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Connections with L^p Bounds on Curvature

Karen K. Uhlenbeck
Department of Mathematics, University of Illinois at Chicago Circle, Chicago, IL 60680, USA

Abstract. We show by means of the implicit function theorem that Coulomb gauges exist for fields over a ball in R^n when the integral $L^{n/2}$ field norm is sufficiently small. We then are able to prove a weak compactness theorem for fields on compact manifolds with L^p integral norms bounded, $p > n/2$.

Introduction

The variational problems for gauge fields arising in physics differ markedly from many other geometric variational problems due to their gauge invariance. This paper provides two technical tools for handling the gauge invariance. First we show the local existence of a "good" gauge (called Lorentz, Hodge or Coulomb) under very weak hypotheses. Secondly, we prove a global theorem on the weak compactness of connections given integral bounds on their curvatures. These technical theorems are very useful for both regularity theorems and direct variational methods. I am particularly indebted to C. Taubes, who pointed out some very important generalizations of the original theorems. The strong form $2p = n$ of Corollary 1.4 and Corollary 2.2 are essentially due to Taubes.

In Sect. 1, we present notation and state the theorems and a few immediate applications. Detailed proofs are in Sect. 2 for the local results, and in Sect. 3 for the global results.

1. Notation and Statement of the Results

In this paper, η is a vector bundle with compact structure group G over a compact Riemannian n-dimensional manifold M. Assume the fibers $\eta_x \cong R^\ell$ carry an inner product and that $G \subset SO(\ell)$ respects this inner product. The bundle Aut η is the automorphism bundle with fiber $(\text{Aut}\,\eta)_x \cong G$. The bundle Ad η is the Lie algebra or adjoint bundle with fiber $(\text{Ad}\,\eta)_x \cong \mathfrak{G}$, the Lie albebra of G. Assume the metric on Aut η and Ad η are compatible with the usual metric on $SO(\ell)$.

Let $\mathfrak{A}$ be the space of smooth connections on η compatible with the structure group. Every such connection D induces a connection on the Ad η bundle which is also called D. In this case we have the Riemannian connection on the tangent bundle TM; therefore $D \in \mathfrak{A}$ induces connections on all bundles associated to

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η and TM. Denote by the symbol D the induced operators (in the sense of exterior differentiation). The space of connections is an affine space. Pick a base connection $D_0 \in \mathfrak{A}$. Then

$$\mathfrak{A} = \{D = D_0 + A : A \in C^\infty(M, \operatorname{Ad}\eta \otimes T^*M)\}.$$

Define as follows the Sobolev space of connections $\mathfrak{A}_k^p$, the L_k^p connections. Here L_k^p is the Sobolev space of functions with k derivatives which are p integrable.

$$\mathfrak{A}_k^p = \{D = D_0 + A : A \in L_k^p(M, \operatorname{Ad}\eta \otimes T^*M)\}.$$

Because of the affine structure, this definition does not depend on the choice of $D_0 \in \mathfrak{A}$.

If $D_0 \in \mathfrak{A}$, then its curvature or field is

$$F(D_0) = D_0^2 \in C^\infty(M, \operatorname{Ad}\eta \otimes T^*M \wedge T^*M).$$

Lemma 1.1. *For $k = 1$ and $2p \geqq \dim M = n$, the curvature map taking a connection onto its curvature extends to a quadratic map*

$$\mathfrak{A}_1^p \to L^p(M, \operatorname{Ad}\eta \otimes T^*M \wedge T^*M).$$

Proof. If $D \in \mathfrak{A}_1^p$, $D = D_0 + A$, $A \in L_1^p(M, \operatorname{Ad}\eta \otimes T^*M)$.

$$F(D) = D^2 = F(D_0) + D_0 A + [A, A].$$

This expression is quadratic in A, $F(D_0) \in C^\infty$ and $A \to D_0 A \in L^p$ is linear. Since $L_1^p \subset L^q$ where $1/q \geqq 1/p - 1/n$ by the Sobolev embedding theorem, then $A \to [A, A] \in L^{q/2}$ in the quadratic term. To get $L^{q/2} \subset L^p$, we need $1/p \geqq 2/q \geqq 2/p - 2/n$. This explains the constraint $2p \geqq n$.

The C^∞ gauge group $\mathscr{D} = C^\infty$ $(\operatorname{Aut}\eta)$ acts on the connections by conjugation. If $s \in \mathscr{D}$

$$s^*(D) = s^{-1} \circ D \circ s = s^{-1}(D_0 + A)s = D_0 + s^{-1} D_0 s + s^{-1} As.$$

The map on the affine section A is

$$A \to s^{-1} D_0 s + s^{-1} As.$$

The gauge group for a connection is logically the sections of Aut η with one more derivative than A. This leads us to define the gauge group for $\mathfrak{A}_k^p$:

$$\mathscr{D}_{k+1}^p = L_{k+1}^p(M, \operatorname{Aut}\eta).$$

For $p(k+1) > \dim M = n$, this is a smooth manifold and Lie group [5]. Also, by the Sobolev theorem, $L_{k+1}^p(M, \operatorname{Aut}\eta) \subset C^0(M, \operatorname{Aut}\eta)$ and these gauge transformations preserve the topological structure of the bundle η. Care must be taken whenever the strict inequality does not hold. We state theorems for k either 0 or 1, only for convenience.

Lemma 1.2. *For $k = 0$ or 1, $(k+1)p > \dim M$, the gauge group $\mathscr{D}_{k+1}^p$ is a smooth Lie group under pointwise multiplication. The induced map*

$$\mathscr{D}_{k+1}^p \times \mathfrak{A}_k^p \to \mathfrak{A}_k^p$$

is smooth. Furthermore, if $D = s^{-1} \circ \tilde{D} \circ s$ *for* $D, \tilde{D} \in \mathfrak{A}_k^p$, *then* $s \in \mathscr{D}_{k+1}^p$.

