

On the $1/n$ Expansion

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Abstract. The $1/n$ expansion is considered for the n -component non-linear σ -model (classical Heisenberg model) on a lattice of arbitrary dimensions. We show that the expansion for correlation functions and free energy is asymptotic, for all temperatures above the spherical model critical temperature. Furthermore, the existence of a mass gap is established for these temperatures and n sufficiently large.

1. Introduction

It was noted by Stanley in 1967 [1] that certain lattice spin systems exhibit considerable simplification as n , the number of spin components, becomes large. In fact formally, as $n \rightarrow \infty$, these models become the so-called spherical model, introduced and solved by Berlin and Kac in 1952 [2]. Their work should in fact be considered the origin of the studies of large n behaviour of multicomponent systems; it also provided motivation for Stanley's work.

In 1973 Wilson [3] in the context of quantum field theory and Abe [4] and Brezin-Wallace [5], in the context of spin systems, found that there is a systematic way to expand in powers of $1/n$. Subsequently these “ $1/n$ expansions” were used to compute a variety of objects of interest, such as critical temperatures and exponents.

Soon several other theories came into the realm of $1/n$ expansion. 1974 't Hooft [6] gave the solution of 2-dimensional QCD with $SU(N)$ gauge group, as $N \rightarrow \infty$, and Gross and Neveu [7] studied the $1/N$ expansion for 2-dimensional four-fermion interactions (these had already been touched in Wilson's work). More recently the $1/n$ expansion has been applied ([8, 9]) to $\mathbb{C}P^n$ and related σ -models and it has been also suggested to be useful in 4-dimensional QCD [10].

There are several reasons why the $1/n$ expansion has aroused such an interest. It is a non-perturbative expansion, typically each term being a (formal) sum of infinitely many orders of ordinary perturbation theory. Already the zeroth order

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reveals non-trivial structure, e.g. in the case of the non-linear σ -model the formal $n \rightarrow \infty$ limit is the spherical model, which exhibits a phase transition. In scale invariant theories, such as QCD with zero fermion masses and various σ -models, there is no natural parameter in which to perturb and so the $1/n$ expansion is the only expansion available. Finally $1/n$ expansion is expected to be valid near the critical point, providing thus among other things a computational tool alternative to the ε -expansion.

Very little has been rigorously proved about the $1/n$ expansion. The only result known to this author is by Kac and Thompson [11], who proved that the limit of the free energy of spin systems with spins of fixed length is that of the spherical model as $n \rightarrow \infty$. However, their method does not generalize to higher orders in $1/n$, nor does one get any information about the critical temperature, fall-off of correlations etc.

In this paper, we will be considering the $1/n$ expansion for n -component non-linear σ -models on a d -dimensional lattice with $d \geq 2$ near the critical point. As mentioned above, the formal $n \rightarrow \infty$ limit of these models is the spherical model, which has non-zero critical temperature T_S in more than two dimensions and in $d=2$, $T_S=0$. We will study the $1/n$ expansion for T above T_S and show that it is an asymptotic expansion for correlation functions arbitrary near T_S . This is achieved by obtaining an explicit expression for the remainder to k orders of the expansion, which we show to have a bound proportional to n^{-k-1} (Sect. 6). Moreover, spatial behaviour of the remainder can be studied, and we will establish exponential falloff in the odd sector and certain parts of the even sector. This proves the existence of a massgap (in these sectors) for all temperatures T above T_S as $n > n(T)$ (Sect. 5).

We note, that it has been previously shown using infrared bounds [12], that the critical temperature $T^{(n)}$ for spontaneous magnetization satisfies $T^{(n)} \geq T_S$ in dimensions $d \geq 3$. If we define $T_1^{(n)}$ to be the smallest temperature above which there is massgap, our results imply that (to be precise, only in the above mentioned sectors) $T_1^{(n)} \leq T_S + an^{-b}$ where $a, b > 0$. We also note, that the known upper bounds for $T^{(n)}$ differ from T_S by a numerical factor [13].

By the Mermin-Wagner theorem, there is no long range order in two dimensions for our models. It is a conjecture, that for $n > 2$ $T_1^{(n)}$ is zero. However no rigorous results exist. Our results imply that $T_1^{(n)} \leq \alpha(\log n)^{-\beta}$, $\alpha, \beta > 0$.

The proofs of our results rely on the use of a “dual” representation of the model (Sect. 2), in which the n -dependence is transparent. On the other hand most of the estimates depend on the fact that in the original representation we can use reflection positivity in the form of chessboard estimates. The interplay of the two representations is the underlying philosophy of our approach.

In the dual representation the $1/n$ expansion turns out to be an expansion about a saddle point of a certain infinite dimensional integral. The technical problems with it are twofold. First of all, the integration measure is complex and non-local. Secondly, local expectations in the original representation are non-local in the dual representation. The second problem is solved using random walk ideas of Brydges and Federbush [14] to reduce to local expectations. Their methods play an important role in our work, allowing us to reduce the problem of proving clustering to that of obtaining sharp pressure estimates. Due to the fact that the measure is complex and non-local these are however nontrivial to prove. We

control the behaviour in the neighborhood of the saddle point by using chessboard estimates in the spin representation as a lower bound, which allows us to reduce to finite volume in the dual representation. The integral away from the saddle point is estimated using chessboard estimates in the dual representation (Sect. 7).

The main results can be found in Sect. 3.

Finally, we remark that results similar to those proved in this paper can be proved for the $(\phi^2)^2$ Euclidean Quantum Field Theory (continuum) in two dimensions. This will be discussed in a separate publication (see also [20]).

2. The Model and the Formal Expansion

We will start by defining the model and giving a formal derivation of the expansion.

Let Λ be a simple, cubic, d -dimensional periodic lattice obtained from $(-l + \frac{1}{2}, l + \frac{1}{2})^d \subset \mathbb{Z}^d + (\frac{1}{2}, \dots, \frac{1}{2})$ by identifying points $(i, \dots, l + \frac{1}{2}, i_{j+1}, \dots, i_d)$ and $(i_1, \dots, -l + \frac{1}{2}, i_{j+1}, \dots, i_d), j = 1, \dots, d$. We occasionally imagine Λ imbedded in the (flat) torus T_Λ obtained from $[-l + \frac{1}{2}, l + \frac{1}{2}] \subset \mathbb{R}^d$. The non-linear σ -model (the classical Heisenberg model) is described by the random variables (spins) $\phi : \Lambda \rightarrow \mathbb{R}^n$ whose joint distribution is

$$d\mu_0^{(\Lambda)}(\phi) = Z_0(\Lambda, n, \beta)^{-1} \exp\left(\frac{\beta}{2} \sum_{|i-j|=1} \phi_i \cdot \phi_j\right) \cdot \prod_{i \in \Lambda} \delta(\phi_i^2 - n) d^n \phi_i, \tag{1a}$$

where $Z_0(\Lambda, n, \beta)$, the partition function, is chosen so that μ_0 is a probability measure on $\mathbb{R}^{n\Lambda}$. $\beta \equiv T^{-1}$ is the inverse temperature. The normalization of ϕ^2 can be changed by scaling β ; n turns out to be natural since it leads to a nontrivial limit as $n \rightarrow \infty$. It is also convenient to rescale $\phi \rightarrow \beta^{-1/2} \phi$, since in $d=2$ we shall be considering $\beta \rightarrow \infty$. Thus we will study the following measure

$$d\mu^{(\Lambda)}(\phi) = Z(\Lambda, n, \beta)^{-1} \exp\left(\frac{1}{2} \sum_{|i-j|=1} \phi_i \cdot \phi_j\right) \prod_{i \in \Lambda} \delta(\phi_i^2 - n\beta) d^n \phi_i. \tag{1b}$$

We will usually suppress Λ in $d\mu^{(\Lambda)}$ and other expressions since the $\Lambda \rightarrow \mathbb{Z}^d$ limit of $Z(\Lambda)$ and correlation functions exists by standard arguments (see e.g. [15]) and our bounds are uniform in $|\Lambda|$.

