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Constellations and Projective Classical Groups

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Abstract. The constellation concept is recalled (geometrical description of a ray in a vector space). The groups $\mathrm{PO}(n+1, \mathbb{C})$ or $P\,\mathrm{Sp}(n+1, \mathbb{C})$ are shown to preserve "harmonic conjugation" between two constellations. The action of the Lorentz subgroup and its rotation subgroup is described. Finally, a theorem concerning Clebsch-Gordan product of constellations is proved.

Introduction

The concept of *constellation* has been introduced a few years ago [1] as a convenient geometrical tool to classify orbits[1] of the rotation group SO(3) acting on states of spin $\frac{n}{2}$, i.e. on rays of the $(n+1)$-dimensional Hilbert space or the projective space $P_n(\mathbb{C})$. Each state of spin $\frac{n}{2}$ can be represented by a constellation of order n on the sphere S^2, that is by a set of n points – not necessarily distinct – on S^2 (this generalizes the well known property valid for $n=1$).

Constellations[2] on S^2 have many applications [1–6], the sphere S^2 having various interpretations, namely $P_1(\mathbb{C})$ or Riemann sphere, the Poincaré sphere [7, 2] (set of polarization states of an electromagnetic plane wave), the set of polarization states of the electron, the Bloch sphere [8, 2, 6] or the celestial sphere itself for which the word *constellation* is self justified.

According to the Klein Erlangen programm [9], *the geometry of constellations must involve some group.* Obviously for the spin states the group is the rotation group SO(3). [The action of SO(3) on S^2 is the trivial action.] For the

1 A mistake has been found in this classification by Michel (private communication). The results of reference 1 must be modified as follows: the orbit $\mathrm{SO}(3)/T$ is present in all representations of integral spin except spins 0, 1, and 3. The mistake was due to the fact that I forgot to take into account the possibility of interlacing octahedrons and tetrahedrons having T as a symmetry group

2 The word *constellation* has been suggested by A. Grossmann and appeared for the first time in [2]

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celestial sphere, the group is the Lorentz group: when we are looking at the sky from the Earth, the celestial sphere is submitted to *rotations* (rotation of 2π radians in 23 h 56 min) and infinitesimal consecutive *boosts* (responsible of the so-called phenomenon of *aberration of fixed stars*). Due to the well-known isomorphism between the Lorentz group and the Møbius group (made of homographic and antihomographic transformations on the Riemann sphere), if four stars are cocyclic in the sky they will be cocyclic at any time[3].

To be complete about the bibliographical applications of constellations, we must mention the works by Shaw on Petrov classification of space-time [11] and on Wigner $3j$-symbols [12].

When this work was completed, I became aware of a paper by Majorana [13] who introduced the constellation description of spin and used it for Stern-Gerlach experiment. It is surprising that the Majorana work has not been explored furthermore.

1. Definition of Constellations

Consider a real manifold Γ (carrier space) and denote by $(z_1, z_2, \ldots, z_n)$ an element of the Cartesian product $V = \Gamma \times \Gamma \times \ldots \times \Gamma$ of $n\Gamma$-copies. Two elements $(z_1, z_2, \ldots, z_n)$ and $(z'_1, z'_2, \ldots, z'_n)$ will be said to be equivalent if they are identical up to a permutation σ, i.e. if

$$z'_k = z_{\sigma k}.$$

The quotient of V by this equivalence relation defines the sky $\mathscr{C}_n(\Gamma)$ of constellations of *order n*. If the z_i's are not all distinct, the constellation is said to be *degenerate*. Then, its *apparent order* is less than n; it is the number of distinct z_i.

An equivalent definition is as follows: a constellation of order n is a set of n elements of Γ, *not necessarily distinct*[4].

Now, an interacting property is the following one: for $\mathscr{C}_n(\Gamma)$ to be a manifold, it is *necessary* for Γ to be of dimension *two* [14]. Up to now, we found some interest in considering as carrier spaces the real plane $\mathbb{R}^2$ (or complex line $\mathbb{C}$), the real sphere S^2 or extended complex line $P_1(\mathbb{C})$ and $P_2(\mathbb{R})$, the real projective plane.

Examples

1. The set of complex polynomials of degree n can be mapped on the set of constellations or order n when the carrier space is the *complex line* $\mathbb{C}$ (Newton projection of a given polynomial on the constellation of its roots).
2. The set of complex polynomials of degree $\leqq n$ can be mapped on the set of constellations of order n when the carrier space is the *extended complex line* $P_1(\mathbb{C})$. For this, we complete the Newton projection by adding the root $z = \infty$ as many times as we need to obtain a constellation of order n.

