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Cohomology and Massless Fields*

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Introduction

The geometry of twistors was first introduced in Penrose [28]. Since that time it has played a significant role in solutions of various problems in mathemetical physics of both a linear and nonlinear nature (cf. Penrose [29], Penrose [35], Ward [48], and Atiyah–Hitchin–Drinfeld–Manin [2], see Wells [52] for a recent survey of the topic with a more extensive bibliography). The major role it has played has been in setting up a general correspondence which translates certain important physical field equations in space-time into holomorphic structures on a related complex manifold known as *twistor space*. The purpose of this paper is to give a rigorous discussion of this correspondence for the case of the linear massless free-field equations, including Maxwell's source-free equations, the wave equation, the Dirac–Weyl neutrino equations, and the linearized (weak-field limit of) Einstein's vacuum equations. These equations may also be analyzed from this point of view on a background provided by the nonlinear Yang–Mills or Einstein equations in the (anti-) self-dual case. The correspondence is effected by an integral-geometric transform, which transforms complex-analytic data on twistor space to solutions of the linear massless field equations, and is, in fact, a generalization of the classical Radon transform, which is discussed further below.

The motivation for finding such a correspondence in general is that it forms an essential part of the "twistor programme" according to which one attempts to eliminate the equations of physics by deriving them from the rigidity of complex geometry and holomorphic functions (see, e.g. Penrose [38]). It is, in fact, rather remarkable the extent to which it is possible to achieve this. Success apparently comes about because in twistor-space descriptions the information is "stored" nonlocally. The (local) value of a field at a point in space-time depends upon the way that the holomorphic structure in the twistor-space is fitted together in the large. So sheaf cohomology and function theory of several complex variables turn out to be the appropriate tools in the twistor framework. It is hoped that, as part of the general twistor programme, some deeper insights may eventually be gained as to the interrelation between quantum mechanics or quantum field

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theory (which depend crucially upon complex function theory) and classical space-time structure.

Our discussion for the most part will be restricted to the case of analytic linear massless fields on flat space-time and will be carried out in terms of the compactified complexification $\mathbb{M}$ (often denoted $\mathbb{CM}$ elsewhere) of Minkowski-space. For this purpose, $\mathbb{M}$ will simply be *defined* as the Grassmannian of linear 2-spaces in twistor space, which is a complex vector space $\mathbb{T}$ of dimension 4. We obtain a number of theorems relating analytic cohomology of portions of $\mathbb{T}$, or rather of the corresponding projective 3-space $\mathbb{P}$, to the solutions of the relevant field equations in the corresponding regions of $\mathbb{M}$.

In the case of such massless fields the above-mentioned correspondence was first given in terms of a contour integral expression which yields general analytic solutions of the field equations when a holomorphic function, defined on a suitable domain, is inserted into the integrand (for right-handed fields see Penrose [29], [30], [33], and Penrose–MacCallum [32]; for left-handed fields see Penrose [34] or Hughston [23]). (For the wave-equation a very closely related integral expression had been found much earlier by Bateman [4].) Only comparatively recently (Penrose [36]) was it realized that these holomorphic functions should be interpreted as Čech cocycles representing elements of cohomology groups.[1] A definition of the transform (for right-handed fields) using the Dolbeault representation of analytic cohomology is described in Woodhouse [55] and Wells [52]. In Hughston [21] and Ward [49] it was shown how elements of cohomology in the left-handed case give rise to potentials for fields and a method of producing the field itself more directly related to the integral formulae is described in Penrose [37] and more invariantly in Wells [53]. The question of whether the transform is bijective was considered in Penrose [37], Lerner [24], Eastwood [8], and Wells [52]. It was shown recently that the transform, in a special case, is also a special case of a generalization of the Radon transform due to Gindikhin and Henkin (see [15]). Integral formulae for background coupled fields have been given in Ward [47]. In this paper we make no explicit reference to the earlier contour integral descriptions but give our discussion entirely in terms of the mathematically more satisfactory cohomological language.

One example of massless field equations is the linearized Einstein equations (the case of helicity ± 2). By the theory developed in this paper the cohomology group $H^1(\mathbb{P}^+, \mathcal{O}(2))$ represents holomorphic solutions of helicity -2 of these equations on the forward tube $\mathbb{M}^+$ in complexified Minkowski space. This cohomology group is precisely the infinitesimal parameter space for the deformations of structure of the fibration $\mathbb{P}^+ \to \mathbb{P}_1$ which appear in the nonlinear graviton construction (Penrose [35]), and which give the general (local) holomorphic anti-self-dual solution of Einstein's equations. These solutions to the massless field equations in terms of cohomology groups relate in a similar manner to solutions of other nonlinear problems and, it is hoped, will give insight into as yet unsolved nonlinear problems (e.g., the self-dual or general solution of

[1] The second author would like to express his gratitude to M. F. Atiyah for suggestions ultimately leading to this interpretation

Einstein's equations). In addition, there are algorithms for generating solutions of nonlinear field equations with specific cohomological solutions of associated linear equations as a principal part of the algorithm (cf. Atiyah–Ward [3], Ward [50]).

In the recent solution of the self-dual Yang–Mills equations on S^4 a key ingredient was the representation of specific cohomology groups on $\mathbb{P}_3$ in terms of solutions of conformally invariant elliptic differential equations on S^4 (see Atiyah–Hitchin–Drinfeld–Manin [2], Drinfeld–Manin [7], Hitchin [19], and Rawnsley [42]). This is a special case of our general method of representing solutions of linear field equations in terms of cohomology as discussed in Sect. 9.

We will now give a brief outline of the paper. In Sect. 1 we set up the basic twistor correspondence between projective twistor space (denoted by $\mathbb{P}$) and the natural complexification of compactified Minkowski space (denoted by $\mathbb{M}$). In particular we set up an important relative de Rham sequence on the flag manifold $\mathbb{F}$ which gives the correspondence between $\mathbb{P}$ and $\mathbb{M}$ via a double fibration of the following type

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{P} & & \mathbb{M}. \end{tikzcd}$$

The basic integral transform of data on $\mathbb{P}$ to data on $\mathbb{M}$ is achieved by pulling back cohomology classes from (open subsets of) $\mathbb{P}$ to $\mathbb{F}$ by the mapping μ, and then "integrating this pullback class over the fibers" appropriately, obtaining (vector-valued) functions on (open subsets of) $\mathbb{M}$ which will satisfy certain field equations. We do this in several steps. First, we study the pullbacks of local data (inverse image sheaves and the relative deRham sequence in Sect. 1). Then we study the problem of "integration over the fibre" for the mapping ν. This involves direct image sheaves and a fundamental spectral sequence for a fibration due to Leray, but is a direct translation of the more classical notion of fibre-wise integration of a differential form into an algebraic and coordinate-free language which makes the group-invariance properties of this process manifest. The direct images of the pullbacks of the various powers of the hyperplane-section bundle on $\mathbb{P}$ are identified with *spinor sheaves* on $\mathbb{M}$, sheaves whose sections, locally, are classical spinor fields of various kinds. We also identify in some detail the invariant differential operators coming from the relative deRham complex with specific classical differential operators acting on spinor fields on $\mathbb{M}$. It is this identification which allows us to make the transition between complex geometry and mathematical physics. One consequence is that the massless field equations become transparently conformally invariant. In Sect. 3 we study the global problem of preserving information when pulling back cohomology classes from $\mathbb{P}$ to $\mathbb{F}$ by the mapping μ. This turns out to be a purely topological problem involving the fibres of the mapping $\mu: \tilde{W} \to W$, where $W \subset \mathbb{P}$ and $\tilde{W} \subset \mathbb{F}$ are open sets to which the mapping μ is restricted. It turns out that the pullback sheaves and the sheaves to which the direct image theory applies don't coincide, but they are related by the relative deRham exact sequences, and the corresponding cohomology groups are related by a standard spectral sequence. This is discussed in Sect. 4. This completes the set of tools necessary to derive the massless field equations and their solutions. In Sect. 5 we obtain the isomorphism, for $s > 0, \mathscr{P}: H^1(\mathbb{P}^+, \mathcal{O}_{\mathbb{P}}(-2s-2)) \xrightarrow{\cong} \{$holomorphic right-handed massless fields of helicity s on $\mathbb{M}^+\}$. This isomorphism is equivariant with respect to the action of $U(2,2)$ on $\mathbb{T}$. We obtain this also for more general geometric settings. In Sect. 6 we derive a similar result for massless fields of helicity zero, where the second-order wave equation comes in. The spectral sequence theory shows why first-order equations arise for non-zero helicity, while second-order equations arise for helicity zero. In fact, all of the differential equations which are solved are first derived in an invariant form from the twistor geometry (and the spectral sequence on $\mathbb{F}$), then they are compared with the classical (noninvariant) forms, and then the solutions are produced by the integral transform. In Sect. 7 the left-handed solutions are described in terms of potentials which are transforms of certain analytic data. This is compared in Sect. 8 with a direct approach (power series expansions of cohomology classes about a given complex line in projective twistor space) given in Wells [53]. In Sect. 9 we generalize our results to background coupled fields, where the background potential is described in twistor space by means of a holomorphic line bundle (or more generally a holomorphic vector bundle) as was originated by Ward ([46], [47], [48]). Finally, in Sect. 10 we discuss various ramifications and possible extensions of the methods and ideas developed in this paper.

1. The Twistor Correspondence and a Relative de Rham Complex

Let $\mathbb{T}$ be the space of *twistors*. $\mathbb{T}$ is, by definition, a complex vector-space of dimension 4 with an Hermitian form Φ of signature $++--$. We define

$$\mathbb{F}_{d_1,\ldots,d_r} := \{(L_1,\ldots,L_r): L_1 \subset \ldots \subset L_r \text{ is a sequence of linear subspaces of } \mathbb{T} \text{ with } \dim_{\mathbb{C}} L_j = d_j.\}$$

and set

$$\mathbb{F} := \mathbb{F}_{1,2}, \quad \mathbb{P} := \mathbb{F}_1, \quad \mathbb{M} := \mathbb{F}_2.$$

These three compact complex manifolds (called *flag manifolds*, cf. Wells [51]) are naturally linked by the double fibration

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{P} & & \mathbb{M} \end{tikzcd} \tag{1.1}$$

where $\mu(L_1, L_2) = L_1$ and $\nu(L_1, L_2) = L_2$. The *twistor correspondence*[2] is the

2 If $\mathbb{M}$ is embedded in $\mathbb{P}_5$ by the Plücker coordinates then this is often referred to as the *Klein correspondence*

set-theoretic assignment (cf. Wells [52])

$$Z \in \mathbb{P} \longmapsto \hat{Z} := \nu(\mu^{-1}(Z)) \subset \mathbb{M}$$

$$z \in \mathbb{M} \longmapsto L_z := \mu(\nu^{-1}(z)) \subset \mathbb{P}$$

We note that $\mathbb{P} \simeq \mathbb{P}_3(\mathbb{C})$, three-dimensional projective space, $\mathbb{M} \simeq G_{2,4}(\mathbb{C})$, the Grassmannian manifold of 2-planes in 4-space, $L_z \simeq \mathbb{P}_1(\mathbb{C})$, a complex projective line embedded in $\mathbb{P}$, and $\hat{Z} \simeq \mathbb{P}_2(\mathbb{C})$, and is referred to as the *α-plane* in $\mathbb{M}$ determined by Z (the *β-planes* in $\mathbb{M}$ are determined by the projective hyperplanes in $\mathbb{P}$, but won't play a role in this paper). With a slight change in notation (1.1) is the basic double fibration used in [52] and we refer the reader to this article for more details and background references concerning this correspondence. Those already familiar with the basics of twistor theory will recognize $\mathbb{P}$ as *projective twistor-space*, $\mathbb{M}$ as *complexified compactified Minkowski-space*, and the correspondence space $\mathbb{F}$ as the (*dual*) *projective primed spin-bundle* over $\mathbb{M}$ (cf. Penrose–MacCallum [32]).

Our object in this section is to describe certain natural sheaves on $\mathbb{P}$ and $\mathbb{F}$ and how they are linked. These sheaves will in some sense provide the basic local data for the integral-geometric transform developed in the later sections.

The quadratic form Φ determines distinguished open subsets of $\mathbb{P}, \mathbb{F}$, and $\mathbb{M}$, to be denoted by $\mathbb{P}^+$, $\mathbb{M}^-$, etc. just as in [52]. In particular the analysis in [52] concentrated on

$$\begin{tikzcd} & \mathbb{F}^+ \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{P}^+ & & \mathbb{M}^+. \end{tikzcd}$$

We shall refer to these subsets explicitly when we need them but we want to consider more general open sets related in the same manner but not necessarily SU(2, 2)-invariant. So let U be an arbitrary open subset of $\mathbb{M}$ and define

$$U' := \nu^{-1}(U)$$

$$U'' := \mu(U') \quad \text{(also occasionally denoted by } L_U).$$

The analytical and topological properties of these sets will determine the precise nature of the transform to be developed later. We have the open inclusion of the two diagrams

$$\begin{tikzcd} & U' \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & & & \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ U'' & & U & \mathbb{P} & & \mathbb{M} \end{tikzcd} \quad \xhookrightarrow[\text{open}]{} \tag{1.2}$$

with μ and ν surjective mappings of maximal rank.

We now want to introduce on $\mathbb{F}$ a complex of differential forms along the fibres of μ. First we introduce some standard notation.[3] If X is a complex manifold

3 See, e.g. Wells [51]. We refer to this reference for standard concepts concerning complex manifolds used in this paper

we let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X and let Ω_X^p be the sheaf of holomorphic p-forms on $X, p \geqq 0$, noting that $\Omega_X^0 = \mathcal{O}_X$. Then we have

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \longrightarrow \dots,$$

the exact deRham sequence of sheaves where d is the usual exterior differentiation operator. Define the sheaf Ω_μ^1 of *relative 1-forms* with respect to the fibration μ by the exact sequence

$$\mu^*\Omega_{\mathbb{P}}^1 \longrightarrow \Omega_{\mathbb{F}}^1 \xrightarrow{\pi_\mu} \Omega_\mu^1 \longrightarrow 0$$

i.e., Ω_μ^1 is the quotient sheaf and π_μ is the usual quotient mapping. Then we set $d_\mu = \pi_\mu \circ d$. We shall write out explicit representations for the sheaf Ω_μ^1 and the differential operator d_μ shortly, but first we extend this concept to higher order forms and to vector-bundle-valued differential forms. Let Ω_μ^2 be the sheaf of *relative 2-forms* defined by the exact sequence

$$\mu^*\Omega_{\mathbb{P}}^1 \wedge \Omega_{\mathbb{F}}^1 \longrightarrow \Omega_{\mathbb{F}}^2 \longrightarrow \Omega_\mu^2 \longrightarrow 0$$

and let

$$d_\mu : \Omega_\mu^1 \longrightarrow \Omega_\mu^2$$

be the induced exterior derivative. Let $\mu^{-1}\mathcal{O}_{\mathbb{P}}$ denote the topological inverse image of the sheaf $\mathcal{O}_{\mathbb{P}}$ i.e. the sections of $\mu^{-1}\mathcal{O}_{\mathbb{P}}$ are locally simply pullbacks by μ of sections of $\mathcal{O}_{\mathbb{P}}$ or, in other words, $\mu^{-1}\mathcal{O}_{\mathbb{P}}$ is the subsheaf of $\mathcal{O}_{\mathbb{F}}$ consisting of those functions locally constant on the fibres of μ.

Lemma 1.1. *The sequence*

$$0 \longrightarrow \mu^{-1}\mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{F}} \xrightarrow{d_\mu} \Omega_\mu^1 \xrightarrow{d_\mu} \Omega_\mu^2 \longrightarrow 0 \tag{1.3}$$

is exact.

Before we prove this lemma we shall give an easy generalization. Suppose that V is a holomorphic vector-bundle over U, an open subset of $\mathbb{P}$, and let μ^*V be the pullback bundle over $\mu^{-1}(U)$. The bundle μ^*V may be described by transition functions which are constant along the fibres of μ. We can tensor (1.3) with $\mu^{-1}\mathcal{O}_{\mathbb{P}}(V)$, the sheaf of germs of sections of μ^*V which are constant along the fibres of μ, to obtain (since $\mathcal{O}_{\mathbb{F}} \otimes_{\mu^{-1}\mathcal{O}_{\mathbb{P}}} \mu^{-1}\mathcal{O}_{\mathbb{P}}(V)$ is canonically isomorphic to $\mathcal{O}_{\mathbb{F}}(\mu^*V)$)

$$\begin{aligned} 0 \longrightarrow \mu^{-1}\mathcal{O}_{\mathbb{P}}(V) \longrightarrow \mathcal{O}_{\mathbb{F}}(\mu^*V) &\xrightarrow{d_\mu} \mathcal{O}_{\mathbb{F}}(\mu^*V) \otimes_{\mathcal{O}_{\mathbb{F}}} \Omega_\mu^1 \\ &\xrightarrow{d_\mu} \mathcal{O}_{\mathbb{F}}(\mu^*V) \otimes_{\mathcal{O}_{\mathbb{F}}} \Omega_\mu^2 \longrightarrow 0. \end{aligned} \tag{1.4}$$

Note that the differential operators d_μ in (1.4) make sense since they annihilate the transition functions of μ^*V. For convenience of notation we shall write

$$\Omega_\mu^0 := \mathcal{O}_{\mathbb{F}}$$
$$\Omega_\mu^p(V) := \mathcal{O}_{\mathbb{F}}(\mu^*V) \otimes_{\mathcal{O}_{\mathbb{F}}} \Omega_\mu^p$$

and we can rewrite (1.4) as

$$0 \longrightarrow \mu^{-1}\mathcal{O}_{\mathbb{P}}(V) \longrightarrow \Omega_\mu^0(V) \xrightarrow{d_\mu} \Omega_\mu^1(V) \xrightarrow{d_\mu} \Omega_\mu^2(V) \longrightarrow 0 \tag{1.5}$$

For the special case of $V = H^n$, the n-th power of the hyperplane section bundle (cf. Wells [52]) we will write $\Omega_\mu^p(n)$ instead of $\Omega_\mu^p(H^n)$. This is consistent with the usual notation for "twisting" bundles on projective space. Thus we obtain in this case, for $n \in \mathbb{Z}$,

$$0 \longrightarrow \mu^{-1}\mathcal{O}_{\mathbb{P}}(n) \longrightarrow \Omega_\mu^0(n) \xrightarrow{d_\mu} \Omega_\mu^1(n) \xrightarrow{d_\mu} \Omega_\mu^2(n) \longrightarrow 0 \qquad (1.6)$$

This exact sequence of sheaves contains the basic local data for representing solutions of the massless field equations on $\mathbb{M}$. We note that (1.6) remains invariant under the action of $GL(\mathbb{T})$ on $\mathbb{T}$ and its induced action on $\mathbb{P}$, $\mathbb{F}$, and $\mathbb{M}$ as well as on H^n. We shall need to "globalize" this data using cohomology and then "push it down to $\mathbb{M}$" using direct images (cf. Sect. 2).

Now we shall proceed towards the proof of Lemma 1.1 and give explicit representations for these sheaves and differential operators. Actually the statement of Lemma 1.1 is local at a point of $\mathbb{F}$ and the proof will depend only on the fact that μ has maximal rank so that, by the implicit function theorem, one can always choose local coordinates in which μ is simply a coordinate projection (i.e. part of $\mathbb{C}^5 = \mathbb{C}^3 \times \mathbb{C}^2 \to \mathbb{C}^3$). However, it is useful at this stage to introduce specific coordinates tied to the spinor structure of the twistor-space $\mathbb{T}$. The coordinates have important physical interpretations (cf. Penrose–MacCallum [32]) and moreover the maximal rank condition will be obvious. Choose coordinates (Z^0, Z^1, Z^2, Z^3) for $\mathbb{T}$ and set

$$(Z^0, Z^1) = (\omega^0, \omega^1)$$
$$(Z^2, Z^3) = (\pi_{0'}, \pi_{1'})$$

so that $(\omega^{\mathbf{A}}, \pi_{\mathbf{A}'})$ are the components of a pair of *spinors* representing the *twistor* Z^α (cf. [32] and [52]). Indeed, we shall often write $\mathbb{T}^\alpha$ instead of $\mathbb{T}$ and $\mathbb{T}_\alpha$ for the dual of $\mathbb{T}^\alpha$. These coordinates are chosen so that

$$\Phi(Z^\alpha) = \omega^{\mathbf{A}}\bar{\pi}_{\mathbf{A}} + \bar{\omega}^{\mathbf{A}'}\pi_{\mathbf{A}'} \quad \text{(summation convention)}$$

or, equivalently, in matrix form

$$\Phi = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}.$$

We choose *affine coordinates* $z^{\mathbf{a}}$ for $\mathbb{M}^I$, the *affine part* of $\mathbb{M}$, where $\mathbb{M}^I$ is defined by the open inclusion

$$\begin{array}{ccc} \mathbb{C}^4 \cong \mathbb{C}^{2\times 2} & \xrightarrow{\simeq} & G_{2,4}^I(\mathbb{C}) \simeq \mathbb{M}^I \subset \mathbb{M} \\ \Psi & & \\ z^{\mathbf{a}} \leftrightarrow z^{\mathbf{AA}'} & \longmapsto & \operatorname{span}\begin{bmatrix} iz^{\mathbf{AA}'} \\ I_2 \end{bmatrix}, \end{array} \qquad (1.7)$$

under the biholomorphism $G_{2,4}(\mathbb{C}) \simeq \mathbb{M}$. We see that $\mathbb{M}^I$ is defined by the set of 2-dimensional subspaces of $\mathbb{T}^\alpha$ which are linear graphs over the $\{\pi_{\mathbf{A}'}\}$-coordinate plane, i.e. planes in $\mathbb{T}^\alpha$ of the form $\omega^{\mathbf{A}} = iz^{\mathbf{AA}'}\pi_{\mathbf{A}'}$. $\mathbb{M}$ is covered by a finite number of such affine coordinate systems (in fact, five are needed). We note that $\mathbb{M}^+$ is the subset of $\mathbb{M}^I$ where, writing $z^{\mathbf{a}} = x^{\mathbf{a}} - iy^{\mathbf{a}}$, $y^{\mathbf{a}}$ is timelike and future-pointing, and that *real* (*affine*) *Minkowski-space* is given when $y^{\mathbf{a}} = 0$ (or equivalently, when $z^{\mathbf{AA'}} = x^{\mathbf{AA'}}$ is Hermitian) (cf. [52]). We let $\mathbb{F}^I := \nu^{-1}(\mathbb{M}^I)$ and $\mathbb{P}^I := \mu(\mathbb{F}^I)$ be the associated "*affine*" portions of $\mathbb{F}$ and $\mathbb{P}$ respectively (although they are not affine in the sense of algebraic geometry). It is easy to describe the naturally associated coordinates on $\mathbb{P}^I$ and $\mathbb{F}^I$. By varying our choice of affine coordinates for $\mathbb{M}$ we could obtain in this way a complete system of linked coordinate charts on the three manifolds. First we observe that

$$\begin{aligned} &\mathbb{F}^I \simeq \mathbb{M}^I \times \mathbb{P}_1 \text{ with coordinates } (z^{\mathbf{AA'}}, [\pi_{\mathbf{A'}}]) \\ &\mathbb{P}^I = \mathbb{P} - I, \end{aligned}$$

where I is the projective line in $\mathbb{P}$ given by[4]

$$I = \{[\omega^{\mathbf{A}}, \pi_{\mathbf{A'}}] \in \mathbb{P} : \pi_{\mathbf{A'}} = 0\}$$

and, as usual, [ ] denotes homogeneous coordinates on a projective space. Moreover,

$$\begin{aligned} \mathbb{F}^I &= \mathbb{F}^I_0 \cup \mathbb{F}^I_1 \\ \mathbb{P}^I &= \mathbb{P}^I_0 \cup \mathbb{P}^I_1 \end{aligned}$$

where

$$\begin{aligned} \mathbb{P}^I_0 &:= \{[\omega^{\mathbf{A}}, \pi_{\mathbf{A'}}] \in \mathbb{P} : \pi_{0'} \neq 0\} \\ \mathbb{P}^I_1 &:= \{[\omega^{\mathbf{A}}, \pi_{\mathbf{A'}}] \in \mathbb{P} : \pi_{1'} \neq 0\} \end{aligned}$$

and

$$\begin{aligned} \mathbb{F}^I_0 &:= \mu^{-1}(\mathbb{P}^I_0) = \mathbb{M}^I \times (\mathbb{P}_1 - \{\pi_{0'} = 0\}) \\ \mathbb{F}^I_1 &:= \mu^{-1}(\mathbb{P}^I_1) = \mathbb{M}^I \times (\mathbb{P}_1 - \{\pi_{1'} = 0\}). \end{aligned}$$