Let us study the characteristic functional $S(g)$ of μ , i.e. $S(g) = \langle \exp(\phi, g) \rangle_\phi$ where

$$\langle \cdot \rangle_\phi = \int \cdot d\mu \tag{2}$$

and $(\phi, g) = \sum_{i \in \Lambda} \sum_{\alpha=1}^n \phi_i^\alpha g_i^\alpha$. After multiplying $S(g)$ by $1 = AA^{-1}$ where $A = \exp(-d - m^2/2)n\beta|\Lambda|$ and using $\phi^2 = n\beta$, we get

$$S(g) = Z_1^{-1} \int e^{-1/2(\phi, (-A + m^2)\phi) + (\phi, g)} \prod_{i \in \Lambda} \delta(\phi_i^2 - n\beta) d^n \phi_i, \tag{3}$$

where Z_1 is the integral with $g=0$, Δ is the lattice Laplacean, i.e. $\Delta_{ij} = -2d\delta_{ij} + \sum_{|k|=1} \delta_{i, j+k}$, and m^2 is at the moment an arbitrary positive constant. We can

write the numerator in (3) as

$$(2\pi)^{-|A|} \int \prod_i d^n \phi_i \int d^{|A|} a e^{i(a, \phi^2 - n\beta)} e^{-1/2(\phi, (-\Delta + m^2)\phi)} e^{\langle \phi, g \rangle}$$

and a similar formula for the denominator. Now the gaussian integrals can be performed and we end up with

$$S(g) = \langle \exp \frac{1}{2} (g, (-\Delta + m^2 - 2ia)^{-1} g) \rangle_a,$$

where $\langle \cdot \rangle_a$ is the normalized measure

$$\det(-\Delta + m^2 - 2ia)^{-n/2} e^{-in\beta \text{tr} a} d^{|A|} a \equiv e^{nF(a)} d^{|A|} a. \tag{4}$$

Note, that the determinant in (4) doesn't have any zeroes since $-\Delta + m^2$ is a positive definite operator and a is self-adjoint. We see from (4) that as $n \rightarrow \infty$ one could expect that the dominant contribution to this measure comes from the vicinity of the critical points of F . The $1/n$ expansion is a formal loop expansion about such a point.

Let us make a change of variables $a \rightarrow az$ where $z = n^{-1/2}$. Thus defining

$$f(a, z) = -\frac{n}{2} \text{tr} \log(1 - 2izCa) - iz^{-1} \beta \text{tr} a, \tag{5}$$

$$d\hat{\mu}(a) = \left(\int e^{f(a, z)} d^{|A|} a \right)^{-1} e^{f(a, z)} d^{|A|} a, \tag{6}$$

we get

$$S(g) = \int \exp \frac{1}{2} (g, (-\Delta + m^2 - 2iaz)^{-1} g) d\hat{\mu}(a), \tag{7}$$

where we cancelled the constant $\det(-\Delta + m^2)^{-n/2}$ and denoted $(-\Delta + m^2)^{-1}$ by C . We will denote expectations in $d\hat{\mu}$, the dual measure, by $\langle \cdot \rangle$.

The critical points are determined from $\partial f / \partial a_i = 0$, i.e.

$$(-\Delta + m^2 - 2iza)_{ii}^{-1} = \beta \quad \text{for all } i \in A. \tag{8}$$

Let us try to choose m^2 such that $a=0$ is a solution to (8), i.e.

$$(-\Delta + m^2)_{ii}^{-1} = \beta \tag{9a}$$

which in the $A \rightarrow \mathbb{Z}^d$ limit can be written as

$$\int_{[-\pi, \pi]^d} \frac{dp}{(2\pi)^d} \left(2 \sum_{i=1}^d (1 - \cos p_i) + m^2 \right)^{-1} = \beta. \tag{9b}$$

Let $m_0(\beta)$ be the solution of (9b) when it exists. We see, that in dimensions $d \geq 3$ there exists a $\beta_s < \infty$ such that $m_0(\beta_s)^2 = 0$, for β below β_s $m_0(\beta)^2 > 0$, and for $\beta > \beta_s$ there is no solution $m_0(\beta)^2 \geq 0$. On the other hand in two dimensions as $\beta \rightarrow \infty$ we can always satisfy (9b) with some $m^2 > 0$ since the integral has a logarithmic singularity at $m^2 = 0$. (9b) is the equation for the massgap $m_0(\beta)$ of the spherical model [2] and β_s is the inverse critical temperature of that model; so for $d=2$ $T_S = \beta_s^{-1} = 0$.

The $1/n$ expansion can thus be considered as an expansion around the spherical model. We will be studying the region $\beta < \beta_s$ so that $m_0(\beta)^2 > 0$. We are in

particular interested in T being close to T_S , i.e. $m_0(\beta)$ close to 0. By a simple computation one gets

$$\lim_{\beta \rightarrow \infty} e^{4\pi\beta} m_0(\beta)^2 < \infty \quad d=2. \tag{10}$$

In $d=3$ one obtains that $m_0(\beta)^{-2}(T - T_S)$ is bounded as $T \rightarrow T_S$.

The expansion for ϕ -expectations is now formally derived by performing the dual transformation

$$\langle G(\phi) \rangle_\phi = \langle \hat{G}(a, z) \rangle$$

and expanding \hat{G} and e^f in powers of z . Since the Taylor series for f at $a=0$ is

$$f(a, z) \sim -\text{tr}(aC)^2 + \frac{1}{2} \sum_{k=3}^{\infty} (2i)^k k^{-1} z^{k-2} \text{tr}(aC)^k \tag{11}$$

one only has to compute gaussian integrals.

Let us consider e.g. the 2-point function which is

$$\langle \phi_i^1 \phi_j^1 \rangle_\phi = \langle (-\Delta + m_0^2 - 2iaz)_{ij}^{-1} \rangle \equiv \langle H_{ij} \rangle \tag{12}$$

by differentiating (7). Hence in this case

$$\hat{G}(a, z) \sim \sum_{k=0}^{\infty} (2iz)^k [C(aC)^k]_{ij}$$

and we have to compute integrals of the form

$$\int (aC)^k \prod_i \text{tr}(aC)^{l_i} d\varrho(a),$$

where $d\varrho$ is the gaussian measure on $\mathbb{R}^{|A|}$ with inverse covariance defined by the quadratic form

$$2\text{tr}(aC)^2 \equiv 2(a, B^{-1}a).$$

Thus e.g. the $O(1/n)$ terms are

$$-4n^{-1} \int (CaCaC)_{ij} d\varrho(a) + 8(3n)^{-1} \int (CaC)_{ij} \text{tr}(aC)^3 d\varrho(a). \tag{13}$$

In general the k :th order in $1/n$ expansion for the 2-point function can thus be expressed graphically as follows. Consider connected graphs made out of various rings (Fig. 1), with up to $2k+1$ external legs, and one chain (Fig. 2) with up to $2k$ legs, by joining the legs pairwise. A ring with n legs carries a factor $\frac{1}{2}(2i)^n z^{n-2}$ and a chain $(2iz)^n$. Straight lines represent C_{ij} and curly ones $\frac{1}{2}B_{ij}$, where $(B^{-1})_{ij} = (C_{ij})^2$. Sum of such graphs with total power of z $2k$ and multiplied by numerical factors constitutes the k :th order term in the expansion. The expansion has only even orders of z . E.g. the expression (13) is represented by the graphs of Fig. 3. We take



as the starting point of our analysis the dual measure $d\hat{\mu}$. The basic technical properties of this measure are given in Sects. 4 and 7.

3. The Main Results

To state the results we need to introduce some notation. Let A be a finite set of points in \mathbb{Z}^d , same point possibly occurring several times, and $\alpha : A \rightarrow [1, 2, \dots, n]$. Let $\langle \phi_A^\alpha \rangle$ be the correlation function

$$\langle \phi_A^\alpha \rangle = \left\langle \prod_{i \in A} \phi_i^{\alpha(i)} \right\rangle.$$

Recall, that $m_0(\beta)$ is the spherical model mass gap; let us denote $\mu_0(\beta) = \cosh^{-1}(1 + \frac{1}{2}m_0(\beta)^2)$. Theorem 1 proves the existence of mass gap in the odd sector for T arbitrarily near T_s – the spherical model critical temperature – as n is sufficiently large.

Theorem 1. *Let $A = A_1 \cup A_2$ where A_1 and A_2 are odd. Let $d(A_1, A_2)$ be the distance between A_1 and A_2 . Then there exist constants $a_1, a_2 > 0$, depending on the dimension d , such that for all $\beta < \beta_s$ and $n > a_1 m_0(\beta)^{-a_2}$*

$$\langle \phi_{A_1}^{\alpha_1} \phi_{A_2}^{\alpha_2} \rangle \leq R(\beta, |A|) e^{-\mu(\beta, n)d(A_1, A_2)},$$

where $\mu(\beta, n) > 0$ and moreover $\mu(\beta, n) \rightarrow \mu_0(\beta)$ as $n \rightarrow \infty$.

We will show that $a_2 \leq 6(d + 2) + 1$ and also get an explicit bound for R (Sect. 5).

Let now $\sum_{m=0}^{\infty} s_m(A, \alpha) n^{-m}$ be the formal $1/n$ expansion for $\langle \phi_A^\alpha \rangle$, derived in Sect. 2. Theorem 2 gives a bound for the remainder of this expansion.