In the following, all constellations are on $\mathbb{C}$ *or* $P_1(\mathbb{C})$. Constellations on $P_2(\mathbb{R})$ will appear as *real* constellations on $P_1(\mathbb{C})$.

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3 For a pedagogical description of that question, see [10]

4 See [2] for another definition

2. Harmonic Conjugate Constellations

Let $Z=[z_1, z_2, \ldots, z_n]$ and $Z'=[z'_1, z'_2, \ldots, z'_{n'}]$ two constellations of order n where the z_i and z'_i are ordinary complex numbers. Let us define their *permanent product*:

$$(Z', Z)=\frac{1}{n!}\mathrm{Perm}(z_i-z'_j)\,, \tag{2.1a}$$

where $\mathrm{Perm}(z_i-z'_j)$ denotes the permanent[5] of the matrix

$$\begin{vmatrix} z_1-z'_1 & z_1-z'_2 & \ldots & z_1-z'_n \\ z_2-z'_1 & z_2-z'_2 & \ldots & z_2-z'_n \\ \multicolumn{4}{c}{\dotfill} \\ z_n-z'_1 & z_n-z'_2 & & z_n-z'_n \end{vmatrix}. \tag{2.2}$$

Equation (2.1) can also be written

$$(Z', Z)=(-)^n(Z, Z')=\frac{1}{n!}\sum_{\text{permut}}(z_1-z'_{i_1})(z_2-z'_{i_2})\ldots(z_n-z'_{i_n})\,, \tag{2.1b}$$

where the summation extends to all permutations of the z'_i.

It is clear that the permanent product is symmetric in the z_i variables and symmetric in the z'_i variables (a property which is needed for the permanent product to deal with constellations). Therefore, if we denote by

$$S_k(z_1, z_2, \ldots, z_n)=\sum z_{i_1}z_{i_2}\ldots z_{i_k} \tag{2.3}$$

the usual symmetric functions of order $k(k\leqq n)$, we see that (Z', Z) can be written in the form

$$(Z', Z)=\sum_{k=0}^{n}\lambda_k S_k(z'_1, z'_2, \ldots, z'_n)\,S_{n-k}(z_1, z_2, \ldots, z_n)\,, \tag{2.4}$$

where the coefficients λ_k are constants to be determined. They are easily obtained in supposing all z_i equal to z and all z'_i equal to z'. Then from (2.2)

$$(Z', Z)=(z-z')^n=\sum_{k=0}^{n}(-)^k z'^k z^{n-k}\,. \tag{2.5}$$

Moreover

$$S_k(z'_1, z'_2, \ldots, z'_n)=\binom{n}{k}z'^k\,, \tag{2.6}$$

$$S_{n-k}(z_1, z_2, \ldots, z_n)=\binom{n}{k}z^{n-k}\,. \tag{2.7}$$

Putting (2.5)–(2.7) into (2.4) we obtain by identification

$$\lambda_k=(-)^k\Big/\binom{n}{k}\,. \tag{2.8}$$

Therefore

$$(Z', Z)=\sum_{k=0}^{n}\left[(-)^k\Big/\binom{n}{k}\right]S_k(z'_1, z'_2, \ldots, z'_n)S_{n-k}(z_1, z_2, \ldots, z_n)\,. \tag{2.9}$$

Now, we propose the

5 We recall that the permanent is computed as a determinant but by using only *plus* signs

Definition. Two constellations of order n will be said to be *harmonic conjugate* (h.c.) if their permanent product is zero.

Our definition is justified by the fact that it generalizes harmonic conjugation of constellations of order 2, since if

$$Z=[z_1, z_2]; \qquad Z'=[z'_1, z'_2] \tag{2.10}$$

one has

$$2(Z', Z)=(z'_1-z_1)(z'_2-z_2)+(z'_1-z_2)(z'_2-z_1) \tag{2.11}$$

and

$$(Z', Z)=0 \Leftrightarrow \frac{z'_1-z_1}{z'_1-z_2} : \frac{z'_2-z_1}{z'_2-z_2}=-1\,. \tag{2.12}$$

Remarks. 1. Equation (2.12) shows that harmonic conjugation concerns a pair of constellations of order 2 rather than a constellation of order four[6].

2. Two constellations of order n which have more than $\frac{n}{2}$ stars in common are trivially h.c. (This readily follows from the fact that more than a quarter of the matrix $|z_i-z'_j|$ would be composed of zeroes.

