor space $\mathbf{T}$.

Conversely, let C be a holomorphic curve in $\mathbf{T}$. To each point we associate the *osculating section* P, that is the section which is tangent to second order to the curve at that point. In local coordinates we may describe C by a holomorphic function

$$\eta=f(\zeta)$$

at points for which the curve is not tangential to the fibres. A section $\eta=a+b\zeta+c\zeta^2$ then osculates C at the point ζ if

$$\begin{aligned} a+b\zeta+c\zeta^2&=f(\zeta),\\ b+2c\zeta&=f'(\zeta),\\ 2c&=f''(\zeta). \end{aligned} \tag{9.1}$$

This describes a holomorphic curve

$$(a,b,c)=(f-\zeta f'+\tfrac{1}{2}\zeta^2 f'', f'-\zeta f'', \tfrac{1}{2}f'')$$

in $\mathbb{C}^3$, and moreover

$$(b')^2-4a'c'=\zeta^2(f''')^2-2f'''(\tfrac{1}{2}\zeta^2 f''')=0,$$

so the curve is null. Thus, in standard coordinates in $\mathbb{R}^3$ (cf. Sect. 3) we have the minimal surface

$$\begin{aligned} x_1&=\mathrm{Re}(\tfrac{1}{2}(1-\zeta^2)f''+\zeta f'-f),\\ x_2&=\mathrm{Re}\left(\frac{-i}{2}(1+\zeta^2)f''+i\zeta f'-if\right),\\ x_3&=\mathrm{Re}(\zeta f''-f'). \end{aligned}$$

The interpretation of the minimal surface corresponding to the spectral curve S of a monopole is left to the reader's imagination.

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