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Removable Singularities in Yang–Mills Fields

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Abstract. We show that a field satisfying the Yang–Mills equations in dimension 4 with a point singularity is gauge equivalent to a smooth field if the functional is finite. We obtain the result that every Yang–Mills field over R^4 with bounded functional (L^2 norm) may be obtained from a field on $S^4 = R^4 \cup \{\infty\}$. Hodge (or Coulomb) gauges are constructed for general small fields in arbitrary dimensions including 4.

There has been a great deal of mathematical interest in the topological and geometrical methods used to construct the instanton solutions to the Yang–Mills equations [1–3]. More recently several articles treating analytic properties have appeared [6], [8], [10], [14]. We consider properties of the Euclidean (Riemannian, elliptic) equations and derive some standard *a priori* estimates on solutions. The main result is a local regularity theorem in 4-dimensions: A Yang–Mills field with finite energy cannot have isolated singularities if its structure group is compact. Apparent point singularities, *including singularities in the bundle*, may be removed by a gauge transformation. In particular, a Yang–Mills field on a bundle over R^4 extends to a smooth field on a bundle constructed over $R^4 \cup \{\infty\} = S^4$.

For convenience we concentrate on bundles over flat manifolds. For the regularity theory, the curvature of the manifold itself is not particularly important. In this paper we also assume all solutions have smooth curvatures where they are defined. Other references have handled the question of weak solutions in detail [10, 14, 16]. An announcement of the results in this paper has appeared [15] and an outline of the proof also appears in [6]. Parker has generalized these results to coupled systems in 4 dimensions.

We give a brief description of the problem in Sect. 1 to establish our notation. In Sect. 2, we prove a number of tedious technical lemmas on canonical gauges for fields with small curvatures which are necessary later. Standard *a priori* estimates appear in Sect. 3. The proof of the removability of singularities in Sect. 4 is remarkably similar to the proof of the removability of singularities of harmonic maps contained in [12].

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1. Yang–Mills Equations

The differential objects we will be working with are a Riemannian manifold M, a vector bundle η over M with fiber $\eta_x \cong R^\ell$ and *compact* structure group G. Denote the Lie algebra of G by $\mathfrak{G}$ and the adjoint and automorphism bundles by $\mathrm{Ad}\,\eta$ and $\mathrm{Aut}\,\eta$ respectively. Assume also η has a metric compatible with the action of G and an inclusion $G \subset SO(\ell)$. We use the metric on G induced by the trace inner product metric on $SO(\ell)$.

Since our theorems are all local, it is not necessary to work only with these abstract elements. We assume often that $M = \mathscr{U}$ is a coordinate chart, and that some local choice of gauge $\rho : \eta|\mathscr{U} \cong \mathscr{U} \times R^\ell$ has been made. Then $\rho : \mathrm{Aut}\,\eta|\mathscr{U} \cong \mathscr{U} \times G$ and $\rho : \mathrm{Ad}\,\eta \cong \mathscr{U} \times \mathfrak{G}$. We compute in these cross-product structures.

The *gauge group* is $\mathscr{G} = C^\infty(\mathrm{Aut}\,\eta)$, which in our trivial bundle case is $\mathscr{G} \cong C^\infty(\mathscr{U}, G)$. The choice of ρ introduces a flat covariant derivative $d = (\partial/\partial x^1, \dots, \partial/\partial x^n)$. Any *covariant derivative* D is given by $D = d + A = \{\partial/\partial x^i + A_i\}$ where $A_i(x) \in \mathfrak{G}$. One can think of A as a Lie algebra valued 1-form, or locally $A : \mathscr{U} \to R^n \otimes \mathfrak{G}$. Gauge changes $s : \mathscr{U} \to G$ act on $D = d + A$ by

$$s^{-1} \circ D \circ s = d + s^{-1} ds + s^{-1} A s = d + \tilde{A}.$$

This means A and $\tilde{A} = s^{-1} ds + s^{-1} A s$ represent the covariant derivative in different coordinates (or gauges).

The *curvature* or *field* $F = F(D)$ of a connection D measures the extent to which covariant derivatives fail to commute. (We always use the symbols d and D to represent exterior differentiation; the symbol ∇ is used for full covariant differentiation). Then $F = D^2 = dA + [A, A]$ is a section of $T^*M \wedge T^*M \otimes \mathrm{Ad}\,\eta$. Locally in the trivial bundle $\mathrm{Ad}\,\eta \cong \mathscr{U} \times \mathfrak{G}$, we have $F : \mathscr{U} \to R^n \wedge R^n \otimes \mathfrak{G}$ is a Lie algebra valued two-form, $F = \{F_{ij}\} = \{[D_i, D_j]\}$.

$$F_{ij} = \partial/\partial x^i A_j - \partial/\partial x^j A_i + [A_i, A_j] \in \mathfrak{G}.$$

Curvature is actually a section of $T^*M \wedge T^*M \otimes \mathrm{Ad}\,\mathfrak{G}$ and transforms under a gauge transformation $s \in \mathscr{G}$ by $F \to s^{-1} F s$.

The *Yang–Mills equations* are the Euler–Lagrange equations for an action integral

$$\|F\|^2 = \int_M |F|^2 d\mu_M = \int_{\mathscr{U}} g^{ik} g^{jl} g^{-1} \langle F_{ij}, F_{k\ell} \rangle.$$

Here the second integral is in local coordinates, g^{ik} is the induced metric tensor, g^2 its determinant, and $\langle A, B \rangle = \mathrm{tr}\, AB^*$ is the trace inner product in $\mathfrak{G} \subset SO(\ell)$. Usually we assume the metric is flat; $g^{ij} = \delta^{ij}$.

The Yang–Mills equations, or the Euler–Lagrange equations for the integral $\|F\|^2$, are written as $D^*F = 0$. In the case $g^{ij} = \delta_{ij}$ in coordinates on $\mathscr{U}$, this means explicitly,

$$(D^*F)_j = \sum_i (\partial/\partial x^i F_{ij} + [A_i, F_{ij}]) = 0.$$

A similar equation holds in general metrics. We say D is a *Yang–Mills connection* and $F = F(D)$ is a *Yang–Mills field* if $D^*F = 0$. If $s \in \mathscr{G}$ lies in the gauge group $\|s^{-1} F s\|^2 = \|F\|^2$. Therefore the solutions of $D^*F = 0$, as either Yang–Mills connections or fields, are an invariant space under gauge transformation. This is the main difficulty in treating the regularity theory.

The *Bianchi identities* $DF = 0$, are always true for $F = F(D)$. This means in coordinates: $D_i F_{kj} + D_k F_{ji} + D_j F_{ik} = 0$. The abelian case, the case where all brackets are zero, is the basic linear model for the theory. In this case $F = dA$, the Bianchi identites are $dF = d^2 A = 0$, and the Yang–Mills equations are $d^*F = 0$. The system $dF = 0$, $d^*F = 0$ is an elliptic system for F.

The situation is more complicated in the non-abelian case. In the abelian case, $A \to A + du$ under a gauge transformation $s = e^u \in \mathcal{G}$ and $F = dA = dA + d^2 u$ is left invariant. However, in the non-abelian case, F transforms to $s^{-1}Fs$ under a gauge transformation $s \in \mathcal{G}$. If s is not smooth, it can make a smooth field into a discontinuous one. So the choice of good gauges is much more important to the non-linear (non-abelian) theory.

The linearized Yang–Mills equations written for A are $d^*dA = 0$. As noted, this is the exact equation if G is abelian. This single system is not elliptic, and as in the Hodge theory for exact forms on manifolds, one usually adds a second equation such as $d^*A = 0$ to complete the elliptic theory. In the abelian case, this involves solving the linear equation $d^*(\tilde{A} + du) = d^*A = 0$ for $u : \mathcal{U} \to \mathfrak{G}$. Here $\tilde{A}$ is the original connection form and $s = e^u \in \mathcal{G}$ the gauge transformation. The equation d^*A can also be added to the non-linear theory as a method of choosing a good gauge. In the general case, it is a non-linear elliptic equation which must be solved to get this "good" gauge in which $d^*A = 0$.

Such a gauge seems to have many names in the physics literature (Lorentz, Landau, Coulomb). For the purposes of this paper, we use our original mathematical term *Hodge gauge*. The entire chapter following this one deals with the technical problem of constructing the Hodge gauges we will need in Sect. 4 for the main proof.