3. Two constellations of order 1 are h.c. if and only if they are identical.

4. Any constellation of odd order is self h.c. This property follows from the fact that any antisymmetric matrix of odd dimension has a null permanent.

5. Harmonic conjugation can be given a meaning on $P_1(\mathbb{C})$ where the point at infinity is included. Suppose, for instance, that z_1 is infinite. Then the permanent of the matrix (2.2) could be written

$$z_1 \,\mathrm{Perm}\begin{vmatrix} 1-z'_1/z_1 & 1-z'_2/z_1 & \dots & 1-z'_n/z_1 \\ z_2-z'_1 & z_2-z'_2 & \dots & z_2-z'_n \\ & & & \\ z_n-z'_1 & z_n-z'_2 & \dots & z_n-z'_n \end{vmatrix} \tag{2.14}$$

and harmonic conjugation would be associated with the vanishing of this new permanent where z_1 goes to infinity (the first row is made of 1's).

6. *Missing star property.* Given two constellations $Z=[z_1, z_2, \dots, z_n]$ of order n and $Z'=[z'_1, z'_2, \dots, z'_{n-1}]$ of order $n-1$, there exists a unique star z' such that

$$([z'_1, z'_2, \dots, z'_{n-1}, z'], Z)=0\,. \tag{2.15}$$

This follows from the fact that (2.15) is an equation of degree one in z' (the permanent is of degree one in each z_i or z'_j).

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6 A constellation of order 4, say $[a, b, c, d]$ has six distinct cross ratios x, $1-x$, $1/x$, $1-1/x$, $1/1-x$, $1-1/1-x$. These values are distinct except if $x=-1$, 2 or $-1/2$ (harmonic constellation) or if $x=(-1 \pm i\sqrt{3})/2$ (antiharmonic constellation). Therefore the cross ratio is not the invariant function suitable for such constellations. Rather we take

$$(a,b,c,d)=\frac{[(a-b)(b-c)(c-d)(d-a)]^3+[(a-c)(c-b)(b-d)(d-a)]^3+[(a-b)(b-d)(d-c)(c-a)]^3}{[(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)]^2}, \tag{2.13}$$

which is $\dfrac{(x^2-x+1)^3}{27x^2(x-1)^2}-\frac{1}{9}$ if x denotes *any* cross ratio of numbers, a, b, c, d

3. $P_n(\mathbb{C})$ and Constellations of Order n on $P_1(\mathbb{C})$

Let A be an element of $P_n(\mathbb{C})$ represented by a non zero complex column of $\mathbb{C}^{n+1}$

$$A=\begin{vmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{vmatrix} \sim \begin{vmatrix} \lambda a_0 \\ \lambda a_1 \\ \vdots \\ \lambda a_n \end{vmatrix} \qquad (\lambda \neq 0). \tag{3.1}$$

We associate with A the constellation $Z=[z_1, z_2, \ldots, z_n]$ where the z_ι are the roots of the polynomial

$$a_0 z^n - a_1 \sqrt{\binom{n}{1}}\, z^{n-1} + a_2 \sqrt{\binom{n}{2}}\, z^{n-2} - \ldots + (-)^n a_n. \tag{3.2}$$

(If $a_0 = a_1 = \ldots = a_k = 0$ and $a_{k+1} \neq 0$, $k+1$ of these roots will be infinite.) It is clear that we have thus defined a *bijection* between $P_n(\mathbb{C})$ and the sky $\mathscr{C}_n$ of constellations of order n on $P_1(\mathbb{C})$. We have

$$P_n(\mathbb{C}) = \mathscr{C}_n(P_1(\mathbb{C})). \tag{3.3}$$

It is natural to try to interpret the harmonic conjugation in $P_n(\mathbb{C})$. For this purpose, we introduce the antidiagonal matrix $(n+1)\times(n+1)$ matrix g

$$g=\begin{vmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & -1 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & -(-)^n & 0 & \ldots & 0 & 0 \\ (-)^n & 0 & 0 & \ldots & 0 & 0 \end{vmatrix} \tag{3.4}$$

and the bilinear form $\tilde{A}'gA$ associated with it, where $\tilde{A}'$ denotes the transposed of A'.