Theorem 2. *Let $\beta < \beta_s$ and $n > \max(8r, a_1 m_0(\beta)^{-a_2})$, where a_1 and a_2 are as in Theorem 1. Then*

$$\left| \langle \phi_A^\alpha \rangle - \sum_{m=0}^{r-1} s_m(A, \alpha) n^{-m} \right| \leq R(r, \beta, A, \alpha) n^{-r}.$$

Thus, in particular, the $1/n$ expansion is an asymptotic expansion for correlation functions (point-wise for T arbitrarily near the spherical model critical temperature). We will get quite explicit bounds for $R(r, \beta, A, \alpha)$, in particular in the odd sector R has exponential falloff $e^{-\mu(n, \beta)d(A_1, A_2)}$ (see Remark 1 after the proof of Theorem 2).

Also, the free energy is shown to have an asymptotic expansion in powers of $1/n$ (Remark 3 in Sect. 6).

4. Reflection Positivity of $d\hat{\mu}$

We will now prove the basic technical property of $d\hat{\mu}$, reflection positivity. Since in Sect. 7 will also need some related measures, we define in general a two parameter family of complex measures on $\mathbb{R}^{|A|}$

$$d\hat{\mu}_{\kappa, t}(a) = Z_{\kappa, t}^{-1} e^{f(a, z)} \chi_{\kappa}(a) \prod_{i \in A} h_t(a_i) da_i, \tag{14}$$

where $\chi_\kappa(a)$ is the characteristic function of the cube $|a_i| \leq \kappa$, $i \in \Lambda$, and

$$h_t(a) = (2d + m_0^2 - 2iza)^{-t}. \tag{15}$$

$Z_{\kappa,t}$ is defined so that $\int d\hat{\mu}_{\kappa,t} = 1$. The measures will be considered only for those values of κ, t when this is possible. We will later see, that this happens for $t=0$, $\kappa > 0$ and for all t if $\kappa > \kappa(t, \Lambda)$. Expectation in $d\hat{\mu}_{\kappa,t}$ is denoted by $\langle \cdot \rangle_{\kappa,t}$. Note that $d\hat{\mu}_{\infty,0} = d\hat{\mu}$. We prove, that $d\hat{\mu}_{\kappa,t}$, although complex, is reflection positive for $z = (\text{integer})^{-1/2}$. We introduce some notation. Let $\Lambda_+ = \{i \in \Lambda \mid i_1 \in [\frac{1}{2}, l - \frac{1}{2}]\}$, and let $\theta : \Lambda \rightarrow \Lambda$ be the reflection in the $x_1 = 0$ plane. Hence $\Lambda = \Lambda_+ \cup \theta\Lambda_+$. For $X \subset \Lambda$ let $A(X)$ denote the functions $F \in S(\mathbb{R}^{|\Lambda|})$ depending only on the variables $\{a_i\}_{i \in X}$. We define $\tau : A(X) \rightarrow A(X)$ by

$$(\tau F)(a) = \overline{F(-a)}.$$

We now extend $\theta : A(X) \rightarrow A(\theta X)$ by

$$(\theta F)(a) = (\tau F)(\theta a), \quad (\theta a)_i = a_{\theta i}.$$

Then we have

Proposition 3. *Let $z = n^{-1/2}$, $n \in \mathbb{N}$ and $Z_{\kappa,t} \neq 0$. Then*

- (a) $\langle F\theta F \rangle_{\kappa,t} \geq 0$ for $F \in A(\Lambda_+)$ (Reflection positivity).
- (b) Let $f_i \in A(\{i\})$ satisfy $\tau f_i = f_i$. Then

$$\left| \left\langle \prod_{i \in \Lambda} f_i(a_i) \right\rangle_{\kappa,t} \right| \leq \prod_{i \in \Lambda} \left\langle \prod_{j \in \Lambda} f_i(a_j) \right\rangle_{\kappa,t}^{1/|\Lambda|}$$

(Chessboard estimates).

Proof. (a) We transform $\langle F\theta F \rangle_{\kappa,t}$ back to the ϕ -representation. Let

$$\check{F}(\phi) = \int e^{i(a \cdot \phi^2 - n\beta)} F\left(\frac{a}{z}\right) \chi_{z\kappa}(a) \prod_{i \in \Lambda_+} h_t(z^{-1}a_i) da_i,$$

where a_i is defined to be zero for $i \in \Lambda_-$. Then

$$\langle F\theta F \rangle_{\kappa,t} = \langle \check{I}(\theta 1) \check{\gamma} \rangle_0^{-1} \langle \check{F}(\theta F) \check{\gamma} \rangle_0,$$

where $\langle \cdot \rangle_0$ is the normalized measure

$$\exp \frac{1}{2}(\phi, \Delta \phi) \prod_{i \in \Lambda} e^{-\frac{m^2}{2} \phi_i^2} d^n \phi_i.$$

We denote the standard reflection operator in ϕ -representation by θ_ϕ i.e.

$$\theta_\phi h(\phi_i) = h(\phi_{\theta i})^*.$$

We get

$$\begin{aligned}
 (\theta F)^\vee(\phi) &= \int e^{i(a, \phi^2 - n\beta)} (\tau F) \left(\theta \frac{a}{z} \right) \chi_{z\kappa}(a) \prod_{j \in \theta A_+} h_t(z^{-1} a_j) da_j \\
 &= \int e^{i(a, (\theta\phi^2 - n\beta))} F \left(-\frac{a}{z} \right)^* \chi_{z\kappa}(a) \prod_{j \in A_+} h_t(z^{-1} a_j) da_j \\
 &= \int e^{-i(a, (\theta\phi^2 - n\beta))} F \left(\frac{a}{z} \right)^* \chi_{z\kappa}(a) \prod_{j \in A_+} h_t(-z^{-1} a_j) da_j \\
 &= \left[\int e^{i(a, (\theta\phi^2 - n\beta))} F \left(\frac{a}{z} \right) \chi_{z\kappa}(a) \prod_{j \in A_+} h_t(z^{-1} a_j) da_j \right]^* \\
 &= \check{F}(\theta_\phi \phi)^* \\
 &= (\theta_\phi \check{F})(\phi).
 \end{aligned}$$

Thus $\langle F\theta F \rangle_{\kappa,t} = \frac{\langle \check{F}\theta_\phi \check{F} \rangle_0}{\langle \check{1}\theta_\phi \check{1} \rangle_0} \geq 0$

by Reflection positivity of $\langle \cdot \rangle_0$ and by assumption $\langle \check{1}\theta_\phi \check{1} \rangle \neq 0$.

(b) Chessboard estimates are a consequence of (a) by the general theory of reflection positivity [16]. \square

5. Mass Gap

In this section we will establish the existence of a mass gap in the odd sector for all $\beta < \beta_s$ as $n > n(\beta)$.

Let us consider the correlation functions $\langle \phi_A^\alpha \rangle$ defined in Sect. 3. Let $A_i = \alpha^{-1}(i)$. We can transform $\langle \phi_A^\alpha \rangle$ to the dual representation by using (7).

$$\langle \phi_A^\alpha \rangle = \left\langle \prod_{i=1}^n \phi_{A_i}^i \right\rangle = \sum_{\{p_i \in P(A_i)\}} \left\langle \prod_{i=1}^n \prod_{\langle jj' \rangle \in p_i} H_{jj'} \right\rangle, \tag{16}$$

where $P(A_i)$ is the set of pairings of the elements of A_i and the right hand side of (16) is defined to be zero if any A_i is odd. As in (12) $H = (-\Delta + m_0^2 - 2iaz)^{-1}$.

As it turns out all estimates of correlation functions such as the ones occurring on the r.h.s. of (16) can be reduced to pressure estimates, namely upper and lower bounds for the partition functions $Z_{\kappa,t}$ occurring in (14). In this section we will in particular need bounds for

$$\begin{aligned}
 \xi^{(t)}(z) &= \lim_{A \rightarrow \mathbb{Z}^d} Z_{\infty,t}(A)^{1/|A|} \\
 &\equiv \lim_{A \rightarrow \mathbb{Z}^d} \left[\int e^{J(a,z)} \prod_{i \in A} h_t(a_i) da_i \right]^{1/|A|}.
 \end{aligned} \tag{17}$$

Let also ξ denote the formal $z \rightarrow 0$ limit of $\xi^{(0)}(z)$:

$$\xi = \lim_{A \rightarrow \mathbb{Z}^d} \left[\int d^{|A|} a e^{-\text{tr}(aC)^2} \right]^{1/|A|}.$$

The proofs of the following basic results are deferred to Sect. 7. Recall that RP implies that $\xi^{(t)}(z) \geq 0$.

Proposition 4. *There exist constants γ_1 and γ_2 , depending only on d , such that for $n > \gamma_1 \alpha_1(\beta)$ and $\alpha_1(\beta) \equiv m_0(\beta)^{-2(d+6)-1}$*

$$\xi^{(0)}(z) \geq \xi(1 - \gamma_2(\delta_1/n)^{1/2d}),$$

for all $\beta < \beta_s$, where $\delta_1 = m_0 \alpha_1(\log n)^{2d+3}$.