On $\mathbb{P}^I_0$ we introduce affine coordinates (q^0, q^1, r) by

$$q^{\mathbf{A}} := \omega^{\mathbf{A}}/\pi_{0'}, \quad r := \pi_{1'}/\pi_{0'}$$

and on $\mathbb{F}^I_0$ the coordinates

$$\begin{aligned} q^{\mathbf{A}} &:= (iz^{\mathbf{AA'}}\pi_{\mathbf{A'}})/\pi_{0'}, \quad r := \pi_{1'}/\pi_{0'} \\ s^{\mathbf{A}} &:= -z^{\mathbf{A}}_{0'} = z^{\mathbf{A}1'}. \end{aligned}$$

With these coordinates $\mu|_{\mathbb{F}^I_0}$ is reduced to a projection

$$\begin{array}{ccccc} \mathbb{F}^I_0 \simeq & \mathbb{C}^3 \times \mathbb{C}^2 & \ni & (q^{\mathbf{A}}, r, s^{\mathbf{A}}) \\ \downarrow & \downarrow & & \updownarrow \\ \mathbb{P}^I_0 \simeq & \mathbb{C}^3 & \ni & (q^{\mathbf{A}}, r) \end{array}$$

so that $s^{\mathbf{A}}$ may be regarded as coordinates along the fibres of μ and d_μ may be realized explicitly as

$$d_\mu f = \frac{\partial f}{\partial s^0} d_\mu s^0 + \frac{\partial f}{\partial s^1} d_\mu s^1 = \frac{\partial f}{\partial s^{\mathbf{A}}} d_\mu s^{\mathbf{A}}$$

4 By an abuse of notation we sometimes identify points in $\mathbb{P}$, $\mathbb{F}$, etc., with their coordinate representation

which can be thought of as exterior differentiation in the $s^{\mathbf{A}}$ variables with $q^{\mathbf{A}}$, r as parameters. Similarly, relative 2-forms may be thought of as 2-forms with respect to $s^{\mathbf{A}}$ depending parametrically on $q^{\mathbf{A}}$, r. We may perform a similar analysis on $\mathbb{F}_1^I$ using coordinates

$$Q^{\mathbf{A}} := (iz^{\mathbf{AA}'}\pi_{\mathbf{A}'})/\pi_{1'}, \quad R := \pi_{0'}/\pi_{1'}$$
$$S^{\mathbf{A}} := -z^{\mathbf{A}}_{1'} = -z^{\mathbf{A}0'}.$$

Thus, on $\mathbb{F}_1^I$, d_μ is given by

$$d_\mu f = \frac{\partial f}{\partial S^{\mathbf{A}}} d_\mu S^{\mathbf{A}}.$$

On $\mathbb{F}_0^I \cap \mathbb{F}_1^I$ we have

$$S^{\mathbf{A}} = -z^{\mathbf{A}}_{1'} = -z^{\mathbf{A}0'} = -(q^{\mathbf{A}}/i) + (z^{\mathbf{A}1'}\pi_{1'}/\pi_{0'}) = -q^{\mathbf{A}}/i + rs^{\mathbf{A}}$$

and thus our frames $d_\mu s^{\mathbf{A}}$ and $d_\mu S^{\mathbf{A}}$ for Ω^1_μ over $\mathbb{F}_0^I \cap \mathbb{F}_1^I$ are related by

$$d_\mu S^{\mathbf{A}} = r d_\mu s^{\mathbf{A}}. \tag{1.8}$$

Now recall that $r = \pi_{1'}/\pi_{0'}$ is the transition function for $H \to \mathbb{P}_1$, the hyperplane section bundle, so (1.8) implies that

$$\Omega^1_\mu|_{\mathbb{F}^I} \simeq \mathcal{O}_{\mathbb{F}}(1) \oplus \mathcal{O}_{\mathbb{F}}(1)|_{\mathbb{F}^I}. \tag{1.9}$$

Similarly one obtains

$$d_\mu S \wedge d_\mu T = r^2 d_\mu s \wedge d_\mu t$$

from which it follows that

$$\Omega^2_\mu|_{\mathbb{F}^I} \simeq \mathcal{O}_{\mathbb{F}}(2)|_{\mathbb{F}^I}. \tag{1.10}$$

Remark. The isomorphisms (1.9) and (1.10) are not valid on all of $\mathbb{F}$. In the case of (1.9) note that a fibre Y of the fibration $\mathbb{F} \to \mathbb{P}$ is isomorphic to $\mathbb{P}_2(\mathbb{C})$ so that Y has non-trivial holomorphic cotangent bundle. But this is exactly what we obtain by restricting Ω^1_μ as a bundle to Y whereas $\mathcal{O}_{\mathbb{F}}(1) \oplus \mathcal{O}_{\mathbb{F}}(1)$ is trivial over Y. A similar argument holds for (1.10). The reason why these isomorphisms do not extend is really that relation (1.8) involves a spinor index $\mathbf{A}$ which has been ignored in (1.9) and (1.10). This problem will be rectified after first completing the proof of Lemma 1.1.

Proof of Lemma 1.1. In the coordinates introduced above we see that the exactness of (1.3) follows immediately from the exact (deRham) sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_{\mathbb{C}^2} \xrightarrow{d} \Omega^1_{\mathbb{C}^2} \xrightarrow{d} \Omega^2_{\mathbb{C}^2} \longrightarrow 0$$

by considering the relative forms in $\mathbb{C}^5$ as forms in $\mathbb{C}^2$ depending on three additional parameters. □

From (1.8) it follows immediately that

$$\left(\frac{\partial f}{\partial s^{\mathbf{A}}}\right)\pi_{0'} = \left(\frac{\partial f}{\partial S^{\mathbf{A}}}\right)\pi_{1'} \text{ on } \mathbb{F}_0^I \cap \mathbb{F}_1^I. \tag{1.11}$$

Using the chain rule on the formulae for $q^{\mathbf{A}}, r$, and $s^{\mathbf{A}}$ we find

$$\nabla_{\mathbf{A}}^{0'} := -\frac{\partial}{\partial z_{0'}^{\mathbf{A}}} = \frac{\partial}{\partial s^{\mathbf{A}}} - ir\frac{\partial}{\partial q^{\mathbf{A}}}$$

$$\nabla_{\mathbf{A}}^{1'} := -\frac{\partial}{\partial z_{1'}^{\mathbf{A}}} = i\frac{\partial}{\partial q^{\mathbf{A}}}$$

Thus, we may rewrite the left-hand side of (1.11):

$$\left(\frac{\partial}{\partial s^{\mathbf{A}}}\right)\pi_{0'} = (\nabla_{\mathbf{A}}^{0'} + r\nabla_{\mathbf{A}}^{1'})\pi_{0'} = \pi_{0'}\nabla_{\mathbf{A}}^{0'} + \pi_{1'}\nabla_{\mathbf{A}}^{1'} = \pi_{\mathbf{A}'}\nabla_{\mathbf{A}}^{\mathbf{A}'}.$$

Rewriting the right-hand side of (1.11), of course, produces the same answer. This representation shows directly how the twist appears in (1.9). It comes from the $\pi_{\mathbf{A}}$ which may be regarded as sections of $\mathcal{O}_{\mathbb{P}_1}(1)$. It also shows how the spinor index $\mathbf{A}$ appears and, indeed, globally Ω_μ^1 may be canonically identified with the *spinor sheaf* $\mathcal{O}_A(1)[-1]'$. This will be fully explained in the next section. Over $\mathbb{F}^I$, however, this spinor sheaf can be trivialized appropriately and this identification reduces to (1.9). From the above discussion the following lemma is immediate over $\mathbb{F}^I$ and the global statement will follow from the arguments of Sect. 2 ($[AB]$ means, as usual, skew-symmetrization of the indices).

Lemma 1.2. *The relative deRham sequence* (1.3) *on* $\mathbb{F}$ *may be canonically identified with the sequence of spinor sheaves*

$$\begin{array}{ccccc}
\Omega_\mu^0 & \longrightarrow & \Omega_\mu^1 & \longrightarrow & \Omega_\mu^2 \\
\| & & \wr\| & & \wr\| \\
\mathcal{O} & \longrightarrow & \mathcal{O}_A(\underline{1})[-\underline{1}]' & \longrightarrow & \mathcal{O}_{[AB]}(2)[-2]' \\
\cup & & \in \quad \ni & & \cup \\
f & \longmapsto & \pi_{A'}\nabla_A^{A'}f, \quad g_A & \longmapsto & \pi_{A'}\nabla_{[A}^{A'}g_{B]}.
\end{array}$$

2. Spinor Sheaves and Direct Image Sheaves

In order to define correctly the massless fields which are the subject of this paper it is necessary to introduce certain natural holomorphic vector-bundles on $\mathbb{M}$. These bundles are the various *conformally weighted spin-bundles* and the fields of interest on $\mathbb{M}$ are best described as sections thereof satisfying differential equations described in terms of the natural differential operators between these spin-bundles. We shall make use of the identification of a holomorphic vector-bundle with the sheaf of its holomorphic sections, often without explicitly saying so. Thus, our first task is to describe the spinor sheaves on $\mathbb{M}$.

In the previous section we referred to $\mathbb{F}$ as the projective primed spin-bundle. The fibre of $\mathbb{F}$ over a point z in $\mathbb{M}$ is the space of rays in z (regarded as a complex 2-plane in $\mathbb{T}$). Hence, $\mathbb{F}$ is the projective version of the vector bundle over $\mathbb{M}$ which associates with each point z in $\mathbb{M}$ the 2-dimensional vector-subspace of $\mathbb{T}$ which z represents. In topology this bundle is called the *universal bundle* on $\mathbb{M}$ (cf. [26], [51]). As warned above we make no distinction between vector-bundles and locally free sheaves and introduce the notation

$$\mathcal{O}_{A'} := \text{the universal bundle on } \mathbb{M},$$

calling this the *dual primed spin-bundle* so that $\mathbb{F}$ is the projective version of $\mathcal{O}_{A'}$. Its dual, the *primed spin-bundle*, will be denoted $\mathcal{O}^{A'}$. It should be emphasized that $\mathcal{O}_{A'}$ is not a direct sum of two line bundles and that A' is thus not a numerical spinor index but an *abstract index* (cf. [31], [40])[5]. This index is meant to serve as a marker which indicates what sections of this holomorphic vector bundle look like in coordinate patches. For example, in terms of such coordinates, $f^{A'} \in \mathcal{O}^{A'}$ is represented by $f^{\mathbf{A}'} = (f^{0'}, f^{1'}) \in \mathcal{O} \oplus \mathcal{O}$, the map from $f^{A'}$ to $f^{\mathbf{A}'}$ being achieved by use of the basis elements (local frame) $\varepsilon_{A'}^{\mathbf{B}'} = (\varepsilon_{A'}^{0'}, \varepsilon_{A'}^{1'}) \in \mathcal{O}_{A'} \oplus \mathcal{O}_{A'}$, with $f^{\mathbf{B}'} = f^{A'} \varepsilon_{A'}^{\mathbf{B}'}$, the *repeated index* A' denoting the abstract scalar product between a vector and an element of its dual space. The map back from $f^{\mathbf{A}'}$ to $f^{A'}$ is achieved by the dual basis $\varepsilon_{\mathbf{B}'}^{A'} = (\varepsilon_{0'}^{A'}, \varepsilon_{1'}^{A'}) \in \mathcal{O}^{A'} \oplus \mathcal{O}^{A'}$ according to $f^{A'} = f^{\mathbf{B}'} \varepsilon_{\mathbf{B}'}^{A'}$, the repeated index $\mathbf{B}'$ now indicating numerical summation convention. Correspondingly, the relation between $g_{A'} \in \mathcal{O}_{A'}$ and $g_{\mathbf{A}'} \in \mathcal{O} \oplus \mathcal{O}$ is given by $g_{\mathbf{B}'} = g_{A'} \varepsilon_{\mathbf{B}'}^{A'}$ and $g_{A'} = g_{\mathbf{B}'} \varepsilon_{A'}^{\mathbf{B}'}$ and the scalar product between $g_{A'}$ and $f^{A'}$ is $g_{A'} f^{A'} = g_{\mathbf{A}'} f^{\mathbf{A}'} = g_{0'} f^{0'} + g_{1'} f^{1'}$. One can choose bases $\varepsilon_{A'}^{\mathbf{B}'}$ and $\varepsilon_{\mathbf{B}'}^{\mathbf{A}'}$ which effect a trivialization of $\mathcal{O}_{A'}$ and $\mathcal{O}^{A'}$ over the subspace $\mathbb{M}^I$ of $\mathbb{M}$ (by using the coordinates given in (1.7), for instance).

To define the unprimed spin-bundle over $\mathbb{M}$ we observe that $\mathcal{O}_{A'}$ is, by definition, canonically embedded in $\mathcal{O}^\alpha$, the trivial bundle over $\mathbb{M}$ whose fibre at each point is twistor-space $\mathbb{T}^\alpha$. The *unprimed spin-bundle* $\mathcal{O}^A$ is defined to be the quotient (or *complement*). In other words, we have the short exact sequence[6]

$$\mathbb{M};\ 0 \longrightarrow \mathcal{O}_{A'} \longrightarrow \mathcal{O}^\alpha \longrightarrow \mathcal{O}^A \longrightarrow 0 \tag{2.1}$$

(cf. the definition of "local twistors" e.g. Penrose and MacCallum [32]). The dual unprimed spin-bundle is denoted $\mathcal{O}_A$, and each of $\mathcal{O}^A$ and $\mathcal{O}_A$ may be trivialized over $\mathbb{M}^I$ by means of bases, e.g.

$$\mathbb{M}^I;\ \mathcal{O} \oplus \mathcal{O} \ni (f_0, f_1) = f_{\mathbf{A}} \mapsto f_A = f_{\mathbf{B}} \varepsilon_A^{\mathbf{B}} \in \mathcal{O}_A.$$

Taking the dual of (2.1) gives the exact sequence

$$\mathbb{M};\ 0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_\alpha \longrightarrow \mathcal{O}^{A'} \longrightarrow 0 \tag{2.2}$$

The similarity between (2.1) and (2.2) bears further investigation. Let $\mathbb{M}^*$ denote the Grassmannian of 2-planes in $\mathbb{T}_\alpha$, the dual twistor space. Then there is a canonical isomorphism

$$\begin{array}{ccc} i : \mathbb{M} & \xrightarrow{\cong} & \mathbb{M}^* \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ z & \longmapsto & z^\perp := \{W_\alpha \in \mathbb{T}_\alpha : Z^\alpha W_\alpha = 0, \forall Z^\alpha \in z\}. \end{array}$$

5 We adopt the "abstract index" notation (Penrose [31]). Abstract indices are non-numerical and serve only as organizational markers enabling the basically coordinate-free operations of contraction, symmetrization, index permutation, etc. to be expressed in a transparent yet frame-independent way. The abstract indices are simply labels for bundles or sections of a given bundle. Italic and Greek letters will be used in this abstract way while Sans serif letters (**A**, **B**, ...) will be used for the corresponding "normal" indices (i.e. to represent the components in some particular frame). Italic lower case indices will be used for tensor indices, italic upper case for spinor indices, and Greek will be used for abstract twistor indices. Sans serif lower case indices **a**, **b**, **c**, ... will take values in the range $\{0, 1, 2, 3\}$ whilst Sans serif upper case indices **A**, **B**, **C**, .. will take values in the range $\{0, 1\}$.

6 From now on, a space X at the left of a diagram (such as $\mathbb{M}$ in equation (2.1)) indicates that X is the base-space of the various sheaves involved

We can now go through exactly the same process to describe various bundles on $\mathbb{M}^*$. Since $\mathbb{T}_\alpha$ has coordinates $(\eta_{\mathbf{A}}, \xi^{\mathbf{A}'})$, it is natural to denote the universal bundle on $\mathbb{M}^*$ by $\mathcal{O}_A$ and define $\mathcal{O}^{A'}$ by means of a sequence which formally looks just the same as (2.2). This abuse of notation is justified by the observation that there are canonical isomorphisms

$$z \cong (\mathbb{T}_\alpha/z^\perp)^*, \quad \mathbb{T}^\alpha/z \cong (z^\perp)^*, \text{ etc.},$$

giving rise to canonical isomorphisms

$$(\mathcal{O}_{A'} \text{ on } \mathbb{M}) \cong i^*(\mathcal{O}_{A'} \text{ on } \mathbb{M}^*), (\mathcal{O}^A \text{ on } \mathbb{M}) \cong i^*(\mathcal{O}^A \text{ on } \mathbb{M}^*), \text{ etc.}$$

But now observe that any isomorphism $p: \mathbb{T}^\alpha \xrightarrow{\simeq} \mathbb{T}_\alpha$ gives rise to an isomorphism

$$p: \mathbb{M} \xrightarrow{\simeq} \mathbb{M}^*$$

under which universal bundles naturally correspond i.e. $\mathcal{O}_{A'} = p^*\mathcal{O}_A$. Hence we conclude that any isomorphism of $\mathbb{T}^\alpha$ with $\mathbb{T}_\alpha$ naturally extends to an isomorphism of (2.1) with (2.2). Thus $\mathcal{O}_{A'}$ and $\mathcal{O}_A$ are on an equal footing when $\mathbb{M}$ is regarded as an abstract complex manifold rather than a specific Grassmannian. Choosing one of these bundles to be called the primed spin-bundle is, in this sense, arbitrary but it corresponds to making a choice of *complex orientation* for $\mathbb{M}$.

Remark. We want to distinguish in this paper between "isomorphism" and "canonical isomorphism," both used in the above discussion. An intuitive notion of what is canonical ("God-given"), and what is not is common among mathematicians, but it is worthwhile to clarify the point. In general, the meaning of the word depends on the context. The idea is that something is canonical if it transforms correctly when the whole system is subjected to a morphism of the category in which one is working (often not explicitly mentioned). In this paper, our basic given data is the 4-dimensional vector space $\mathbb{T}$, and a canonical isomorphism for us will mean an isomorphism of two quantities dependent upon $\mathbb{T}$ but which is preserved under the action of $GL(\mathbb{T})$ on $\mathbb{T}$. In the discussion above, we see that $i: \mathbb{M} \xrightarrow{\cong} \mathbb{M}^*$ is a canonical isomorphism whereas $p: \mathbb{M} \xrightarrow{\simeq} \mathbb{M}^*$ is not. We use "$\cong$" to denote canonical isomorphism, and "$\simeq$" to refer to an isomorphism, not necessarily canonical (usually where some choice has been made, e.g. $\mathbb{M} \simeq G_{2,4}(\mathbb{C})$ corresponds to a choice of basis in $\mathbb{T}$).