Proof. The Lie group structure is standard. The multiplication theorems in Sobolev spaces give the smoothness of

$$A \to s^{-1} D_0 s + s^{-1} A s.$$

If $D = D_0 + A$ and $\tilde{D} = D_0 + \tilde{A}$, then a gauge transformation carrying one to the other satisfies

$$A = s^{-1} D_0 s + s^{-1} \tilde{A} s.$$

Use $\| \ \|_{p,k}$ to denote a Sobolev L_k^p norm. Then $\| s^{-1} D_0 s \|_{p,0} \leqq \| A \|_{q,0} + \| s^{-1} \tilde{A} s \|_{q,0}$. For $k = 0$, we are done since s^{-1} is orthogonal with norm 1. If $k = 1$, we obtain the same estimate for L_0^q, where q lies in the Sobolev range $1/n - 1/p + 1/q \geqq 0$. Now estimate $D_0 s \in L_1^p$ using the formula

$$D_0 s = sA - \tilde{A} s$$

and the multiplication theorems $L_1^p \otimes L_1^q \to L_1^p$. This same type of estimate appears over and over again in this paper and we assume familiarity with Palais [5, Chap. 9].

We can now state the main theorems. The notation $\| \ \|_{p,k}$ again means a Sobolev L_k^p norm, as it does throughout the paper.

Theorem 1.3. *Let* $M = B^n$, $\eta = B^n \times R^\ell$, G *compact,* $G \subset SO(\ell)$, $2p \geqq n$ *and* $D = d + \tilde{A}$ *for* $\tilde{A} \in L_1^p(B^n, R\ell \times \mathfrak{G})$. *Then there exists* $\kappa(n) > 0$ *and* $c(n) < \infty$ *such that if* $\| F \|_{n/,0}^{n/2} = \| d\tilde{A} + [\tilde{A}, \tilde{A}] \|_{n/2,0}^{n/2} \leqq \kappa(n)$, *then* $d + \tilde{A} = D$ *is gauge equivalent by an element* $s \in L_2^p(B^n, G) = \mathscr{D}_2^p$ *to a connection* $d + A$ *where* A *satisfies*:

$$\text{(i)} \ d^* A = 0$$

$$\text{(ii)} \ \| A \|_{p,1} \leqq c(n) \| F \|_{p,0}.$$

There are more details in Chap. 1. Note that for $2p = n$, the multiplication and inversions involved in gauge transformations *are not continuous.* This borderline case follows from a weak limit argument from that for $2p > n$.

Regularity of solutions of Yang–Mills equations for connections $D \in \mathfrak{G}_1^p$, $2p \geqq \dim M$ follows rather easily from such a theorem. Since $\int_M |F|^{n/2} * 1 < \infty$, one can restrict to a small disk $\int_{B^n} |F|^{n/2} dx \leqq \kappa(n)$. (The size of the disk is not uniform. This is a dilation invariant integral and there always is such a disk. When $2p > \dim M$, the size of the disk can be uniformly determined by Lemma 3.4 from $\int_M |F|^p * 1$.) Then apply Theorem 1.3. The system of equations consisting of the Yang–Mills equation $d^* dA + [A, dA] + [A, [A, A]] = 0$ and $d^* A = 0$ is uniformly elliptic. Now standard techniques apply (Morrey [4], Chap. 6). This technique applies to coupled equations [7].

Corollary 1.4. *If* $D \in \mathfrak{A}_1^p$ *for* $2p \geqq \dim M$ *is a weak solution of the pure Yang–Mills equations then* D *is locally equivalent over a cover* $\{\mathscr{U}\}$ *by gauge transformation* $s \in L_2^p(\mathscr{U}, \operatorname{Aut} \eta | \mathscr{U})$ *to an analytic connection. If* $2p > \dim M$, *uniform estimates depending on* $\int_M |F|^p * 1$ *exist.*

The global theorem is the following. This is proved as Theorem 3.6.

Theorem 1.5 (3.6). *Let $2p > \dim M$ and $D(i) \in \mathfrak{A}_1^p$ be a sequence of connections with $\int_M |F(D(i))|^p * 1 \leqq B$. Assume M and G compact. Then there exists a subsequence $\{i'\} \subset \{i\}$ and gauge transformations $s(i) \in \mathscr{D}_2^p$ such that $s(i')^{-1} \circ D(i') \circ s(i')$ is weakly convergent in $\mathfrak{A}_1^p$. The weak limit D satisfies $\int_M |F(D)|^p * 1 \leqq B$.*

2. The Local Theorem

For this section $M = B^n = \{x \in R^n : |x| \leqq 1\}$. Then a trivialization $\eta = M \times R^\ell$ can be fixed, and we use the parameters of this trivialization on all the associated bundles. Let d be exterior differentiation.

$$\mathfrak{A}_k^p = \{d + A : A \in L_k^p(B^n, \mathfrak{G} \times R^n)\}.$$

$$\mathscr{D}_{k+1}^p = L_{k+1}^p(B^n, G).$$

Define $\mathfrak{A}_{1,\kappa}^p = \{D \in \mathfrak{A}_1^p : \int_{|x| \leqq 1} |F|^{n/2} dx \leqq \kappa\}$.

Theorem 2.1. *Let $n > p > n/2$ and assume G compact. Then there exists $\kappa = \kappa(n) > 0$ and $c = c(n)$ such that every connection $D \in \mathfrak{A}_{1,\kappa}^p$ is gauge equivalent to a connection $d + A \in \mathfrak{A}_1^p$, where A satisfies:*

(a) $d^* A = 0$.

(b) $(x \cdot A) = 0$ on $S^{n-1} = \partial B^n$.

(c) $\|A\|_{n/2,1} \leqq c(n)\left(\int_{|x| \leqq 1} |F(D)|^{n/2} dx\right)^{2/n}$.