Proposition 5. *There exist constants γ_3 and γ_4 such that for $n > \gamma_3 \alpha_2(\beta)$ and $\alpha_2(\beta) \equiv m_0(\beta)^{d-9}$*

$$\xi^{(t)}(z) \leq \xi(2d + m_0^2)^{-t} \left(1 + \gamma_4 \frac{\delta_2}{n} \right),$$

where $\delta_2 = m_0 \alpha_2(\log n)^2$.

Remark. Note that the upper bound, Proposition 5 is much sharper than the lower bound. The lower bound thus determines our bounds for critical temperature, physical mass etc.

We will now bound the expectation in (16). The crucial idea is to expand $H_{ij} = (-\Delta + m_0^2 - 2iza)_{ij}^{-1}$ in powers of the off-diagonal part K of $-\Delta = 2d - K$. Such an expansion has been previously used by Brydges and Federbush [14]. Since $\|K\| = 2d$ and $m_0(\beta)^2 > 0$ for $\beta < \beta_s$, the expansion converges for all a . Using the fact that K generates random walk we get

$$H_{ij} = \sum_{\omega: i \rightarrow j} \prod_{k \in A} (2d + m_0^2 - 2iza_k)^{-n(\omega, k)}, \tag{18}$$

where ω runs through the random walks on A from i to j and $n(\omega, k)$ is the number of times the walk ω visits the site k .

Let us denote

$$n(\beta) = m_0(\beta)^{-6(d+2)-1}. \tag{19}$$

We can now prove

Proposition 6. *There exists a constant γ_5 , depending only on the dimension d , such that for $n > \gamma_5 n(\beta)$*

$$\left\langle \prod_{\alpha=1}^r H_{i_\alpha j_\alpha} \right\rangle \leq \prod_{\alpha=1}^r C_{i_\alpha j_\alpha}(m^2(\beta, n)),$$

where $C(m^2) = (-\Delta + m^2)^{-1}$ and the physical mass $m^2(\beta, n) > 0$. Moreover

$$\lim_{n \rightarrow \infty} m(\beta, n) = m_0(\beta).$$

Proof. Using (17) and recalling (15) we get

$$\begin{aligned} \left\langle \prod_{\alpha=1}^r H_{i_\alpha j_\alpha} \right\rangle &= \sum_{\{\omega_\alpha: i_\alpha \rightarrow j_\alpha\}} \left\langle \prod_{\substack{\alpha=1 \\ k \in A}}^r h_{n(\omega_\alpha, k)}(a_k) \right\rangle \\ &= \sum_{\{\omega_\alpha\}} \left\langle \prod_{k \in A} h_{t(\omega_\alpha, k)}(a_k) \right\rangle, \end{aligned}$$

where $t(\omega, k) = \sum_{\alpha=1}^r n(\omega_\alpha, k)$.

Since $\tau h_m = h_m$ we can apply chessboard estimates (Proposition 3b) to obtain (we take the $A \rightarrow \mathbb{Z}^d$ limit)

$$\left| \left\langle \prod_{\alpha=1}^r H_{i_\alpha j_\alpha} \right\rangle \right| \leq \sum_{\{\omega_\alpha\}} \prod_{k \in \mathbb{Z}^d} \frac{\xi^{t(\omega, k)}(z)}{\xi^{(0)}(z)}. \tag{20}$$

Combining Propositions 4 and 5 we can find a $\gamma_5 > 0$ such that as $n > \gamma_5 n(\beta)$

$$\frac{\xi^{(t)}(z)}{\xi^{(0)}(z)} \leq (2d + m^2(\beta, n))^{-t}$$

for all t , where $m^2(\beta, n) > 0$. We can choose $\lim_{n \rightarrow \infty} m^2(\beta, n) = m_0(\beta)^2$ since

$$\limsup_{z \rightarrow 0} \frac{\xi^{(t)}(z)}{\xi^{(0)}(z)} \leq (2d + m_0^2)^{-t}. \text{ Thus}$$

$$\left\langle \prod_{\alpha=1}^r H_{i_\alpha j_\alpha} \right\rangle \leq \sum_{\{\omega_\alpha\}} \prod_{k, \alpha} (2d + m^2)^{-n(\omega_\alpha, k)} = \prod_{\alpha=1}^r C_{i_\alpha j_\alpha}(m^2). \quad \square$$

As a consequence of Proposition 6 we obtain the

Proof of Theorem 1. In (16) each term in the sum includes at least one H_{jj} such that $j \in A_1$ and $j' \in A_2$. Proposition 6 therefore implies the claim since $C(m^2)$ has the desired exponential falloff. \square

We would in general expect that truncated correlations $\langle \phi_{A_1}^{\alpha_1}; \phi_{A_2}^{\alpha_2} \rangle$ decrease exponentially in the distance $d(A_1, A_2)$, for $\beta < \beta_s$ and $n > \gamma_5 n(\beta)$. Proposition 6 however will not permit us to perform the necessary cancellations required by the truncation. We thus get exponential clustering only when the truncation is trivial, i.e. also in certain parts of the even sector. However it is possible to extend this result to certain non-trivial truncations using *Ward identities* as follows.

Let $\langle \cdot \rangle_{\phi, \varepsilon}$ be the expectation in the (normalized) measure $e^{-1/2(\phi, \varepsilon \phi)} d\mu(\phi)$, where $\varepsilon_{ij} = \varepsilon_i \delta_{ij}$, $\varepsilon_i > 0$, and $\langle \cdot \rangle_\varepsilon$ in the corresponding dual measure. Of course $\langle \cdot \rangle_{\phi, \varepsilon} = \langle \cdot \rangle_\phi$. We thus get ‘‘Ward identities’’

$$0 = \frac{d}{d\varepsilon_k} \langle F(\phi) \rangle_{\phi, \varepsilon} = \frac{d}{d\varepsilon_k} \langle \hat{F}_\varepsilon(a, z) \rangle_\varepsilon$$

which by simple computation can be written in the form $(H^{(\varepsilon)} = (H^{-1} + \varepsilon)^{-1})$

$$\left\langle \frac{d\hat{F}_\varepsilon}{d\varepsilon_k} \right\rangle_\varepsilon - \frac{n}{2} \langle H_{kk}^{(\varepsilon)}; \hat{F}_\varepsilon \rangle_\varepsilon = 0. \tag{21}$$

In particular let $\hat{F} = H_{ij}$ so

$$\left. \frac{d\hat{F}^{(\varepsilon)}}{d\varepsilon_k} \right|_{\varepsilon=0} = -H_{ik} H_{kj}$$

and thus inserting in (21)

$$\langle H_{kk}; H_{ij} \rangle = -\frac{2}{n} \langle H_{ik} H_{jk} \rangle.$$

Hence

$$\begin{aligned} \langle (\phi_k^{(1)})^2; \phi_i^{(1)} \phi_j^{(1)} \rangle &= \langle H_{kk}; H_{ij} \rangle + 2 \langle H_{ik} H_{jk} \rangle \\ &= 2 \left(1 - \frac{1}{n} \right) \langle H_{ik} H_{jk} \rangle \end{aligned} \tag{22}$$

and we were able to perform explicitly the cancellations in this non-trivial truncation. The right hand side of (22) can be bounded as before by $2 \left(1 - \frac{1}{n} \right) C_{ik}(m^2) C_{jk}(m^2)$ and we get the desired exponential falloff.

In a similar way we can deal with the expectations $\langle (\phi_k^{(1)})^2; \prod_j \phi_j^{\alpha_j} \rangle$ and we get exponential falloff.

6. The 1/n Expansion and a Bound for the Remainder

We shall in this section derive the 1/n expansion in a nonperturbative way and bound the remainder.

The expansion is generated by “resolvent” expanding H ’s in (16) and integrating factors of a by parts in the dual measure. We rewrite the f of (5) as

$$f(a, z) = -\text{tr}(aC)^2 + g(a, z) \equiv -(a, B^{-1}a) + g(a, z) \tag{23}$$

thus defining g and B^{-1} . We need the following elementary Lemma, whose proof is given in the end of this section:

Lemma 7. B^{-1} defined in (23) is a strictly positive operator, whose inverse B obeys

$$|B_{ij}| \leq b m_0^2 e^{-\alpha \mu_0 |i-j|}$$

with α and b constants and $\mu_0 = \cosh^{-1} \left(1 + \frac{m_0^2}{2} \right)$.