From these two basic bundles we can generate lots more by applying standard tensor operations. The following notation is natural and essentially self-explanatory[7]:

7 Here we mean tensor products as $\mathbb{C}$-vector bundles, i.e. the fibre-wise tensor product of the vector bundle fibres as complex vector spaces. It may be remarked, also, that the abstract index notation entails that $\mathcal{O}_{A'}$ and $\mathcal{O}_{B'}$ are canonically isomorphic copies of one another, but not, strictly speaking, the same space. The same applies to $\mathcal{O}_{C'}$, etc. The identification of $\mathcal{O}_{A'B'}$ with $\mathcal{O}_{A'} \otimes \mathcal{O}_{B'}$ rather than with $\mathcal{O}_{B'} \otimes \mathcal{O}_{A'}$ involves the notion of an alphabetical ordering for the abstract indices. In fact, $\mathcal{O}_{B'A'} \cong \mathcal{O}_{A'B'} = \mathcal{O}_{A'} \otimes \mathcal{O}_{B'} \cong \mathcal{O}_{B'} \otimes \mathcal{O}_{A'} \cong \mathcal{O}_{A'C'} = \mathcal{O}_{A'} \otimes \mathcal{O}_{C'}$, etc. (cf. [31], [40])

$$\mathcal{O}_{A'B'} := \mathcal{O}_{A'} \otimes \mathcal{O}_{B'}, \quad \mathcal{O}_{AA'} := \mathcal{O}_{A} \otimes \mathcal{O}_{A'}$$
$$\mathcal{O}_{(A'B')} := \mathcal{O}_{A'} \odot \mathcal{O}_{B'} \text{ where } \odot \text{ means symmetric product}$$
$$\mathcal{O}_{[A'B']} := \mathcal{O}_{A'} \wedge \mathcal{O}_{B'} = \det(\mathcal{O}_{A'}), \quad \mathcal{O}_{A}^{A'} := \mathcal{O}_{A} \otimes \mathcal{O}^{A'}, \text{etc., etc.}$$

Of particular interest are the two natural line bundles $\mathcal{O}_{[AB]}$ and $\mathcal{O}_{[A'B']}$. We introduce an alternative notation:

$$\mathcal{O}_{[AB]} := \mathcal{O}[-1] \quad \text{and} \quad \mathcal{O}_{[A'B']} := \mathcal{O}[-1]'.$$

This is in line with the notation of $\mathcal{O}(-1)$ for the universal bundle on a projective space. The number -1 in the square brackets may be regarded, with suitable normalization, as the first Chern class of these bundles. Just as for projective space, $\mathcal{O}[1]$ (resp. $\mathcal{O}[1]'$) will denote the dual of $\mathcal{O}[-1]$ (resp. $\mathcal{O}[-1]'$) and, more generally,

$$\mathcal{O}[\mathrm{k}] := \{\mathcal{O}[1]\}^{k} \quad \text{and} \quad \mathcal{O}[k]' := \{\mathcal{O}[1]'\}^{k}.$$

We can now rewrite, for example,

$$\mathbb{M};\ \mathcal{O}_{A[A'B']} = \mathcal{O}_{A} \otimes \mathcal{O}_{[A'B']} = \mathcal{O}_{A} \otimes \mathcal{O}[-1]' := \mathcal{O}_{A}[-1]',$$

and write $\mathcal{O}_{(AB)}[-1][2]'$ to mean $\mathcal{O}_{(AB)} \otimes \mathcal{O}[-1] \otimes \mathcal{O}[2]'$, etc. Our notation has some redundancies. For example, there is the canonical isomorphism

$$\mathbb{M};\ \mathcal{O}_{[C'D']}[1]' := \mathcal{O}_{[C'D']} \otimes \mathcal{O}^{[A'B']} \cong \mathcal{O}.$$

Under this isomorphism we shall denote by $\varepsilon_{A'B'}$ the section of $\mathcal{O}_{[A'B']}[1]'$ corresponding to $1 \in \mathcal{O}$. Similarly, we have

$$\varepsilon_{AB} \in \mathcal{O}_{[AB]}[1],\ \varepsilon^{A'B'} \in \mathcal{O}^{[A'B']}[-1]',\ \varepsilon^{AB} \in \mathcal{O}^{[AB]}[-1].$$

These are to be regarded as the usual conformally weighted ε's used for raising and lowering spinor indices. Thus, we see that the numbers in the square brackets may be regarded as *conformal weights*, with $\varepsilon_{A'B'}$, for example, having conformal weight 1. This entails, roughly speaking, that a metric g_{ab} is replaced by $\lambda\lambda' g_{ab}$ when $\varepsilon_{A'B'}$ is replaced by $\lambda'\varepsilon_{A'B'}$ and ε_{AB} by $\lambda\varepsilon_{AB}$ (whereas $\varepsilon^{A'B'}$ must be replaced by $(\lambda')^{-1}\varepsilon^{A'B'}$ and ε^{AB} by $\lambda^{-1}\varepsilon^{AB}$). Moreover, we observe that these ε's really can be used for raising and lowering spinor indices on sheaves. For example:

$$\mathbb{M};\ \begin{array}{ccccccc} \mathcal{O}^{A'} & \cong & \mathcal{O}^{A'} \otimes \mathcal{O} & \cong & \mathcal{O}^{A'} \otimes \mathcal{O}_{[A'B']}[1]' & \cong & \mathcal{O}_{B'}[1]' \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\ f^{A'} & \longleftrightarrow & f^{A'} \otimes 1 & \longleftrightarrow & f^{A'} \otimes \varepsilon_{A'B'} & \longleftrightarrow & f^{A'}\varepsilon_{A'B'} \end{array}$$

All these bundles may be pulled back to $\mathbb{F}$ and we will often use the same notation to denote the pullbacks. Thus, we write $\mathcal{O}_{A}$ instead of $\nu^{*}\mathcal{O}_{A}$ if it is clear that we are discussing bundles on $\mathbb{F}$ rather than $\mathbb{M}$. Also, we can pull the bundles $\mathcal{O}(k)$ on $\mathbb{P}$ back to $\mathbb{F}$. Again we make no notational distinction and naturally write

$$\mathcal{O}_{A}(1)[-1]' \text{ instead of } \nu^{*}\mathcal{O}_{A}[-1]' \otimes \mu^{*}\mathcal{O}(1) \text{ etc.}$$

This makes sense of all the sheaves in Lemma 1.2 and our next job is to make sense of the differential operators. We claim that, on $\mathbb{M}$,

$$\nabla_{AA'} : \mathcal{O} \longrightarrow \mathcal{O}_{AA'}$$

is a well-defined intrinsic differential operator. Indeed, we claim that it may be canonically identified with the exterior derivative

$$d : \mathcal{O} \longrightarrow \Omega^1 .$$

To make full use of the abstract index notation [31], we should also write this as

$$\nabla_a : \mathcal{O} \longrightarrow \mathcal{O}_a .$$

In local coordinates d and $\nabla_{AA'}$ certainly look identical. The problem is to see that Ω^1 is canonically isomorphic to $\mathcal{O}_{AA'}$. Equivalently, the problem is to show that the holomorphic tangent bundle can be identified naturally with $\mathcal{O}^{AA'}$. By definition,

$$\mathbb{M}\,;\ \mathcal{O}^{AA'} = \mathcal{O}^A \otimes \mathcal{O}^{A'} = \mathcal{O}^A \otimes (\mathcal{O}_{A'})^* \cong \operatorname{Hom}(\mathcal{O}_{A'}, \mathcal{O}^A).$$

It is well known, however, that the tangent bundle of a Grassmannian manifold is isomorphic to the bundle of homomorphisms from the universal bundle to its complement (cf. [26]). Other differential operators are defined by using ε's to raise indices. For example, $\nabla_A^{A'}$ is defined as the composition

$$\mathbb{M}\,;\ \mathcal{O} \xrightarrow{\nabla_{AB'}} \mathcal{O}_{AB'} \xrightarrow{\varepsilon^{A'B'}} \mathcal{O}_A^{A'}[-1]' .$$

The exterior derivative $d : \Omega^1 \to \Omega^2$ may also be written out in terms of spinor sheaves. Ω^2 splits into its self-dual and anti-self-dual parts: $\Omega^2 = \Omega^2_+ \oplus \Omega^2_-$. Decomposing a 2-form F_{ab} as $F^+_{ab} + F^-_{ab}$ is easily written in spinor notation (cf. [52], [40]):

$$F_{ab} = F_{AA'BB'} = \phi_{A'B'}\varepsilon_{AB} + \psi_{AB}\varepsilon_{A'B'} ,$$

where $\phi_{A'B'}$ and ψ_{AB} are symmetric. Alternatively, in terms of bundles,

$$\begin{aligned} \mathbb{M}\,;\ \Omega^2 \cong \mathcal{O}_{[ab]} \cong \Omega^2_+ \oplus \Omega^2_- &\cong \mathcal{O}_{(A'B')[AB]} \oplus \mathcal{O}_{(AB)[A'B']} \\ &= \mathcal{O}_{(A'B')}[-1] \oplus \mathcal{O}_{(AB)}[-1]' \end{aligned}$$

and the exterior derivative may be written

$$\mathbb{M}\,;\ \begin{array}{ccccc} \mathcal{O}_{AA'} & \longrightarrow & \mathcal{O}_{(A'B')}[-1] & \oplus & \mathcal{O}_{(AB)}[-1]' \\ \Uparrow & & & & \Uparrow \\ \phi_{AA'} & \longmapsto & \nabla^A_{(B'}\phi_{A')A} & + & \nabla^{A'}_{(B}\phi_{A)A'} \end{array}$$

More generally, the whole deRham sequence may be written:

$$\mathbb{M}\,;\ \begin{array}{ccccccccc} \Omega^0 & \longrightarrow & \Omega^1 & \longrightarrow & \Omega^2 & \longrightarrow & \Omega^3 & \longrightarrow & \Omega^4 \\ \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| \\ \mathcal{O} & \xrightarrow{\nabla_{AA'}} & \mathcal{O}_{AA'} & \underset{\nabla^{A'}_{B}}{\xrightarrow{\nabla^{A'}_{B'}}} & \begin{matrix} \mathcal{O}_{(A'B')}[-1] \\ \oplus \\ \mathcal{O}_{(AB)}[-1]' \end{matrix} & \underset{\nabla^A_{B'}}{\xrightarrow{\nabla^{A'}_{B}}} & \mathcal{O}_{BB'}[-1][-1]' & \xrightarrow{\nabla_{BB'}} & \mathcal{O}[-2][-2]' \end{array} \qquad (2.3)$$

These operators also act between other pairs of spinor sheaves as we will see when we discuss direct images.

To make sense of the differential operator

$$\mathbb{F};\ \pi_{A'}\nabla_A^{A'}:\mathcal{O}\longrightarrow\mathcal{O}_A(1)[-1]'$$

which appears in Lemma 1.2 we first observe that we can invariantly define a bundle homomorphism

$$\mathbb{F};\ \pi_{A'}:\mathcal{O}^{A'}\longrightarrow\mathcal{O}(1).$$

Indeed, it is equivalent to define the dual

$$\mathbb{F};\ {}^t\pi_{A'}:\mathcal{O}(-1)\longrightarrow\mathcal{O}_{A'}$$

and geometrically this can be seen as follows. By definition, the fibres of $\mathcal{O}(-1)$ and $\mathcal{O}_{A'}$ over $(L_1, L_2)\in\mathbb{F}$ are precisely L_1 and L_2 respectively so that ${}^t\pi_{A'}$ may be defined over (L_1, L_2) as the inclusion $L_1\hookrightarrow L_2$. One could hope, perhaps, that $\pi_{A'}\nabla_A^{A'}$ might be defined as the composition

$$\mathbb{F};\ \mathcal{O}\xrightarrow{\nabla_A^{A'}}\mathcal{O}_A^{A'}[-1]'\xrightarrow{\pi_{A'}}\mathcal{O}_A(1)[-1]'$$

but this is not the case as $\nabla_{AA'}$ does not make sense on $\mathbb{F}$. If, however, we choose a specific trivialization of $\mathbb{F}$ over a neighbourhood of a point in $\mathbb{M}$, then, of course, $\nabla_{AA'}$ does make sense as differentiation in the base direction. We claim that then $\pi_{A'}\nabla_A^{A'}$ is independent of this choice of trivialization. Certainly this should be the case since $\pi_{A'}\nabla_A^{A'}$ is supposed to be d_μ which is, by definition, differentiation along the fibres of μ and these fibres are mapped isomorphically by v to the α-planes in $\mathbb{M}$ [52]. Thus we can think of d_μ as first restricting to a fibre of μ, then identifying this fibre with an α-plane in $\mathbb{M}$, and finally effecting the exterior derivative on this α-plane. This last step can certainly be expressed using just $\nabla_{AA'}$, the exterior derivative on $\mathbb{M}$. We can therefore justify our claim by showing that the map from the cotangent bundle of $\mathbb{M}$ restricted to some α-plane to the cotangent bundle of the α-plane itself is given by $\pi^{A'}$. This is easy to see geometrically from the definitions of the spaces involved.

It is, perhaps, worth while to give an alternative way of describing d_μ as $\pi_{A'}\nabla_A^{A'}$ more directly. If we choose local coordinates $(x^{\mathbf{AA}'}, \pi_{\mathbf{A}'})$ on $\mathbb{F}$ then $\pi^{\mathbf{A}'}\nabla_{\mathbf{AA}'}$ makes perfectly good sense and, by our calculations of Sect. 1, does indeed represent d_μ. Actually, it is easy to see that if f is a function which is constant on the fibres of μ i.e. f depends on $\omega^{\mathbf{A}}(:=ix^{\mathbf{AA}'}\pi_{\mathbf{A}'})$ and $\pi_{A'}$ alone, then f is annihilated by $\pi^{\mathbf{A}'}\nabla_{\mathbf{AA}'}$:

$$\pi^{\mathbf{A}'}\nabla_{\mathbf{AA}'}f(\omega^{\mathbf{B}},\pi_{\mathbf{B}'})=\pi^{\mathbf{A}'}\left(i\pi_{\mathbf{A}'}\frac{\partial f}{\partial\omega_{\mathbf{A}}}(\omega^{\mathbf{B}},\pi_{\mathbf{B}'})\right)=0.$$

The invariance of $\pi^{A'}\nabla_{AA'}$ may then be checked directly, by changing from one local coordinate system to another.

It may be helpful, at this stage, to point out precisely the lack of invariance that can arise in expressions involving the operator $\nabla_{AA'}$. Each of the trivializations of $\mathbb{F}$ that we have been considering is associated with a particular flat connection $\nabla_{AA'}$ on a portion of $\mathbb{M}$ (e.g. on the $\mathbb{M}^I$ of Sect. 1). The non-invariance that may arise in the use of $\nabla_{AA'}$ can be seen in explicit transformation formulae relating one such connection to another. It is helpful, moreover, to examine such transformation formulae in the broader context of general torsion-free connections on $\mathbb{M}$ that are compatible with the (complex) conformal structure of $\mathbb{M}$. We require $\nabla_{AA'}$ to act on spinor fields on some open $W \subset \mathbb{M}$ and, correspondingly, to provide a bundle connection on $W' \subset \mathbb{F}$. We suppose g_{ab} and $\hat{g}_{ab}$ are two (not necessarily flat) complex metrics on W compatible with the given conformal structure, with spinor representations given as

$$g_{AA'BB'} = \varepsilon_{AB}\varepsilon_{A'B'}, \hat{g}_{AA'BB'} = \hat{\varepsilon}_{AB}\hat{\varepsilon}_{A'B'}$$

where

$$\hat{g}_{ab} = \Omega\tilde{\Omega}g_{ab}, \hat{\varepsilon}_{AB} = \Omega\varepsilon_{AB}, \hat{\varepsilon}_{A'B'} = \tilde{\Omega}\varepsilon_{A'B'},$$

the Ω and $\tilde{\Omega}$ being arbitrary non-vanishing holomorphic scalar functions on W. For spinor fields on $\mathbb{M}$, the relevant formulae are essentially given in [31], [32], [40]:

$$\begin{aligned}(\hat{\nabla}_{AA'} - \nabla_{AA'})\chi^{P\cdots S'\cdots}_{B\cdots F'\cdots} &= \varepsilon_A{}^P Y_{XA'}\chi^{X\cdots S'\cdots}_{B\cdots F'\cdots} + \ldots + \varepsilon_{A'}{}^{S'} Y_{AX'}\chi^{P\cdots X'\cdots}_{B\cdots F'\cdots} + \ldots \\ &\quad - Y_{BA'}\chi^{P\cdots S'\cdots}_{A\cdots F'\cdots} - \ldots - Y_{AF'}\chi^{P\cdots S'\cdots}_{B\cdots A'\cdots} - \ldots + k\Pi_{AA'}\chi^{P\cdots S'\cdots}_{B\cdots F'\cdots} \quad (2.4)\end{aligned}$$

where

$$\begin{aligned}k &= (\text{no. of upper unprimed and lower primed indices}) \\ &\quad - (\text{no. of upper primed and lower unprimed indices})\end{aligned}$$

and where

$$2Y_{AA'} = \Omega^{-1}\nabla_{AA'}\Omega + \tilde{\Omega}^{-1}\nabla_{AA'}\tilde{\Omega}, \quad 4\Pi_{AA'} = \Omega^{-1}\nabla_{AA'}\Omega - \tilde{\Omega}^{-1}\nabla_{AA'}\tilde{\Omega},$$

these formulae being uniquely characterized by the additivity, Leibniz and torsion-free properties, and the conditions

$$\begin{aligned}\nabla_{AA'}\varepsilon_{BC} &= 0 = \hat{\nabla}_{AA'}\hat{\varepsilon}_{BC} \\ \nabla_{AA'}\varepsilon_{B'C'} &= 0 = \hat{\nabla}_{AA'}\hat{\varepsilon}_{B'C'}.\end{aligned}$$

The formula (2.4) is useful for checking the invariance of many of the operations of this paper, for example, the massless field equations (of appropriate weights) considered in Sect. 5–Sect. 9, where a quantity $\chi^{\cdots}_{\cdots}$ of weights $[p][q]'$, when acted upon by $\hat{\nabla}_{AA'}$, is replaced by

$$\hat{\chi}^{\cdots}_{\cdots} = \Omega^p\tilde{\Omega}^q\chi^{\cdots}_{\cdots}.$$

In fact, the difference between the primed and unprimed weights does not seem to play much of a role here, as it turns out that a $GL(\mathbb{T})$ transformation always produces an Ω and $\tilde{\Omega}$ which are constant multiples of one another, so here $\Pi_{AA'} = 0$.

For a scalar field f on $\mathbb{F}$ we find, with $\Pi_{AA'} = 0$,

$$(\hat{\nabla}_{AA'} - \nabla_{AA'})f = \pi_{A'}Y_{AC'}\frac{\partial f}{\partial \pi_{C'}}, \tag{2.5}$$

where the fibre-wise derivative $\partial f/\partial \pi_{C'}$ has an obvious invariant meaning. The general form of the right-hand side of (2.5), i.e. with $\pi_{A'}Y_{AC'}$ replaced by some $Q_{AA'C'}$ independent of f, follows from the fact that each of $\hat{\nabla}_{AA'}$ and $\nabla_{AA'}$ gives simply d on $\mathbb{M}$. The particular expression given in (2.5) follows from (2.4), for example, by taking

$$f = f^{P'\cdots R'}\pi_{P'} \dots \pi_{R'},$$

where $f^{P'\cdots R'}$ is a spinor field on $\mathbb{M}$. We can treat expressions with both spinor indices and π-dependence by adding the appropriate term (2.5) to the expression (2.4).

The invariance of $\pi^{A'}\nabla_{AA'}$ when acting on unweighted scalar quantities on $\mathbb{F}$ is now immediately obvious.

Note that we may use ε^{AB} to rewrite the relative deRham sequence in Lemma 1.2 as

$$\mathbb{F};\quad \begin{array}{ccccc} \Omega^0_\mu & \xrightarrow{d_\mu} & \Omega^1_\mu & \xrightarrow{d_\mu} & \Omega^2_\mu \\ \wr\| & & \|\wr & & \wr\| \\ \mathcal{O} & \xrightarrow{\pi_{A'}\nabla^{A'}_A} & \mathcal{O}_A(1)[-1]' & \xrightarrow{\pi_{A'}\nabla^{AA'}} & \mathcal{O}(2)[-1][-2]' \end{array}$$

or alternatively (with a change of sign which we suppress)

$$\mathbb{F};\ \mathcal{O} \xrightarrow{\pi_{A'}\nabla^{AA'}} \mathcal{O}^A(1)[-1][-1]' \xrightarrow{\pi_{A'}\nabla^{A'}_A} \mathcal{O}(2)[-1][-2]'.$$

We now want to consider sheaves on $\mathbb{M}$ derived from these sheaves Ω^p_μ and more generally from $\Omega^p_\mu(V)$ for a holomorphic vector bundle V over a portion of $\mathbb{P}$. We shall use the notion of *direct image sheaves* (cf. Bredon [5], Godement [14]) which are defined as follows. If $f: X \to Y$ is a continuous mapping of topological spaces, and $\mathscr{F}$ is a sheaf of Abelian groups on X then we define the direct images $f^q_*\mathscr{F}$ on Y by means of the presheaf

$$U \longmapsto H^q(f^{-1}(U), \mathscr{F}), \quad \text{for } U^{\text{open}} \subseteq Y$$

with the obvious restriction mapping. Thus, we see that the stalk of $f^q_*\mathscr{F}$ at $y \in Y$ is given by

$$(f^q_*\mathscr{F})_y = \varinjlim_{U \ni y} H^q(f^{-1}(U), \mathscr{F}),$$

i.e. the "cohomology along the fibre" of the sheaf $\mathscr{F}$. In the most natural case when $V = H^n$ the direct images $v^q_*(\Omega^p_\mu(n))$ turn out to be various spin-bundles on $\mathbb{M}$ as already defined in this section. This identification will be deduced from Lemma 1.2 and the following (letting $v_* := v^0_*$).