2. Canonical Choices of Gauge

This section treats the problem of finding a gauge in a domain $\mathcal{U}$ for a connection in which $D + d + A$ and $d^*A = 0$ when $\|F\|_\infty = \max_{x \in \mathcal{U}} |F(x)|$ is sufficiently small. We prove this in three cases: when $\mathcal{U} = S^{n-1} = \{x \in R^n : |x| = 1\}$,

$$\mathcal{U} = B^n = \{x \in R^n : |x| \leqq 1) \quad \text{and} \quad \mathcal{U} = \mathfrak{A} = \{x \in R^n : 1 \leqq |x| \leqq 2\}.$$

Assume a gauge is given in which $D = d + \tilde{A}$. Then it is an elementary calculation that the equations $d^*A = d^*(s^{-1}ds + s^{-1}\tilde{A}s) = 0$ for $s \in \mathcal{G}$ are Euler–Lagrange equations for the integral

$$J(s) = \|A\|_2^2 = \int_{\mathcal{U}} |A|^2 = \int_{\mathcal{U}} |s^{-1}ds + s^{-1}\tilde{A}s|^2.$$

There is a relationship between finding Hodge gauges and the existence of harmonic maps from $\mathcal{U}$ to G. The two equations agree in their top orders. We know quite a lot about harmonic maps. In particular, we know we do not have a good global theory [12], but we do have good local theories [7].

First we find some gauge which is not too large when $\|F\|_\infty$ is not too large

(Lemmas 2.2–2.4). Then we use the implicit function theorem to find a Hodge gauge $d^*A=0$ with estimates (Theorems 2.5–2.8). Assume throughout that G is compact and every connection has some gauge in which it is continuously differentiable.

The simplest geometric method of choosing a local gauge is to fix a fiber over x_0 and identify nearby fibers in a geodesic ball by setting $(x'(t)\cdot A(x(t)))=0$ along all geodesics $x(t)$ emanating from x_0 (meaning $x(0)=x_0$). This fixes gauge in all balls within the cut locus of M. In a Euclidean ball B^n with $x_0=0$, this corresponds to $A(0)=0$ and $\sum_j x^j A_j(x)=A_r=0$. We call such a gauge an *exponential gauge.*

A word on notation. We use the coordinate change $x=\{x_i\}_{i=1}^n \cong (r,\psi)=(|x|,\{x^i/|x|\})$ for $\psi=x/|x|\in S^{n-1}$ as transformation from Euclidean to spherical coordinates. The one-form $A=\{A_i\}\cong(A_r,A_\psi)$ splits into radial and spherical parts. The two form $F=\{F_{ij}\}\cong(F_{r\psi},F_{\psi\psi})$ splits into two pieces also (note $F_{rr}=0$ because of anti-symmetry). Here $F_{\psi\psi}$ is a two-form along S^{n-1}. In the sphere S^{n-1}, we sometimes change coordinates on $S^{n-1}, \psi\cong(\varphi,\theta)$ from spherical to "polar" coordinates. Here $\varphi\in(0,\pi)$ is the polar angle, $\theta\in S^{n-2}$ and $\psi=(\cos\varphi, \sin\varphi\theta)=x/|x|$. Again on S^{n-1}, $A=\{A_\psi\}=(A_\varphi,A_\theta)$ and $F=\{F_{\psi\psi}\}=(F_{\varphi\theta},F_{\theta\theta})$ in a natural way. This notation is used throughout the paper.

Lemma 2.1. *In an exponential gauge in R^n,*

$$|A(x)|\leqq 1/2\cdot|x|\cdot \max_{|y|\leqq|x|}|F(y)|. \tag{2.1-a}$$

Proof. Assume a gauge is given in which $D=d+\tilde{A}; d\tilde{A}+[\tilde{A},\tilde{A}]=\tilde{F}$. Solve the ordinary differential equation in t with ψ fixed.

$$\frac{d}{dt}\sigma(t,\psi)=-\tilde{A}_r(t,\psi)\sigma(t,\psi),$$

with initial condition $\sigma(0,\psi)=I\in G\subset SO(\ell)$. Then the gauge transformation $s(x)=\sigma(|x|,x/|x|)\in C^1(B^n,G)$ if $\tilde{A}$ is continuously differentiable. Also $s^{-1}\partial/\partial_r s\in C^1(B^n,G)$. By construction, in the gauge changed by s, $D=d+A$, where $A=s^{-1}ds+s^{-1}\tilde{A}s$, and $\sum_k x^k A_k(x)=A_r(x)=0$. Note that A is not necessarily as differentiable as $\tilde{A}$, but that $F=s^{-1}\tilde{F}s$ exists.

We compute easily from the equation $\sum_k x^k A_k=0$.

$$\begin{aligned}\sum_k x^k F_{kj}(x)&=\sum_k(x^k\partial/\partial x^k A_j - x^k\partial/\partial x^j A_k + x^k[A_k,A_j])\\ &=r\partial/\partial r A_j + A_j=\partial/\partial r(rA_j).\end{aligned}$$

By integrating we get

$$A_j(x)=\int_0^1\sum_k \tau x^k F_{kj}(\tau x)d\tau.$$

$$|A_j(x)|\leqq\left(\int_0^1\tau d\tau\right)|x|\max_{y\leqq|x|}|F(y)|.$$

The same argument is carried out to get an exponential gauge in S^{n-1} based at the north or south pole. Only the estimate is slightly different due to the curvature. Here in this gauge $(A_\psi) = (A_\varphi, A_\theta) = (0, A_\theta)$.

$$|A_\theta(\varphi, \theta)| = \left| \int_0^\varphi F_{\varphi,\theta}(\tau, \theta) d\tau \right|.$$

However, in T^*S^{n-1} we use the correct norms

$$\|A_\theta(\varphi, \theta)\| = (\sin \varphi)^{-1} |A_\theta(\varphi, \theta)| \text{ (etc.)}.$$

$$\begin{aligned} \|A_\theta(\varphi, \theta)\| &= \csc \varphi |A_\theta(\varphi, \theta)| \\ &\leqq \csc \varphi \left(\int_0^\varphi \sin \tau d\tau \right) \max [|F_{\varphi,\theta}(\tau, \theta)| \csc \varphi] \\ &= \frac{1 - \cos \varphi}{\sin \varphi} \|F\|_\infty . \end{aligned}$$

This gives the estimate on S^{n-1} for exponential gauges:

$$\|A(\varphi, \theta)\| \leqq \tan \varphi/2 \|F\|_\infty . \tag{2.1-b}$$

At the cut locus from $\varphi = 0$, as $\varphi \to \pi$, the estimate blows up and the exponential gauge becomes singular.

Finally, given a gauge for $\eta | S^{n-1}$, we may also extend it with $A_r = 0$, into a collar neighborhood. The integral formula is

$$A_j(x) = 1/|x| A_j(x/|x|) + \sum_j \int_{1/|x|}^{1} \tau x^k F_{kj}(x\tau) d\tau .$$

Call these normal exponential gauges *transverse*. For these transverse gauges off S^{n-1} we get the estimate

$$|A(x)| \leqq 1/|x| \, |A_\psi(x/|x|)| + ||x| + 1/|x|| \max_{|x| \leqq |y| \leqq 1} |F(y)|. \tag{2.1-c}$$

The next three lemmas are proved in precisely the same way. In each case, two exponential or transverse gauges are matched by a rotation.

Lemma 2.2. *There exists $\alpha_0 > 0$ and $\kappa < \infty$ depending on G, such that if D is a connection in a bundle over S^{n-1} in which*

$$\max_{\psi \in S^{n-1}} |F| = \|F\|_\infty < \alpha_0,$$

then there exists a gauge $\rho : \eta \cong S^{n-1} \times R^\ell$ in which $\|A\|_\infty \leqq \kappa \|F\|_\infty$.