(d) $\|A\|_{p,1} \leqq c(n)\left(\int_{|x| \leqq 1} |F(D)|^{p/2} dx\right)^{1/p}$.

Before we give the proof, we state a corollary.

Corollary 2.2. *Suppose $\tilde{A} \in L_1^{n/2}(B^n, R^n \times \mathfrak{G})$ and $F(\tilde{A}) = F(d + \tilde{A}) = d\tilde{A} + [\tilde{A}, \tilde{A}]$ satisfies*

$$\int_{|x| \leqq 1} |F(\tilde{A})|^{n/2} dx < \kappa(n).$$

Then there exists $s \in L_2^{n/2}(B^n, G)$ such that $A = s^{-1} ds + s^{-1} \tilde{A} s$ satisfies (a)–(c).

Proof. Approximate $\tilde{A}$ by smooth $\tilde{A}_i \to \tilde{A}$ in $L_1^{n/2}(B^n, R^n \times \mathfrak{G})$ with $\int_{|x| \leqq 1} |F(\tilde{A}_i)|^{n/2} dx \leqq \kappa(n)$. Then we may apply Theorem 2.1 to the $\tilde{A}_i \in L_1^p(B^n, R^n \times \mathfrak{G})$. Since Lemma 2.4 holds for $p = n/2$, we are finished.

The outline of the proof of Theorem 2.1 is straightforward. We show $\mathfrak{A}_{1,\kappa}^p$ is connected. The set of connections in $\mathfrak{A}_{1,\kappa}^p$ satisfying (a)–(d) of Theorem 2.1 is both open and closed. This is quite different from the approach used for solving the Dirichlet problem [8].

Lemma 2.3. *$\mathfrak{A}_{1,\kappa}^p \subset \mathfrak{A}_1^p$ is connected $n > p > n/2$.*

Proof. Let $D = d + A$. Define the one-parameter family $D_\sigma = d + \sigma A(\sigma x)$ for $0 \leqq \sigma \leqq 1$. Then the curvature $F(D_\sigma)$ has the formula

$$F(D_\sigma)(x) = \sigma^2 (dA)(\sigma x) + [A(\sigma x), A(\sigma x)] = \sigma^2 F(D)(\sigma x).$$

$$\| F(D_\sigma) \|_{n/2,0}^{n/2} = \int_{|x| \leqq 1} |F(D_\sigma)|^{n/2} dx = \int_{|x| \leqq \sigma} |F(D)|^{n/2} dx.$$

This formula is perhaps easier to understand by observing that it comes from the pull-back of D under the map $x \to \sigma x$. Clearly $D_\sigma \in \mathfrak{A}^p_{1,\kappa}$ for $0 \leqq \sigma \leqq 1$ if $2p \geqq n$ and $D \in \mathfrak{A}^p_{1,\kappa}$. For fixed D, $D_\sigma = d + \sigma A(\sigma)$ is a continuous curve in $\mathfrak{A}^p_{1,\kappa}$. $D_1 = D$ and $D_0 = d$.

Lemma 2.4. *The set of* $D \in \mathfrak{A}^p_{1,k}$ *satisfying* (a)–(d) *is closed for* κ *sufficiently small and* $n > p \geqq n/2$.

Proof. Let $D_i = d + \tilde{A}_i \to d + \tilde{A} \in \mathfrak{A}^p_1$ be a sequence of connections convergent in $\mathfrak{A}^p_1$ such that D_i is gauge equivalent to $d + A_i$, where conditions (a)–(d) hold on A_i. Choose $A =$ weak limit of A_i in $L^p_1(B^n, R^n \times \mathfrak{G})$. Conditions (a)–(d) are preserved under weak limits, provided we can show a gauge transformation from $\tilde{A}$ to A exists.

$$s_i^{-1} ds_i + s_i^{-1} \tilde{A}_i s_i = A_i \text{ or}$$

$$ds_i = s_i A_i - \tilde{A}_i s_i.$$

For $1/n - 1/p + 1/q = 0$, since s_i is orthogonal,

$$\| ds_i \|_{q,0} \leqq \| A_i \|_{q,0} + \| \tilde{A}_i \|_{q,0} \leqq c \| A_i \|_{p,1} + \| \tilde{A}_i \|_{p,1}.$$

Since G is compact, $\| s_i \|_{q,1}$ is uniformly bounded and we can pick a subsequence $s_i \rightharpoonup s$ in $L^q_1(B^n, G)$. The equation

$$ds_i = s_i A_i - \tilde{A}_i s_i$$

is preserved under weak limits.

$$ds = sA - \tilde{A} s.$$

From Lemma 1.2 $s \in L^p_2(B^n, G) = \mathcal{D}^p_2$.

The next step is to show that (c) and (d) (which are closed conditions) are a priori valid estimates on solutions to equations (a) and (b).

Lemma 2.5. *There exists* $\not{k}(n) > 0$ *such that if* $\| A \|_{n,0} \leqq \not{k}(n)$ *and* (a)–(b) *are satisfied, then for* $n > p \geqq n/2$

$$\| A \|_{p,1} \leqq c(n) \left(\int_{|x| \leqq 1} |F(A)|^p dx \right)^{1/p}.$$

Proof. The pair of equations ($d^*A = 0$, $dA + [A, A] = F(A)$) form a non-linear overdetermined elliptic system for which the Neumann boundary condition $(x \cdot A) | S^{n-1} = 0$ is elliptic. By simple integration by parts, if $d^*A = 0$ and $(x \cdot A) | S^{n-1} = 0$, then

$$\int_{|x| \leqq 1} |\nabla A|^2 dx + \int_{|x| = 1} |A|^2 dx = \int_{|x| \leqq 1} |dA|^2 dx.$$