Proposition 2.1. *The direct images $v^q_*\mathcal{O}(k)$ of the sheaves $\mathcal{O}(k)$ on $\mathbb{F}$ may be canonically identified as:*

$$\mathbb{M};\quad \begin{array}{ll} \text{a) if } n \geqq 0: v_*\mathcal{O}(n) \cong \mathcal{O}^{(A'B'\cdots D')} & (n \text{ indices}) \\ \qquad v^q_*\mathcal{O}(n) = 0 \quad \forall q \geqq 1 & \\ \text{b)} \qquad v^q_*\mathcal{O}(-1) = 0 \quad \forall q & \\ \text{c) if } n \geqq 0: v^1_*\mathcal{O}(-n-2) \cong \mathcal{O}_{(A'B'\cdots D')}[-1]' & (n \text{ indices}) \\ \qquad v^q_*\mathcal{O}(-n-2) = 0 \quad \forall q \neq 1. & \end{array}$$

Proof. By definition, our task is to compute $H^q(v^{-1}(U), \mathcal{O}(k))$ for open sets U in

$\mathbb{M}$ but since there are arbitrarily small Stein open sets[8] it suffices to restrict our attention to the case U Stein and where $\nu^{-1}(U)$ may be trivialized: $U' := \nu^{-1}(U) \simeq U \times \mathbb{P}_1$. Choose coordinates $[\pi_{0'}, \pi_{1'}]$ for $\mathbb{P}_1$. We first look at cohomology on $\mathbb{P}_1$ and then consider the additional parameters in U. If we cover $\mathbb{P}_1$ with

$$V_0 := \{[\pi_{0'}, \pi_{1'}] \in \mathbb{P}_1 : \pi_{0'} \neq 0\}$$
$$V_1 := \{[\pi_{0'}, \pi_{1'}] \in \mathbb{P}_1 : \pi_{1'} \neq 0\},$$

then $V_0 \cong V_1 \cong \mathbb{C}$ which is Stein. We may therefore use the Mayer–Vietoris sequence (cf. Bredon [5])

$$\begin{array}{l} 0 \longrightarrow \Gamma(\mathbb{P}_1, \mathcal{O}(k)) \longrightarrow \Gamma(V_0, \mathcal{O}(k)) \oplus \Gamma(V_1, \mathcal{O}(k) \longrightarrow \Gamma(V_0 \cap V_1, \mathcal{O}(k)) \\ \longrightarrow H^1(\mathbb{P}_1, \mathcal{O}(k)) \longrightarrow \underset{\substack{\| \\ 0}}{H^1(V_0, \mathcal{O}(k))} \oplus \underset{\substack{\| \\ 0}}{H^1(V_1, \mathcal{O}(k))} \longrightarrow \dots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad : (\text{since } V_0, V_1 \text{ are Stein}), \end{array}$$

to compute $H^q(\mathbb{P}_1, \mathcal{O}(k))$. The computation consists of expanding elements of $\Gamma(V_0, \mathcal{O}(k)), \Gamma(V_1, \mathcal{O}(k))$, and $\Gamma(V_0 \cap V_1, \mathcal{O}(k))$ in Laurent series and comparing coefficients (cf. Griffiths and Adams [16]). Now since $U' \simeq U \times \mathbb{P}_1$ we may cover U' by two Stein subsets

$$U'_1 \cong U \times V_0$$
$$U'_1 \cong U \times V_1$$

and proceed to compute $H^q(U', \mathcal{O}(k))$ in terms of this Leray covering by exactly the same technique of Laurent series expansion. The only difference is that the coefficients of the expansions will be holomorphic functions on U. This follows since the coefficients may be defined by means of integral formulae on $\mathbb{P}_1$. Thus we obtain easily

$$H^q(U, \mathcal{O}(k)) = \{\text{holomorphic functions} : U \longrightarrow H^q(\mathbb{P}_1, \mathcal{O}\ (k))\}. \tag{2.6}$$

The vanishing statements in the proposition now follow from (2.6) and the well-known vanishing theorem for $\mathbb{P}_1$ (cf. Wells [51])

$$\begin{aligned} H^q(\mathbb{P}_1, \mathcal{O}(n)) &= 0 \quad \text{for } n \geqq 0, \quad q \geqq 1 \\ H^q(\mathbb{P}_1, \mathcal{O}(-1)) &= 0 \quad \text{for all } q \\ H^q(\mathbb{P}_1, \mathcal{O}(-n-2)) &= 0 \quad \text{for } n \geqq 0, \quad q \neq 1\,. \end{aligned} \tag{2.7}$$

Alternatively, these statements follow from the Laurent series expansion argument. It is also clear from (2.6) that the direct images $\nu^q_* \mathcal{O}(k)$ in general represent vector bundles whose fibre at $z \in \mathbb{M}$ may be canonically identified with $H^q(\nu^{-1}(z), \mathcal{O}_{\nu^{-1}(z)}(k))$. The problem is to see how these fibres fit together. The simplest case is a), to identify $\nu_* \mathcal{O}(n)$. Clearly $\nu_* \mathcal{O} = \mathcal{O}$ because the fibres of ν are compact so admit only constant holomorphic functions. Thus, let us consider $\nu_* \mathcal{O}(1)$. For any point

8 See Gunning and Rossi [18] and Hörmander [20] for the theory of holomorphic functions on open complex manifolds, in particular Stein manifolds

$z \in \mathbb{M}$ the fibre $\nu^{-1}(z)$ is by definition the projective space $\mathbb{P}(\mathbb{C}_{A'})$, where $\mathbb{C}_{A'}$ is the fibre over z of the primed spin-bundle $\mathcal{O}_{A'}$. Now, $\mathcal{O}_{\mathbb{P}(\mathbb{C}_{A'})}(1)$ may be defined invariantly as the line-bundle which associates to every line in $\mathbb{C}_{A'}$ (i.e. to every point in $\mathbb{P}(\mathbb{C}_{A'})$) its dual (as a complex vector space). Every element of $\mathbb{C}^{A'}$, the dual of $\mathbb{C}_{A'}$, may be restricted to the lines in $\mathbb{C}_{A'}$ thus giving a section of $\mathcal{O}_{\mathbb{P}(\mathbb{C}_{A'})}(1)$. Thus, we have a canonical homomorphism

$$\mathbb{C}^{A'} \longrightarrow \Gamma(\mathbb{P}(\mathbb{C}_{A'}), \mathcal{O}_{\mathbb{P}(\mathbb{C}_{A'})}(1))$$

and the Laurent series argument shows easily that this is an isomorphism. Hence, we can identify $H^0(\nu^{-1}(z), \mathcal{O}_{\nu^{-1}(z)}(1))$ with $\mathbb{C}^{A'}$, the fibre of $\mathcal{O}^{A'}$ over z, in a completely coordinate free manner. It follows that $\nu_*\mathcal{O}(1) \cong \mathcal{O}^{A'}$ as required. The general result that $\nu_*\mathcal{O}(n) \cong \mathcal{O}^{(A'B'\cdots D')}$ follows in the same sort of way. Now we come to case c), to identify $\nu^1_*\mathcal{O}(-n-2)$. This is not quite so simple as the case $\nu^1_*\mathcal{O}(-2)$ already shows. The problem is that, although $H^1(\mathbb{P}_1, \mathcal{O}(-2))$ is isomorphic to $\mathbb{C}$, this isomorphism is not canonical but depends on a choice of coordinates for $\mathbb{P}_1$, for example. What is true is that there is a canonical isomorphism $H^1(\mathbb{P}_1, \Omega^1) \cong \mathbb{C}$ (given by integration: Serre duality [43]) and a noncanonical isomorphism: $\mathcal{O}(-2) \simeq \Omega^1$. The isomorphism of Serre duality may be given as follows. Identifying $\mathbb{P}_1$ with $\mathbb{P}(\mathbb{C}_{A'})$ as before we have the exact sequence

$$\mathbb{P}_1;\ 0 \longrightarrow \Omega^1 \longrightarrow \mathcal{O}^{A'}(-1) \xrightarrow{\pi_{A'}} \mathcal{O} \longrightarrow 0 \tag{2.8}$$

where $\pi_{A'}: \mathcal{O}^{A'} \to \mathcal{O}(1)$ is defined as the dual of ${}^t\pi_{A'}: \mathcal{O}(-1) \to \mathcal{O}_{A'}$ whose definition is coordinate free (as $\mathcal{O}(-1)$ is the universal bundle which, by definition, is a sub-bundle of $\mathcal{O}_{A'}$, the trivial bundle with fibre $\mathbb{C}_{A'}$). Indeed (2.8) may be taken as defining Ω^1. The long exact sequence on cohomology which arises from (2.8) gives

$$\begin{array}{ccccccc} \Gamma(\mathbb{P}_1, \mathcal{O}^{A'}(-1)) & \longrightarrow & \Gamma(\mathbb{P}_1, \mathcal{O}) & \longrightarrow & H^1(\mathbb{P}_1, \Omega^1) & \longrightarrow & H^1(\mathbb{P}_1, \mathcal{O}^{A'}(-1)) \\ \| & & & & & & \| \\ 0\ (\text{by } (2.7)) & & & & & & 0\ (\text{by } (2.7)) \end{array}$$

which shows $H^1(\mathbb{P}_1, \Omega^1)$ is isomorphic to $\Gamma(\mathbb{P}_1, \mathcal{O})$. But now $\Gamma(\mathbb{P}_1, \mathcal{O})$ is canonically isomorphic to $\mathbb{C}$ by evaluation at any point. To mimic this proof for $\mathcal{O}(-2)$ we use instead of (2.8) the exact sequence (isomorphic to (2.8), but not canonically):

$$\mathbb{P}_1;\ 0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\pi_{A'}} \mathcal{O}_{A'}(-1) \xrightarrow{\pi_{B'}} \mathcal{O}_{[A'B']} \longrightarrow 0. \tag{2.9}$$

This shows that $H^1(\nu^{-1}(z), \mathcal{O}_{\nu^{-1}(z)}(-2))$ is canonically isomorphic to $\mathcal{O}_{[A'B']}$ and thus $\nu^1_*\mathcal{O}(-2) \cong \mathcal{O}_{[A'B']} = \mathcal{O}[-1]'$, as required. Actually, we could deduce directly from (2.9), considered as a sequence on $\mathbb{F}$, that $\nu^1_*\mathcal{O}(-2) \cong \nu_*\mathcal{O}_{[A'B']} \cong \mathcal{O}_{[A'B']}$ just by using the long exact sequence of direct images [45]. This would avoid explicit use of (2.6). To identify $\nu^1_*\mathcal{O}(-n-2)$ we replace (2.9) with

$$\mathbb{P}_1;\ 0 \longrightarrow \mathcal{O}(-n-2) \xrightarrow{\pi_{A'}\cdots\pi_{E'}} \mathcal{O}_{(A'B'\cdots D'E')}(-1) \xrightarrow{\pi_{F'}} \mathcal{O}_{(A'B'\cdots D')[E'F']} \longrightarrow 0 \tag{2.10}$$

to conclude $\nu^1_*\mathcal{O}(-n-2) = \mathcal{O}_{(A'B'\cdots D')[E'F']} = \mathcal{O}_{(A'B'\ldots D')}[-1]'$. □

In the proof of the above proposition we showed that if U is an open Stein subset of $\mathbb{M}$ then $H^q(U', \mathcal{O}(k))$ is isomorphic to $\Gamma(U, \nu^q_* \mathcal{O}(k))$. More generally we have:

Proposition 2.2. *Let $U \subseteq \mathbb{M}$ be any open set. Then, for all p,*

a) if $n \geqq 0: H^p(U', \mathcal{O}(n)) \cong H^p(U, \mathcal{O}^{(A'B' \cdots D')})$
b) $H^p(U', \mathcal{O}(-1)) = 0$
c) if $n \geqq 0: H^p(U', \mathcal{O}(-n-2)) \cong H^{p-1}(U, \mathcal{O}_{(A'B' \ldots D')}[-1]')$.

Proof. These statements follow immediately from the Leray spectral sequence [14] and Proposition 2.1 which shows that $E_2^{p,q} = E_\infty^{p,q}$. Alternatively one can argue directly using Čech cohomology and a Stein cover of U. □

From Proposition 2.1 and Lemma 1.2 it follows immediately that

$$\begin{aligned}
&\nu_* \Omega^0_\mu(n) \cong \nu_* \mathcal{O}(n) \cong \mathcal{O}^{(A'B' \cdots D')} \quad (n \text{ indices}) \quad \text{if } n \geqq 0 \\
&\nu_* \Omega^1_\mu(n) \cong \nu_* \mathcal{O}_A(n+1)[-1]' \cong \mathcal{O}_A^{(A'B' \cdots E')}[-1]' \quad (n+1 \text{ indices}) \quad \text{if } n \geqq -1 \\
&\nu_* \Omega^2_\mu(n) \cong \nu_* \mathcal{O}(n+2)[-1][-2]' \cong \mathcal{O}^{(A'B' \cdots F')}[-1][-2]' \quad (n+2 \text{ indices}) \\
\mathbb{M};\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \geqq -2 \\
&\nu^1_* \Omega^0_\mu(-n-2) \cong \nu^1_* \mathcal{O}(-n-2) \cong \mathcal{O}_{(A'B' \cdots D')}[-1]' \quad (n \text{ indices}) \quad \text{if } n \geqq 0 \\
&\nu^1_* \Omega^1_\mu(-n-2) \cong \nu^1_* \mathcal{O}_A(-n-1)[-1]' \cong \mathcal{O}_{A(B' \cdots D')}[-2]' \quad (n-1 \text{ indices}) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \geqq 1 \\
&\nu^1_* \Omega^2_\mu(-n-2) \cong \nu^1_* \mathcal{O}(-n)[-1][-2]' \cong \mathcal{O}_{(C' \cdots D')}[-1][-3]' \quad (n-2 \text{ indices}) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \geqq 2
\end{aligned} \tag{2.11}$$

and all other possibilities vanish. The obvious analogue of Proposition 2.2 holds. Our next task is to interpret the effect of the differential operators d_μ on direct images. For ν_* we can argue as follows. Let $f^{A'B' \cdots D'}$ be a local section of $\mathcal{O}^{(A'B' \cdots D')}$. The differential operator

$$\mathbb{F};\ \pi_{E'} \nabla_A^{E'} : \mathcal{O}(n) \longrightarrow \mathcal{O}_A(n+1)[-1]'$$

gives

$$\pi_{A'} \pi_{B'} \ldots \pi_{D'} f^{A'B' \cdots D'} \longmapsto \pi_{A'} \pi_{B'} \ldots \pi_{D'} \pi_{E'} \nabla_A^{E'} f^{A'B' \cdots D'}$$

which induces on $\nu_* \mathcal{O}(n) = \mathcal{O}^{(A'B' \cdots D')}$

$$\nabla_A^{E'} : f^{A' \cdots D'} \longmapsto \nabla_A^{(E'} f^{A'B' \cdots D')}.$$

Hence, Lemma 1.2 and (2.11) give

$$\begin{array}{ccccc}
\nu_* \Omega^0_\mu(n) & \xrightarrow{d_\mu} & \nu_* \Omega^1_\mu(n) & \xrightarrow{d_\mu} & \nu_* \Omega^2_\mu(n) \\
\wr\| & & \wr\| & & \wr\| \\
\mathbb{M};\ \mathcal{O}^{(A'B' \cdots D')} & \longrightarrow & \mathcal{O}_A^{(A'B' \cdots E')}[-1]' & \longrightarrow & \mathcal{O}^{(A'B' \cdots F')}[-1][-2]' \\
\cup & & \cup \qquad \cup & & \cup \\
f^{A'B' \cdots D'} & \longmapsto & \nabla_A^{(E'} f^{A'B' \cdots D')},\ g_A^{A'B' \cdots E'} & \longmapsto & \nabla^{A(F'} g_A^{A'B' \cdots E')}
\end{array} \tag{2.12}$$

Notice that this shows automatically that the operator $\nabla_A^{E'}$ from $\mathcal{O}^{(A'B' \cdots D')}$ to $\mathcal{O}_A^{(A'B' \cdots E')}[-1]'$ is well-defined. To compute the effect of d_μ on the first direct images we can use (2.10) and write locally, after choosing coordinates $\pi_{A'}$:

$$
\mathbb{F};\quad
\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{O}(-n-2) & \xrightarrow{\pi_{A'}\ldots\pi_{F'}} & \mathcal{O}_{(A'B'\cdots D'E')}(-1) & \xrightarrow{\pi^{E'}\varepsilon_{F'G'}} & \mathcal{O}_{(A'B'\cdots D')[F'G']} & \longrightarrow 0 \\
& \downarrow{\scriptstyle \pi_{A'}\nabla_A^{A'}} & & \downarrow{\scriptstyle \nabla_A^{A'}} & & \downarrow{\scriptstyle \nabla_A^{A'}} & \\
0 \longrightarrow & \mathcal{O}_A(-n-1)[-1]' & \xrightarrow{\pi_{B'}\ldots\pi_{E'}} & \mathcal{O}_{A(B'\cdots D'E')}(-1)[-1]' & \xrightarrow{\pi^{E'}\varepsilon_{F'G'}} & \mathcal{O}_{A(B'\cdots D')[F'G']}[-1]' & \longrightarrow 0
\end{array}
$$

Hence

$$
\mathbb{M};\quad
\begin{array}{ccccccc}
\nu_*^1\mathcal{O}(-n-2) & \cong & \nu_*\mathcal{O}_{(A'B'\cdots D')[E'F']} & = & \mathcal{O}_{(A'B'\cdots D')[E'F']} & = & \mathcal{O}_{(A'B'\cdots D')}[-1]' \\
\downarrow{\scriptstyle \nu_*^1(\pi_{A'}\nabla_A^{A'})} & & \downarrow{\scriptstyle \nu_*(\nabla_A^{A'})} & & \downarrow{\scriptstyle \nabla_A^{A'}} & & \downarrow{\scriptstyle \nabla_{A'}^{A}} \\
\nu_*^1\mathcal{O}_A(-n-1)[-1]' & \cong & \nu_*\mathcal{O}_{A(B'\cdots D')[E'F']}[-1]' & = & \mathcal{O}_{A(B'\cdots D')[E'F']}[-1]' & \cong & \mathcal{O}_{A(B'\cdots D')}[-2]
\end{array}
$$

and so the analogue of (2.12) for ν_*^1 is

$$
\begin{array}{lcccc}
& \nu_*^1\Omega_\mu^0(-n-2) & \xrightarrow{d_\mu} & \nu_*^1\Omega_\mu^1(-n-2) & \xrightarrow{d_\mu} & \nu_*^1\Omega_\mu^2(-n-2) \\
& \wr\Vert & & \wr\Vert & & \wr\Vert \\
\mathbb{M}; & \mathcal{O}_{(A'B'\cdots D')}[-1]' & \longrightarrow & \mathcal{O}_{A(B'\cdots D')}[-2]' & \longrightarrow & \mathcal{O}_{(C'\cdots D')}[-1][-3]' \\
& \in & & \in\quad\ni & & \in \\
& \phi_{A'B'\cdots D'} & \longmapsto & \nabla_A^{A'}\phi_{A'B'\cdots D'},\ \psi_{AB'C'\cdots D'} & \longmapsto & \nabla^{AB'}\psi_{AB'C'\cdots D'}.
\end{array}
\tag{2.13}
$$

We emphasize that the formulae of (2.12) and (2.13) arise quite naturally just from the basic twistor geometry of (1.1). These formulae give rise to massless fields as we shall see. In other words, the massless field equations are implicitly generated by the twistor geometry. This would seem to be a significant simplification.

3. The Topology of the Mapping μ

We want to be able to interpret analytic information on subsets of $\mathbb{P}$ (for instance, certain cohomology groups) in terms of similar objects defined on subsets of $\mathbb{F}$. To do this we need to investigate some further properties of $\mu: \mathbb{F} \to \mathbb{P}$.

Let $\tilde{W}$ be an open subset of $\mathbb{F}$ and $W = \mu(\tilde{W})$. W will also be open. We want to study the behaviour of sheaves and cohomology on W and $\tilde{W}$, determining conditions on $\tilde{W}$ so that we can transform, without loss of information, data on W to data on $\tilde{W}$. If $\mathscr{S}$ is a sheaf on W then we may form the *topological inverse image sheaf* $\mu^{-1}\mathscr{S}$ on $\tilde{W}$. It may be defined by the presheaf $V \mapsto \Gamma(\mu(V), \mathscr{S})$ for V open in $\tilde{W}$ with the obvious restriction mappings, and it is characterized by the requirement that $(\mu^{-1}\mathscr{S})_q \cong \mathscr{S}_{\mu(q)}$ for all $q \in \tilde{W}$. This isomorphism gives rise to a map

$$\mu^* : \Gamma(W, \mathscr{S}) \longrightarrow \Gamma(\tilde{W}, \mu^{-1}\mathscr{S}),$$

defined by $\mu^* f = f \circ \mu$ (where, by abuse of notation, we have identified $(\mu^{-1}\mathscr{S})_q$ with $\mathscr{S}_{\mu(q)}$). The sheaf $\mu^{-1}\mathcal{O}(V)$ introduced in Sect. 1 is an example of this construction. The pull-back μ^* may be extended to cohomology in the following manner (cf. Godement [14]). We may calculate $H^p(W, \mathscr{S})$ by choosing a suitable resolution of $\mathscr{S}$, i.e. an exact sequence

$$W;\ 0 \longrightarrow \mathscr{S} \xrightarrow{\ \delta\ } \mathscr{R}^0 \xrightarrow{\ \delta\ } \mathscr{R}^1 \xrightarrow{\ \delta\ } \mathscr{R}^2 \xrightarrow{\ \delta\ } \ldots$$

which we'll abbreviate to

$$W;\ 0 \longrightarrow \mathscr{S} \longrightarrow \mathscr{R}^{\boldsymbol{\cdot}}, \tag{3.1}$$

where "suitable" can be taken to mean injective, flabby, fine or soft (often, in practice, a resolution by sheaves of smooth differential forms). Then the abstract deRham theorem (cf. Wells [51]) says that there is a canonical isomorphism

$$\begin{aligned} H^p(W,\mathscr{S}) &= H^p(\Gamma(W,\mathscr{R}^{\cdot})) \\ &:= \frac{\ker\delta:\Gamma(W,\mathscr{R}^p)\longrightarrow\Gamma(W,\mathscr{R}^{p+1})}{\operatorname{im}\delta:\Gamma(W,\mathscr{R}^{p-1})\longrightarrow\Gamma(W,\mathscr{R}^p)} \end{aligned} \tag{3.2}$$

The resolution of $\mathscr{S}$ in (3.1) gives rise to a resolution

$$W;\ 0\longrightarrow\mu^{-1}\mathscr{S}\longrightarrow\mu^{-1}\mathscr{R}^{\cdot}$$

of $\mu^{-1}\mathscr{S}$. By writing this as a collection of short exact sequences and looking at the corresponding long exact sequences on cohomology (or by using some universal property of resolutions) it may easily be seen that there is a canonical homomorphism

$$H^p(\Gamma(\tilde{W},\mu^{-1}\mathscr{R}^{\cdot}))\longrightarrow H^p(\tilde{W},\mu^{-1}\mathscr{S}). \tag{3.3}$$

The mapping μ^* on sections gives rise to a map of complexes

$$\mu^*:\Gamma(W,\mathscr{R}^{\cdot})\longrightarrow\Gamma(\tilde{W},\mu^{-1}\mathscr{R}^{\cdot})$$

and hence a map on cohomology which we can combine with (3.2) and (3.3) to define μ^* on $H^p(W,\mathscr{S})$ by means of the commutative diagram

$$\begin{array}{ccc} H^p(W,\mathscr{S}) & \xrightarrow{\mu^*} & H^p(\tilde{W},\mu^{-1}\mathscr{S}) \\ \| & & \uparrow \\ H^p(\Gamma(W,\mathscr{R}^{\cdot})) & \longrightarrow & H^p(\Gamma(\tilde{W},\mu^{-1}\mathscr{R}^{\cdot}). \end{array}$$

It is easy to check that this construction is independent of the choice of resolution, $\mathscr{R}^{\cdot}$. It is also not difficult to see how to define μ^* naturally in terms of Čech theory.