Proof. Let $D = d^0 + A^0$ in the exponential gauge from the north pole ($\varphi = 0$) and $D = d^\pi + A^\pi$ in the exponential gauge from the south pole ($\varphi = \pi$). From (2.1-b)

$$\begin{aligned} \|A^0(\varphi, 0)\| &= \csc \varphi |A^0(\varphi, \theta)| \leqq \tan \varphi/2 \|F\|_\infty \\ \|A^\pi(\varphi, \theta)\| &= \csc \varphi |A^\pi(\varphi, \theta)| \leqq \tan (\pi - \varphi)/2 \|F\|_\infty . \end{aligned}$$

Because $D = d^0 + A^0 = d^\pi + A^\pi$, A^0 and A^π are related by a gauge change

$$d^0 - d^\pi = s^{-1} ds = A^\pi - A^0.$$

$$\partial/\partial\varphi s = s(A^\pi_\varphi - A^0_\varphi) = 0.$$

Therefore $s(\varphi, \theta) = \tilde{s}(\theta)$ for $\tilde{s}: S^{n-2} \to G$. Moreover $|d\tilde{s}(\theta)| = |ds(\pi/2, \theta)| = |A^0_\theta(\pi/2, \theta) - A^\pi_\theta(\pi/2, \theta)| \leqq 2\|F\|_\infty < 2\alpha_0$. If $\alpha_0 < 1/4$ (length of the shortest non-minimizing geodesic in (G), then $\tilde{s}(\theta) = s_0 \exp u(\theta)$, where $u: S^{n-2} \to (\mathfrak{G})$. By assuming $\int_{S^{n-1}} u = 0$, we have

$$\|du\|_\infty \leqq c(G)\|d\tilde{s}\|_\infty \leqq 2c(G)\|F\|_\infty.$$

Define a new gauge by multiplying the exponential gauge from the north pole by $h: S^{n-1} - \{\pi, \theta\} \to G$.

$$h(\varphi, \theta) = s_0 \exp(\sin^2(\varphi/2) u(\theta)).$$

This is the same gauge defined by rotating the exponential gauge from the south pole by $q: S^{n-1} - \{0, \theta\} \to G$.

$$q(\varphi, \theta) = \exp(-\cos^2 \varphi/2\, u(\theta)).$$

This new gauge is defined on all of S^{n-1}, and if $D = d + A$ in this gauge

$$A = h^{-1} A^0 h + h^{-1} dh = q^{-1} A^\pi q + q^{-1} dq.$$

On the entire sphere

$$\begin{aligned} |A_\varphi(\varphi, \theta)| &= |h(\varphi, \theta) \sin \varphi/2 \cos \varphi/2\| u(\theta)| \\ &= |q(\varphi, \theta) \sin \varphi/2 \cos \varphi/2\| u(\theta)| \\ &= \tfrac{1}{2} \sin \varphi |u(\theta)| \leqq c(G)\|F\|_\infty. \end{aligned}$$

One way to estimate $\|A_\theta(\varphi, \theta)\|$ is

$$\csc \varphi |A_\theta(\varphi, \theta)| \leqq \csc \varphi(|A^0(\varphi, \theta)| + |d_\theta h|).$$

The other way is by

$$\csc \varphi |A_\theta(\varphi, \theta)| \leqq \csc \varphi(|A^\pi(\varphi, \theta)| + |d_\theta h|).$$

The first gives the estimate for $\varphi \leqq \pi/2$, the second for $\varphi \geqq \pi/2$.

The next two lemmas involve fixing gauge on the boundary, as in Dirichlet boundary conditions. Note we actually allow these boundary gauges to rotate by a constant element s_0 of G. Later on we shall see this extra degree of freedom makes the appropriate Dirichlet problem on an annular region $\mathfrak{U}$ confusing. This particular problem is a familiar annoyance in gauge theory.

Lemma 2.3. *Let η be a bundle with a covariant derivative D over B^n and curvature $\|F\|_\infty \leqq \alpha$. Assume a gauge is fixed on $\eta|S^{n-1} = \eta|\partial B^n$ in which $D_\psi = d_\psi + \tilde{A}_\psi$, $|\tilde{A}_\psi(1, \psi)| \leqq \alpha$ all $\psi \in S^{n-1}$. Then there exists $\alpha_1 = \alpha_1(G) > 0$ such that if $\alpha < \alpha_1$, there exists a gauge on $\eta|B^n$ in which $D = d + A$, $\tilde{A}_\psi = A_\psi$ on $\eta|S^{n-1}$ and $\|A\|_\infty \leqq \kappa\alpha$.*

Proof. Consider the exponential gauge from zero, $D = d^\circ + A^\circ$ and match by rotation with the transverse gauge off the unit sphere $D = d^1 + A^1$ which fixes

$\tilde{A}_\psi$. From (2.1-a)

$$|A^0(x)| \leqq 1/2|x| \, \|F\|_\infty \leqq 1/2|x|\alpha.$$

From (2.1-c) we have the inequality

$$|A^1(x)| \leqq |x|^{-1} \|\tilde{A}_\psi\| + (|x|^{-1} + |x|) \|F\|_\infty \leqq 3|x|^{-1}\alpha.$$

Then the two gauges are related by $s = s_0 \exp \tilde{u}(\psi)$. Change gauge from the exponential gauge at zero by $s_0(\exp r^2 \tilde{u}(\psi))$. Estimate as in the previous lemma.

Lemma 2.4. *Let η be a bundle with a covariant derivative D over $\mathfrak{U} = \{x : 1 \leqq |x| \leqq 2\}$ and curvature $\|F\|_\infty \leqq \alpha$. Let $S_t^{n-1} = \{x : |x| = t\}$. Suppose gauges are chosen on $\eta | S_t^{n-1}$ in which $D_\psi = d_\psi^t + \tilde{A}_\psi^t$ with $|\tilde{A}_\psi^t(t, \psi)| \leqq \alpha$ for $t = (1, 2)$. Then there exists $\alpha_2 > 0$ such that if $\alpha < \alpha_2$, there is a gauge on $\eta | \mathfrak{U}$ in which $D = d + A$, $\tilde{A}_\psi^t = A_\psi$ on S_t^{n-1}, and $\|A\|_\infty \leqq \kappa\alpha$.*

Proof. Match transverse gauges from the boundary sphere S_t^{n-1}, $t = (1, 2)$ exactly as before.

At this point we are in a position to apply the ordinary implicit function theorem in Banach spaces to solve the non-linear elliptic system

$$d^*(s^{-1} ds + s^{-1} \tilde{A} s) = d^* A = 0$$

for s when $\tilde{A}$ is small enough. Unfortunately, the exponential and transverse gauges used to construct a connection $\tilde{A}$ from small curvatures F in Lemmas 2.2–2.4 produce estimates on $\tilde{A}$ of the same differentiability as F. Intuitively, we should be able to get one more derivative on $\tilde{A}$ than on F. Since a method for doing this has not appeared so far, the applications of this entire procedure are limited. The trick used in [16] to circumvent the construction of $\tilde{A}$ does not work here. We use Sobolev spaces L_k^p of connections or maps in the k derivatives in L^p. We are restricted to using the implicit function theorem on the equation $d^* A = 0$ as a mapping on Sobolev spaces $L_1^p \to L_{-1}^p$ because $\tilde{A}$ is not smooth enough to use the more usual map $L_2^p \to L_0^p$. Also, the difference in the behavior of the boundary conditions on S^{n-1}, B^n and $\mathfrak{U}^n$ leads us to state the theorems separately, although the proofs follow the same line of argument.

Theorem 2.5. *Let η be a bundle over S^{n-1} with a covariant derivative D, curvature $F = F(D)$. There exists $\gamma_0 > 0$ such that if $\|F\|_\infty \leqq \gamma_0$, then there exists a gauge $\rho : \eta \cong S^{n-1} \times R^\ell$ in which $D = d + A$ and $d^* A = 0$. Furthermore, $\|A\|_\infty \leqq K \|F\|_\infty$. The choice of gauge is unique up to constant multiplication by an element of G.*

Proof. From Lemma 2.2, if $\gamma_0 \leqq \alpha_0$ we can construct a gauge in which $D = d + \tilde{A}$, $\|\tilde{A}\|_\infty \leqq \kappa \|F\|_\infty \leqq \kappa\gamma_0$. Fix any $\infty > p > n - 1$. The expression

$$Q(u, B) = d^*[\exp(-u) d \exp u + \exp(-u) B u]$$

induces a C^∞ map on $u \in L_1^p(S^{n-1}, \mathfrak{G})$, $B \in L^p(S^{n-1}, \mathfrak{G} \otimes R^n)$.

$$Q : L_1^p(S^{n-1}, \mathfrak{G}) \times L^p(S^{n-1}, \mathfrak{G} \otimes R^n) \to L_{-1}^p(S^{n-1}, \mathfrak{G}).$$

The image actually lies in

$$L_{-1}^{p,\perp}(S^{n-1}, \mathfrak{G}) = \{\xi \in L_{-1}^p(S^{n-1}, \mathfrak{G}) : \langle \xi, u_0 \rangle = 0, u_0 \in \mathfrak{G}\}.$$