It is a simple matter to show that

$$\begin{aligned}(A', A) = \tilde{A}'gA &= \sum_{k=0}^{n} (-)^k a'_k a_{n-k} \\ &= \sum_{k=0}^{n} (-)^k a_0 a'_0 S_k(z'_1, z'_2, \ldots, z'_n) S_{n-k}(z_1, z_2, \ldots, z_n) \Big/ \binom{n}{k}\end{aligned} \tag{3.5}$$

by use of the relationship between the coefficients of the polynomial (3.2) and the symmetric functions of its roots. If we compare now (3.5) with (2.9) we readily see that

$$(A', A) = a'_0 a_0 (Z', Z). \tag{3.6}$$

It follows that orthogonality in the sense of (3.4) is equivalent to harmonic conjugation. Moreover, the linear group which preserves the harmonic conjugation in $P_n(\mathbb{C})$ is identical with the linear group on $\mathbb{C}^{n+1}$ which preserves the bilinear form g. This group is isomorphic to $\mathrm{PO}(n+1, \mathbb{C})$ if n is even, to $P\,\mathrm{Sp}(n+1, \mathbb{C})$ if n is odd. They are the projective complex orthogonal group and the projective complex symplectic group, respectively.

4. The Møbius (Lorentz) Group as a Subgroup of $\mathrm{PO}(n+1, \mathbb{C})$ or $P\,\mathrm{Sp}(n+1, \mathbb{C})$

The connected Møbius group is the group of *homographic transformations*

$$z \mapsto \frac{az+b}{cz+d}, \tag{4.1}$$

that is the projective linear group $\mathrm{PGL}(2, \mathbb{C})$ acting on $P_1(\mathbb{C})$.

The full Møbius group is two-sheeted. It contains homographic and *antihomographic transformations:*

$$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}, \tag{4.2}$$

where $\bar{z}$ is the complex conjugate of z. It is the group of all holomorphic and antiholomorphic mappings which map circles and straightlines into themselves.

It is easy to show[7] that the Lorentz group including parity *or* time-reversal is geometrically related to the full Møbius group and that they are *isomorphic.*

Any Møbius transformation can be considered as a sequence of transformations of the following kinds

$$\begin{array}{lll} \text{translations:} & z \mapsto z+\alpha & \\ \text{dilations:} & z \mapsto \lambda z & (\lambda \neq 0) \\ \text{inversion:} & z \mapsto 1/z & \\ \text{complex conjugation:} & z \mapsto \bar{z}\,. & \end{array} \tag{4.3}$$

It is now a very simple matter to verify that such transformations on constellations preserve the harmonic conjugation. Moreover, it follows from results of the last section that, the homographic transformations – i.e. the connected Lorentz group – act linearly on $P_n(\mathbb{C})$ as a subgroup of $\mathrm{PO}(n+1, \mathbb{C})$ or $P\,\mathrm{Sp}(n+1, \mathbb{C})$. The representation is irreducible as it follows from the classical work [15] on the rotation group and polynomials. This representation is often denoted by D_{j0} with $n=2j$.

Clearly the antihomographic transformations act *antilinearly* on $P_n(\mathbb{C})$. Therefore D_{j0} can be considered as a *corepresentation*[8].

Remark. For lower values of n, we have the following isomorphisms

$$P\,\mathrm{Sp}(2, \mathbb{C}) \sim \mathrm{PO}(3, \mathbb{C}) \sim \text{Lorentz}, \tag{4.4}$$

$$P\,\mathrm{Sp}(4, \mathbb{C}) \sim \mathrm{PO}(5, \mathbb{C})\,. \tag{4.5}$$

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7 The Lorentz group acts linearly on space-time and preserves the light-cone $x^2+y^2+z^2-t^2=0$. Its action goes to the projective space-time. If we set $X=x/t$, $Y=y/t$, $Z=z/t$, we can restrict its action to the sphere $X^2+Y^2+Z^2=1$. The linear character of the group has as a consequence that circles are transformed into circles. If we make a stereographic projection on the complex line, we arrive at the full Møbius group. The isomorphism follows

8 A corepresentation is a representation where some elements of the group are represented antilinearly and the other ones linearly

5. The Projective Unitary Group PU($n+1$) and the Rotation Group SO(3)

Let us consider a Hilbert space $\mathscr{H}$ of dimension $n+1$. The pure states associated with it are rays. They form the *projective space* $P(\mathscr{H})$ isomorphic to $P_n(\mathbb{C})$. The canonical transformations of $\mathscr{H}$ are the unitary operators; they form a group $U(\mathscr{H})$ isomorphic to $U(n+1)$. It is the group of linear transformations which preserves the Hermitian scalar product $\langle\psi_1|\psi_2\rangle$ *up to a phase* or, in other words, which preserves the expression $|\langle\psi_1|\psi_2\rangle|$ or, equivalently, the quantity

$$\frac{|\langle\psi_1|\psi_2\rangle|}{\sqrt{(\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle)}}. \tag{5.1}$$

The group action of $U(\mathscr{H})$ on $\mathscr{H}$ (or of $U(n+1)$ *on* $\mathbb{C}^{n+1}$) induces an action on $P_n(\mathbb{C})$. The group which acts on $P_n(\mathbb{C})$ is usually denoted PU($n+1$). It can be defined either in saying that it preserves (5.1) or in saying that *it preserves the orthogonality of states*[9].