If $\mu:\tilde{W}\to W$ has connected fibres, it is clear that

$$\Gamma(W,\mathscr{S})=\Gamma(\tilde{W},\mu^{-1}\mathscr{S}).$$

The situation is not so simple for the higher cohomology,

$$\mu^*:H^p(W,\mathscr{S})\longrightarrow H^p(\tilde{W},\mu^{-1}\mathscr{S}),\quad p\geqq 1. \tag{3.4}$$

As it turns out, the higher order topology of the fibres of μ must enter into the discussion. The basic condition we shall use is that the mapping

$$\mu:\tilde{W}\longrightarrow W$$

have fibres which are connected and have vanishing first Betti number. In other words, for all $Z\in W$, $\mu^{-1}(Z)$ is connected and $H^1(\mu^{-1}(Z),\mathbb{C})=0$. We shall refer to such a mapping by saying it is *elementary*.

Remark. If $U\subseteq\mathbb{M}$ and U', U'' are as in Sect. 1 the fibre $\mu^{-1}(Z)$ of $\mu:U'\to U''$ is biholomorphic to $\hat{Z}\cap U$ in $\mathbb{M}$. Here $\hat{Z}$ is the α-plane determined by the (projective) twistor $Z\in\mathbb{P}$ (see Sect. 1). Our principal interest later will be where $W=U''$

and $\tilde{W} = U'$ so this observation may be used to check whether $\mu: \tilde{W} \to W$ is elementary.

Returning to the question of the nature of the mapping (3.4), we consider the special case of $\mathscr{S} = \mathcal{O}(V)$ for some holomorphic vector-bundle V on W. We may define μ^* by using the Dolbeault resolution

$$W;\ 0 \longrightarrow \mathcal{O}(V) \longrightarrow \mathscr{E}^{0,0}(V) \xrightarrow{\bar\partial} \mathscr{E}^{0,1}(V) \xrightarrow{\bar\partial} \mathscr{E}^{0,2}(V) \xrightarrow{\bar\partial} \dots$$

where $\mathscr{E}^{p,q}(V)$ denotes the sheaf of forms of type (p,q) with coefficients in V (cf. [51] or [43]). If V is a smooth (i.e. C^∞) vector-bundle we let $\mathscr{E}(V)$ be the sheaf of smooth sections of V (i.e. $\mathscr{E}(V) = \mathscr{E}^{0,0}(V)$).

Lemma 3.1. *Suppose that $\mu: \tilde{W} \to W$ is elementary, and that V is a smooth vector bundle over W. Then*

$$H^1(\tilde{W}, \mu^{-1}\mathscr{E}(V)) = 0.$$

Proof. By analogy with the holomorphic case of Sect. 1 we may define a sheaf of smooth relative p-forms $\mathscr{E}^p_\mu$ on $\tilde{W}$ and differential operators d_μ so that we obtain a fine resolution

$$\tilde{W};\ 0 \longrightarrow \mu^{-1}\mathscr{E}(V) \longrightarrow \mathscr{E}^0_\mu(V) \xrightarrow{d_\mu} \mathscr{E}^1_\mu(V) \xrightarrow{d_\mu} \mathscr{E}^2_\mu(V) \xrightarrow{d_\mu} \mathscr{E}^3_\mu(V) \xrightarrow{d_\mu} \mathscr{E}^4_\mu(V) \longrightarrow 0 \quad (3.5)$$

(cf. (1.5)). Suppose that $Z \in W$. We want to show that there is a neighbourhood U_Z of Z so that $H^1(\mu^{-1}(U_Z) \cap \tilde{W}, \mu^{-1}\mathscr{E}(V)) = 0$. First, we may choose U_Z so that $\mu^{-1}(U_Z)$ is trivialized:

$$\begin{tikzcd} \mathbb{F} \supseteq \mu^{-1}(U_Z) \arrow[d, "\mu"] \arrow[r, phantom, "\simeq"] & U_Z \times \mathbb{P}_2 \arrow[d] \\ \mathbb{P} \supseteq U_Z & U_Z \end{tikzcd}$$

Next, we choose any $s_0 \in \mathbb{P}_2$ such that $(Z, s_0) \in \tilde{W}$ (where, by abuse of notation we have identified $\mu^{-1}(U_Z)$ and $U_Z \times \mathbb{P}_2$) and shrink U_Z, if necessary, to ensure that $X \in U_Z$ implies $(X, s_0) \in \tilde{W}$. We may also suppose that V is trivial on $\tilde{U}_Z$, where $\tilde{U}_Z$ denotes $\mu^{-1}(U_Z) \cap \tilde{W}$. We have

$$H^1(\tilde{U}_z, \mu^{-1}\mathscr{E}(V)) \cong H^1(\tilde{U}_z, \mu^{-1}\mathscr{E})^r,$$

where r is the rank of V and $\mathscr{E}$ is the sheaf of smooth functions on $\mathbb{P}$. As a special case of (3.5) we see that $H^1(\tilde{U}_z, \mu^{-1}\mathscr{E}) = 0$ if and only if

$$\Gamma(\tilde{U}_z, \mathscr{E}^0_\mu) \xrightarrow{d_\mu} \Gamma(\tilde{U}_z, \mathscr{E}^1_\mu) \xrightarrow{d_\mu} \Gamma(\tilde{U}_z, \mathscr{E}^2_\mu)$$

is exact. So suppose $\omega \in \Gamma(\tilde{U}_z, \mathscr{E}^1_\mu)$ with $d_\mu \omega = 0$. Then ω may be considered as a family of smooth closed 1-forms $\omega(X)$ in the s-variables ($s \in \mathbb{P}_2$) smoothly parametrized by $X \in U_z$. Thus, we may define $f \in \Gamma(\tilde{U}_z, \mathscr{E}^0_\mu)$ by means of the integral formula

$$f(X, s) = \int_{(X, s_0)}^{(X, s)} \omega(X)$$

where the integral is taken over any path in $\mu^{-1}(X)\cap\tilde{W}$ joining (X, s_0) to (X, s). This is possible and independent of the choice of path since $\mu:\tilde{W}\to W$ is elementary. By construction $d_\mu f=\omega$ so $H^1(\tilde{U}_z, \mu^{-1}\mathscr{E})=0$. Hence $H^1(\tilde{U}_z, \mu^{-1}\mathscr{E}(V))=0$ i.e.

$$\Gamma(\tilde{U}_z, \mathscr{E}^0_\mu(V)) \xrightarrow{d_\mu} \Gamma(\tilde{U}_z, \mathscr{E}^1_\mu(V)) \xrightarrow{d_\mu} \Gamma(\tilde{U}_z, \mathscr{E}^2_\mu(V)) \tag{3.6}$$

is exact. Choose a covering of W by sets U_z of the above form and a partition of unity subordinate to this covering. When pulled back to $\tilde{W}$ this partition of unity is constant along the fibres of μ, i.e. is annihilated by d_μ. It may therefore be used to patch solutions to the equation "(3.6) is exact" to obtain the exactness of

$$\Gamma(\tilde{W}, \mathscr{E}^0_\mu(V)) \xrightarrow{d_\mu} \Gamma(\tilde{W}, \mathscr{E}^1_\mu(V)) \xrightarrow{d_\mu} \Gamma(\tilde{W}, \mathscr{E}^2_\mu(V))$$

or, in other words, (by (3.5)) $H^1(\tilde{W}, \mu^{-1}\mathscr{E}(V))=0$, as desired. □

Remark. If $W=U''$, $\tilde{W}=U'$, where $U\subset\mathbb{M}$ is Stein, then the above argument can be used directly on the sequence (1.6) to prove the following theorem in the special case $V=H^n, n\geqq -1$. For $n=0$ (i.e. V is the trivial bundle) the proof is essentially given in Ward [47].

Theorem 3.2. *Suppose $\tilde{W}\subseteq\mathbb{F}$ is an open set and let W denote $\mu(\tilde{W})$. If $\mu:\tilde{W}\to W$ is elementary then, for any holomorphic vector-bundle V on W,*

$$\mu^*: H^1(W, \mathcal{O}(V)) \longrightarrow H^1(\tilde{W}, \mu^{-1}\mathcal{O}(V))$$

is a canonical isomorphism.

Proof. The sheaves $\mathscr{E}^{p,q}(V)$ are of the form $\mathscr{E}(\tilde{V})$ for $\tilde{V}=\Lambda^{p,q}T^*(W)\otimes V$, where $\Lambda^{p,q}T^*(W)$ is the smooth bundle whose sections are forms of type (p,q). Thus by Lemma 3.1 $H^1(\tilde{W}, \mu^{-1}\mathscr{E}^{p,q}(V))=0$ for all p,q. Actually we only need $H^1(\tilde{W}, \mu^{-1}\mathscr{E}(V))=0$ to conclude by simple diagram chasing that the canonical map (3.3)

$$H^1(\Gamma(\tilde{W}, \mu^{-1}\mathscr{E}^{0,\cdot}(V)) \longrightarrow H^1(\tilde{W}, \mu^{-1}\mathcal{O}(V))$$

is an isomorphism (it's always injective). Now, $\mu:\tilde{W}\to W$ has connected fibres so

$$\Gamma(\tilde{W}, \mu^{-1}\mathscr{E}^{0,q}(V))=\Gamma(W, \mathscr{E}^{0,q}(V)).$$

Hence, μ^* is the composition of three isomorphisms

$$\begin{aligned} H^1(W, \mathcal{O}(V)) &\xrightarrow{\cong} H^1(\Gamma(W, \mathscr{E}^{0,\cdot}(V))) \\ &\xrightarrow{\cong} H^1(\Gamma(\tilde{W}, \mu^{-1}\mathscr{E}^{0,\cdot}(V))) \xrightarrow{\cong} H^1(\tilde{W}, \mu^{-1}\mathcal{O}(V)) \end{aligned}$$

and the result follows. □

The above theorem indicates how the topology of the fibres of μ affects the properties of μ^*. So far we have:

A. $\mu^*:\Gamma(W, \mathcal{O}(V)) \longrightarrow \Gamma(W, \mu^{-1}\mathcal{O}(V))$ is always injective.

B. If $\mu:\tilde{W} \longrightarrow W$ has connected fibres then
$\mu^*:\Gamma(W, \mathcal{O}(V)) \longrightarrow \Gamma(\tilde{W}, \mu^{-1}\mathcal{O}(V))$ is an isomorphism and
$\mu^*:H^1(W, \mathcal{O}(V)) \longrightarrow H^1(\tilde{W}, \mu^{-1}\mathcal{O}(V))$ is injective.

C. If, in addition, the first Betti number of each fibre of $\mu:\tilde{W} \longrightarrow W$ vanishes (in other words, $\mu:\tilde{W} \longrightarrow W$ is elementary), then $\mu^*:H^1(W,\mathcal{O}(V)) \longrightarrow H^1(\tilde{W},\mu^{-1}\mathcal{O}(V))$ is an isomorphism and $\mu^*:H^2(W,\mathcal{O}(V)) \longrightarrow H^2(\tilde{W},\mu^{-1}\mathcal{O}(V))$ is injective (this last assertion following easily from Lemma 3.1 by a similar argument to the proof of Theorem 3.2). We expect this to be part of a general pattern but a proof of the general case is lacking*. It is, however, fairly straightforward to show that if all of the fibres of $\mu:\tilde{W} \to W$ are convex, then $\tilde{W}$ is homeomorphic to a product $W \times$ fibre, and one can deduce via a higher degree version of Lemma 3.1 that

$$\mu^*:H^p(W,\mathcal{O}(V)) \xrightarrow{\cong} H^p(\tilde{W},\mu^{-1}\mathcal{O}(V))$$

for all p. An alternative proof of this in the case $W = \mathbb{P}^+$, $\tilde{W} = \mathbb{F}^+$ is given in Eastwood [9]. For the applications in this paper, however, we shall be concerned primarily with the action of μ^* on H^1 so Theorem 3.2 will suffice.

4. The General $\mathscr{P}$-Transform

The aim of this section is to finish constructing a general analytical machine which transforms elements of analytic cohomology groups $H^p(U'',\mathcal{O}(V))$ into solutions of certain differential equations on U, an open subset of $\mathbb{M}$. The complete process will be called the *$\mathscr{P}$-transform*[9]. The pieces of this transform already constructed include the isomorphism.

$$\mu^*:H^1(U'',\mathcal{O}(V)) \xrightarrow{\cong} H^1(U',\mu^{-1}\mathcal{O}(V)) \tag{4.1}$$

when $\mu:U' \to U''$ is elementary (Sect. 3), and the canonical direct image isomorphisms (for instance):

$$\begin{array}{lccc}
\text{a)} & H^1(U',\Omega^0_\mu(-n-2)) & \xrightarrow{d_\mu} & H^1(U',\Omega^1_\mu(-n-2)) \\
& \wr\Vert & & \wr\Vert \\
\text{b)} & \Gamma(U,\mathcal{O}_{(A'B'\cdots D')}[-1]') & \xrightarrow{\nabla^{A'}_{A}} & \Gamma(U,\mathcal{O}_{A(B'\cdots D')}[-2]')
\end{array} \tag{4.2}$$

discussed in Sect. 2. What is missing in this case is a relation between $H^1(U',\mu^{-1}\mathcal{O}(-n-2))$ and $H^1(U',\Omega^p_\mu(-n-2))$. This will be a special case of what we will establish in this section (Theorem 4.1). If $n \geqq 1$, it's not difficult to show that

$$H^1(U',\mu^{-1}\mathcal{O}(-n-2) \cong \ker d_\mu \text{ in } (4.2\text{a}) \tag{4.3}$$

---

* *Note added in proof.* The general case has recently been proved by N. P. Buchdahl

9 The transform we wish to describe has been referred to elsewhere (e.g. in Wells [53]) as the *Penrose transform*. It has also been referred to (e.g. in Rawnsley [42]) as the *twistor transform*. We prefer to avoid the term twistor transform in this context since it is already in common use as a name for the isomorphism

$$\mathscr{T}:H^1(\mathbb{P}^+,\mathcal{O}(-n-2) \xrightarrow{\cong} H^1(\mathbb{P}^{*-},\mathcal{O}(n-2))$$

(cf. Penrose-MacCallum [32], where this is expressed in pre-cohomological language) and thus we have the integral transform (assuming $\mu: U' \to U''$ is elementary)

$$\mathscr{P}: H^1(U'', \mathcal{O}(-n-2)) \xrightarrow{\cong} \ker \nabla_A^{A'} \text{ in (4.2b)} \tag{4.4}$$

The details are in Sect. 5 but this is the basic idea. The general form of (4.3) will be a *spectral sequence* (cf. Eastwood [9], in case $U = \mathbb{M}^+$) linking $H^r(U', \mu^{-1}\mathcal{O}(V))$ and $H^q(U', \Omega_\mu^p(V))$. Depending on the geometry and the bundles involved the spectral sequence will give the right relationships between these cohomology groups (e.g. (4.3) above). In the remaining sections of this paper we shall work out various special cases of the $\mathscr{P}$-transform. In this section we will only formulate the general method.

In general terms the transform works as follows. An element ω in $H^p(U'', \mathcal{O}(V))$ is first pulled-back to $\mu^*\omega$ in $H^p(U', \mu^{-1}\mathcal{O}(V))$. We then restrict the cohomology class $\mu^*\omega$ to a fibre $\nu^{-1}(z)$ for $z \in U$, giving a "value" in the stalk of an appropriate direct-image sheaf. As z varies this is interpreted as a spinor-field. Thus,

$$(\mathscr{P}\omega)(z) = \omega \text{ "evaluated" on } L_z .$$

Indeed, the original integral formulae are of this type without explicit reference to the intermediate space $U' \subseteq \mathbb{F}$. However, it turns out that to break the transform into two steps via $\mathbb{F}$ is a good way of being able the analyze the analytical behaviour of $\mathscr{P}$.

We now have the principal result of this section:

Theorem 4.1. *Let X be an open subset of $\mathbb{F}$. Then there is a spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega_\mu^p(V)) \Longrightarrow H^{p+q}(X, \mu^{-1}\mathcal{O}(V))$$

where the differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ are induced by $d_\mu : \Omega_\mu^p(V) \to \Omega_\mu^{p+1}(V)$,

Proof. The exact sequence (1.5):

$$\tilde{W};\ 0 \longrightarrow \mu^{-1}\mathcal{O}(V) \longrightarrow \Omega_\mu^0(V) \longrightarrow \Omega_\mu^1(V) \longrightarrow \Omega_\mu^2(V) \longrightarrow 0$$

provides a resolution of $\mu^{-1}\mathcal{O}(V)$. The spectral sequence called for in the theorem therefore follows a standard construction (for a "differential sheaf" in a special case cf. Bredon [5]). □

Remark. One way of constructing the spectral sequence in Theorem 4.1 is to write (1.5) as a pair of short exact sequences, consider the resulting long exact sequences on cohomology, and do some elementary diagram chasing. The same diagram chasing can, of course, be used to prove directly anything which can be deduced from the spectral sequence. The spectral sequence terminology does produce, however, a more illuminating and unifying way of viewing the results of the next few sections, as well as a means of keeping track of sometimes complicated information.

5. The Twistor Description of Right-handed Fields

Let $\mathcal{O}_{(A'B'\cdots D')}[-1]'$ be the spinor sheaf on $\mathbb{M}$ with $n(\geqq 1)$ indices as defined in Sect. 2. Let

$$\mathbb{M};\ \mathscr{Z}_n' := \ker \nabla_A^{A'} : \mathcal{O}_{(A'B'\cdots D')}[-1]' \longrightarrow \mathcal{O}_{A(B'\cdots D')}[-2]'. \tag{5.1$'$}$$

We shall call $\mathscr{Z}'_n$ the sheaf of *holomorphic right-handed massless free fields of helicity* $n/2$. The sections of $\mathscr{Z}'_n$ over an open subset U contained in $\mathbb{M}$,

$$\Gamma(U, \mathscr{Z}'_n) = \{\phi_{A'B'\cdots D'} \in \Gamma(U, \mathcal{O}_{(A'B'\cdots D')}[-1]') : \nabla_A^{A'} \phi_{A'B'\cdots D'} = 0\}$$

are the *holomorphic massless fields on* U *of helicity* $n/2$ (Dirac [6], Fierz [12]). Note that we do not (as is sometimes done) reserve the term "right-handed" (or equivalently "positive helicity") for the case $U = \mathbb{M}^+$. For helicity $= \frac{1}{2}$ these equations are the Dirac–Weyl equations of the anti-neutrino, for helicity $= 1$ we have Maxwell's equations for a self-dual field (a right-handed photon), and for helicity $= 2$ the self-dual linearized Einstein equations. More precisely the sections of $\mathscr{Z}'_n$ are the holomorphic right-handed massless free fields of helicity $n/2$ with *primed conformal weight*. On $\mathbb{M}^I$ these fields coincide with the classical massless fields first introduced by Dirac. We could also envisage fields with unprimed conformal weight, i.e. we could replace (5.1)′ by

$$\mathbb{M}\,;\ \mathscr{Z}_n := \ker \nabla_A^{A'} : \mathcal{O}_{(A'B'\cdots D')}[-1] \longrightarrow \mathcal{O}_{A(B'\cdots D')}[-1][-1]' \qquad (5.1)$$

which also agrees with Dirac's definition for $\mathbb{M}^I$. The only distinction here arises when we consider non-unimodular $GL(\mathbb{T})$ transformations, under which $\mathscr{Z}_n$ and $\mathscr{Z}'_n$ are not *canonically* isomorphic. The physical significance of this is yet unclear, but it turns out that the $\mathscr{P}$-transform acting on $\mathbb{P}$ produce naturally sections of $\mathscr{Z}'_n$. The distinction between $\mathscr{Z}'_n$ and $\mathscr{Z}_n$ would disappear (as would that between primed and unprimed weights generally) if we had been concerned with the action of $SL(\mathbb{T})$ rather than $GL(\mathbb{T})$.

We now have the following theorem which represents right-handed holomorphic massless fields on $\mathbb{M}$ in terms of holomorphic data on $\mathbb{P}$.

Theorem 5.1. *For U open in $\mathbb{M}$ and $n \geqq 1$, there is a canonical linear transformation*

$$\mathscr{P} : H^1(U'', \mathcal{O}(-n-2)) \longrightarrow \Gamma(U, \mathscr{Z}'_n).$$

If $\mu : U' \to U''$ has connected fibres, then $\mathscr{P}$ is injective and if $\mu : U' \to U''$ is elementary, then $\mathscr{P}$ is an isomorphism.

Before we proceed with the proof of this theorem we recall that $\mu : U' \to U''$ has connected fibres if and only if every α-plane in $\mathbb{M}$ intersects U in a connected set and that $\mu : U' \to U''$ is elementary if and only if all these intersections are connected and have vanishing first Betti number.

Proof of Theorem 5.1. We already know from Sect. 3 that

$$\mu^* : H^1(U'', \mathcal{O}(-n-2)) \longrightarrow H^1(U', \mu^{-1}\mathcal{O}(-n-2))$$

is injective or bijective according to the hypotheses given in the theorem. Thus it suffices to show that

$$H^1(U', \mu^{-1}\mathcal{O}(-n-2)) \cong \Gamma(U, \mathscr{Z}'_n)$$

in a natural way. We do this by using the spectral sequence of Theorem 4.1. By the formulae (2.11) (and the analogue of Proposition 2.2) we compute:

$$E_1^{p,0} = H^0(U', \Omega_\mu^p(-n-2)) = 0 \quad \text{for all } p$$

$$E_1^{0,1} = H^1(U', \Omega_\mu^0(-n-2)) \cong \Gamma(U, \mathcal{O}_{(A'B'\cdots D')}[-1]') \quad (n \text{ indices})$$
$$E_1^{1,1} = H^1(U', \Omega_\mu^1(-n-2)) \cong \Gamma(U, \mathcal{O}_{A(B'\cdots D')}[-2]')$$
$$\text{etc.}$$

Thus the E_1-level of the spectral sequence of Theorem 4.1 is isomorphic to

$$\begin{array}{l} \uparrow q \\ H^1(U, \mathcal{O}_{(A'B'\cdots D')}[-1]') \longrightarrow \ldots \\ | \\ \Gamma(U, \mathcal{O}_{(A'B'\cdots D')}[-1]') \xrightarrow{\nabla_A^{A'}} \Gamma(U, \mathcal{O}_{A(B'\cdots D')}[-2]') \xrightarrow{\nabla^{AB'}} \Gamma(U, \mathcal{O}_{(C'\cdots D')}[-1][-3]') \longrightarrow . \\ | \\ 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow p \end{array} \tag{5.2}$$

Thus $E_2^{0,1} \cong \ker \nabla_A^{A'} : \Gamma(U, \mathcal{O}_{(A'B'\cdots D')}[-1]') \longrightarrow \Gamma(U, \mathcal{O}_{A(B'\cdots D')}[-2]')$
$= \Gamma(U, \mathscr{Z}_n')$.