Likewise define

$$L_1^{p,\perp}(S^{n-1}, \mathfrak{G}) = \{u \in L_1^p(S^{n-1}, \mathfrak{G}) : \int_{S^{n-1}} u = 0\}.$$

Then $d_1 Q_{(0,0)} : L_1^{p,\perp}(S^{n-1}, \mathfrak{G}) \to L_{-1}^{p,\perp}(S^{n-1}, \mathfrak{G})$ is an isomorphism. (Note $d_1 Q_{(0,0)} u = \Delta u$). The ordinary implicit function theorem in Banach spaces now says we may solve

$$Q(u, \tilde{A}) = d^*(s^{-1} ds + s^{-1} \tilde{A} s) = d^* A = 0$$

if $\tilde{A} \in L^p(S^{n-1}, \mathfrak{G} \otimes R^{n-1})$ is sufficiently small. Here $s = \exp u \in L_1^p(S^{n-1}, G)$ and $u \in L_1^{p,\perp}(S^{n-1}, \mathfrak{G})$. By taking γ_0 small $\|\tilde{A}\|_p \leqq c_{n,p} \|\tilde{A}\|_\infty < \kappa\gamma_0$ can be assumed small. Since the norm $\|u\|_{p,1}$ is also small, $\|A\|_p \leqq (1+\kappa_1)\|\tilde{A}\|_\infty$ where $A = s^{-1} ds + s^{-1}\tilde{A}s$. Finally, $d^*A = 0$ and $dA + [A, A] = F = s^{-1}\tilde{F}s$. Consequently $\|A\|_{q,1} \leqq \kappa_2(p)(\|dA\|_q) \leqq \kappa_2(p)(\|F\|_q + \||A|^2\|_q + \|A\|_q)$. Let $q = p/2$ to get an estimate on $\|A\|_{p,1}$. An estimate on $\|A\|_{q,1}$ leads by the Sobolev theorems to an estimate on $\|A\|_{q'}$ for $1/n - 1/q + 1/q' = 0$. Once $2q > n$ this is an improvement and we get estimates on all $\|A\|_{q,1}$ norms.

Corollary 2.6. *Under the hypotheses of* (2.5) *with* $n = 4$, *we have*

$$(2 - K\|F\|_\infty)^2 \int_{S^3} |A|^2 \leqq \int_{S^3} |F|^2.$$

Proof. Since A is a co-closed Lie-algebra valued one-form on S^3, $\lambda \int_{S^3} |A|^2 \leqq \int_{S^3} |dA|^2$ for λ the first eigenvalue of the Laplace operator on co-closed one forms on S^3. This can be computed to be 4 from [10]. Using the formula $F = dA + [A, A]$, we estimate the error.

$$\left(4\int_{S^3} |A|^2\right)^{1/2} \leqq \left(\int_{S^3} |dA|^2\right)^{1/2} \leqq \left(\int_{S^3} |F|^2\right)^{1/2} + \left(\int_{S^3} |A|^4\right)^{1/2}$$
$$\leqq \left(\int_{S^3} |F^2|\right)^{1/2} + K\|F\|_\infty \left(\int_{S_3} |A|^2\right)^{1/2}.$$

Theorem 2.7. *Let D be a covariant derivative in a bundle over B^n. There exists $\gamma_1 > 0$ such that if $\|F\|_\infty \leqq \gamma_1$, then there exists a gauge for η over B^n such that if $D + d + A$ in this gauge, then $d^*A = 0$ in B^n and $d_\psi^* A_\psi = 0$ on S^{n-1}. Furthermore $\|A\|_\infty \leqq \tilde{\kappa}_1 \|F\|_\infty$.*

Proof. If $\gamma_1 < \gamma_0$, we may apply Theorem 2.5 to fix the gauge on $S^{n-1} = \partial B^n$ with $d_\psi^* A_\psi = 0$ and $\|A_\psi\|_\infty \leqq K\|F_{\psi\psi}\|_\infty$. By Lemma 2.3, if $K\gamma_1 \leqq \alpha_1$ and $\gamma_1 \leqq \alpha_1$, we may construct an appropriate gauge over B^n such that if $\tilde{D} = d + \tilde{A}$ in this gauge, $\|\tilde{A}\|_\infty \leqq \kappa_1\gamma_1$. We can now solve

$$Q(u, A) = d^*A = d^*(s^{-1}ds + s^{-1}\tilde{A}s) = 0,$$

for $s = e^u$, $u|S^{n-1} = 0$ by the implicit function theorem. The formula for Q induces a smooth map on $u \in L_{1,0}^p(B^n, \mathfrak{G})$, $\tilde{A} \in L^p(B^n, R^n \otimes \mathfrak{G})$ for $p > n$.

$$Q : L_{1,0}^p(B^n, \mathfrak{G}) \otimes L^p(B^n, R^n \otimes \mathfrak{G}) \to L_{-1}^p(B^n, \mathfrak{G}).$$

Since we are using Dirichlet boundary conditions, the linearization

$$d^1Q(0,0)=\Delta:L^p_{1,0}(B^n,\mathfrak{G})\to L^p_{-1}(B^n,\mathfrak{G})$$

is an isomorphism. Also, since $\|\tilde{A}\|_\infty \leqq \kappa_1\gamma_1$, we clearly make $\|\tilde{A}\|_p$ arbitrarily small by choosing γ_1 small. This procedure produces $s=e^u\in L^p_1(B^n,G)$ and the regularity argument is exactly as in Theorem 2.5.

Theorem 2.8. *Let D be a covariant derivative in a bundle η over $\mathfrak{U}=\{x:1\leqq|x|\leqq 2\}$. There exists $\gamma'>0$ such that if $\|F\|_\infty\leqq\gamma'$, then there exists a gauge in which $D=d+A$, $d^*A=0$, $d^*_\psi A_\psi=0$ on S^{n-1}_1 and S^{n-1}_2, and $\int_{|x|=t} A_r=0$ for all $t\in[1,2]$. Moreover, $\|A\|_\infty\leqq K'\|F\|_\infty$.*

Proof. Apply Theorem 2.5 to D on the boundary spheres $S^{n-1}_t(t=1,2)$ and construct $\tilde{A}$ using Lemma 2.4. Again we shall use the implicit function theorem to solve the equation

$$Q(u,\tilde{A})=d^*A=d^*(s^{-1}ds+s^{-1}\tilde{A}s)=0$$

for $s=\exp u$. This is the variational equation for the problem of minimizing $\int_{\mathfrak{U}}|A|^2=\int_{\mathfrak{U}}|s^{-1}ds+s^{-1}\tilde{A}s|^2$, subject to the appropriate boundary conditions. In fact, to preserve the condition $d^*_\psi A_\psi=0$, we shall require that s be constant on each component of $\partial\mathfrak{U}=S^{n-1}_1\cup S^{n-1}_2$ (although we do not specify the values of these constants). Thus we set

$$L^{p,\perp}_1(\mathfrak{U},\mathfrak{G})=\{u\in L^p_1(\mathfrak{U},\mathfrak{G}):u \text{ is constant on } S^{n-1}_t \text{ for } t=1,2 \text{ and } u \text{ is } L^2\text{-perpendicular to the constants of } \mathfrak{G}\}.$$

Then for $p>n$, Q induces a map

$$Q:L^{p,\perp}_1(\mathfrak{U},\mathfrak{G})\otimes L^p(\mathfrak{U},\mathfrak{G}\otimes\mathbb{R}^n)\to L^p_{-1}(\mathfrak{U},\mathfrak{G}).$$

However, the linearization has a $\mathfrak{G}$-dimensional kernel (corresponding to the constant gauge transformations). This allows us to add a $\mathfrak{G}$-valued function

$$f:L^{p,\perp}_1(\mathfrak{U},\mathfrak{G})\otimes L^p(\mathfrak{U},\mathfrak{G}\otimes\mathbb{R}^n)\to\mathfrak{G}$$

given by $f(u,\tilde{A})=\int_{\mathfrak{U}}A_r=\int_{\mathfrak{U}}s^{-1}(\partial/\partial r)s+s^{-1}A_rs\in\mathfrak{G}$. Then the linearization of (Q,f)

$$(d^1Q(0,0),d^1f(0,0)):L^{p,\perp}_1(\mathfrak{U},\mathfrak{G})\to L^p_{-1}(\mathfrak{U},\mathfrak{G})\otimes\mathfrak{G}$$

is an isomorphism, and we can solve $d^*A=0$ and $\int_{\mathfrak{U}}A_r=0$ when $\tilde{A}\in L^p(\mathfrak{U},\mathfrak{G}\otimes\mathbb{R}^n)$ is sufficiently small. The regularity is proved as in Theorem 2.5.