Here ∇ denotes the full derivative of the one-form A. This can be seen by noting that

$$\int_{|x|\leqq 1} (\tfrac{1}{2}|dA|^2 - |\nabla A|^2)dx = \sum_{i,j} - \int_{|x|\leqq 1} \left(\frac{\partial}{\partial x^i}A_j\frac{\partial}{\partial x^j}A_i\right)dx$$

$$= \sum_{i,j} - \int_{|x|=1} x^i\left(A_j\frac{\partial}{\partial x^j}A_i\right)dx = \sum_j \int_{|x|=1} |A_j|^2 dx$$

$$- \sum_{i,j} \int_{|x|=1} A_j\frac{\partial}{\partial x^j}(A_i x^i)dx.$$

Since $\sum_i A_i x^i = 0$ on $|x|=1$, $\sum_{i,j} A_j\frac{\partial}{\partial x^j}(A_i x^i) = \sum_{i,j,k} (x^k A_k)\left(x^j\frac{\partial}{\partial x^j}\right)(x^i A_i) = 0$,

and the equality is proved. Elliptic systems are well-behaved on Sobolev spaces. So if $d^*A = 0$ and $(x\cdot A)|S^{n-1} = 0$, for the closed range $n \geqq p \geqq n/2$

$$\|A\|_{p,1} \leqq k'(n)\|dA\|_{p,0}.$$

From the equation $F = dA + [A, A]$ for curvature

$$\|dA\|_{p,0} \leqq \|F\|_{p,0} + \|A\|^2_{2p,0}.$$

The number q given by $1/p = 1/n + 1/q$ is identical in the Hölder inequality

$$\|A\|^2_{2p,0} \leqq \|A\|_{n,0}\|A\|_{q,0}$$

and the Sobolev inequality

$$\|A\|_{q,0} \leqq k''(n)\|A\|_{p,1}.$$

The last four inequalities combine naturally to give the inequality (putting the quadratic estimate on the left)

$$(1 - k'(n)k''(n)\|A\|_n)\|A\|_{p,1} \leqq k'(n)\|F\|_{p,0}.$$

It is sufficient to choose $\|A\|_n \leqq \kappa(n) = 1/2(k''(n)k'(n))^{-1}$.

The next two lemmas are preparation for the openness result. Lemma 2.6 is probably well-known.

Lemma 2.6. *There exists a linear operator $P: L_1^p(B^n) \to L_2^p(B^n)$ such that if $f \in L_1^p(B^n)$, $P(f) \in L_2^p(B^n)$, $P(f)|S^{n-1} = 0$ and $(x\cdot dP(f) - (f))|S^{n-1} = 0$.*

Proof. Let $P(f)$ be the solution of an inhomogeneous heat equation $0 < r \leqq 1$ with zero initial conditions at $r = 1$ multiplied by a smooth cut-off function φ with $\varphi(0) = 0$, $\varphi(x) = 1$ near $|x| = 1$. Invert the heat operator with r as time, S^{n-1} as space.

$$P(f) = \varphi[\partial/\partial r - \Delta_{S^{n-1}}]^{-1} f.$$

$$P(f)|_{r=1} = 0.$$

The regularity theorem gives $P(f) \in L_2^p(B^n - \{0\})$ for $f \in L_1^p(B^n - \{0\})$ [2]

The equation for $s \in \mathcal{D}_2^p = L_2^p(M, G)$, the gauge transformation, is

$$d^*A = d^*(s^{-1}ds + s^{-1}\tilde{A}s) = 0.$$

To get $(x\cdot A)|S^{n-1}=0$ (assume for the moment $x\cdot\tilde{A}|S^{n-1}=0$), we use the Neumann conditions $(x\cdot ds)=\partial_\nu s=0$ on S^{n-1}.

$$L^p_{1,\nu}=\{\lambda\in L^p_1(B^n,R^n\otimes\mathfrak{G}):x\cdot\lambda|S^{n-1}=0\},$$

$$\mathscr{D}^p_{2,\nu}=\{s\in L^p_2(B^n,G):(x\cdot ds)|S^{n-1}=0\}.$$

Lemma 2.7. *Suppose* $d^*A=0$, $d+A\in\mathfrak{A}^p_1\subset\|A\|_{n,0}\leqq k(n)$. *Then there exists* $\varepsilon>0$ *such that for* $\|\lambda\|_{p,1}\leqq\varepsilon$, $\lambda\in L^p_{1,\nu}$ *the non-linear equation*

$$d^*(s^{-1}ds+s^{-1}(A+\lambda)s)=0$$

has a solution $s(\lambda)\in\mathscr{D}^p_{2,\nu}\subset\mathscr{D}^p_2$. *The solution* s *depends smoothly on* $\lambda\in L^p_{1,\nu}$.

Proof. Define the spaces $L^p_{1,\nu}$ as above.

$$L^{p\perp}_{2,\nu}=\{U\in L^p_2(B^n,\mathfrak{G}):\int_{B^n}U\,dx=0,\ x\cdot dU|S^{n-1}=0\}.$$

$$L^{p\perp}_0=\{V\in L^p(B,\mathfrak{G}):\int_{B^n}V\,dx=0\}.$$

Then the operator

$$(U,\lambda)\to d^*(e^{-U}de^U+e^{-U}(A+\lambda)e^U)$$

is a smooth map

$$L^{p\perp}_{2,\nu}\otimes L^p_{1,\nu}\to L^{p\perp}_0.$$

Moreover at $(U,\lambda)=(0,0)$, the self-adjoint linearization

$$H\psi=d^*(d\psi+[A,\psi])=d^*d\psi+[A,d\psi]$$

is an isomorphism from $L^{p\perp}_{2\nu}$ to $L^{p\perp}_0$ if $\|A\|_{n,0}$ is sufficiently small. (For the same q in Hölder and Sobolev inequalities as before,

$$\|H(\psi)\|_{p,0}\geqq\|d^*d\psi\|_p-\|A\|_n\|d\psi\|_q\geqq\|d\psi\|_{p,1}(k'(n)-\|A\|_nk''(n)).$$

Now we may apply the implicit function theorem to get the result.