In a general spectral sequence argument we would now have to consider $d_2 : E_2^{0,1} \to E_2^{2,0}$ but $E_2^{1,0} = E_1^{1,0} = 0$ so this is just the zero map. Hence $E_2^{0,1} = E_\infty^{0,1}$. Also $E_1^{1,0} = 0$ so $E_\infty^{1,0} = 0$. In general the spectral sequence of Theorem 4.1 gives an exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1(U', \mu^{-1}\mathcal{O}(V)) \longrightarrow E_\infty^{0,1} \longrightarrow 0.$$

In our case, however, $E_\infty^{1,0} = 0$ so we can conclude that

$$H^1(U', \mu^{-1}\mathcal{O}(-n-2)) \cong E_\infty^{0,1} \cong E_2^{0,1} \cong \Gamma(U, \mathscr{Z}_n')$$

as required.

The group $GL(\mathbb{T})$ of linear automorphisms of $\mathbb{T}$ (isomorphic to $GL(4, \mathbb{C})$) clearly acts on $\mathbb{P}$ and $\mathbb{M}$ in such a way that the twistor correspondence (1.1) is preserved. Thus, if $\varphi \in GL(\mathbb{T})$ and U is an open subset of $\mathbb{M}$ then $\varphi(U)'' = \varphi(U'')$. φ also preserves $\mathcal{O}(-n-2)$ and $\mathscr{Z}_n'$. Hence it makes sense to ask that the diagram

$$\begin{tikzcd} \mathscr{P} : H^1(U'', \mathcal{O}(-n-2)) \arrow[r] & \Gamma(U, \mathscr{Z}_n') \\ \mathscr{P} : H^1(\varphi(U''), \mathcal{O}(-n-2)) \arrow[r] \arrow[u, "\varphi"] & \Gamma(\varphi(U), \mathscr{Z}_n') \arrow[u, "\varphi"] \end{tikzcd} \tag{5.3}$$

be commutative. This is what it means for $\mathscr{P}$ to be canonical. It is, moreover, clear that (5.3) is commutative since the proof of Theorem 5.1 is free from any choice of basis for $\mathbb{T}$. □

We carried out the argument in the above proof in rather too much detail. For anyone familiar with the language of spectral sequences a glance at (5.2) would suffice. For other examples and applications of spectral sequences see Bredon [5] or Godement [14]. As remarked earlier, the same result easily follows directly from (1.6) and (2.11) without the use of this seemingly cumbersome machinery.

Massless fields are often described by means of a Fourier integral formula (Fierz [12]). It is then possible, in terms of this formula, to define what is meant by a *positive-frequency* (or equivalently, *positive-energy*) field. Wave-functions occurring in nature are positive-frequency. A real-analytic positive-frequency field on real Minkowski space automatically extends to a holomorphic field on an open neighbourhood of $\overline{\mathbb{M}^+}$ (often taken as the definition of positive-frequency). If $\mathscr{P}$ were an isomorphism for a suitable collection of such neighbourhoods then we'd be able to take a direct limit over all such neighbourhoods to obtain an isomorphism

$$\mathscr{P}: H^1(\overline{\mathbb{P}^+}, \mathcal{O}(-n-2)) \xrightarrow{\cong} \{\text{real-analytic positive-frequency helicity } n/2 \text{ massless fields on real Minkowski-space}\}$$

In fact, $\mathscr{P}$ is an isomorphism for suitable arbitrarily small open neighbourhoods of $\overline{\mathbb{M}^+}$. The proof is essentially the same as for $\mathbb{M}^+$ itself (see Corollary 5.2, below), the usual choice for U (in Penrose [37] and Wells [52] for example). There is another reason why $\mathbb{M}^+$ and $\mathbb{M}^-$ are of particular interest, namely, these are just the type of set required to define hyperfunctions on real Minkowski-space. Therefore it is reasonable to expect that the $\mathscr{P}$-transform can be extended to include hyperfunction fields and this is indeed possible (see Wells [53] and [54]). The result for $\mathbb{M}^+$ is worth stating separately.

Corollary 5.2. *For* $n \geqq 1$,

$$\mathscr{P}: H^1(\mathbb{P}^+, \mathcal{O}(-n-2)) \longrightarrow \Gamma(\mathbb{M}^+, \mathscr{Z}'_n)$$

is a canonical isomorphism.

Proof. $\mathbb{M}^+$ is a convex subset of $\mathbb{M}^I \simeq \mathbb{C}^4$. Thus, every α-plane intersects $\mathbb{M}^+$ in a convex and, hence, simply-connected piece. $\square$

It is also clear that Theorem 5.1 describes every right-handed field locally since each point of $\mathbb{M}$ has convex neighbourhoods to which the theorem applies.

6. The Wave Equation

The classical scalar *wave equation* in affine Minkowski space $\mathbb{M}^I$ is

$$\square\phi = 0, \tag{6.1}$$

where ϕ is a scalar field (a function) and $\square := \nabla^a\nabla_a = \nabla^{AA'}\nabla_{AA'}$. Every component of a massless field satisfies the wave equation and indeed, (6.1) is also called the *helicity zero massless field equation.* Globally a massless field has to have conformal weight -1. In other words ϕ is to be regarded as a section of $\mathscr{Z}_0$ or $\mathscr{Z}'_0$ where

$$\mathscr{Z}_0 := \ker \square : \mathcal{O}[-1] \longrightarrow \mathcal{O}[-2][-1]'$$
$$\mathscr{Z}'_0 := \ker \square : \mathcal{O}[-1]' \longrightarrow \mathcal{O}[-1][-2]'. \tag{6.2}$$

We shall see that $\square$ arises naturally from the twistor geometry. It is interesting to see how a second order operator appears whereas $\nabla_A^{A'}$ in Sect. 5 was first order. The spectral sequence shows how this transition occurs.

Theorem 6.1. *For U open in $\mathbb{M}$ there is a canonical linear transformation*

$$\mathscr{P}: H^1(U'', \mathcal{O}(-2)) \longrightarrow \Gamma(U, \mathscr{Z}'_0).$$

If $\mu: U' \to U''$ has connected fibres, then $\mathscr{P}$ is injective and if $\mu: U' \to U''$ is elementary, then $\mathscr{P}$ is an isomorphism.

Proof. As in the proof of Theorem 5.1 it suffices to compute $H^1(U', \mu^{-1}\mathcal{O}(-2))$ by means of Theorem 4.1. Using (2.11) it follows that

$$E_1^{1,q} = H^q(U', \Omega_\mu^1(-2)) = 0 \quad \text{for all } q.$$

Thus $E_2^{p,q} \cong E_1^{p,q}$ and (2.11) shows that the E_2-level is given by

$$\begin{tikzcd}
q \\
H^1(U, \mathcal{O}[-1]') \arrow[u] \arrow[drr] & 0 & H^2(U, \mathcal{O}[-1][-2]') \\
\Gamma(U, \mathcal{O}[-1]') \arrow[u, dash] \arrow[drr, "D"] & 0 & H^1(U, \mathcal{O}[-1][-2]') \\
0 \arrow[u, dash] \arrow[r, dash] & 0 \arrow[r, dash] & \Gamma(U, \mathcal{O}[-1][-2]') \arrow[r] & p
\end{tikzcd}$$

Thus $H^1(U', \mu^{-1}\mathcal{O}(-2)) \cong \ker D : \Gamma(U, \mathcal{O}[-1]') \to \Gamma(U, \mathcal{O}[-1][-2]')$. Since the differentials of the first level of this spectral sequence all vanish and this is the second level, it follows that D is a second order differential operator. To identify D we proceed as follows. Recall that to identify $H^1(U', \mathcal{O}(-2))$ with $\Gamma(U, \mathcal{O}[-1]')$ we use the sequence (cf. (2.9))

$$\mathbb{F};\ 0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\pi_{A'}} \mathcal{O}_{A'}(-1) \xrightarrow{\pi^{A'}} \mathcal{O}[-1]' \longrightarrow 0$$

and the fact that $\Gamma(U', \mathcal{O}[-1]') \cong \Gamma(U, \mathcal{O}[-1]')$. The map of complexes

$$\begin{tikzcd}
& \mathcal{O}(-2) \arrow[r, "\pi_{A'}"] \arrow[d] & \mathcal{O}_{A'}(-1) \arrow[r, "\pi^{A'}"] \arrow[d, "\nabla_A^{A'}"] & \mathcal{O}[-1]' \arrow[d, "\frac{1}{2}\Box"'] \\
\mathbb{F}; & \mathcal{O}(-2) \arrow[r, "\pi_{A'}\nabla_A^{A'}"] & \mathcal{O}_A(-1)[-1]' \arrow[r, "\pi_{A'}\nabla^{AA'}"] & \mathcal{O}[-1][-2]'
\end{tikzcd}$$

gives rise to a map of spectral sequences which on the E_2-level is

$$\begin{tikzcd}
q \\
H^1(U', \mathcal{O}(-2)) \arrow[u] \arrow[drr, "\cong"] \arrow[ddd, "\|"] & & \\
0 \arrow[u, dash] \arrow[r, dash] & 0 \arrow[r, dash] & \Gamma(U', \mathcal{O}[-1]') \arrow[r] \arrow[dd, "\frac{1}{2}\Box"] & p \\
q \\
H^1(U', \mathcal{O}(-2)) \arrow[u] \arrow[drr, "D"] & & \\
0 \arrow[u, dash] \arrow[r, dash] & 0 \arrow[r, dash] & \Gamma(U', \mathcal{O}[-1][-2]') \arrow[r] & p
\end{tikzcd}$$

Hence, $D = \frac{1}{2}\Box$ and the result follows. $\Box$

7. Left-handed Fields via Potentials

A classical *holomorphic left-handed massless free field* on an open subset U of $\mathbb{M}$ is a holomorphic spinor field

$$\psi_{AB\cdots D} \in \Gamma(U, \mathcal{O}_{(AB\cdots D)}[-1]') \quad \text{satisfying} \tag{7.1}$$
$$\nabla^{DD'}\psi_{AB\cdots D} = 0,$$

and the field is said to be of helicity $-n/2$ if there are n indices. For helicity -1 these are the spinor form of Maxwell's equations for an anti-self-dual field. We shall denote the sheaf of such by $\mathscr{Z}'_{-n}$. There is also the isomorphic sheaf $\mathscr{Z}_{-n}$ defined as the kernel of $\nabla^{DD'}: \mathcal{O}_{(AB\ldots D)}[-1] \to \mathcal{O}_{(AB\ldots C)}[-2][-1]'$.

One way of representing these fields is via analytic cohomology on portions of dual projective twistor-space $\mathbb{P}^*$. This would follow exactly the construction of Sect. 5 and would yield sections of $\mathscr{Z}_{-n}$. Left-handed fields (sections of $\mathscr{Z}'_{-n}$) may, however, be represented on portions of $\mathbb{P}$ itself. This representation may be accomplished via the contour integral approach (cf. Penrose [34]) or by "power series expansion" of cohomology classes about the lines L_z, for $z \in \mathbb{M}$ (cf. Wells [53] and the discussion in the next section). The method which comes out of Theorem 4.1, however, is to use potentials.

By a *potential* for a field $\psi_{AB\ldots D} \in \Gamma(U, \mathscr{Z}'_{-n})$ we mean a spinor field $\phi_A^{B'\ldots D'} \in \Gamma(U, \mathcal{O}_A^{(B'\ldots D')}[-1]')$ such that

$$\begin{cases} \nabla^{A(A'}\phi_A^{B'\cdots D')} = 0 \\ \nabla_{D'(D\cdots}\nabla_{B'B}\phi_{A)}^{B'\cdots D'} = \psi_{AB\cdots D}\,. \end{cases} \tag{7.2}$$

(The symmetrization on the left-hand side refers only to the unprimed indices) It is straightforward but tedious to check (cf. (7.6) and (7.7)) that if $\psi_{AB\cdots D}$ is given by (7.2) then the field equations (7.1) are automatically satisfied. There is clearly some "gauge freedom" in choice of potential for it is easy to see that if $\phi_A^{B'\cdots D'}$ is a potential for a field $\psi_{AB\cdots D}$, then so is

$$\phi_A^{B'\cdots D'} + \nabla_A^{(B'}\gamma^{C'\cdots D')} \tag{7.3}$$

for any spinor field $\gamma^{C'\cdots D'} \in \Gamma(U, \mathcal{O}^{C'\cdots D'})$. There are other related notions of potentials. A useful concept, for example, is that of a Hertz potential (cf. Penrose [27]).

Locally it is always possible to find a potential for a given massless field and (7.3) is the only freedom allowed in such a choice. This follows from constructions in Penrose [37] and Ward [49], which involve explicit formulae for representations of the fields and the potentials, respectively, as transforms of the cohomology classes.

According to our definition, a potential for a neutrino field (helicity $-1/2$) is the field itself, so the simplest case is the Maxwell field:

$$\nabla^{BB'}\psi_{AB} = 0. \tag{7.4}$$

If we regard ψ_{AB} as an anti-self-dual 2-form $F_{ab} = \psi_{AB}\varepsilon_{A'B'}$, then (7.4) becomes $dF = 0$, the desired potential is a 1-form ω such that $d\omega = F$, and the gauge freedom is simply that ω may be replaced by $\omega + d\gamma$ for any holomorphic function γ. Hence, the local equivalence of the field potential modulo gauge follows from the exactness of the deRham sequence

$$\mathbb{M}\,;\ \Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \Omega^3 .$$

In terms of spinors this portion of the deRham sequence, up to constant factors, is (cf. (2.3))

$$\mathbb{M}\,,\ \mathcal{O} \xrightarrow{\nabla_A^{B'}} \mathcal{O}_A^{B'}[-1]' \xrightarrow[\nabla_{BB'}]{\nabla^{AA'}} \begin{matrix} \mathcal{O}^{(A'B')}[-1][-2]' \\ \oplus \\ \mathcal{O}_{(AB)}[-1]' \end{matrix} \xrightarrow[\nabla^{AA}]{-\nabla_{BB'}} \mathcal{O}_B^{A'}[-1][-2]'. \tag{7.5}$$

Thus, one way of seeing that a left-handed Maxwell field is locally equivalent to a potential modulo gauge, is by studying the following commutative diagram (cf. Eastwood [9]).

$$
\begin{tikzcd}
 & 0 & 0 & & \\
 & \mathcal{O} \arrow[u] \arrow[r, "\nabla_A^{B'}"] & \mathcal{O}_A^{B'}[-1]' \arrow[u] \arrow[r, "\nabla^{AA'}"] & \mathcal{O}^{(A'B')}[-1][-2]' & \\
\mathbb{M}, & \mathcal{O} \arrow[u] \arrow[r, "\nabla_A"] & \mathcal{O}_A^{B'}[-1]' \arrow[u] \arrow[r, "\nabla^{AA'}", "\nabla_{BB'}"'] & \begin{matrix}\mathcal{O}^{(A'B')}[-1][-2]' \\ \oplus \\ \mathcal{O}_{(AB)}[-1]'\end{matrix} \arrow[u] \arrow[r, "\nabla_{BB}", "\nabla^{AA'}"'] & \mathcal{O}_B^{A'}[-1][-2]'. \\
 & & 0 \arrow[u] \arrow[r] & \mathcal{O}_{(AB)}[-1]' \arrow[u] \arrow[r, "'\nabla^{BB'}"] & \mathcal{O}_A^{B'}[-1][-2]' \arrow[u] \\
 & & & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}
\tag{7.6}
$$

The point is that all the columns are exact and the middle row is (7.5) and hence exact. Therefore, simple diagram chasing shows that the cohomology of the first row (potential modulo gauge) is isomorphic to the cohomology of the last row (fields). The reason for rewriting the argument this way is that it generalizes locally to apply to all helicities. The precise details are rather messy and will be omitted. The diagram for helicity $-n/2$ contains $n-1$ intermediate rows between the potential and the field descriptions, the jth intermediate row being the deRham sequence shunted $j-1$ steps to the right and tensored by $\mathcal{O}_{(A_1 \cdots A_{j-1})}^{(A'_1 \cdots A'_{n-j-1})}$.

We now consider what happens globally in the helicity -1 case. To investigate the global question of whether fields are equivalent to potentials modulo gauge on a given open subset U of $\mathbb{M}$ we take sections of (7.6) over U. For example, if we were dealing with smooth forms, then sections of (7.5) over U would yield an exact sequence if and only if $H^1(U, \mathbb{C}) = H^2(U, \mathbb{C}) = 0$. Indeed, (7.6) would give rise to an exact sequence

$$
H^1(U, \mathbb{C}) \longrightarrow \frac{\{\text{Potentials}\}}{\{\text{Gauge}\}} \longrightarrow \left\{ \begin{matrix} \text{Left-handed Maxwell} \\ \text{fields on } U \end{matrix} \right\} \longrightarrow H^2(U, \mathbb{C}) \tag{7.7}
$$

and so we would conclude that if $H^1(U, \mathbb{C}) = H^2(U, \mathbb{C}) = 0$, then we'd have an isomorphism on U:

$$
\frac{\{\text{Potentials}\}}{\{\text{Gauge}\}} \xrightarrow{\cong} \left\{ \begin{matrix} \text{Left-handed} \\ \text{Maxwell fields} \end{matrix} \right\}.
$$

In the case of holomorphic forms, however, it is easy to check that in addition we need

$$
\nabla_A^{B'} : H^1(U, \mathcal{O}) \longrightarrow H^1(U, \mathcal{O}_A^{B'}[-1]') \tag{7.8}
$$

to be injective. If $U \subseteq \mathbb{M}^I$ is convex then $H^1(U, \mathbb{C}) = H^2(U, \mathbb{C}) = 0$ and U is Stein so that $H^1(U, \mathcal{O}) = 0$. Hence, a potential description is always valid for such a U and, in particular, for $U = \mathbb{M}^+$. The investigation of the general helicity case is similar and will be omitted from this paper. The main result of this section is

Theorem 7.1. *Suppose that $n \geqq 1$, U is open in $\mathbb{M}$, and*

$$\nabla_A^{B'} : H^1(U, \mathcal{O}^{\overbrace{(C' \cdots D')}^{n-2}}) \longrightarrow H^1(U, \mathcal{O}_A^{(B'C' \cdots D')}[-1]') \tag{7.9}$$

is injective, where this condition is taken to be vacuous if $n = 1$. Then there is a canonical linear transformation

$$\mathscr{P} : H^1(U'', \mathcal{O}(n-2)) \longrightarrow \left\{ \begin{matrix} \text{Potential for helicity } -n/2 \\ \text{holomorphic massless fields on } U \end{matrix} \right\} \Big/ \left\{ \begin{matrix} \text{Gauge} \\ \text{Freedom} \end{matrix} \right\} \tag{7.10}$$

If $\mu : U' \to U''$ has connected fibres, then $\mathscr{P}$ is injective and if $\mu : U' \to U''$ is elementary, then $\mathscr{P}$ is an isomorphism

Proof. As in the proofs of Theorems 5.1 and 6.1, $\mathscr{P}$ is constructed by first mapping $\mu^* : H^1(U'', \mathcal{O}(n-2)) \to H^1(U', \mu^{-1}\mathcal{O}(n-2))$, and then using the spectral sequence of Theorem 4.1 to compute $H^1(U', \mu^{-1}\mathcal{O}(n-2))$. Thus it suffices to show that there is a natural isomorphism

$$H^1(U', \mu^{-1}\mathcal{O}(n-2)) \cong \{\text{Potentials}\}/\{\text{Gauge}\}.$$

Using the isomorphisms (2.11) we compute the E_1-level of the spectral sequence:

$$\begin{array}{l} q \uparrow \\ H^1(U, \mathcal{O}^{\overbrace{(C' \cdots D')}^{n-2}}) \xrightarrow{\nabla_A^{B'}} H^1(U, \mathcal{O}_A^{(B' \cdots D')})[-1]') \longrightarrow \cdots \\ \Gamma(U, \mathcal{O}^{(C' \cdots D')}) \xrightarrow{\nabla_A^{B'}} \Gamma(U, \mathcal{O}^{(B' \cdots D')})[-1]') \xrightarrow{\nabla^{AA'}} \Gamma(U, \mathcal{O}^{(A'B' \cdots D')}[-1][-2]') \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow[p]{} \cdots \end{array}$$

The injectivity of (7.9) is therefore precisely that $E_2^{0,1} = 0$. Moreover, by definition

$$E_2^{1,0} = \{\text{Potentials}\}/\{\text{Gauge}\}.$$

$E_\infty^{0,1} = E_2^{0,1} = 0$ and $E_\infty^{1,0} \cong E_2^{1,0}$ so the exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1(U', \mu^{-1}\mathcal{O}(n-2)) \longrightarrow E_\infty^{0,1} \longrightarrow 0$$

gives the desired result. □

The hypothesis in Theorem 7.1 that (7.9) be injective may seem a little strange. It is a very weak condition and, in particular, Theorem 7.1 applies to $\mathbb{M}^+$ and applies locally in the sense that any point has arbitrarily small neighbourhoods for which the result holds. Nevertheless, we observe that in case $n = 2$ (i.e. left-handed photons) (7.8) and (7.9) coincide. In other words, the extra hypothesis required in the theorem is precisely the additional analytic condition needed, in addition to topological conditions, to ensure the existence of potentials. This seems to indicate that if we drop this condition then it still may be possible to produce the field, but not necessarily a potential (compare also Penrose [37]). In the remainder of this section we shall discuss only the case $n = 2$.