The integration by parts formula is (we denote $\partial_j \equiv \partial/\partial a_j$)

$$\langle a_i F \rangle = \frac{1}{2} \sum_j B_{ij} \langle (\partial_j + \partial_j g) F \rangle. \tag{24}$$

$\partial_j g$ is easily computed from (5):

$$\begin{aligned} \partial_j g &= iz^{-1} \text{tr} P_j H - iz^{-1} \beta + 2 \text{tr} P_j CaC \\ &= 4iz \text{tr} P_j CaHaC, \end{aligned} \tag{25}$$

where P_j is the projection onto the j :th coordinate and we used the saddle point condition $C_{ii} = \beta$. $\partial_j g$ thus has an explicit z factor. Note that in perturbation theory $\partial_j g$ is $O(z)$. In (24) we may also encounter $\partial_j H$:

$$\partial_j H = 2izHP_jH \tag{26}$$

The “resolvent” formula for H is:

$$H = C + 2izCaH. \tag{27}$$

As an example of above let us derive the $O(1/n)$ reminder for the two-point function $\langle H_{ij} \rangle$. We first expand $\langle H \rangle = C + 2izC\langle aH \rangle$ and then integrate the a by parts to obtain

$$\langle H \rangle = C + \frac{R^{(1)}}{n},$$

where

$$R^{(1)} = 4 \sum_{k,l} CP_k B_{kl} \langle H \operatorname{tr}(P_l CaHaC) \rangle - 2 \sum_{k,l} CB_{kl} \langle HP_l H \rangle.$$

In the general case (16) we proceed in the same way. We first expand an H using (27). In terms having a 's we always integrate by parts. In terms having no a 's we expand an H . When no H are left we have generated a term in the expansion. The remainder has an explicit n^{-k} factor and involves products of C_{ij} 's and B_{kl} 's contracted to expectations of the form

$$\left\langle \prod_{i \in I} a_i \prod_{\langle k,l \rangle} H_{kl} \right\rangle \tag{28}$$

which we now bound.

Lemma 8. *Let $n \geq \max(4I, \gamma_s n(\beta))$. Then*

$$\left| \left\langle \prod_{i \in I} a_i \prod_{\alpha=1}^r H_{k_{\alpha l} \alpha} \right\rangle \right| \leq \frac{|I|!}{2} (b)^{|I|} \prod_{\alpha} C_{k_{\alpha l} \alpha}(m^2),$$

where b is a constant and m^2 is as in Proposition 6.

Proof. Using (18) and recalling (15) we get

$$\left| \left\langle \prod_j a_j \prod_{\alpha} H_{k_{\alpha l} \alpha} \right\rangle \right| \leq \sum_{\{\omega_{\alpha}\}} \left| \left\langle \prod_j (ia_j) \prod_{l \in A} h_{t(\omega,l)}(a_l) \right\rangle \right|,$$

where we inserted a factor $i^{|I|}$. The summand is again of the form $\left\langle \prod_l G_l(a_l) \right\rangle$, and since $G_l(a)$ is $(ia)^s h_t(a)$ for some s and t we have $\tau G_l = G_r$. Using chessboard estimates it is bounded by $\prod_l \left| \left\langle \prod_r G_r(a_r) \right\rangle \right|^{1/|A|}$. If $s=0$ we can use Proposition 6. Thus it suffices to consider $s \neq 0$. Then

$$\left| \left\langle \prod_r G_r(a_r) \right\rangle \right| \leq \left| \int e^f da \right|^{-1} (2d + m_0^2)^{-i} \int e^{\operatorname{Re} f} \prod_r |a_r^s| da_r.$$

The last factor equals

$$\begin{aligned} & \int da \prod_r |a_r^s| |\det(1 - 2izCa)|^{-n/2} \\ &= \int da \prod_r |a_r^s| \det(1 + 4z^2 C^{1/2} aCaC^{1/2})^{-n/4}. \end{aligned} \tag{28}$$

Since $C^{1/2}aCaC^{1/2} \geq c_0 C^{1/2}a^2 C^{1/2}$ where $c_0 = \inf \text{spec } C = (2d + m_0^2)^{-1}$ and since by minimax principle, if $0 \leq A \leq B$, $\det A \leq \det B$, we get

$$\begin{aligned} & \det(1 + 4z^2 C^{1/2} a C a C^{1/2})^{-n/2} \\ & \leq \det(1 + 4c_0 z^2 C^{1/2} a^2 C^{1/2})^{-n/4} = \det(1 + 4c_0 z^2 a C a)^{-n/4} \\ & \leq \det(1 + 4c_0^2 z^2 a^2)^{-n/4} = \prod_{i \in A} \left(1 + \frac{4}{n} c_0^2 a_i^2\right)^{-n/4}. \end{aligned}$$

Since $s \leq |I| \leq \frac{n}{4}$ (28) is now by explicit computation bounded by $\left[\frac{s}{2}\right]! b_1^s$ where $b_1 = (2d + m_0^2) b_2$ and b_2 is independent on β . We can now undo $\sum_{\{\omega\}}$ as in Proposition 6, and we get the claim since $\prod_{\Sigma s_k = |I|} (1/2s_k)! \leq \frac{|I|!}{2}$ and by Proposition 4 $[\int e^J]^{-1/A}$ is bounded by a constant ($\det B < \infty$ even at β_s). \square

We are now in a position to prove Theorem 2:

Proof of Theorem 2. From Lemma 8 and the discussion preceeding it we see, that the estimate for the remainder to $k-1$ orders of the expansion of (16) is a sum of terms of the form

$$\frac{1}{n^k} d(k, \beta) \sum_{\{l_j(k_j)\}} A(l_j(k_j)) \prod_{\langle jj' \rangle} \prod_{k_j=0}^{p_j} C_{l_j(k_j)l_j(k_j+1)}^{(k_j)}, \tag{29}$$

where $C^{(k_j)}$ is either $C(m_0^2)$ or $C(m^2)$, $\{l_j(k_j)\}$ run through $\bigtimes^P \mathbb{Z}^d$ and $A(l)$ is a graph with $P = \sum p_j$ external legs and lines B_{ij} and C_{ij} with CC and CCB vertices, such that each connected component of A contains a leg $l_j(k_j)$. We defined $l_j(0) = j$ and $l_j(p_j + 1) = j'$. Since C and B have exponential falloff, $\sup_{(l)} A(l) < \infty$. The claim follows. \square

Remark 1. The r and β dependence of $R(r, \beta, A, \alpha)$ are easy to study. Note that if we formally let $z \rightarrow 0$ in our remainder, all H 's turn into C 's and we are left with expectations $\left\langle \prod_{i \in I} a_i \right\rangle_0$ in the gaussian measure with covariance $\frac{1}{2}B$, which produce $e^{c|I|} \Gamma\left(\frac{|I|}{2}\right)$ terms. The remainder then turns into the corresponding term of the expansion. Thus from Lemma 7

$$R(r, \beta, A, \alpha) \leq R_1(r) R_2(\beta, A, \alpha),$$

where $R_1(r) \leq b_1^r R_3(r)$, R_3 being the number of graphs in the r :th order term.

Remark 2. As for the β dependence, we use the fact that the remainder consists of graphs of the expansion with some B lines removed. E.g. for the two-point function the largest terms are those with maximal number of a factors, graphs such as in

Fig. 4, where dots denote a 's. These are proportional to

$$(C_{ii}^3)^{2r+1} \left(\sum_j |B_{ij}| \right)^{2r+1} \sim m_0^{-2(2r+1)}$$

by using Lemma 7. These claims can be established by an analysis similar to the one sketched in Sect. 8.

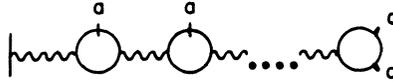


Fig. 4

Remark 3. Theorem 2 allows us to prove that the expansion is asymptotic to the

free energy $p = \lim_{A \rightarrow \mathbb{Z}_\beta^d} p_A$, where

$$p_A = |A|^{-1} \log \int e^{\sum \phi_i \cdot \phi_j} \prod_{i \in A} A_n^{-1} \delta(\phi_i^2 - n) d^n \phi_i$$

and A_n is the area of $n-1$ dimensional sphere of radius n . Namely

$$p_A(\bar{\beta}) = \int_0^{\bar{\beta}} d\beta \frac{dp_A(\beta)}{d\beta} = n d \int_0^{\bar{\beta}} \frac{d\beta}{\beta} \langle \phi_0^1 \phi_{e_1}^1 \rangle \tag{30}$$

and we can now expand the right hand side of (30) in powers of $1/n$. As $\beta \rightarrow 0$ (high temperature) we only need to note that $\beta^{-1} m_0(\beta)^2 \rightarrow \text{constant}$ [from (9)] and that the b in Lemma 8 is bounded by $\alpha_1 + \alpha_2 m_0^2$. It is now easy to verify that the remainder to $k-1$ orders is bounded by $R_k(\bar{\beta}) \beta n^{-k}$ uniformly for $\beta \in [0, \bar{\beta}]$.