Lemma 2.8. *Suppose* $D\in\mathfrak{A}^p_{1,\kappa}$ *is gauge equivalent to* $d+A$, *where* A *satisfies* (a)–(d). *Then if* κ *is sufficiently small, there exists an open neighborhood of* $D\in\mathfrak{A}^p_{1,\kappa}$ *satisfying* (a)–(d) *of Theorem 2.1.*

Proof. We show there is a neighborhood of $d+A$ which satisfies (a)–(d) and pull it back to a neighborhood of D by the gauge transformation taking D to $d+A$. We would like to apply *Lemma 2.7*. However, we cannot assume $x\cdot\lambda=0$ on S^{n-1}. First we use Lemma 2.6 to get the problem into a framework where we can apply Lemma 2.7.

Let $U=U(\lambda)=P(x\cdot\lambda)$ where P is the linear operator constructed in Lemma 2.6. Make the gauge transformation

$$e^{-U}(d+A+\lambda)e^U=d+e^{-U}de^U+e^{-U}Ae^U+e^{-U}\lambda e^U=d+A+\tilde{\lambda}.$$

Since $\tilde{\lambda}=e^{-U}Ae^U-A+e^{-U}de^U+e^{-U}\lambda e^U$ and $\|U\|_{p,2}\leqq\tilde{c}\|x\cdot\lambda\|_{p,1}$, it is possible to make $\|\tilde{\lambda}\|_{p,1}$ as small as we need by taking $\|\lambda\|_{p,1}$ sufficiently small. Since $U=0$ on S^{n-1}, $de^U=dU$ on S^{n-1} and $x\cdot\tilde{\lambda}=0$ on S^{n-1}

We may now apply Lemma 2.7 to $d + A + \tilde{\lambda}$. This completes the proof.

The proof of this Lemma 2.8 could be done more elegantly using boundary value spaces. However, these are not as well-known as the methods of treating the usual Dirichlet and Neumann boundary value problems. For this reason we avoided them.

3. Construction of Global Gauge Transformations

In the previous section we proved an essentially local theorem, which we now piece together. To do this, we work with connections presented in terms of the local trivializations obtained from Theorem 2.1 in an open cover $\{\mathcal{U}_\alpha\}$ of M. We extend a theorem on the equivalence of bundles with C^0 close overlap functions from the topological category to the Sobolev category. We use this to go from the local trivializations obtained in Theorem 2.1 to global gauge transformations. A certain amount of effort was spent in finding a more elegant procedure; so far this has failed.

Let M be a compact manifold and $\{\mathcal{U}_\alpha\}$ a fixed finite set of smooth open disk neighborhoods covering M. Then any set of continuous maps

$$g_{\alpha,\beta}: \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$$

satisfying the consistency conditions

$$g_{\alpha,\beta} g_{\beta,\alpha} = 1 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta,$$
$$g_{\alpha,\beta} g_{\beta,\gamma} = g_{\alpha,\gamma} \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma,$$

gives a topological description of a principle bundle. If the $g_{\alpha,\beta}$ are C^∞ maps, the bundle is smooth. We are interested in the intermediate Sobolev case $g_{\alpha,\beta} \in L_2^p(\mathcal{U}_\alpha \cap \mathcal{U}_\beta, G)$ for $2p > \dim M$.

Given the total space of a vector bundle in the abstract, the overlap description is obtained from a set of trivializations (Gauss maps)

$$\sigma_\alpha : \eta | \mathcal{U}_\alpha \cong R^\ell \times \mathcal{U}_\alpha.$$

Then $g_{\alpha,\beta}(x) = \sigma_\alpha(x, \sigma_\beta^{-1}(x, \;)): R^\ell \to R^\ell$ for $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Here the inclusion $G \subset SO(\ell) \subset GL(\ell)$ in a fixed canonical representation is assumed, so $g_{\alpha,\beta}(x) \in G \subset GL(\ell) \subset R^\ell \times R^\ell$ is a natural identification.

Two sets of overlap functions $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$ can represent the same bundle (or equivalently, they are the overlap functions for different trivializations of the same bundle). This is true if there exists a subcover $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$, $M \subset \bigcup_\alpha \mathcal{V}_\alpha$, and $\rho_\alpha : \mathcal{V}_\alpha \to G$ satisfying $h_{\alpha\beta} = \rho_\alpha g_{\alpha\beta} \rho_\beta^{-1}$.

We first prove a technical lemma. Fix a neighborhood $\tilde{G}$ of 1 in G in the domain of $\exp^{-1}$, where $\exp: \mathfrak{G} \to G$ is the usual exponential map in the group. The notation $\exp^{-1} g$ implies $g \in \tilde{G}$.

Lemma 3.1. *Let G be a compact group with an equivariant metric. Then there exists $f_0 > 0$ such that if $h, g, \rho \in G$, $|\exp^{-1} hg| \leqq f_0$ and $|\exp^{-1} \rho| < f_0$, then $h\rho g \in \tilde{G}$ and*

$$|\exp^{-1} h\rho g| \leqq 2(|\exp^{-1} hg| + |\exp^{-1} \rho|).$$

Proof. The map Q given by the formula

$$\exp(Q(k,u)) = \exp k \exp u$$

is defined and smooth for (k,u) in a neighborhood of 0 in $\mathfrak{G}$. We have $Q(0,0)=0$ and $|dQ(0,0)|=1$. Choose $\mathcal{O}=\{x\in\mathfrak{G}:|x|\leqq f_0\}$ such that $|dQ(k,u)|\leqq 2$ for $k\in\mathcal{O}, u\in\mathcal{O}$. Since $\mathcal{O}$ is convex, by the mean value theorem $|Q(k,u)|\leqq 2(|k|+|u|)$ for $|k|\leqq f_0, |u|\leqq f_0$. The lemma follows if we set $k=\exp^{-1}(hg)$ and $u=\operatorname{Ad} g(\exp^{-1}\rho)$.

$$Q(k,u)=\exp^{-1}(hg\exp(\operatorname{Ad} g(\exp^{-1}\rho)))=\exp^{-1}(hg\rho).$$

$$|Q(k,u)|\leqq 2(|\exp^{-1}(hg)|+|\operatorname{Ad} g(\exp^{-1}\rho)|)=2(|\exp^{-1}hg|+|\exp^{-1}\rho|).$$

In the following proposition, the finite open cover $\{\mathcal{U}_\alpha\}$ is fixed and has ℓ elements $\{\alpha\}=\{1,2,\dots,\ell\}$. We prove this proposition carefully so the proof extends to the Sobolev category.