We recall that $\mathscr{P}$ of Theorem 7.1 is constructed by first applying $\mu^* : H^1(U'', \mathcal{O}) \to H^1(U', \mu^{-1}\mathcal{O})$, and then using the spectral sequence of Theorem 4.1 to interpret

$H^1(U', \mu^{-1}\mathcal{O})$ in terms of data on U. If we define a complex of sheaves $\mathcal{S}^\cdot$ on $\mathbb{M}$ by

$$\mathbb{M};\quad \begin{tikzcd} \ldots \arrow[r] & \mathcal{S}^{-1} \arrow[r] \arrow[d, equal] & \mathcal{S}^{0} \arrow[r] \arrow[d, equal] & \mathcal{S}^{1} \arrow[r] \arrow[d, equal] & \mathcal{S}^{2} \arrow[r] \arrow[d, equal] & \mathcal{S}^{3} \arrow[r] \arrow[d, equal] & \ldots \\ \ldots \arrow[r] & 0 \arrow[r] & \mathcal{O} \arrow[r, "\nabla_A^{B'}"] & \mathcal{O}_A^{B'}[-1]' \arrow[r, "\nabla^{AA'}"] & \mathcal{O}^{(A'B')}[-1][-2]' \arrow[r] & 0 \arrow[r] & \ldots \end{tikzcd}$$

then the spectral sequence of Theorem 4.1 is exactly the second hypercohomology spectral sequence of $\mathcal{S}^\cdot$ (cf. Swan [54], or the spectral sequence of a differential sheaf in Bredon [5] or Godement [14]). Thus, the interpretation of $H^r(U', \mu^{-1}\mathcal{O})$ as data on U is, in general, given by the isomorphism

$$H^r(U', \mu^{-1}\mathcal{O}) \cong \mathbb{H}^r(U, \mathcal{S}^\cdot). \tag{7.11}$$

Now define a second complex $\mathcal{R}^\cdot$ by

$$\mathbb{M};\quad \begin{tikzcd} \ldots \arrow[r] & \mathcal{R}^{1} \arrow[r] \arrow[d, equal] & \mathcal{R}^{2} \arrow[r] \arrow[d, equal] & \mathcal{R}^{3} \arrow[r] \arrow[d, equal] & \mathcal{R}^{4} \arrow[r] \arrow[d, equal] & \mathcal{R}^{5} \arrow[r] \arrow[d, equal] & \ldots \\ \ldots \arrow[r] & 0 \arrow[r] & \mathcal{O}_{(AB)}[-1]' \arrow[r, "\nabla_B^{B'}"] & \mathcal{O}_A^{B'}[-1][-2]' \arrow[r, "\nabla_{B'}^{A}"] & \mathcal{O}[-2][-2]' \arrow[r] & 0 \arrow[r] & \ldots \end{tikzcd}$$

Identifying the various spinor sheaves in terms of holomorphic forms, we can complete (7.6) to the exact sequence of complexes

$$\mathbb{M};\ 0 \longrightarrow \mathcal{R}^\cdot \longrightarrow \Omega^\cdot \longrightarrow \mathcal{S}^\cdot \longrightarrow 0$$

and hence obtain a long exact sequence on hypercohomology

$$\ldots \longrightarrow \mathbb{H}^1(U, \Omega^\cdot) \longrightarrow \mathbb{H}^1(U, \mathcal{S}^\cdot) \longrightarrow \mathbb{H}^2(U, \mathcal{R}^\cdot) \longrightarrow \mathbb{H}^2(U, \Omega^\cdot) \longrightarrow \ldots \tag{7.12}$$

Since $\Omega^\cdot$ is a resolution of $\mathbb{C}$,

$$\mathbb{H}^r(U, \Omega^\cdot) = H^r(U, \mathbb{C}). \tag{7.13}$$

The hypercohomology group $\mathbb{H}^2(U, \mathcal{R}^\cdot)$ is easily calculated by the first hypercohomology spectral sequence, or from the definition of hypercohomology, and we find that

$$\mathbb{H}^2(U, \mathcal{R}^\cdot) = \Gamma(U, \mathcal{Z}'_{-2}), \quad \mathbb{H}^1(U, \mathcal{R}^\cdot) = 0 \tag{7.14}$$

Substituting (7.11), (7.13) and (7.14) into (7.12) we obtain the exact sequence of (cf. Eastwood [9])

$$0 \longrightarrow H^1(U, \mathbb{C}) \longrightarrow H^1(U', \mu^{-1}\mathcal{O}) \longrightarrow \Gamma(U, \mathcal{Z}'_{-2}) \longrightarrow H^2(U, \mathbb{C}) \longrightarrow \ldots \tag{7.15}$$

and $\mathcal{P}$ may be defined as the composition

$$H^1(U'', \mathcal{O}) \xrightarrow{\mu^*} H^1(U', \mu^{-1}\mathcal{O}) \longrightarrow \Gamma(U, \mathcal{Z}'_{-2}).$$

It is now easy to see when

$$\mathcal{P}: H^1(U'', \mathcal{O}) \longrightarrow \Gamma(U, \mathcal{Z}'_{-2})$$

is an isomorphism. Firstly, there are the usual topological conditions on $\mu: U' \to U''$ ensuring that μ^* be an isomorphism (Theorem 3.2). Secondly, (7.15) shows that $H^1(U', \mu^{-1}\mathcal{O}) \to \Gamma(U, \mathcal{Z}'_{-2})$ is an isomorphism if $H^1(U, \mathbb{C})$ and $H^2(U, \mathbb{C})$ vanish. Thus, the topological obstructions to existence and uniqueness of potentials remain whereas the analytic condition has disappeared. This suggests that perhaps

$\mathbb{H}^1(U, \mathscr{S}^{\cdot})$ should be taken as a definition of potential modulo gauge freedom in the case of holomorphic forms.

8. A Direct Approach to Left-handed Fields

In Wells [53] a cohomological method was described for producing a left-handed field of helicity $-n/2$ directly from an element of $H^1(U'', \mathcal{O}(n-2))$. For this range of homogeneities the integral formulae (cf. Penrose [34]) involve differentiation in the integrand and this shows up in the cohomological method by using powers of the conormal bundle of a line in U'' to try to expand an element of $H^1(U'', \mathcal{O}(n-2))$ as a power series about the line. It is therefore quite straightforward to see that when an integral representation is possible, it agrees with this more abstract approach. What is not so clear is that this approach agrees with that described in the previous section of the present paper. In this section we shall first review the construction of Wells [53], describing it invariantly[10] with the aid of the spinor sheaves of Sect. 2. Then we shall use the process of *raising and lowering helicity* (cf. Eastwood [10] and Penrose [33], [39]) to show that it does indeed agree with the twistor description via potentials given in Sect. 7. This also follows by comparison of the constructions given in Penrose [37] and Ward [49].

Consider the bundle $\mathcal{O}^A(1)(:=\mathcal{O}(1)\otimes\mathcal{O}^A)$ on $\mathbb{P}\times\mathbb{M}$. The fibre of this bundle over $(L_1, L_2)\in\mathbb{P}\times\mathbb{M}$ is $L_1^*\otimes(\mathbb{T}/L_2)$. Choose $\ell\in L_1$. Let ℓ^* denote the dual element of L_1^* (i.e. $\ell^*(\ell)=1$), and let $[\ell]$ denote the image of ℓ under the quotient $\mathbb{T}\to\mathbb{T}/L_2$. Then $\ell^*\otimes[\ell]\in L_1^*\otimes(\mathbb{T}/L_2)$ is independent of the choice of ℓ and hence defines a section of $\mathcal{O}^A(1)$ which we will call ω^A. Observe that $\omega^A(L_1, L_2)=0$ if and only if $L_1\subset L_2$ and indeed ω^A defines $\mathbb{F}$ without multiplicitly as a subvariety of $\mathbb{P}\times\mathbb{M}$. In other words, if $\mathscr{J}$ denotes the ideal sheaf of $\mathbb{F}$ as a subvariety of $\mathbb{P}\times\mathbb{M}$, then

$$\mathbb{P}\times\mathbb{M};\quad \begin{array}{ccc} \mathcal{O}_A(-1) & \longrightarrow & \mathscr{J} \\ \cup\!\!\!\!\!\!\in & & \cup\!\!\!\!\!\!\in \\ f_A & \longmapsto & f_A\omega^A \end{array}$$

is surjective. This also gives the following exact sequence

$$\mathbb{P}\times\mathbb{M};\quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{J}^2 & \longrightarrow & \mathscr{J} & \longrightarrow & {}_{\mathbb{F}}\mathcal{O}_A(-1) & \longrightarrow & 0 \\ & & & & \cup\!\!\!\!\!\!\in & & \cup\!\!\!\!\!\!\in & & \\ & & & & f_A\omega^A & \longmapsto & f_A|_{\mathbb{F}} & & \end{array} \tag{8.1}$$

which identifies the conormal bundle $\mathscr{J}/\mathscr{J}^2$ as $\mathcal{O}_A(-1)$ on $\mathbb{F}$. Here, we have allowed ourselves to write ${}_X\mathcal{O}$ instead of $\mathcal{O}_X$ to avoid confusion with spinor indices. Now suppose U is an open Stein subset of $\mathbb{M}$ and consider the above construction

---

10 The construction itself is manifestly invariant but we repeat it here to show how it fits in with the spinor sheaves of Sect. 2. The use of the ideal sheaves in what follows is a device to make sense of $\partial/\partial\omega^A$ in an invariant manner. Knowing $\partial/\partial\omega^A$ is the same as knowing the location of I, the line in $\mathbb{P}$ corresponding to the vertex of the null-cone at infinity of $\mathbb{M}$. If this is fixed then it is easy to check that the method in Wells [53] agrees with that described by Penrose [37].

applied to U' as a subvariety of $U'' \times U$. We claim that

$$H^1(U'' \times U, \mathscr{I}^n(n-2)) \longrightarrow H^1(U'' \times U, \mathcal{O}(n-2)) \qquad (8.2)$$

is an isomorphism. The general case is no more difficult than the helicity -1 $(n=2)$ case so, for notational simplicity, we will restrict our proofs to this case. From the exact sequence of sheaves

$$U'' \times U;\ 0 \longrightarrow \mathscr{I} \longrightarrow \mathcal{O} \longrightarrow {}_{U'}\mathcal{O} \longrightarrow 0$$

we obtain an exact sequence on cohomology

$$\Gamma(U'' \times U, \mathcal{O}) \longrightarrow \Gamma(U', \mathcal{O}) \longrightarrow H^1(U'' \times U, \mathscr{I}) \longrightarrow H^1(U'' \times U, \mathcal{O}) \longrightarrow H^1(U', \mathcal{O}).$$

The holomorphic functions on $U'' \times U$ and U' are the holomorphic functions on U so $\Gamma(U'' \times U, \mathcal{O}) \to \Gamma(U', \mathcal{O})$ is surjective. Also, by Proposition 2.2, $H^1(U', \mathcal{O}) \cong H^1(U, \mathcal{O}) = 0$, since U is Stein. Hence

$$H^1(U'' \times U, \mathscr{I}) \longrightarrow H^1(U'' \times U, \mathcal{O})$$

is an isomorphism. Now consider the exact sequence (8.1) on $U'' \times U$. The corresponding long exact sequence on cohomology includes the portion

$$\Gamma(U', \mathcal{O}_A(-1)) \longrightarrow H^1(U'' \times U, \mathscr{I}^2) \longrightarrow H^1(U'' \times U, \mathscr{I}) \longrightarrow H^1(U', \mathcal{O}_A(-1))$$

and Proposition 2.2 shows that the first and last terms vanish. Hence we may conclude

$$H^1(U'' \times U, \mathscr{I}^2) \xrightarrow{\cong} H^1(U'' \times U, \mathscr{I}) \xrightarrow{\cong} H^1(U'' \times U, \mathcal{O})$$

as required. Intuitively, (8.2) being an isomorphism says that the information contained within an element of $H^1(U'' \times U, \mathcal{O}(n-2))$ depends only on normal derivatives to U' of order at least n. To evaluate the field we in effect just take the n^{th} normal derivative by factoring out $\mathscr{I}^{n+1}(n-2)$. This also has the required effect of restricting to U' since $\mathscr{I}^n(n-2)/\mathscr{I}^{n+1}(n-2)$ is supported on U'. It follows from the exact sequence (cf. 8.1)

$$\begin{array}{ccccccccc} \mathbb{P} \times \mathbb{M};\ 0 & \longrightarrow & \mathscr{I}^{n+1} & \longrightarrow & \mathscr{I}^n & \longrightarrow & {}_{U'}\mathcal{O}_{(AB\cdots D)}(-n) & \longrightarrow & 0 \\ & & & & \Psi & & \Psi & & \\ & & & & f_{AB\cdots D}\omega^A\omega^B\ldots\omega^D & \longmapsto & f_{AB\cdots D}|_{U'} & & \end{array}$$

that $\mathscr{I}^n(n-2)/\mathscr{I}^{n+1}(n-2)$ may be canonically identified with $\mathcal{O}_{(AB\cdots D)}(-2)$ on U'. The $\mathscr{P}$-transform in Wells [53] is defined as the composition

$$\begin{array}{ccccc} H^1(U'', \mathcal{O}(n-2)) & \longrightarrow & H^1(U'' \times U, \mathcal{O}(n-2)) & \xleftarrow{\cong} & H^1(U'' \times U, \mathscr{I}^n(n-2)) \\ & & & & \downarrow \\ & & & & H^1(U', \mathcal{O}_{(AB\cdots D)}(-2)) \\ & & & & \wr\| \\ & & & \searrow & \Gamma(U, \mathcal{O}_{(AB\cdots D)}[-1]') \end{array} \qquad (8.3)$$

where the last isomorphism follows from Proposition 2.2. To show that (8.2) was an isomorphism we assumed that U was Stein (though we only used $H^1(U, \mathcal{O}) = 0$ in case $n=2$). This indicates strongly that the construction is really via potentials especially as, in case $n=1$, the assumption is unnecessary. To get around this problem we simply perform the transform (8.3) locally. It is clear that these local transforms will agree on overlaps and hence patch together to give a well-defined transform for any U. An alternative approach is to apply the argument to a fixed line L_z, for $z \in U$, as a subvariety of U'' instead of U' as a subvariety of $U'' \times U$. This is the approach that we shall adopt to show that $\mathscr{P}$ defined by (8.3) agrees (apart from a combinatorial constant factor) with $\mathscr{P}$ defined by Theorem 7.1.

Again we will restrict our discussion to the helicity -1 case. Choose a point of U. We shall call this point the origin and introduce coordinates $z^{\mathbf{AA'}}$ as in Sect. 1 so that this origin really is the origin of $\mathbb{M}^I$. Also introduce coordinates $(\omega^{\mathbf{A}}, \pi_{\mathbf{A'}})$ on $\mathbb{T}$ as in Sect. 1. Then the line L in $\mathbb{P}$ corresponding to the origin is defined by the vanishing of $\omega^{\mathbf{A}}$. If $\mathscr{J}$ denotes the ideal sheaf of L as a subvariety of U'' then the value of $\mathscr{P}$ at the origin according to (8.3) is given by the composition

$$H^1(U'', \mathcal{O}) \xleftarrow{\cong} H^1(U'', \mathscr{J}^2) \longrightarrow H^1(L, \mathscr{J}^2/\mathscr{J}^3) \cong H^1(L, {}_L\mathcal{O}_{(\mathbf{AB})}(-2)) \quad (8.4)$$

where the last space is regarded as the fibre of $\mathcal{O}_{(\mathbf{AB})}[-1]'$ over the origin. To compare this with Sect. 7 we proceed as follows. Since $\mathscr{J}^2$ is generated by $\omega^{\mathbf{A}}\omega^{\mathbf{B}}$ we have the exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathcal{O}_{(\mathbf{AB})}(-2) \xrightarrow{\omega^{\mathbf{A}}\omega^{\mathbf{B}}} \mathscr{J}^2 \longrightarrow 0 \quad (8.5)$$

where $\mathscr{K}$ is some coherent analytic sheaf. Consider the following commutative diagram

$$\begin{array}{ccccc} H^1(U'', \mathcal{O}_{(\mathbf{AB})}(-2)) & \xrightarrow{\omega^{\mathbf{A}}\omega^{\mathbf{B}}} & H^1(U'', \mathscr{J}^2) & \xrightarrow{\cong} & H^1(U'', \mathcal{O}) \\ \downarrow & & \downarrow & & \\ H^1(L, {}_L\mathcal{O}_{(\mathbf{AB})}(-2)) & \cong & H^1(U'', \mathscr{J}^2/\mathscr{J}^3) & & \end{array} \quad (8.6)$$

Tracing this diagram from $H^1(U'', \mathcal{O})$ to $H^1(L, {}_L\mathcal{O}_{(\mathbf{AB})}(-2))$ is the proposed construction (8.4) whereas the restriction $H^1(U'', \mathcal{O}_{(\mathbf{AB})}(-2)) \to H^1(L, {}_L\mathcal{O}_{(\mathbf{AB})}(-2))$ is the construction of the value of a field at the origin according to Sect. 6. The interpretation of

$$\omega^{\mathbf{A}}\omega^{\mathbf{B}} : H^1(U'', \mathcal{O}_{(\mathbf{AB})}(-2)) \longrightarrow H^1(U'', \mathcal{O})$$

on the corresponding massless fields (via Sect. 6 and Sect. 7) can be explicitly computed (Eastwood [10] and Penrose [39]) and turns out to be

$$\phi_{\mathbf{AB}} \longmapsto -[z^{\mathbf{CA'}}z^{\mathbf{DB'}}\nabla_{\mathbf{AA'}}\nabla_{\mathbf{BB'}}\phi_{\mathbf{CD}} + 4z^{\mathbf{CA'}}\nabla_{\mathbf{A'(A}}\phi_{\mathbf{B)C}} + 2\phi_{\mathbf{AB}}].$$

Hence, the value of the field at the origin is simply multiplied by -2. If $\omega^{\mathbf{A}}\omega^{\mathbf{B}}$: $H^1(U'', \mathcal{O}_{(\mathbf{AB})}(-2)) \to H^1(U'', \mathscr{J}^2)$ were surjective, then we would be able to conclude from (8.6) that, apart from the factor of -2, the method of Sect. 7 agreed with (8.4). Then, all the conclusions of Sect. 7 would carry over. If $H^2(U'', \mathscr{K}) = 0$ then (8.5) would imply the required surjectivity. For suitable U this is true (cf. Andreotti and Grauert [1]) but we can circumvent this more complicated argument as follows. The value of the field at the origin is unchanged if we shrink U (and hence U''). We claim it is always possible to shrink U to V, say, so that the corresponding map $H^2(U'', \mathscr{K}) \to H^2(V'', \mathscr{K})$ is the zero map. This would certainly suffice to complete the comparison and is a consequence of the following lemma.

Lemma 8.1. *Any line L in $\mathbb{P}$ has arbitrarily small open neighbourhoods W with $H^2(W,\mathscr{S})=0$ for all coherent analytic sheaves $\mathscr{S}$.*

Proof. Choose two standard coordinate patches on $\mathbb{P}$ whose union contains L. In each of these patches there are arbitrarily small Stein sets containing the part of L lying in the patch. Thus a neighbourhood W may be chosen which is a union of two Stein sets. The Mayer–Vietoris theorem and Cartan's theorem B (cf. Hörmander [20] or Gunning and Rossi [18]) show that $H^2(W,\mathscr{S})=0$ for all coherent analytic sheaves $\mathscr{S}$. □

9. Background Coupled Fields

In [46], [47] Ward gave a method involving the "twisted photon" for describing certain background coupled fields by means of twistor integral formulae. In this section we shall show how the general machine of Sect. 4 may be combined with the results of Sect. 7 to interpret this result cohomologically.

Suppose $\Phi_A^{A'}$ is a potential for a left-handed photon (Maxwell field) on U in the sense of Sect. 7. Define a differential operator $D_A^{A'}$ on U by

$$D_A^{A'}:=\nabla_A^{A'}+2\pi i\Phi_A^{A'}:\mathscr{O}_{(A'B'\cdots D')}[-1]\longrightarrow\mathscr{O}_{A(B'\cdots D')}[-1][-1]'$$

Then (cf. Gasiorowicz [13]), a holomorphic right-handed massless field on U *minimally coupled into the electromagnetic background defined by* $\Phi_A^{A'}$ is a section of $\mathscr{O}_{(A'B'\cdots D')}[-1]$ satisfying

$$D_A^{A'}\phi_{A'B'\cdots D'}=0$$

If there are n indices, then the field is said to be of helicity $n/2$. We will denote the sheaf of such by $\mathscr{Z}_n(\Phi)$. If $\Phi_A^{A'}=0$ then, of course, we just have $\mathscr{Z}_n$ as before in (5.1). Also, there is the background coupled version of (5.1)′ which will be denoted by $\mathscr{Z}'_n(\Phi)$.

If we replace $\Phi_A^{A'}$ by an equivalent potential (cf. (7.3))

$$\Phi_A^{A'}+\nabla_A^{A'}\gamma,$$

then the background coupled fields are essentially undisturbed since

$$\nabla_A^{A'}+2\pi i\Phi_A^{A'}=e^{2\pi i\gamma}(\nabla_A^{A'}+2\pi i(\Phi_A^{A'}+\nabla_A^{A'}\gamma))e^{-2\pi i\gamma}$$

Hence, the potential is only important up to the usual gauge freedom.

In Sect. 7 we described how left-handed photons may be represented under suitable conditions by elements of $H^1(U'',\mathscr{O})$. Since this representation was via potentials modulo gauge freedom we may expect it to be useful in describing electromagnetically coupled fields. This is indeed the case. The representation of a left-handed photon as an element of $H^1(U'',\mathscr{O})$ is called the *passive description*. The *active description* is in terms of a line-bundle over U'', namely the *twisted photon* (Ward [46], [47]).[11] To obtain this bundle from the passive description

11 The "active" and "passive" terminology used here differs from that used in Penrose and MacCallum [32].

consider the short exact sequence of sheaves

$$\mathbb{P};\ 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp 2\pi i} \mathcal{O}^* \longrightarrow 0$$

where $\mathbb{Z}$ denotes the constant sheaf of integers and $\mathcal{O}^*$ the sheaf of nowhere-vanishing holomorphic functions. The long exact sequence on cohomology contains the portion

$$H^1(U'',\mathbb{Z}) \longrightarrow H^1(U'',\mathcal{O}) \longrightarrow H^1(U'',\mathcal{O}^*) \xrightarrow{c} H^2(U'',\mathbb{Z}) \qquad (9.1)$$

and the *twisted photon* L is defined as the image of the passive description under the exponential map

$$H^1(U'',\mathcal{O}) \longrightarrow H^1(U'',\mathcal{O}^*).$$

The possible line-bundles obtainable by this process are, by (9.1), precisely those which are topologically trivial (i.e. $c(L)=0$). Moreover, recall (Sect. 7) that there are conditions on U which must be satisfied for the description of a left-handed Maxwell field as an element of $H^1(U'',\mathcal{O})$ to be valid including

a) $$H^1(U,\mathbb{C})=0, \qquad (9.2)$$

b) $$\mu: U' \to U'' \text{ is elementary.}$$

It is straightforward to show that conditions (9.2) imply $H^1(U'',\mathbb{Z})=0$ so that left-handed photons are precisely in $1-1$ correspondence with topologically trivial line-bundles on U''. This is connected with "charge integrality" (see Penrose [36]).