We now prove Lemma 7.

Proof of Lemma 7. Positivity of B^{-1} follows from

$$(a, B^{-1}a) = \text{tr } Ca^\dagger Ca = \text{tr } C^{1/2} a^\dagger Ca C^{1/2} \geq 0$$

since $a^\dagger Ca$ is a positive operator by the positivity of C .

As for the exponential falloff, recall that

$$B_{ij} = \int \frac{dk}{(2\pi)^d} \left(\int_{I_d} dp \hat{C}(p) \hat{C}(p-k) \right)^{-1} e^{ik(i-j)},$$

where

$$\hat{C}(p) = \left(2 \sum_{i=1}^d (1 - \cos p_i) + 2(\cosh \mu_0(\beta) - 1) \right)^{-1}$$

and $I_d = [-\pi, \pi]^d$. Now $(\hat{C} * \hat{C})(k)$ is periodic in $\text{Re } k$, and analytic in k_1 for $|\text{Im } k_1| \leq \mu_0(\beta)$ because $\text{Re}[\hat{C}(p)\hat{C}(p-k)]$ is strictly positive there. Thus we can shift the k_1 integration to $[-\pi, \pi] \pm i(\mu_0 - \varepsilon)$ and hence obtain falloff in the 1-direction and by symmetry to all coordinate directions. Taking geometrical means and estimating the overall constant gives the claim. \square

7. Bounds for the Dual Measure

This section proves the basic pressure estimates, Propositions 4 and 5.

For large n (small z) we expect our measures $d\hat{\mu}_{\kappa,t}$ to get their main contribution from the vicinity of $a=0$, the critical point. Thus we expect $Z_{\kappa,t}$ to depend very little on κ as the following lemma shows.

Lemma 9. *Let $\xi_{\kappa,t}(A, z) = Z_{\kappa,t}(A, z)^{1/|A|}$. Then*

$$\left| \frac{d\xi_{\kappa,t}}{d\kappa} \right| \leq 2 \left(1 + \frac{4\kappa^2}{n(2d+m_0^2)^2} \right)^{-n/4} (2d+m_0)^{-t}.$$

Proof. Let κ be such that $\xi_{\kappa,t} \neq 0$ (if $\xi_{\kappa,t} = 0$ for all κ we are done; however, e.g. $\xi_{\infty,t} > 0$ since in the ϕ -representation the corresponding single spin measure is positive). Recall that $\xi_{\kappa,t} \geq 0$ by Proposition 3.

Defining

$$\delta_{\kappa}^+(a) = \delta(a + \kappa) + \delta(a - \kappa)$$

we compute

$$\begin{aligned} \frac{d\xi_{\kappa,t}}{d\kappa} &= |A|^{-1} \xi_{\kappa,t} \left\langle \sum_{i \in A} \delta_{\kappa}^+(a_i) \right\rangle_{\kappa,t} \\ &= \xi_{\kappa,t} \cdot \langle \delta_{\kappa}^+(a_i) \rangle_{\kappa,t}, \end{aligned}$$

where we used translation invariance in the second step. Since $\tau\delta^+ = \delta^+$, Proposition 3 gives (by a limiting argument)

$$\begin{aligned} \left| \frac{d\xi_{\kappa,t}}{d\kappa} \right| &\leq \xi_{\kappa,t} \left| \left\langle \prod_{l \in A} \delta_{\kappa}^+(a_l) \right\rangle_{\kappa,t} \right|^{1/|A|} \\ &= \left| \int \prod_l \delta_{\kappa}^+(a_l) h_l(a_l) e^{f(a,z)} \chi_{\kappa}(a) da \right|^{1/|A|} \\ &\leq 2 \sup_{a_i = \pm \kappa} \left| e^{f(a,z)} \prod_{l \in A} h_l(a_l) \right|^{1/|A|}, \end{aligned} \tag{31}$$

where in the second step we cancelled $\xi_{\kappa,t}$ and in the third step noticed that there is no integration left. Recalling that

$$e^{f(a,z)} = \det(1 - 2izCa)^{-n/2} e^{-iz^{-1}\beta \text{tr} a}$$

we get

$$|e^{f(a,z)}|^2 \leq \det(1 + 4z^2 C^{1/2} a C a C^{1/2})^{-n/2}$$

and using minimax principle as in Lemma 8

$$\det(1 + 4z^2 C^{1/2} a C a C^{1/2})^{-n/2} \leq \left(1 + \frac{4\kappa^2 z^2}{(2d+m_0^2)^2} \right)^{-n|A|/2} \tag{32}$$

since $a^2 = \kappa^2$. Inserting (32) to (31) we get the claim for $\xi_{\kappa,t} \neq 0$ since $|h_l(a_l)| \leq (2d+m_0)^{-t}$. By continuity the claim holds for all κ . \square

Lemma 9 enables us to reduce the study of $\xi^{(t)}$ in Propositions 4 and 5 to the study of the cutoff partition functions $Z_{\kappa,t}$

$$Z_{\kappa,t}(A, z) = \int e^{f(a,z)} \chi_{\kappa}(a) \prod_{l \in A} h_l(a_l) da_l.$$

We wish to bound $Z_{\kappa,t}(A, z)$ in terms of $\lim_{z \rightarrow 0} Z_{\kappa,t}(A, z)$. Let us define for $\kappa \in [0, \infty]$

$$\xi_{\kappa}(A, z) = Z_{\kappa,0}(A, z)^{1/|A|}, \tag{33a}$$

$$\xi_{\kappa}(z) = \lim_{A \rightarrow \mathbb{Z}^d} \xi_{\kappa}(A, z), \tag{33b}$$

$$\xi_{\kappa}(A) = \lim_{z \rightarrow 0} \xi_{\kappa}(A, z) = \left(\int e^{-\text{tr}(aC_A)^2} \chi_{\kappa}(a) da \right)^{1/|A|}, \tag{33c}$$

$$\xi_{\kappa} = \lim_{A \rightarrow \mathbb{Z}^d} \xi_{\kappa}(A). \tag{33d}$$

In (33c) we denoted explicitly the A dependence of $C_A = (-\Delta_A + m_0(\beta, A)^2)^{-1}$, where we recall that m_0^2 is also A dependent since it is a solution of the saddle point condition $(C_A)_{ii} = \beta$ (9a).

Proposition 5 is now rather easy to prove:

Proof of Proposition 5. We can estimate $\xi_{\kappa,t}$ by taking absolute values:

$$|\xi_{\kappa,t}(A, z)|^{|A|} \leq (2d + m_0^2)^{-t|A|} \int e^{\text{Re } f(a,z)} \chi_{\kappa}(a) da. \tag{34}$$

Now from (5)

$$\begin{aligned} \text{Re } f(a, z) &= \text{Re} \left(\frac{n}{2} \int_0^1 ds \text{tr} [2izCa(1 - 2izsCa)^{-1}] - iz^{-1} \beta \text{tr } a \right) \\ &= -\text{tr}(aC)^2 + 8z^2 \text{Re} \int_0^1 ds s^3 \text{tr}(Ca)^4 (1 - 2izsCa)^{-1}, \end{aligned} \tag{35}$$

where in the last step we used $C_{ii} = \beta$. Inserting (35) to (34) and choosing $z \leq \frac{m_0}{4}$ we obtain

$$|\xi_{\kappa,t}(A, z)| \leq \xi_{\infty}(A) (2d + m_0^2)^{-t} \exp 4z^2 \kappa^4 |A|^{-1} \text{tr } C_A^4. \tag{36}$$

Using the Fourier representation of C we get

$$\frac{1}{|A|} \text{tr } C_A^4 \leq b_1 m_0^{d-8}, \tag{37}$$

where b_1 is independent on A . Let us now choose κ such that $\frac{1}{4} \left(\frac{\kappa}{2d + m_0^2} \right)^2 = \log n$.

Then from Lemma 9 we deduce

$$\begin{aligned} |\xi_{\kappa}(z) - \xi_{\infty}(z)| &\leq \limsup_{A \rightarrow \mathbb{Z}^d} |\xi_{\kappa}(A, z) - \xi_{\infty}(A, z)| \\ &\leq b_2 \exp \left(-\frac{1}{2} \frac{\kappa^2}{(2d + m_0^2)^2} \right) = \frac{b_2}{n^2}. \end{aligned} \tag{38}$$

(36)–(38) give now the claim, since $n^{-1}(\log n)^2 m_0^{d-8} < \varepsilon$ can be solved by $n > \gamma m_0^{d-9}$ uniformly in β . \square

We will now turn to the lower bound, which is more subtle because of the complexity of the measure.