Proposition 3.2. *Let $h_{\alpha,\beta}:\mathcal{U}_\alpha\cap\mathcal{U}_\beta\to G$ and $g_{\alpha,\beta}:\mathcal{U}_\alpha\cap\mathcal{U}_\beta\to G$ be two sets of continuous functions describing vector bundles over M. Then there exists f_ℓ such that if*

$$m=\max_{\substack{(\alpha,\beta)\\ x\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta}}|\exp^{-1}h_{\alpha,\beta}(x)g_{\beta,\alpha}(x)|\leqq f_\ell$$

the following holds:

There exists a smaller cover $\mathcal{V}_\alpha\subset\mathcal{U}_\alpha$, $M\subset\bigcup_\alpha\mathcal{V}_\alpha$ and continuous $\rho_\alpha:\mathcal{V}_\alpha\to G$ such that $h_{\alpha,\beta}=\rho_\alpha g_{\alpha,\beta}\rho_\beta^{-1}$ on $\mathcal{V}_\alpha\cap\mathcal{V}_\beta$. Moreover, $\max_{x\in\mathcal{V}_\alpha}|\exp^{-1}\rho_\alpha|\leqq c_\ell m$.

Proof. The proof is inductive on the number of elements in the cover. To start the induction, let $\rho_1=1\in G$. Suppose we have constructed $\mathcal{U}_{\alpha,k}\subset\mathcal{U}_\alpha$ and $\rho_\alpha:\mathcal{U}_{\alpha,k}\to G$ satisfying $h_{\alpha,\beta}=\rho_\alpha g_{\alpha,\beta}\rho_\beta^{-1}$ on $\mathcal{U}_{\alpha,k}\cap\mathcal{U}_{\beta,k}$ for $1\leqq\alpha\leqq k$, $1\leqq\beta\leqq k$. Furthermore, assume $M\subset\left(\bigcup_{\alpha\leqq k}\mathcal{U}_{\alpha,k}\right)\bigcup\left(\bigcup_{\alpha>k}\mathcal{U}_\alpha\right)$ and $|\exp^{-1}\rho_\alpha|\leqq c_k m$. If m is sufficiently small, we claim we may continue the construction from $j=k$ to $j=k+1$. This will prove the proposition by induction.

Use the equation $u_j=\exp^{-1}(h_{j,\alpha}\rho_\alpha g_{\alpha,j})$ to define a continuous $u_j:\mathcal{U}_{\alpha,k}\cap\mathcal{U}_j\to\mathfrak{G}$ for $\alpha\leqq k=j-1$. If $m\leqq f_0/c_k$, we have $|\exp^{-1}\rho_\alpha(x)|\leqq c_k m\leqq f_0$ and $|\exp^{-1}h_{j,\alpha}(x)g_{\alpha,j}(x)|\leqq m\leqq f_0$. Lemma 3.1 shows that u_j exists and $|u_j(x)|\leqq 2(1+c_k)m=c_j m$. It follows algebraically from the consistency conditions that u_j is consistently defined on $\mathcal{U}_j\cap\left(\bigcup_{\alpha\leqq k}\mathcal{U}_{\alpha,k}\right)$.

Choose a smooth C^∞ partition of unity φ_j on M which is 0 on $\mathcal{U}_j-\bigcup_{\alpha\leqq k}\mathcal{U}_{\alpha,k}$. This can be done in such a way so that the sets

$$\mathcal{U}_{\alpha,j}=\mathcal{U}_{\alpha,k}\cap\text{interior}\{x:\varphi_j(x)=1\}$$

cover $M-\overline{\bigcup_{\alpha>k}\mathcal{U}_\alpha}$. Define $\rho_j=\exp\varphi_j u_j$ on $\mathcal{U}_j\cap\left(\bigcup_{\alpha\leqq k}\mathcal{U}_{\alpha,k}\right)$ and $\rho_j=1$ on $\mathcal{U}_j-\bigcup_{\alpha\leqq k}\mathcal{U}_{\alpha,k}$. Then $|\exp^{-1}\rho_j(x)|\leqq|\varphi_j(x)u_j(x)|\leqq 2(1+c_k)m=c_j m$. The continuous map ρ_j and the sets $\mathcal{U}_{\alpha,j}$ have the listed properties for $j=k+1$. Note that by iteration $c_{k+1}=2(1+c_k)$ can be explicitly computed.