Finally, since $d^*A=0$, the integral $\int_{|x|=t}A_r$ is independent of t and the condition $\int_{\mathfrak{U}}A_r=0$ implies that $\int_{|x|=t}A_r=0$, $\forall t\in[1,2]$.

Corollary 2.9. (*For $n>2$*). *There exists a constant λ_n such that if D is a covariant derivative $D=d+A$ in $\mathfrak{A}$ with curvature $\|F\|_\infty<\gamma'$, $d^*A=0$, $d^*_\psi A^*_\psi=0$ and*

$\int_{|x|=r} A_r = 0$, *then*

$$(\lambda_n - \kappa' \| F \|_\infty^2) \int_{\mathfrak{A}} |A|^2 \leqq \int_{\mathfrak{A}} |F|^2.$$

Proof. As in the proof of Corollary 2.2, λ_n is constructed as

$$\lambda_n = \min \frac{\int |df|^2}{\int |f|^2}$$

for $f \in L_1^2(T^*\mathfrak{U})$, $d^*f = 0$, $d_\psi^* f_\psi | S_t^{n-1} = 0$, $\int_{\mathfrak{U}} f_r = 0$. The problem is elliptic, so to show $\lambda_n > 0$, we need only show that the value $\lambda_n = 0$ is not taken on. Suppose $\lambda_n = 0$ is taken on. Then there exists $f \neq 0$ satisfying the conditions with $df = 0$. But $\mathfrak{A}$ is simply-connected; therefore $f = dg$. However, on the compact boundary spheres $d_\psi^* f_\psi = d_\psi^* d_\psi g = 0$ and g is constant on $S_t^{n-1} t = (1, 2)$. Since $d^* f = d^* dg = 0$, g is a harmonic function on $\mathfrak{U}$ which is constant on the two boundaries, or $g = c_1 + c_2 r^{2-n}$. However, $\int_{|x|=t} g_r = 0$ implies $c_2 = 0$, or $f = dg = 0$. The rest of the proof is identical to the proof of Corollary 2.2. Note that the condition that $\int_{|x|=t} A_r = 0$ which gave us so much trouble in the proof of Theorem 2.7 is very important in showing $\lambda_n > 0$.

3. Basic A Priori Estimates

We assume all covariant derivatives D are smooth in some gauge, since regularity theorems now appear elsewhere. The basic inequality of Lemma 3.1 is more carefully discussed by Bourguignon and Lawson [5]. We assume the metric on M is flat for convenience. The difference between the flat case and the case where curvature is not zero contains a lower order term which would be relatively unimportant in our calculations. In this section $B(x, a) = B^n(x, a) = \{ y \in R^n : |x - y| \leqq a \}$.

Lemma 3.1. *If F is a Yang–Mills field, then*

$$|F| \Delta |F| \geqq 2(F.[F, F]) = 2 \sum_{i,j,k} \langle F_{ij}, [F_{jk}, F_{ki}] \rangle$$

$$\Delta |F| \geqq -4 |F|^2.$$

Proof. We give a brief outline of the computation [5]. From the Yang–Mills equations $D^*F = 0$ and the Bianchi identities $DF = 0$ we have $(D^*D + DD^*)F = 0$. The Laplacian $D^*D + DD^* = \Delta$ on one–forms differs from the full covariant derivative Laplacian $\nabla^* \nabla = \nabla^2$ by a curvature term

$$(\nabla^2 - \Delta)\psi = [F, \psi] = \left\{ \sum_j [F_{ij}, \psi_{jk}] - [\psi_{kj}, F_{ij}] \right\}.$$

The full covariant Laplacian can be used to estimate a scalar Laplacian on the norm.

$$|\psi| \Delta |\psi| = \langle \psi, \nabla^2 \psi \rangle + \langle \nabla \psi, \nabla \psi \rangle - |d|\psi||^2 \geqq \langle \psi, \nabla^2 \psi \rangle.$$

These three equations combine to give the first inequality. The second inequality follows from the rough estimate

$$2\langle F,[F,F]\rangle \leqq 2|F||[F,F]| \leqq 4|F|^3.$$

We now regard $-4|F|=b$ as a fixed function and write the inequality as

$$\Delta f \geqq -bf$$

for $f=|F|$. If $b\in L^{n/2+\mu}$ for any $\mu>0$, then a theorem of Morrey (see [9], Theorem 5.3.1) applies to this problem. We state the case of the theorem which applies here.

Theorem 3.2. *Let b be bounded in $L^q(B(x_0,a_0))$ for $q>n/2, f\geqq 0$, and $f^\gamma\in L_1^2(B(x_0,a_0))$ for $1/2<\gamma\leqq 1$. Suppose also that in a weak sense*

$$-\Delta f \leqq bf.$$

Then f is bounded on domains interior to $B(x_0,a_0)$ and for $B(x,a)\subset B(x_0,a_0)$

$$|f^\gamma(x)|^2 \leqq \mathscr{K}_1 a^{-r} \int_{B(x_0,a_0)} |f^\gamma(y)|^2.$$

Moreover, the constant $\mathscr{K}_1$ depends uniformly on n,q,γ and $a_0^{q-n/2} \int_{B(x_0,a_0)} |b|^q$.

Proof. If $b\in L^q$, we have the inequality

$$\int_{|x-y|\leqq a} b^{n/2} \leqq \left(\int_{|x-y|\leqq a} (b^q)\right)^{n/2q} a^\mu \text{ for } \mu>0.$$

We may then apply Theorem 5.3.1 of Morrey [9]. The requirement $f\geqq 1$ is not necessary here. We can prove the estimate for $f_1=Nf+1$, which implies the same inequality for $f_2=f+1/N$, and then let $N\to\infty$ in R. The uniform dependence of $\mathscr{K}_1$ can be computed by dilating $B(x_0,a_0)$ to the unit ball.

We wish to make minor extensions of this theorem. These are derived from the basic *a priori* integral inequality used by Morrey in proving Theorem 3.2.

Lemma 3.3. *Let $\mathscr{U}\subset R^n, f\in L^2_{1,\mathrm{loc}}(\mathscr{U})\cap L^\infty_{\mathrm{loc}}(\mathscr{U}), f\geqq 0, \infty>p>1/2$ $v=2n/n-2$ and $u\in C_0^\infty(\mathscr{U})$. Then if*

$$-\Delta f\leqq bf,$$

$$\int_{\mathscr{U}} |d(uf^p)|^2 \leqq \int |p-1|p(2p-1)|\Delta u^2| + (du)^2]f^{2p} + p^2/2p-1\left(\int_{\mathscr{U}} b^{2/n}\right)^{n/2}\left(\int_{\mathscr{U}} (uf^p)^v\right)^{2/v}.$$

Proof. We may replace f by $f+\varepsilon$, prove the estimate for $f+\varepsilon$ and let $\varepsilon\to 0$. Take u^2f^{2p-1} as a test function. Then

$$\int_{\mathscr{U}} d(u^2f^{2p-1})\cdot df = -\int_{\mathscr{U}} u^2f^{2p-1}\Delta f \leqq \int_{\mathscr{U}} bu^2f^{2p}.$$

On the left-hand side, rearrange the integrand algebraically to

$$(2p-1)/p^2|d(uf^p)|^2 - (p-1)/p(du^2\cdot df^{2p}) - (2p-1)/p^2|du|^2f^{2p}.$$

The right-hand side can be estimated using Hölder's inequality.

$$\int_{\mathscr{U}} b(uf^p)^2 \leqq \int_{\mathscr{U}} b^{n/2}\left(\int_{\mathscr{U}} (uf^p)^v\right)^{2/v}.$$

This gives us

$$\begin{aligned}(2p-1)/p^2 \int_{\mathscr{U}} |d(uf^p)|^2 \leqq & |p-1|/p \int_{\mathscr{U}} du^2 \cdot d(f^{2p}) \\ & + (2p-1)/p^2 \int_{\mathscr{U}} |du|^2 f^{2p} \\ & + \left(\int_{\mathscr{U}} b^{2/n}\right)^{n/2}\left(\int_{\mathscr{U}} (u^2 f^p)^v\right)^{2/v}.\end{aligned}$$

Integrate the first term on the right by parts and multiply the entire equation by $p^2/(2p-1)$ to get the inequality of the lemma.