Proof of Proposition 4. We start by bounding $\xi_\kappa(z)$ and then use Lemma 9. Let us transform $\xi_\kappa(\Lambda, z)$ to the ϕ -representation :

$$\xi_\kappa(\Lambda, z) = \left\langle \prod_{i \in \Lambda} g_\kappa(\phi_i^2) \right\rangle_{0, \Lambda}^{1/|\Lambda|},$$

where the expectation is in the gaussian measure on $\mathbb{R}^{n|\Lambda|}$ with the covariance

$$\bigoplus_{i=1}^n C_\Lambda \equiv \bigoplus_{i=1}^n (-\Delta_\Lambda + m_0(\beta, \Lambda)^2)^{-1}$$

and

$$g_\kappa(\phi_i^2) = \int_{-\kappa}^{\kappa} da e^{iaz^{-1}(\phi_i^2 - n\beta)}. \tag{39}$$

Let $A_0 \subset \Lambda$ be $[-L + \frac{1}{2}, L + \frac{1}{2}]^d$. We will choose L later. Since g_κ is real and $|A_0|$ even, we obtain using chess-board estimates in the reverse direction (by RP everything is positive)

$$\xi_\kappa(\Lambda, z) \geq \left\langle \prod_{i \in A_0} g_\kappa(\phi_i^2) \right\rangle_{0, \Lambda}^{1/|A_0|}.$$

Hence

$$\xi_\kappa(z) = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \xi_\kappa(\Lambda, z) \geq \left\langle \prod_{i \in A_0} g_\kappa(\phi_i^2) \right\rangle_0^{1/|A_0|}, \tag{40}$$

where the covariance of $\langle \cdot \rangle_0$ is now $\bigoplus C_{\mathbb{Z}^2}$ with mass $m_0(\beta) = \lim_{\Lambda \rightarrow \mathbb{Z}^d} m_0(\beta, \Lambda)$. To get lower bound for (40) we transform the right hand side of (40) to the a -representation :

$$\left\langle \prod_{i \in A_0} g_\kappa(\phi_i^2) \right\rangle_0 = \int d^{|\Lambda_0|} a \chi_\kappa(a) \det(1 - 2izCa)^{-n/2} e^{-i\frac{\beta}{z} \text{tr} a},$$

where a is the matrix

$$\begin{aligned} a_{ij} &= a_i \delta_{ij} & \text{if } i \in \Lambda_0 \\ a_{ij} &= 0 & \text{if } i \notin \Lambda_0. \end{aligned}$$

Recall that by the saddlepoint condition (9) $C_{ii} = \beta$ for all $i \in \mathbb{Z}^d$. We can thus compute :

$$\begin{aligned} & \left\langle \prod_{i \in \Lambda_0} g_\kappa(\phi_i^2) \right\rangle_0 \\ &= \int d^{|\Lambda_0|} a \chi_\kappa(a) \exp\left(\frac{n}{2} \int_0^1 ds \text{tr} 2izCa(1 - 2izsCa)^{-1} - iz^{-1} \beta \text{tr} a\right) \\ &= \int d^{|\Lambda_0|} a \chi_\kappa(a) \exp\left(-\text{tr}(aC)^2 - 4iz \int_0^1 ds s^2 \text{tr}(Ca)^3 (1 - 2izsCa)^{-1}\right). \end{aligned}$$

To estimate the integral in the exponent, let $2\|C\|z\kappa < \frac{1}{2}$ i.e. $z\kappa \leq \frac{m_0^2}{4}$. Then

$$\left| \int_0^1 ds s^2 \operatorname{tr}(Ca)^3 (1 - 2izsCa)^{-1} \right| \leq \frac{2}{3} \kappa^3 \operatorname{tr}(P_0C)^3.$$

We denoted the projection onto A_0 by P_0 . Since (as in the previous proof)

$$\operatorname{tr}(P_0C)^3 \leq b_3 \beta^{-3} m_0^{d-6} |A_0|$$

we get by fixing $|A_0|$ such that

$$\frac{2}{3} b_3 \beta^{-3} m_0^{d-6} |A_0| z \kappa^3 = \frac{\pi}{4} \tag{41}$$

the result

$$\xi_\kappa(z) \geq 2^{-1/2|A_0|} \left(\int \chi_\kappa(a) e^{-\operatorname{tr}(aC)^2} d^{|A_0|} a \right)^{1/|A_0|}. \tag{42}$$

Note, that the last factor in (42) is not $\xi_\kappa(A_0)$ since it involves C instead of C_{A_0} – the periodic covariance in A_0 with mass $m_0^2(\beta, A_0)$. Let C be the periodic covariance in A_0 with mass $m_0^2(\beta)$. Since by the method of images (see [17], Chap. IX)

$$C_{ij}^{(A_0)} = \sum_{k \in \mathbb{Z}^d} C_{ij+2Lk}$$

we get denoting $(\delta C)_{ij} = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} C_{ij+2Lk}$ that

$$\int \chi_\kappa e^{-\operatorname{tr}(aC)^2} d^{|A_0|} a \geq e^{-\kappa^2(\operatorname{tr}(\delta C)^2 + 2\operatorname{tr}C\delta C)} \int \chi_\kappa e^{-\operatorname{tr}(aC^{(A_0)})^2} da. \tag{43}$$

Now for $L > m_0^{-1}$ we estimate

$$0 \leq \operatorname{tr}(\delta C)^2 + 2\operatorname{tr}C\delta C \leq b_4 m_0^{-2} |\partial A_0| = b_5 m_0^{-2} \frac{|A_0|}{L}. \tag{44}$$

(42), (43), and (44) give

$$\xi_\kappa(z) \geq \exp\left(-\frac{\log 2}{2|A_0|} - b_5 \kappa^2 m_0^{-2} L^{-1}\right) \left(\int \chi_\kappa e^{-\operatorname{tr}(aC^{(A_0)})^2} da \right)^{\frac{1}{|A_0|}}. \tag{45}$$

Note, that $C^{(A_0)}$ has the same mass as C . In Lemma 10 below we show that

$$\begin{aligned} & \left| \xi_\infty - \left[\int \chi_\kappa e^{-\operatorname{tr}(aC^{(A_0)})^2} d^{|A_0|} a \right]^{1/|A_0|} \right| \\ & \leq b_6 \left[e^{-\left(\frac{\kappa}{2d+m_0^2}\right)^2} + \frac{\xi_\infty}{m_0 L} \right]. \end{aligned} \tag{46}$$

Choosing κ as in (38) and combining (45), (46), (41)

$$\xi_\infty(z) > \xi_\infty (1 + b_7 \varepsilon_0^{1/d}), \tag{47}$$

where $\varepsilon_0 = m_0^{-6-d} (\log n)^{d+3/2} z$. The claim is now obtained as in the previous proof. \square

Let us denote $(\int \chi_\kappa(a) e^{-\text{tr}(aC^{(A_0)})^2} d^{|A_0|} a)^{1/|A_0|}$ by $\eta_\kappa(A_0)$. Only (46) remains to be proved:

Lemma 10. *There exists a constant b_6 such that*

$$|\xi_\infty - \eta_\kappa(A_0)| \leq b_6 \left(e^{-\left(\frac{\kappa}{2d+m_0^2}\right)^2} + \xi_\infty (m_0 L)^{-1} \right).$$

Proof. By translation invariance

$$\frac{d\eta_\kappa(A_0)}{d\kappa} = \eta_\kappa(A_0) \langle \delta_\kappa^+(a_0) \rangle_{\kappa, \infty}. \tag{48}$$

In (48) $\langle \cdot \rangle_{\kappa, \infty}$ is the expectation in the normalized measure $\exp(-\text{tr}(aC^{(A_0)})^2) \chi_\kappa(a) d^{|A_0|} a$ and as in Lemma 9, $\delta_\kappa^+(a) = \delta(a + \kappa) + \delta(a - \kappa)$. In order to be able to use chessboard estimates we define the expectation

$$\begin{aligned} \langle \cdot \rangle_{\kappa, n} &= \eta_\kappa(A_0, n)^{-|A_0|} \int \cdot \det(1 - 2izC^{(A_0)} a)^{-n/2} \\ &\quad \cdot e^{-iz^{-1} \beta \text{tr} a C^{(A_0)}} d^{|A_0|} a \end{aligned} \tag{49}$$

defined for n large enough. We now write

$$\langle \delta_\kappa^+(a_0) \rangle_{\kappa, \infty} = \lim_{n \rightarrow \infty} \langle \delta_\kappa^+(a_0) \rangle_{\kappa, n}. \tag{50}$$