Corollary 3.3. *Let $h_{\alpha,\beta}$ and $g_{\alpha,\beta}$ be two sets of L_2^p overlap functions on $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$ for $2p > \dim M$, $g_{\alpha,\beta} \in L_2^p(\mathscr{U}_\alpha \cap \mathscr{U}_\beta, G)$, $h_{\alpha,\beta} \in L_2^p(\mathscr{U}_\alpha \cap \mathscr{U}_\beta, G)$. Suppose*

$$m = \max_{\substack{(\alpha,\beta) \\ x \in \mathscr{U}_\alpha \cap \mathscr{U}_\beta}} |\exp^{-1} h_{\alpha,\beta}(x)\, g_{\beta,\alpha}(x)| \leqq f_\ell.$$

Then the ρ_α constructed in Proposition 3.2 satisfies $\rho_\alpha \in L_2^p(\mathscr{V}_\alpha, G)$. Furthermore if $\|h_{\alpha,\beta}|\mathscr{U}_\alpha \cap \mathscr{U}_\beta\|_{p,2} \leqq m'$ and $\|g_{\alpha,\beta}|\mathscr{U}_\alpha \cap \mathscr{U}_\beta\|_{p,2} \leqq m'$ for all pairs (α,β), then there exists $k(m')$ such that

$$\|\exp^{-1} \rho_\alpha | \mathscr{V}_\alpha\|_{p,2} \leqq k(m').$$

(Note that restrict means both derivatives and integrals are restricted to the open set named.)

Proof. We simply note that by the rules of multiplication and composition of Sobolev L_2^p functions in the range $2p > \dim M$, that inductively

$$\rho_j = \exp(\varphi_j \exp^{-1} h_{j,\alpha}\, \rho_\alpha\, g_{\alpha,j})$$

can be bounded in $L_2^p(\mathscr{U}_j, G)$. The bound could be made explicit in norms of $\varphi_j \in C^\infty(M)$, $\rho_\alpha \in L_2^p(\mathscr{U}_{\alpha,k}, G)$, $h_{j,\alpha} \in L_2^p(\mathscr{U}_j \cap \mathscr{U}_\alpha, G)$ and $g_{\alpha,j} \in L_2^p(\mathscr{U}_j \cap \mathscr{U}_\alpha, G)$.

It is now possible to proceed with the main business of this section. Fix $p > n/2$ and assume $\int_M |F|^p * 1 < B$ is a fixed uniform L^p bound on the curvature of a set of connections. The next lemma refers back to the proof of Lemma 2.3.

Lemma 3.4. *There exists a finite cover $\mathscr{U}_\alpha$ of M depending on p $(2p > \dim M)$ and B such that $B^n \cong \mathscr{U}_\alpha$. Under this coordinate identification $\int_{|x| \leqq B^n} |F|^p\, dx \leqq \kappa'(n)$.*

Proof. Choose exponential balls about each point $x_0 \in M$. If the balls are small enough, the Riemannian norms in M compare uniformly to the Euclidean norms. Using the construction and dilation of Lemma 2.3, we can assume every $x_0 \in M$ lies in a ball $\int_{x \in B^n} |F|^p\, dx \leqq \kappa'$. Since x is compact, a finite subcover of these coordinate geodesic patches cover M. The choice is independent of D, but depends on B, κ' and $2p - n > 0$.

Choose $\kappa' = \kappa(n)$ of Theorem 2.1. Apply this theorem to any connection D restricted to each $\mathscr{U}_\alpha$ of the cover if $\int_M |F(D)|^p * 1 < B$. This theorem then essentially chooses a trivialization $\sigma_\alpha(D): \eta | \mathscr{U}_\alpha \cong R^\ell \times \mathscr{U}_\alpha$. We have already used several times that bounds on the connection forms give bounds on the gauge transformations, (or overlap functions) relating them (in Lemma 1.2 and Lemma 2.4). We apply this to a sequence of connections.

Lemma 3.5. *Let $D(i)$ be a sequence of connections in $\mathfrak{A}_1^p$ and assume $\int_M F(D(i)) * 1 \leqq B$. Then there exists a fixed open cover $\{\mathscr{U}_\alpha\}$ of M and trivializations $\sigma_\alpha(i)$: $\eta | \mathscr{U}_\alpha \cong R^\ell \times \mathscr{U}_\alpha$ which induce the connection forms $\sigma_\alpha(i)\,(D(i)|\mathscr{U}_\alpha)\sigma_\alpha^{-1} = d + A(i,\alpha)$. These trivializations satisfy the properties*

(a) *Conditions* (a)–(d) *of Theorem 2.1 are satisfied by the $A = A(i,\alpha)$ on $\mathscr{U}_\alpha$.*

(b) *The overlap functions $g_{\alpha,\beta}(i) = \sigma_\alpha(i) \circ \sigma_\beta(i)^{-1}$ are uniformly bounded in $L_2^p(\mathscr{U}_\alpha \cap \mathscr{U}_\beta, G)$.*

(c) *For a subsequence, we have weak convergence*

$$A(i',\alpha) \rightharpoonup A(\alpha) \text{ in } L_1^p(\mathscr{U}_\alpha, \mathfrak{G} \times R^n)$$

$$g_{\alpha,\beta}(i') \rightharpoonup g_{\alpha,\beta}(\infty) \text{ in } L_2^p(\mathscr{U}_\alpha \cap \mathscr{U}_\beta, G).$$

(d) *The $A(\alpha)$ represents a connection D on η presented in terms of a trivialization of η whose overlap functions are given by $g_{\alpha,\beta}(\infty)$.*

Proof. Condition (a) follows from Lemma 3.4, (b) from the computations in Lemmas 1.2 and 2.4 which we will not repeat and (c) from weak compactness of $L_1^p(M)$ and $L_2^p(M)$. Because the consistency conditions are preserved under weak limits, it is clear $A(\alpha)$ represents a connection in a bundle presented in terms of $g_{\alpha,\beta}(\infty)$. That this bundle is topologically η follows from Theorem 3.2. We go into this in greater detail in the proof of our main theorem which follows.