Lemma 3.4. *Assume the conditions of Lemma 3.3, and suppose for $q \geqq 1$ there exists a constant c_n such that if $B(x_0, a_0) \subset \mathscr{U}$,*

$$c_n - \left(\int_{\mathscr{U}} |b|^{2/n}\right)^{n/2} (q^2/2q-1) > \gamma > 0.$$

Then for all $B(x, 2a) \subset \mathscr{U}$, we have $uf^q \in L_1^2(B(x,a))$ with

$$a^{-n+2} \int_{B(x,a)} (df^q)^2 \leqq c_\gamma a^{-n} \int_{B(x,2a)} f^2$$

$$\left(a^{-n} \int_{B(x,a)} f^{qv}\right)^{2/qv} \leqq c'_\gamma a^{-n} \int_{B(x,2a)} f^2.$$

Furthermore, c_γ and c'_γ depend only on γ, q and n.

Proof. Lemma 3.3 applies with $\mathscr{U} = B(x, 2a)$ and $1 \leqq p \leqq q$. We may assume by dilation $a = 1$. For convenience, $\mathscr{K}(u,p) = \max|p-1|p/(2p-1)|\Delta u^2| + (du)^2$. By applying the Sobolev inequality as well with $v = 2n/(n-2)$,

$$\begin{aligned}c_n\left(\int_{\mathscr{U}} (uf^p)^v\right)^{2/v} \leqq \int_{\mathscr{U}} |d(uf^p)|^2 \leqq & \mathscr{K}(u,p) \int_{\mathscr{U}} f^{2p} \\ & + p^2/(2p-1)\left(\int_{\mathscr{U}} b^{n/2}\right)^{2/n}\left(\int_{\mathscr{U}} (uf^p)^v\right)^{2/v}.\end{aligned}$$

From the hypotheses of this lemma

$$\gamma\left(\int_{\mathscr{U}} (uf^p)^v\right)^{2/v} \leqq \mathscr{K}(u,p) \int_{\mathscr{U}} f^{2p}$$

as well as

$$\gamma/c^n \int_{\mathscr{U}} |d(uf^p)|^2 \leqq \mathscr{K}(u,p) \int_{\mathscr{U}} f^{2p} \leqq \mathscr{K}(u,q) \int_{\mathscr{U}} f^{2p}.$$

We get a bound on the L^{pv} norm on interior domains in terms of the L^{2p} norm on a domain. By interation we obtain the result for $p_i = (n/n-2)^i = vp_{i-1}/2$ which gives us the estimate for q in a finite number of steps.

We can now prove the main result of this section.

Theorem 3.5. *There exists a constant c'_n such that if F is a Yang–Mills field in $B(x_0, 2a_0)$ and $\int_{B(x_0,2a_0)} |F|^{n/2} < c'_n$, then $|F(x)|$ is uniformly bounded in the interior of $B(x_0, 2a_0)$ and*

$$|F(x)|^2 \leqq a^{-n}\mathscr{K}_n \int_{B(x,a)} |F|^2$$

for all $B(x, a) \subset B(x_0, a_0)$.

Proof. Let $b = 4|F|$ and $|F| = f$. Choose $c'_n = c_n/(4n)$ where c_n is the constant of Lemma 3.4. Then Lemma 3.4 applies for $q = n$, $\gamma = c_n/3$, and

$$\left(\int_{B(x_0,2a_0)} b^{n/2}\right)^{2/n} = 4\left(\int_{B(x_0,2a_0)} |F|^{n/2}\right)^{2/n}.$$

Apply Lemma 3.4 to get a bound on $\int_{B(x,a)} |F|^n$. Now Theorem 3.2 applies. Since $\left(\int_{B(x,a)} |F|^{n/2}\right)^{2/n}$ is invariant under dilation, the size of the ball does not affect the constants $\mathscr{K}_n$ or c_n.

Theorem 3.6. *Let F be a smooth Yang–Mills field in a punctured ball $\mathscr{U} = B(x_0, a) - \{x_0\}$ such that $\int_{\mathscr{U}} |F|^q < \infty$ for $q > \max(n/n-2, n/2)$. Then $|F|$ is uniformly bounded in the interior of $B(x_0, a_0)$.*

Proof. Apply Lemmas 3.3 and 3.4 with $b = 4|F|, f = |F|$ and $\mathscr{U} = B(x_0, a) - \{x_0\}$. Here we let u be a cut-off function $u = v + v'$ where v is a cut-off function which is zero at x_0 and $v' \in C_0^\infty(B(x_0, a))$. We fix v' and let $v(x - x_0) = \varphi(x/\varepsilon)$ where φ has support in the unit ball. Check the growth of the error on the right in Lemma 3.3 (or $\mathscr{K}(u, p)$ of Lemma 3.4) as $\varepsilon \to 0$. The contribution from v' is fixed. So $K(u, p) \sim \mathscr{K}(v, p) = \varepsilon^{-2}K(\varphi, p)$ and we have

$$\begin{aligned}\int [|dv|^2 + (p-1)p/2p\text{-}1\ \Delta(v^2)] f^{2p} &\\ \leqq \varepsilon^{-2}\mathscr{K}(p, \varphi) \int_{|x-x_0|\leqq\varepsilon} f^{2p} &\\ \leqq \mathscr{K}(\varphi, p)\varepsilon^{n(1-2p/q)-2}\left(\int f^q\right)^{2p/q}. &\end{aligned}$$

The error term contribution to the inequality from the singularity approaches zero if $(1 - 2p/q) - 2/n > 0$, or for $p < (n-2)/2nq$. Clearly p may be chosen greater than 1/2 if and only if $q > n/(n-2)$. Then we have $f^p \in L_1^2(B(x_0, a_0))$ by Lemmas 3.3 and 3.4. If, in addition, $q > n/2$ we may apply Theorem 3.2.

This theorem can in fact be improved for $n = 2, 3$. The proof in two dimensions is simple because we can use the *first* order equations valid only for $n = 2$,

$$d|{*F}|^2 = 2({*F}, D{*F}) = 0.$$

The proof in three dimensions is considerably more difficult [12]. The differential inequality $-\Delta|F| \leqq 4|F|^2$ is insufficient due to the fact that the fundamental solution to the Laplacian in R^3 is $1/|x|$, of smaller growth than $1/|x|^2$.

The construction of Yang–Mills fields with point singularities can be accomplished by what is in effect a separation of variables. Let D be a connection in a bundle η over S^{n-1} and let $f: B^n - \{0\} \to S^{n-1}$ be given by $f(x) = x/|x|$. Then any connection D on η can be pulled back via f to a connection f^*D on the bundle $f^*\eta$. It is an easy calculation that if D is Yang–Mills on η over S^{n-1}, then f^*D is Yang–Mills over $f^*\eta$. Moreover, the curvature of $f^*\eta$ grows *exactly like* $1/|x|^2$. Since S^2, S^3 and S^4 are known to have non-trivial Yang–Mills fields (in some bundles) this produces examples of isolated singularities at 0 of Yang–Mills fields in dimensions 3, 4 and 5. The curvature grows like $1/|x|^2$ about the singularity $x = 0$, so the integral $\int_{D^n} |f^*(F)|^q$ is finite for $q < n/2$, but infinite for $q \geqq n/2$.

4. Removability of Singularities

In this section we complete the proof of our main theorem.

Theorem 4.1. *Let D be a Yang–Mills connection in a bundle η over $B^4 - \{0\}$. If the L^2 norm of the curvature F of D is finite, $\int_{B^4} F^2 < \infty$, then there exists a gauge in which the bundle η extends to a smooth bundle $\bar{\eta}$ over B^4 and the connection D extends to a smooth Yang–Mills connection $\bar{D}$ in B^4.*

We have the immediate corollary.

Corollary 4.2. *Let D be a Yang–Mills connection in a bundle η over an exterior region $\mathscr{U} = \{x \in R^4 : |x| \geqq N\}$. If $\int_{\mathscr{U}} F^2 < \infty$, then $|F| \leqq C|x|^{-4}$ for some constant (not uniform). Moreover, if we map $B^4 - \{0\} \to \mathscr{U}$ by $f(x) = N\frac{x}{|x|^2}$, there exists a gauge change in η such that $f^*\eta$ and f^*D extend to a smooth bundle $\overline{f^*\eta}$ and a smooth Yang–Mills connection $\overline{f^*D}$ over all of B^4.*

Proof. The map f is conformal, so f^*D is Yang–Mills in $B^4 - \{0\}$. Also

$$\int_{B^4} |f^*F|^2 = \int_{B^4} |F(f^*D)|^2 = \int_{\mathscr{U}} |F|^2 < \infty.$$

Here F is the curvature of D and $f^*F = F(f^*D)$ is the curvature of the pull-back connection. We may now apply Theorem 4.1 to $f^*(D)$. The growth at infinity is obtained from the change in variables of two-forms under conformal operations.

$$\begin{aligned} |F(f(x))| &= |f^*F(x)||df(x)|^{-2} \\ &\leqq \max_{x \in B^4} |f^*F|(x)(N/|x|^2)^{-2} \\ &= (C'N^2)|x|^{-4}. \end{aligned}$$

The global form of this corollary can be more simply stated.