We need not impose such restrictions, however, just to obtain a cohomological description of background coupled fields. We need only suppose that we are given a twisted photon (i.e. a topologically trivial line-bundle on U'') which happens to correspond to a potential on U in a sense which may be made precise as follows. Recall the exact sequence (1.3)

$$\mathbb{F};\ 0 \longrightarrow \mu^{-1}\mathcal{O} \longrightarrow \mathcal{O} \xrightarrow{d_\mu} \Omega^1_\mu \xrightarrow{d_\mu} \Omega^2_\mu \longrightarrow 0$$

Define a sheaf $\mathcal{K}$ on $\mathbb{F}$ by

$$\mathcal{K} := \ker d_\mu : \Omega^1_\mu \longrightarrow \Omega^2_\mu$$

Then we obtain an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \Gamma(U',\mathcal{K}) & \longrightarrow & \Gamma(U',\Omega^1_\mu) & \xrightarrow{d_\mu} & \Gamma(U',\Omega^2_\mu) \\ & & \wr\| & & \wr\| \\ & & \Gamma(U,\mathcal{O}^{A'}_A[-1]') & \xrightarrow{\nabla^{AB}} & \Gamma(U,\mathcal{O}^{(A'B')}[-1][-2]') \end{array}$$

so $\Gamma(U',\mathcal{K})$ is isomorphic to the space of potentials on U. The exact sequence

$$\mathbb{F};\ 0 \longrightarrow \mu^{-1}\mathcal{O} \longrightarrow \mathcal{O} \xrightarrow{d_\mu} \mathcal{K} \longrightarrow 0$$

gives rise to an exact sequence on cohomology

$$\begin{array}{ccccccc} \ldots \longrightarrow \Gamma(U',\mathcal{O}) & \xrightarrow{d_\mu} & \Gamma(U',\mathcal{K}) & \xrightarrow{\delta} & H^1(U',\mu^{-1}\mathcal{O}) \longrightarrow H^1(U',\mathcal{O}) \longrightarrow \ldots \\ \wr\| & & \wr\|_I & & \\ \Gamma(U,\mathcal{O}) & \xrightarrow{\nabla^{A'}_A} & \{\text{potentials on } U\}. & & \end{array} \qquad (9.3)$$

Thus, an element in $H^1(U', \mu^{-1}\mathcal{O})$ gives rise to a potential on U if and only if its image in $H^1(U', \mathcal{O})$ is zero. Furthermore, the potential is unique up to the usual gauge freedom. We shall say that a *potential* $\Phi_A^{A'}$ *corresponds to a twisted photon* L if and only if we can find $\omega \in H^1(U'', \mathcal{O})$ so that $L = \exp 2\pi i\omega$ and $\mu^*\omega = \delta\Phi$, where $I(\Phi) = \Phi_A^{A'}$ under the direct image isomorphism I in (9.3). If the hypotheses of Theorem 7.1 are satisfied then this agrees with taking the passive description of that theorem and then exponentiating.

The cohomological description of coupled fields is obtained by replacing H^n by $L \otimes H^n$ in the usual description of uncoupled fields (cf. Theorem 5.1).

Theorem 9.1. *Suppose that* L *is a twisted photon corresponding to a potential* $\Phi_A^{A'}$ *on* U *for a left-handed Maxwell field. Then, for* $n \geqq 1$, *there is a canonical linear transformation*

$$\mathcal{P} : H^1(U'', \mathcal{O}(L)(-n-2)) \longrightarrow \Gamma(U, \mathscr{Z}'_n(\Phi)).$$

If $\mu : U' \to U''$ *has connected fibres, then* $\mathcal{P}$ *is injective and if* $\mu : U' \to U''$ *is elementary, then* $\mathcal{P}$ *is an isomorphism.*

Proof. The proof follows that of Theorem 5.1 and $\mathcal{P}$ is constructed by first applying

$$\mu^* : H^1(U'', \mathcal{O}(L)(-n-2)) \longrightarrow H^1(U', \mu^{-1}\mathcal{O}(L)(-n-2)).$$

It now suffices to show that

$$H^1(U', \mu^{-1}\mathcal{O}(L)(-n-2)) \cong \Gamma(U, \mathscr{Z}'_n(\Phi)).$$

We do this by using Theorem 4.1, as usual. Hence, we must compute $H^q(U', \Omega_\mu^p(L)(-n-2))$ and the differentials

$$d_\mu : H^q(U', \Omega_\mu^p(L)(-n-2)) \longrightarrow H^q(U', \Omega_\mu^{p+1}(L)(-n-2)). \tag{9.4}$$

Since L corresponds to a potential $\Phi_A^{A'}$ we have the following commutative diagram (cf. (9.3)).

$$\begin{array}{ccccccc}
\Gamma(U', \mathscr{K}) & \longrightarrow & H^1(U', \mu^{-1}\mathcal{O}) & \longrightarrow & H^1(U', \mathcal{O}) & \longrightarrow & H^1(U', \mathcal{O}^*) \\
\cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\
\Phi & \longmapsto & \mu^*\omega & \longmapsto & 0 & \longmapsto & 0 \\
 & & \uparrow & & & & \uparrow \\
 & & \omega & & \longmapsto & & L \\
 & & \cap\!\!\!\shortmid & & & & \cap\!\!\!\shortmid \\
 & & H^1(U'', \mathcal{O}) & & \longrightarrow & & H^1(U'', \mathcal{O}^*)
\end{array} \tag{9.5}$$

Thus, μ^*L is trivial as a holomorphic line bundle on U', whence

$$\Omega_\mu^p(L)(-n-2) \cong \Omega_\mu^p(-n-2).$$

Hence, the terms in the E_1-level of the spectral sequence of Theorem 4.1 are exactly as in the case of Theorem 5.1 where L is trivial. The only difference is in the differentials where we expect $\nabla_A^{A'}$ to be replaced by $D_A^{A'}$ throughout. The differentials are given by (9.4) and so it suffices to show that if we trivialize μ^*L in a natural manner (using $\Phi_A^{A'}$), then we have the following commutative diagram

$$\begin{array}{ccc} & \Omega^0_\mu(L) \xrightarrow{d_\mu} \Omega^1_\mu(L) & \\ & \wr\| \qquad\qquad \wr\| & \\ \mathbb{F}; & \mathcal{O} \longrightarrow \mathcal{O}_A(1)[-1]' & \\ & \Cup \qquad\qquad \Cup & \\ & f \longmapsto \pi_{A'} D_A^{A'} f & \end{array} \tag{9.6}$$

for this is the appropriate generalization of Lemma 1.2. The trivialization of μ^*L is given implicitly by (9.5). To make this explicit we use the Čech definition of cohomology. Let $\{U_j\}_{j\in J}$ be an open cover of U'' such that ω has a representation as a Čech cocycle:

$$\omega_{jk} \in \Gamma(U_j \cap U_k, \mathcal{O}).$$

$\mu^*\omega$ is then defined by the cocycle

$$\omega_{jk} \circ \mu \in \Gamma(\mu^{-1}(U_j) \cap \mu^{-1}(U_k), \mu^{-1}\mathcal{O}),$$

with respect to the cover $\{\mu^{-1}(U_j)\}$. From (9.5), $\delta\Phi = \mu^*\omega$, i.e. there is:

1) An open cover $\{V_j\}_{j\in J}$ of U' (after perhaps redefining the indexing of $\{U_j\}$) so that $V_j \subseteq \mu^{-1}(U_j)$)
2) $h_j \in \Gamma(V_j, \mathcal{O})$ so that $\begin{cases} d_\mu h_j = \Phi|_{V_j} \\ \omega_{jk} \circ \mu = h_j - h_k \quad \text{on } V_j \cap V_k. \end{cases}$

Hence, from this last equation

$$(\exp 2\pi i \omega_{jk} \circ \mu) \exp 2\pi i h_k = \exp 2\pi i h_j \text{ on } V_j \cap V_k$$

But $\{\exp 2\pi i \omega_{jk} \circ \mu\}$ are transition functions for the bundle μ^*L so $\{\exp 2\pi i h_j\}$ is a nowhere-vanishing section of μ^*L which therefore trivializes it:

$$\begin{array}{ccc} & \mathcal{O} = \mathcal{O}(\mu^*L) := \Omega^0_\mu(L) & \\ \mathbb{F}; & \Cup \qquad\qquad \Cup & \\ & f \longmapsto \{f \exp 2\pi i h_j\}. & \end{array}$$

Now we apply $d_\mu : \Omega^0_\mu(L) \to \Omega^1_\mu(L)$ to this section:

$$\begin{aligned} d_\mu(f \exp 2\pi i h_j) &= d_\mu f \exp 2\pi i h_j + f(2\pi i d_\mu h_j) \exp 2\pi i h_j \\ &= [(d_\mu + 2\pi i \Phi) f] \exp 2\pi i h_j. \end{aligned}$$

Identifying $\Omega^1_\mu(L)$ with $\mathcal{O}_A(1)[-1]'$ using the same trivialization, we obtain (9.6) as required. □

The above theorem interprets $H^1(U', \mathcal{O}(L)(k))$ for $k < -2$. For $k = -2$, an analogue of Theorem 6.1 holds, and for $k > -2$, an analogue of Theorem 7.1. All that is necessary is to replace $\nabla_A^{A'}$ by $D_A^{A'}$ in the statements and proofs of the theorems. Note that $\nabla^{A(A'}\Phi_A^{B')} = 0$ is equivalent to saying that $D_A^{(B'}D^{A')A} = 0$.

Hence, replacing the operator $\nabla_A^{A'}$ by the deformed operator $D_A^{A'}$ is equivalent to deforming the trivial bundle to obtain L. This point of view, of regarding L as obtained by deforming the trivial bundle, is how the twisted photon was originally constructed by Ward [46], [47] (though the definition given there has a twist of -1 also). Actually, it is best to regard $D_A^{A'}$ as a connection for the trivial line bundle on U. The twisted photon construction then generalizes as a $1-1$ correspondence (provided $\mu: U' \to U''$ is elementary) between holomorphic line-bundles on U with anti-self-dual connection and holomorphic line-bundles on U'' which are trivial when restricted to lines L_z for $z \in U$. Theorem 9.1 is easily generalized to cover this. Indeed, the general case is immediate by patching the statement for trivial bundles.

For suitable U (e.g. convex in $\mathbb{M}^I$) we have $H^2(U'', \mathbb{Z}) \cong \mathbb{Z}$ so all line-bundles are of the form $L(k)$ for a suitable twisted photon L, and $c(L(k)) = k$ where c is the Chern class mapping: $H^1(U'', \mathcal{O}^*) \to H^2(U'', \mathbb{Z})$. Thus, the Chern class of a line-bundle V is simply related to the helicity of the field represented by $H^1(U'', \mathcal{O}(V))$. Specifically,

$$\text{helicity} = (-c(V) - 2)/2.$$

In [43] and [44] Ward also described how the twisted photon construction may be generalized to non-Abelian gauge groups. Left-handed Yang–Mills fields are then described by holomorphic vector-bundles on U''. If we take V to be one of these bundles in $H^1(U', \mathcal{O}(V))$, then we generate fields coupled to Yang–Mills potentials (see Rawnsley [42] and Hitchin [19]).

10. Further Remarks

The results of Sect. 5 through Sect. 9 are concerned with the interpretation of the first cohomology of U'' with coefficients in various analytic sheaves in terms of corresponding differential equations on U. The general $\mathscr{P}$-transform (Theorem 4.1), however, is not limited to first cohomology and we should ask what it says about $H^p(U'', \mathcal{O}(n))$ for p other than 1. We consider first $H^0(U'', \mathcal{O}(n)) = \Gamma(U'', \mathcal{O}(n))$, and we find, for any open set $U \subset \mathbb{M}$:

$$\mathscr{P}: \Gamma(U'', \mathcal{O}(n)) \xrightarrow{\cong} \ker \nabla_A^{E'} : \Gamma(U, \mathcal{O}^{(A'B'\cdots D')}) \longrightarrow \Gamma(U, \mathcal{O}_A^{(A'B'\cdots E')}), \quad \text{if } n \geqq 0.$$

$$\Gamma(U'', \mathcal{O}(n)) = 0, \quad \text{if } n \leqq -1.$$

For U sufficiently nice, (e.g. convex $U \subset \mathbb{M}^I$) one finds that U'' satisfies the hypotheses of Andreotti–Grauert's vanishing theorem: $H^q(U'', \mathscr{S}) = 0, q \geqq 2$, for any coherent sheaf $\mathscr{S}$ on U'', so the $\mathscr{P}$-transform of these higher degree cohomology groups would not yield too much information (cf. Lemma 8.1). One could readily compute what the image was, and it would in general be of the form [potentials]/[gauge freedom], and the $\mathscr{P}$-transform being zero would simply say that each potential was gauge equivalent to zero in this case. We remark that $H^3(U'', \mathscr{S}) = 0$ for *any* U'' and a coherent sheaf $\mathscr{S}$ on U'' (Siu [44]; Malgrange [25], in the case of a locally free sheaf as we have), so there are no nontrivial fields generated by these cohomology groups for any geometry.

It is easy to show that every section of $\mathcal{O}(n)$ over U'' extends uniquely to a section of $\mathcal{O}(n)$ over all of $\mathbb{P}$. Thus, we have canonical isomorphisms

$$\Gamma(U'', \mathcal{O}(n)) = \Gamma(\mathbb{P}, \mathcal{O}(n)) = \begin{cases} \odot^n \mathbb{T}^* & \text{if } n \geqq 0. \\ \odot & \text{if } n \leqq -1. \end{cases}$$

In particular ($n = 1$), this shows that a dual twistor $(\eta_A, \xi^{A'})$ may be regarded as a solution of the equation

$$\nabla_A^{(B'}\zeta^{A')} = 0 \tag{10.1}$$

Indeed, it is easy to show that the general solution of (10.1) is

$$\zeta^{A'} = \xi^{A'} + iz^{AA'}\eta_A$$

for constant spinors $\xi^{A'}$ and η_A. Thus, (10.1) may be regarded as defining $\mathbb{T}^*$. This equation is called the *dual twistor equation*. Similarly, twistors may be regarded as solutions of the *twistor equation* (cf. Penrose [28])

$$\nabla_{A'}^{(B}\omega^{A)} = 0. \tag{10.2}$$

We conclude with some general remarks concerning the methods of this paper. It is clear that our approach is not limited only to the usual twistor correspondence (1.1). All we need is some general correspondence

$$\begin{tikzcd} & C \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ X & & Y \end{tikzcd} \tag{10.3}$$

with ν having nice fibres. In case of (1.1) the fibres of ν are Riemann spheres so that any vector-bundle splits as a direct sum of line-bundles which are characterized by their Chern classes (cf. Grothendieck [17]). Thus it is possible to identify the appropriate direct images and produce a transform starting with any vector bundle V over a portion of $\mathbb{P}$. In general, things may be more complicated but we can still handle many interesting special cases of (10.3) where the fibres of ν are higher-dimensional projective spaces (or flag manifolds). For example, it is possible to give a cohomological interpretation of the integral formulae described by Hughston [22], in connection with *generalized twistors*. Another possibility is to replace U'', U', U used extensively in this paper by $X \subset \mathbb{P}$, $Y \subset \mathbb{M}$, and $C \subset \mathbb{F}$ so that (10.3) is a surjective double fibration. It need not be the case that $X = Y''$, for instance, and more general phenomena can occur.[12] A particularly interesting generalization is to the case when (1.1), or rather (1.2), is *deformed*. The resulting deformation of twistor space is called the *non-linear graviton construction* (Penrose [35]) and the corresponding space-time is automatically right-flat. The methods of this paper give a cohomological interpretation of the integral formulae of Penrose and Ward [41]. In this right-flat space-time the twistor description of left-handed fields via potentials is no longer equivalent to the field equations. This suggests that the correct description of left-handed fields in a right-flat background is as potentials modulo gauge freedom (see Hitchin [19] and Penrose and Ward [41] for details). There are a least two more examples where the methods of this paper apply. One is to obtain a cohomological understanding of the twistor integral formulae for massive fields (cf. Penrose [33] and Hughston [23]). In

12 cf. Hodges A.P.: Cohomological wave functions. Twistor Newsletter 7, 18, Oxford University, June 1978

the case of two twistors see Eastwood [11]. The other example is in the application of twistor theory to the classification of instantons given by Atiyah, Hitchin, Drinfeld and Manin [2]. If we identify $\mathbb{T}$ with $\mathbb{H}^2$, the 2-dimensional quaternionic vector-space then we obtain an inclusion

$$S^4 = \mathbb{P}_1(\mathbb{H}) \subset \mathbb{M}.$$

In [2] the $\mathscr{P}$-transform is used to identify $H^1(U'', \mathcal{O}(E)(-2))$ with solutions of a certain differential equation (the Yang–Mills coupled wave-equation) on an open subset U of S^4, where E is an "instanton bundle" on $\mathbb{P}$. To establish this result using the methods of this paper we thicken S^4 a little inside $\mathbb{M}$, using the ellipticity of the wave-equation to extend the solution to this thickening. Theorem 3.2 then applies to show μ^* is an isomorphism and the argument continues as in Sect. 6.

Finally, we should remark that the seemingly abstract arguments of this paper can be made quite explicit to give integral formulae for $\mathbb{P}$ and its inverse. The original integral formulae for constructing massless fields (cf. Penrose [29] etc.) really use a Čech representation for the cohomology. This is very useful as it allows the residue theorem of one-variable complex analysis to be applied to produce specific examples of massless fields (especially the "elementary states" (cf. Penrose [33])). For more general arguments this method is not so good, since the choice of contour depends on the choice of covering and can therefore become quite involved (using "branched-contours"). To avoid this one can use the Dolbeault representation of cohomology and integrate over the whole lines L_z in $\mathbb{P}$ rather than contours upon these "lines" (really Riemann spheres). This method is described by Woodhouse [55] and Wells [52]. Both methods really come down to identifying the cohomology $H^1(\mathbb{P}_1(\mathbb{C}), \mathcal{O}(k))$ by means of Serre duality [43]. This integration has been replaced by taking the first direct image $v^1_*\mathcal{O}(k)$ in (2.11). To describe the inverse transform $\mathscr{P}^{-1}$ explicitly we must use the integral formula in Lemma 3.1 to reverse μ^* as in Theorem 3.2, together with an explicit method of inverting the direct image v^1_*. Both of these steps involve some choice but this is only to be expected as the Dolbeault form produced in this way is supposed to be only a representative (one of many) for a cohomology class. A particularly natural way of inverting v^1_* is to choose harmonic representatives for the cohomology of the fibres of v. This is described explicitly by Woodhouse [55] for massless fields and by Rawnsley [42] for instanton bundles. For the potential description of left-handed fields (Sect. 7) integral formulae can be avoided for $\mathbb{P}$ since we are only using Liouville's theorem that $\mathbb{P}_1(\mathbb{C})$ supports only constant holomorphic functions (cf. Penrose and Ward [41]).

In our discussion of fields in $\mathbb{M}$ we have been careful to maintain a distinction between primed and unprimed weights. The physical significance of this distinction is not altogether clear, and its role played in this paper is not a crucial one. However, it may be pointed out that if a local rescaling of the metric is made, as was discussed in Sect. 2 (cf. (2.4), (2.5)), according to which $\varepsilon_{AB} \mapsto \Omega\varepsilon_{AB}$ and $\varepsilon_{A'B'} \mapsto \tilde{\Omega}\varepsilon_{A'B'}$ with Ω and $\tilde{\Omega}$ independent scalar functions on (subsets of) $\mathbb{M}$ then the field equations are *locally* invariant for helicity $\frac{1}{2}r$ (where r is a positive or negative integer) if the weights are taken to be $[-\frac{1}{4}r-\frac{1}{2}]\,[\frac{1}{4}r-\frac{1}{2}]'$. This works also for

$r=0$ if the wave equation is taken in the form $(\Box + \frac{1}{6}R)\phi = 0$, where R is the scalar curvature, the metric being rescaled according to $g_{ab} \to \Omega\tilde{\Omega}g_{ab}$. The potential description of Sect. 7 is invariant for helicity $-\frac{1}{2}n(n>1)$ if the weights are $[-\frac{1}{4}n]$ $[\frac{1}{4}n-1]'$. When the ratio $\Omega/\tilde{\Omega}$ is constant over $\mathbb{M}$ (as is the case for the global GL($\mathbb{T}$) transformations considered here) local conformal invariance depends only on the sum of the primed and unprimed weights (cf. (2.4)), and for massless fields the value -1 for this sum, as obtained in Sects. 5–8, agrees with the above local invariance.

The role of the more general conformal invariance that these fields exhibit when $\Omega/\bar{\Omega}$ need not be constant is obscure, but the specific weights that arise, as mentioned above, seem to be related to the phenomenon of Grgin [15a] according to which it is only the fields of *odd* integral helicity that can satisfy the field equations globally on *real* compactified Minkowski space, assuming that they are to be represented as tensor fields in the ordinary way. Such fields of even integral helicity would require a twofold cover of this space and those of half-odd helicity, a fourfold cover. This arises from the fact that the fields change by a factor of $(-1)^{(1/4)r-1/2}$ across infinity. It is, however, not necessary to take these covering spaces if the fields are regarded as cross-sections of the appropriate twisted vector bundles (see Lerner [24a]). In fact, this interpretation is implicit in the descriptions of this paper. The definition of the spin-bundles $\mathcal{O}_{A'}$ and $\mathcal{O}_A$ that were introduced in Sect. 2 incorporate this twist in a natural way (cf. also [32]).

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