Note, that $\langle \cdot \rangle_{\kappa, n}$ is none of our earlier expectations since $C_{ii}^{(A_0)} \neq \beta$ in general. However, we can still perform the transformation to the ϕ -representation:

$$\langle \delta_\kappa^+(a_0) \rangle_{\kappa, n} = \langle d_\kappa^+(\phi_0^2) \rangle_{\phi, \kappa, n}, \tag{51}$$

where $\langle \cdot \rangle_{\phi, \kappa, n}$ is a reflection positive measure. We can proceed as in Lemma 9 to deduce from (50) and (51)

$$\begin{aligned} \eta_\kappa(A_0) \langle \delta_\kappa^+(a_0) \rangle &\leq \lim_{n \rightarrow \infty} 2 \left(1 + \frac{4\kappa^2 n^{-1}}{(2d+m_0^2)^2} \right)^{-1/4n} \\ &= 2 \exp\left(\frac{-\kappa^2}{(2d+m_0^2)^2} \right). \end{aligned} \tag{52}$$

(48) and (52) hence give us

$$|\eta_\kappa(A_0) - \eta_\infty(A_0)| \leq b \exp - \left(\frac{\kappa}{2d+m_0^2} \right)^2. \tag{53}$$

We are left with estimating

$$|\eta_\infty(A_0) - \xi_\infty| = \xi_\infty |1 - \eta_\infty \xi_\infty^{-1}|. \tag{54}$$

Note that η_∞ and ξ_∞ involve gaussian integrals; denoting as before $(C_{ij}^{(A)})^2 = (B^{(A)})_{ij}^{-1}$ we have

$$\begin{aligned} \eta_\infty(A_0)\xi_\infty^{-1} &= \lim_{A \rightarrow \mathbb{Z}^d} (\det B^{(A)})^{-\frac{1}{2|A|}} (\det B^{(A_0)})^{\frac{1}{2|A_0|}} \\ &= \exp \frac{1}{2(2\pi)^d} \left[\int dp \log \int dk C(k)C(p-k) - \sum_p \left(\frac{\pi}{L}\right)^2 \log \sum_k \left(\frac{\pi}{L}\right)^d C(k) \right. \\ &\quad \left. C(p-k) \right], \end{aligned} \tag{55}$$

where the sums run through $\left\{ \frac{\pi m}{L} \mid m \in \mathbb{Z}^d, |m_\alpha| \leq L \right\}$. The difference between a Riemann sum and integral in (55) is easily estimated. We add and subtract in the exponent

$$[2(2\pi)^d]^{-1} \int dp \log \sum_k \left(\frac{\pi}{L}\right)^d C(k)C(p-k).$$

Since $C(p) \sim (p^2 + m_0^2)^{-1}$ for p small, we can choose c_1 such that for $L > c_1 m_0^{-1}$ e.g. the term

$$\begin{aligned} |\delta_p| &\equiv \left| \int dk C(k)C(p-k) - \sum_k \left(\frac{\pi}{L}\right)^d C(k)C(p-k) \right| \\ &\leq \frac{C}{L} \int dk |\nabla_k(C(k)C(p-k))| \end{aligned}$$

and thus easily

$$\frac{1}{2(2\pi)^d} \int dp |\delta_p| \leq c_2 m_0^{-1} L^{-1}.$$

Hence

$$\begin{aligned} \frac{1}{2(2\pi)^d} \left| \int dp (\log \int dk C(k)C(p-k) - \log \left(\frac{\pi}{L}\right)^d \sum_k C(k)C(p-k)) \right| \\ \leq c_3 m_0^{-1} L^{-1}. \end{aligned}$$

The remaining terms in (55) are similar and we obtain for $m_0 L < 1$

$$|1 - \eta_\infty(A_0)\xi_\infty^{-1}| \leq c_3 (m_0 L)^{-1}. \tag{56}$$

(53), (54), and (56) imply the claim. \square

8. Discussion

The bounds for the remainder of the expansion derived in Sect. 6 are in dimensions $d \geq 3$ qualitatively what one would expect; our remainder is approximately the sum of the absolute values of the next order graphs (slightly different graphs however, since some lines are missing). In 2 dimensions we have reasons to expect that the true behaviour is qualitatively different from this because of certain cancellations in the expansion, which we will now explain.

Let us consider first the $O(1/n)$ term for the self energy (mass correction), Fig. 3. Near $T_s=0$ we can for simplicity assume the propagator is $C(p)=(p^2+m_0^2(\beta))^{-1}$, with some β -independent cutoff. The discussion also applies to the renormalization of the formal expansion for the continuum theory, where m_0^2 is fixed and we vary the cutoff. We use both languages freely.

Note that both graphs of Fig. 3 have quadratic divergence in the cutoff, since the ultraviolet behaviour of $B(k)$ is k^2 . In other words the graphs are $O(1)$ in $m_0^{-2} \sim e^{4\pi\beta}$ (10). Thus it appears that the physical mass $m^2(\beta)=m_0^2(\beta)(1+n^{-1}O(e^{4\pi\beta}))$ and hence $\log n \gg \beta$ would be the region where the expansion could be expected to be valid. This is also essentially our bound since we require $n^{-1}m_0^{-\alpha}$ to be small. However the graphs of Fig.3 cancel almost completely. This can be seen as follows.

Let G_k be the "self energy insertion" $B_{kl}C_{kl}$ occurring in the graphs of Fig. 3 and $G_{kl}^s = G_{kl} - \delta_{kl} \hat{G}(0)$ the subtracted G i.e. in momentum space

$$\hat{G}^s(p) = \hat{G}(p) - \hat{G}(0).$$

Then the second graph can be written as

$$\begin{aligned} \sum_{j,k,l} B_{ij}C_{ij}C_{jl}G_{kl} &= \sum_{j,k,l} B_{ij}C_{jk}C_{jl}G_{kl}^s \\ &+ \sum_{j,k} B_{ij}C_{jk}C_{jk}\hat{G}(0) = \sum_{j,k,l} B_{ij}C_{jk}C_{jl}G_{kl}^s + \hat{G}(0), \end{aligned}$$

where we used $(B^{-1})_{ij}=(C_{ij})^2$. Thus the difference of the graphs is that of Fig. 5 i.e. G is replaced by G^s . The quadratically divergent G has been subtracted at zero momentum; G^s is only logarithmically divergent. The expansion thus regularizes itself!

In terms of the field theory logarithmic divergencies are what one on formal grounds expects, since the only counterterm for the Lagrangean $L=(\partial_\mu\phi)^2$ with $\phi^2=\text{constant}$ is L itself; there should only be wavefunction and coupling constant renormalization. We thus expect the difference of the graphs to be proportional to $m_0^2 \log m_0^2 \sim \beta m_0^2$, which is confirmed by explicit calculation. Hence the relevant small parameter would seem to be β/n and not $e^{4\pi\beta}/n$.

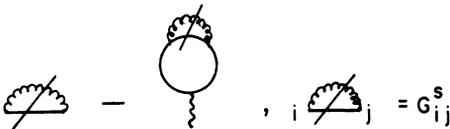


Fig. 5

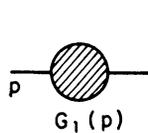


Fig. 6

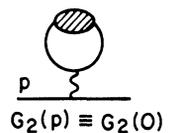


Fig. 7

It is in fact easy to prove that these cancellations occur in all orders of $1/n$. Namely, by powercounting we see, that the only quadratically divergent subgraphs are the self energies in Fig. 6. We also note that each such insertion also occurs in the graph of Fig. 7. Let us check the combinatorics and relative signs. Let G_1 be an insertion to $\text{tr}(aC)^n$, $n > 3$ and hence G_2 is an insertion to $\text{tr}(aC)^{n-1}$; all other parts of the graphs are the same. Recall, that $\text{tr}(aC)^k$ carries the factor $\frac{1}{2}(2i)^k k^{-1}$. There are $n(n-1)$ ways to insert G_1 (G_2). Thus the graphs have the

factors $n \cdot n^{-1}(2i)^n(\frac{1}{2})^3$ and $(n-1)(n-1)^{-1}(2i)^{n+2\frac{1}{3}} \cdot 3 \cdot (\frac{1}{2})^5$ where we recall that each B has to be multiplied by $\frac{1}{2}$. Hence the two graphs have same combinatorial factors, but opposite signs. For $n=3$ G_1 is a part of G_2 and for insertions to $(Ca)^n C$ same argument works. Thus repeating the argument above

$$\hat{G}_1(p) - \hat{G}_2(0) = \hat{G}_1^s(p) - \hat{G}_2^s(0),$$

where

$$\hat{G}_1^s(p) = \hat{G}_1(p) + \hat{G}_1(0) \quad \text{and} \quad \hat{G}_2^s$$

includes \hat{G}_1^s instead of G_1 . The G_1 insertion is subtracted at zero momentum. We can hence assume that all such self-energy insertions appear in subtracted form. Standard renormalization theory can now be applied to show that the terms in the expansion have only logarithmic divergence.

Finally we remark that there is further heuristic evidence that β/n is the correct small parameter in $d=2$. Renormalization group calculations [18, 19] give explicit n, β dependence for the mass. If this is expanded in $1/n$, β/n appears naturally.

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