Corollary 4.3. *Let D be a Yang–Mills connection on a bundle η over R^4 with finite L^2 norm of its curvature $\int_{R^4} |F|^2 < \infty$. Then if $f: S^4 - \{0\} \to R^4$ is a stereographic projection, f^*D is a Yang–Mills field on $f^*\eta$ over $S^4 - \{0\}$ which extends in some gauge to a Yang–Mills connection $\overline{f^*D}$ on a bundle $\overline{f^*\eta}$ over S^4.*

The proof of Theorem 4.1 proceeds in two steps. First we need to find a useful gauge in $B^4 - \{0\}$. We do this by piecing together Hodge gauges in the annuli $\mathfrak{U}_\ell = \{x : 2^{-\ell-1} \leqq |x| \leqq 2^{-\ell}\}$. Then we use the Yang–Mills equations in this broken Hodge gauge to show that $|F|$ actually has better growth near the singularity $x = 0$ than $|x|^{-2}$. In fact, we are luckily able to show directly that $|F|$ is bounded, although by Theorem 3.6 any growth $\sim |x|^{-2+\varepsilon}$ for $\varepsilon > 0$ would have been sufficient. Once we know $|F|$ is bounded, we may find a Hodge gauge by applying Theorem 2.7 directly in B^n.

The construction of the broken Hodge gauge can be carried out in any dimension, under suitable hypotheses (for example $\int F^{n/2} < \infty$). However, the second step is strictly a four dimensional argument.

Lemma 4.4. *If the hypotheses of Theorem 4.1 hold, given any* $\varepsilon > 0$, *we may assume* $\int_{B(0,2)} |F|^2 \leqq \varepsilon^2$.

Proof. If $\int_{B^4} F^2 < \infty$, then $\lim_{r \to 0} \int_{|x| \leqq r} |F|^2 = 0$. Assume then, that $\int_{|x| \leqq \rho} |F|^2 \leqq \varepsilon^2$. Change coordinates by $y = 2x/\rho$. Then $F(x)$ pulls back to a Yang–Mills field $\tilde{F}(y)$ on $\{y : 0 < |y| \leqq 2/\rho\}$ and

$$\int_{B(0,2)} |\tilde{F}|^2 = \int_{|x| \leqq \rho} |F|^2 \leqq \varepsilon^2.$$

The truth of Theorem 4.1 for $\tilde{F}$ implies its correctness for F. Note, however, that the uniformity of the estimates is lost in passing from $\tilde{F}$ back to F.

Lemma 4.5. *Under the hypotheses of Theorem 4.1, if* $\int_{B(0,2)} F^2 \leqq C'$, *then*

$$|F(x)|^2 \leqq |x|^{-4} k \int_{B(0,2|x|)} F^2$$

for $|x| \leqq 1$. *Here* $C' = C'_4$ *and* $k = k_4$ *are the constants of Theorem 3.5 with* $n = 4$.

Proof. If $|x| \leqq 1$, $B(x, |x|) \subset B(0, 2)$ and $\int_{B(x,|x|)} F^2 \leqq \int_{B(0,2)} F^2 \leqq C'_4 2$.
We may then apply Theorem 3.5.

We are now directly in a position to construct the broken Hodge gauges. We break B^4 up into annuli

$$\mathfrak{U}_\ell = \{x : 2^{-\ell-1} \leqq |x| \leqq 2^{-\ell}\} \text{ for } \ell = \{0, 1, 2, \ldots\}$$
$$S_\ell = \{S = \{x : |x| = 2^{-\ell}\} \quad \text{for } \ell = \{0, 1, 2, \ldots\}.$$

Definition. A broken Hodge gauge for a connection D in a bundle η over $B^n - \{0\} = \bigcup_{\ell=0}^{\infty} \mathfrak{U}_\ell$ is a gauge related continuously to the original gauge in which $D = d + A$ and $A | \mathfrak{U}_\ell = A(\ell)$ have the following properties for *all* $\ell \geqq 0$:

(a) $d^* A(\ell) = 0$ in $\mathfrak{U}_\ell$.

(b) $A_\psi(\ell) | S_\ell = A_\psi(\ell - 1) | S_\ell$

(c) $d_\psi^* A_\psi(\ell) = 0$ on S_ℓ and $S_{\ell+1}$

(d) $\int_{S_\ell} A_r(\ell) = \int_{S_{\ell+1}} A_r(\ell) = 0.$

Note that (a) means the gauge is Hodge in $\mathfrak{U}_\ell$, but not necessarily consistent across S_ℓ. Condition (b) implies that the induced connection on the pull-back bundle $\eta | S_\ell$ is the same from the gauges given in $\mathfrak{U}_\ell$ and $\mathfrak{U}_{\ell-1}$. This is actually insured by the condition that the gauge is continuous. Condition (c) says to choose gauges over the $\eta | S_\ell$ which are Hodge, and condition (d) allows us to apply Theorem 2.7 and its Corollary 2.8.

The following theorem is true in all dimensions.

Theorem 4.6. *There exists* $\gamma'(=\gamma_n)>0$ *such that if* D *is a smooth connection in* $B^n-\{0\}$, *and the growth of the curvature satisfies* $|F(x)||x|^2 \leqq \gamma \leqq \gamma'$, *then there exists a broken Hodge gauge in* $B^n-\{0\}$ *satisfying*

$$\text{(e)}\quad |A(\ell)(x)| \leqq \kappa' \| F(\ell) \|_\infty 2^{-\ell} \leqq \kappa' \gamma 2^{\ell+1}$$

$$\text{(f)}\quad (\lambda_n - k^2\omega^2) \int_{\mathfrak{U}(\ell)} |A(\ell)|^2 \leqq 2^{-2\ell} \int_{\mathfrak{U}(\ell)} |F|^2.$$

Proof. The dilation $y = x\,2^{\ell+1}$ carries $\mathfrak{U}_\ell$ into the standard annulus $\mathfrak{U}$ of Theorem 2.8. Moreover, the inequality $|F(x)||x|^2 \leqq \gamma$ translates into the inequality $|\tilde{F}(y)||y|^2 \leqq \gamma$, $\|\tilde{F}\|_\infty \leqq \gamma$ on the curvature $\tilde{F}$ in the new variables. So we may apply Theorem 2.8 to $D = D(\ell)$ in the annulus $\mathfrak{U}_\ell$ to get a gauge over $\mathfrak{U}(\ell)$ in which (a), (b), (c) and (d) are true. At first, it is not clear that the gauge changes across the spheres S_ℓ are continuous. However, recall from Theorem 2.5 that gauges for $\eta | S_\ell$ in which $d_\psi^* A_\psi = 0$ are unique up to multiplication by constant elements in G. Therefore, the gauge chosen on S_ℓ from the construction on $\mathfrak{U}_\ell$ differs by a constant element $g_\ell \in G$ from that chosen by the construction on $\mathfrak{U}_{\ell-1}$. Rotate the gauge on $\mathfrak{U}_\ell$ by the constant element $h_\ell = g_\ell, \ldots, g_1$. Now the choice of gauge is continuous across S_ℓ. The inequality $\|A\|_\infty \leqq \kappa' \|F\|_\infty$ translates into (e) in the coordinates of $\mathfrak{U}_\ell$ rather than $\mathfrak{U}$. Likewise, Corollary 2.9 becomes (f) under the same dilation.

We now restrict our attention to 4-dimensions again. Our main result follows from the following differential inequality.