Theorem 3.6 (1.5). *Let $2p > \dim M$ and $D(i)$ be a sequence of connections in $\mathfrak{A}_1^p$ such that $\int\limits_M |F(D(i))|^p * 1 \leqq B$. Then there exists a subsequence $\{i'\} \subset \{i\}$ and gauge transformations $s(i) \in \mathscr{D}_2^p = L_2^p(M, \text{Aut } \eta)$ such that*

$$s(i')^{-1} \circ D(i') \circ s(i') \rightharpoonup D \text{ in } \mathfrak{A}_1^p.$$

Proof. We assume the situation described in Lemma 3.5 has been constructed. Renumber so $i' = i$. Because $L_2^p(M) \subset C^0(M)$ is a compact embedding for $2p > n$, $g_{\alpha,\beta}(i) \to g_{\alpha,\beta}(\infty)$ (strongly) in $C^0(\mathscr{U}_\alpha \cap \mathscr{U}_\beta, G)$. Therefore there exists a fixed j, such that for $\infty \geqq i > j$ we may apply Theorem 3.2 and Corollary 3.3 to $g_{\alpha,\beta}(j) = g_{\alpha,\beta}$ and $g_{\alpha,\beta}(i) = g_{\alpha,\beta}$.

There exists a cover of M by the open sets $\mathscr{V}_\alpha \subset \mathscr{U}_\alpha$ such that for $\infty \geqq i > j$, $\rho_\alpha(i) \in L_2^p(\mathscr{V}_\alpha, G)$ and

$$g_{\alpha,\beta}(i) = \rho_\alpha(i)\, g_{\alpha,\beta}(j)\, \rho_\beta(i)^{-1}.$$

Moreover, $\rho_\alpha(i) \in L_2^p(\mathscr{V}_\alpha, G)$ is bounded and converges to $\rho_\alpha(\infty)$ in $C^0(\mathscr{V}_\alpha, G)$, which is equivalent to $\rho_\alpha(i) \rightharpoonup \rho_\alpha(\infty)$ in $L_2^p(\mathscr{V}_\alpha, G)$.

Define the global gauge transformation $s(i) \in \mathscr{D}_2^p$ on $\mathscr{U}_\alpha$ by the formula

$$s(i) = \sigma_\alpha^{-1}(i)\, \rho_\alpha(i)\, \sigma_\alpha(j).$$

On $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$, the consistency condition

$$\sigma_\alpha^{-}(i)\, \rho_\alpha(i)\, \sigma_\alpha(j) = \sigma_\beta^{-1}(i)\, \rho_\beta(i)\, \sigma_\beta(j)$$

is algebraically

$$\rho_\alpha(i)\, \sigma_\alpha(j)\, \sigma_\beta(j)^{-1}\, \rho_\beta(i)^{-1} = \sigma_\alpha(i)\, \sigma_\beta^{-1}(i).$$

From the definition of the overlap functions this is precisely the condition we used to choose $\rho_\alpha(i)$.

$$\rho_\alpha(i)\, g_{\alpha,\beta}(j)\, \rho_\beta(i)^{-1} = g_{\alpha,\beta}(i).$$

We have still to show that $s(i)^{-1}\, D(i)\, s(i)$ is weakly convergent. The fixed trivialization $\sigma_\alpha(j): \eta | \mathscr{V}_\alpha \to R^\ell \times \mathscr{V}_\alpha$ does lie in L_2^p (although we have no bound on norms *because* we have no natural choice of norm in the affine space $\mathfrak{A}_1^p$). It is sufficient to show that the induced connection forms in this trivialization over

$\mathscr{V}_\alpha$ converge weakly in $L_2^p(\mathscr{V}_\alpha, R^\ell \times \mathfrak{G})$. However,

$$\sigma_\alpha(j)\, s(i)^{-1} \circ D(i) \circ s(i)\, \sigma_\alpha^{-1}(j)$$

is algebraically

$$\rho_\alpha^{-1}(i)\, \sigma_\alpha(i) \circ D(i) \circ \sigma_\alpha^{-1}(i)\, \rho_\alpha(i).$$

The trivializations $\sigma_\alpha(i)$ were chosen to make $\sigma_\alpha(i) \circ D(i) \circ \sigma_\alpha^{-1}(i) = d + A(\alpha, i)$ satisfy weak convergence, (c) of Lemma 3.5. In our present trivialization, the connection $s(i)^{-1} \circ D(i) \circ s(i)$ is now

$$\rho_\alpha^{-1}(i) \circ (d + A(\alpha, i)) \circ \rho_\alpha(i) = d + \rho_\alpha^{-1}(i)\, d\rho_\alpha(i) + \rho_\alpha^{-1}(i)\, A(\alpha, i) \rho_\alpha(i).$$

Because $A(\alpha, i)$ is weakly convergent in $L_1^p(\mathscr{V}_\alpha, R^\ell \times \mathfrak{G})$ and $\rho_\alpha(i)$ in $L_2^p(\mathscr{V}_\alpha, G)$ by the rules of multiplication, this connection converges weakly in $L_1^p(\mathscr{V}_\alpha, R^\ell \times \mathfrak{G})$.

References

1. Bourguignon, J. P. Lawson, H. B., Jr.: Commun. Math. Phys. **79**, 189–230 (1981)
2. Hamilton, R.: Harmonic maps of manifolds with boundary. In: Lecture Notes in Mathematics, Vol. 471 Berlin, Heidelberg, New York: Springer 1975
3. Husemoller, D.: Fibre bundles (Chap. 5). In: Graduate Texts in Mathematics, Vol. 20 Berlin, Heidelberg, New York: Springer 1966
4. Morrey, C. B., Jr.: Multiple integrals in the calculus of variations. Berlin, Heidelberg, New York: Springer 1966
5. Palais, R. S.: Foundations of global non-linear analysis. New York: Benjamin, 1968
6. Steenrod, N.: The topology of fibre bundles (Part I). Princeton, New Jersey: Princeton University Press, 1951
7. Taubes, C.: Existence of multimonopole solutions to the static SU (2) Yang–Mills–Higgs equations in the Prasad–Summerfield limit. See Jaffe, A. and Taubes, C., Vortices and Monopoles, Boston: Birkhäuser 1980
8. Uhlenbeck, K.: Removable singularities in Yang–Mills fields, Commun. Math. Phys. **83**, 11–29 (1982)

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