Proposition 4.7. *Let* $n=4$. *Then there exists* $\varepsilon > 0$ *such that if* D *is a Yang–Mills connection in* $B(2,0)-\{0\}$ *and* $\int_{B(2,0)} F^2 \leqq \varepsilon^2$ *then*

$$\left(1-\omega\left(\int_{|x|\leqq 2r} F^2\right)^{1/2}\right)\left(\int_{|x|\leqq r} F^2\right) \leqq 1/4r \int_{|x|=r} F^2.$$

Proof. From Lemma 4.5, we get $|F(x)|^2 \leqq |x|^{-4} k \int_{B(0,2|x|)} F^2$, by choosing $\varepsilon^2 \leqq C'$. If we choose $\varepsilon^2 k \leqq \sqrt{\gamma'}$, we may apply Theorem 4.6. We now estimate $\int_{\mathfrak{U}} F^2$ in integration by parts in the Hodge gauge. Assume $F = F(\ell)$, $A = A(\ell)$ in the broken Hodge gauge over $\mathfrak{U}_\ell$.

$$\begin{aligned}\int_{\mathfrak{U}_\ell} F^2 &= \int_{\mathfrak{U}_\ell} \langle dA(\ell) + [A(\ell), A(\ell)], F(\ell)\rangle \\ &= \int_{\mathfrak{U}_\ell} \langle (DA(\ell) - [A(\ell), A(\ell)]), F(\ell)\rangle\end{aligned}$$

$$= \int_{\mathfrak{U}_\ell} \langle A(\ell), -(D^*F(\ell) + [A(\ell), F(\ell)])\rangle$$
$$+ \int_{S_\ell} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle - \int_{S_{\ell+1}} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle .$$

Sum this equality over $\ell \geqq 0$. The boundary terms cancel, except for those over S_0, since $A_\psi(\ell) = A_\psi(\ell-1)|S_\ell$ and the curvature F is continuous across S_ℓ. The other boundary terms become negligible as $\ell \to \infty$; $\lim_{\ell\to\infty} \int_{S_{\ell+1}} \langle A_\psi(\ell), F_{r\psi}(\ell)\rangle = 0$. This follows because the estimate of Lemma 4.5, $|F(x)|^2|x|^{-4} \leqq k' \int_{B(0,|x|)} F^2$ improves as $x \in \mathfrak{U}_\ell$, $\ell \to \infty$. The term $D^*F(\ell) = 0$ disappears because D is Yang–Mills. We now have

$$\sum_{\ell\geqq 0} \int_{\mathfrak{U}_\ell} \langle F(\ell), (F(\ell) + [A(\ell), A(\ell)])\rangle$$
$$= \int_{S_0} \langle A_\psi(0), F_{r\psi}(0)\rangle$$
$$\leqq \left(\int_{S_0} (|A_\psi|^2)\right)^{1/2} \left(\int_{S_0} |F_{r\psi}|^2\right)^{1/2}.$$

Apply Corollary 2.6 to the connection $D_\psi + A_\psi$ on $\eta|S_0$. Note $D_\psi^* A_\psi = 0$. We have

$$(2 - K\|F\|_\infty)^2 \int_{S_0} |A_\psi|^2 \leqq \int_{S_0} |F_{\psi\psi}|^2.$$

Corollary 2.9 is used to estimate the error

$$\left|\int_{\mathfrak{U}_\ell} \langle F(\ell), [A(\ell), F(\ell)]\rangle\right| \leqq \|F(\ell)\|_\infty \int_{\mathfrak{U}_\ell} |A(\ell)|^2$$
$$\leqq 2^{-2\ell} \|F(\ell)\|_\infty (\lambda_4 - \kappa' 2^{-2\ell} \|F(\ell)\|_\infty^2)^{-1} \int_{\mathfrak{U}_\ell} |F(\ell)|^2.$$

Here the factor $2^{-2\ell}$ arises from the dilations between $\mathfrak{U}_\ell$ and the standard annulus used to state Corollary 2.9. By Lemma 4.5, we have

$$2^{-2l}\|F(\ell)\|_\infty \leqq k\left(\int_{|x|\leqq 2^{-\ell+1}} |F(\ell)|^2\right)^{1/2} \leqq k\left(\int_{|x|\leqq 2} F^2\right)^{1/2} \leqq k\varepsilon.$$

Assume $\kappa' k\varepsilon \leqq \lambda 4/2$. The above estimate simplifies to

$$\left|\int_{\mathfrak{U}_\ell} \langle F(\ell), [A(\ell), A(\ell)]\rangle\right| \leqq k\, 2/\lambda_4 \left(\int_{|x|\leqq 2} |F|^2\right)^{1/2} \int_{\mathfrak{U}_\ell} |F(\ell)|^2.$$

We go back to the main inequality $*$ and put in the estimates for the right-hand side.

$$\int_{B=\Sigma\mathfrak{U}_\ell} F^2 \leqq k\, 2/\lambda_4 \left(\int_{|x|\leqq 2} |F|^2\right)^{1/2} \int_{B=\Sigma\mathfrak{U}_\ell} F^2 +$$
$$+ \left(2 - kK\left(\int_{|x|\leqq 2} F^2\right)^{1/2}\right)^{-1} \left(\int_{|x|=1} |F_{\psi\psi}|^2\right)^{1/2} \left(\int_{|x|=1} |F_{r\psi}|^2\right)^{1/2}.$$

Rearranging terms:

$$\left(1-k^2/\lambda_4\left(\int_{|x|\leq 2} F^2\right)^{1/2}\right)\left(2-kK\left(\int_{|x|\leq 2} F^2\right)^{1/2}\right)\int_{|x|\leq 1} F^2$$

$$\leq\left(\int_{|x|=1} |F_{\psi\psi}|^2\right)^{1/2}\int_{|x|=1} |F_{r\psi}|^2 \leq 1/2 \int_{|x|=1} F^2.$$

Let $\omega = k(2\lambda_4^{-1} + kK/2)$. Then

$$\left(1-\omega\left(\int_{|x|\leq 2} F^2\right)^{1/2}\right)\int_{|x|\leq 1} F^2 \leq 1/4 \int_{|x|=1} F^2.$$

This result for $r=1$ implies the inequality for arbitrary r by dilation.

Theorem 4.8. *Let $n=4$. Then there exists $\varepsilon>0$ such that if D is a Yang–Mills connection in $B(2,0)-\{0\}$ satisfying $\int_{|x|\leq 2} F^2 \leq \varepsilon^2$ then $\|F\|_\infty$ is bounded in $|x|\leq 2$.*

Proof. Using the same ε of Proposition 4.7, (assuming in addition $1-\omega\varepsilon=\gamma>0$)

$$(1-\omega\varepsilon)\int_{|x|\leq r} F^2 \leq 1/4r \int_{|x|=r} F^2.$$

Let $f(r)=\int_{|x|\leq r} F^2$ so $f'(r)=\int_{|x|=r} F^2$. Since

$$\frac{4(1-\omega\varepsilon)}{r}\leq\frac{f'(r)}{f(r)}$$

by integration

$$f(r)\leq r^{4(1-\omega\varepsilon)}f(1)$$

or

$$\int_{|x|\leq r} F^2 \leq r^{4\gamma}\varepsilon^2.$$

Replace the inequality of Proposition 4.7 by

$$4(1-\omega(2r)^{2\gamma}\varepsilon)f(r)\leq rf'(r),$$

Integration of this differential inequality gives

$$f(r)\leq r^4 e^{4\omega\varepsilon/\gamma}f(1).$$

Finally, by Lemma 4.5 again,

$$|F(x)|^2 \leq |x|^{-4}k\int_{B(0,2|x|)} F^2 = |x|^{-4}kf(2|x|)$$

$$\leq e^{4\omega\varepsilon/\gamma}2^4kf(1).$$

Our last step is to show the existence of a gauge in which the bundle and covariant derivative are smooth. The dimension becomes unimportant again.

Theorem 4.9. *Let D be Yang–Mills in $B^n(0,1)=\{x\in R^n: 0<|x|\leq 1\}$. Assume in addition that $F=F(D)$ is pointwise bounded in norm. Then there exists a gauge in which the bundle η extends smoothly to $\bar\eta$ over $x=0$ and D extends to a smooth $\bar D$ in $\bar\eta$ which is Yang–Mills.*

Proof. As in all the previous calculations, we may assume (by dilation, if necessary) that Proposition 4.7 holds in $|x| \leqq 1$. However, in this Hodge gauge, $|A(\ell)| \leqq \kappa' \| F(\ell) \|_{\infty} 2^{-\ell}$. Equivalently $|A(x)| \leqq 2x\kappa' \| F \|_{\infty}$. Now apply the implicit function theorem described in the proof of Theorem 2.7. In the new gauge, $d^*A = 0$. The general regularity theorem gives the regularity of $D = d + A$ in this gauge [10, 14, 16].

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