It is therefore necessary to know how the orthogonality of states is described in the constellation language. For this purpose, we first define the notion of *antipodal constellation.*

Definition 1. The antipodal constellation of $[z_1, z_2, \ldots, z_n]$ is the constellation

$$[-\bar{z}_1^{-1}, -\bar{z}_2^{-1}, \ldots, -\bar{z}_n^{-1}].$$

If we make the stereographic projection relating $P_1(\mathbb{C})$ and S^2, the definition is simple: *two constellations are antipodal if they are symmetric with respect to the center of the sphere.*

We will need another definition.

Definition 2. A constellation will be said to be *real*[10] if it is equal to its antipodal.

It readily follows that *a real constellation is necessarily* of even order.

Proposition. *Two constellations Z and Z′ of order n are associated with orthogonal states if and only if Z′ is harmonic conjugate of $\tilde{Z}$, where $\tilde{Z}$ denotes the antipodal of Z (or, equivalently, if Z is h.c. of $\tilde{Z}'$).*

Proof. Let $|\psi\rangle$ be a representative of Z in $\mathbb{C}^{n+1}$ and $|\psi'\rangle$ a representative of Z'.

$$|\psi\rangle = \begin{vmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{vmatrix}, \quad |\psi'\rangle = \begin{vmatrix} a'_0 \\ a'_1 \\ \vdots \\ a'_n \end{vmatrix}. \tag{5.2}$$

According to (3.2) we have

$$\langle\psi'|\psi\rangle = 0 \Leftrightarrow \sum_{k=0}^{n} \bar{a}'_k a_k = 0 \Leftrightarrow \sum_{k=0}^{n} \frac{S_k(z'_1, \ldots, z'_n) S_{n-k}(z_1, \ldots, z_n)}{\binom{n}{k}}. \tag{5.3}$$

9 PU($n+1$) is isomorphic to SU($n+1$)/Z_{n+1}

10 This denomination follows from the following property: if we identify antipodal points on S^2, we get the projective space $P_2(\mathbb{R})$. Then, any real constellation of order $2n$ will appear as a constellation of order n on $P_2(\mathbb{R})$. This means that $P_{2n}(\mathbb{R}) = \mathscr{C}_n(P_2(\mathbb{R}))$

Now,

$$S_k(\bar{z}'_1, \ldots, \bar{z}'_n) = (-)^{n-k} S_{n-k}(-\bar{z}'^{-1}_1, \ldots, -\bar{z}'^{-1}_n)\bar{z}'_1\bar{z}'_2 \ldots \bar{z}'_n . \tag{5.4}$$

Therefore, (5.3) reads

$$\langle \psi' | \psi \rangle = 0 \Leftrightarrow \sum_{k=0}^{n} \frac{S_{n-k}(-\bar{z}'^{-1}_1, \ldots, -\bar{z}'^{-1}_n) S_k(z_1, \ldots, z_n)}{\binom{n}{k}} = 0 . \tag{5.5}$$

Our proposition readily follows.

Remark 1. In the case $n=1$, to be h.c. means to be equal. Therefore the orthogonality coincides with antipodality (as it is well known for spin$\frac{1}{2}$ states as well as for polarization states of the photon on the Poincaré sphere).

Remark 2. Orthogonality relation can also be written in the permanent form:

$$\langle Z', Z \rangle = 0 \Leftrightarrow \mathrm{Perm}[1 + \bar{z}'_i z_j] = 0 . \tag{5.6}$$

Remark 3. The group which preserves *both* harmonic conjugation and orthogonality is the intersection of $\mathrm{PU}(n+1)$ with $\mathrm{PO}(n+1, \mathbb{C})$ or $P\,\mathrm{Sp}(n+1, \mathbb{C})$. It is isomorphic to $\mathrm{PO}(n+1, \mathbb{R})$ or $P\,\mathrm{Sp}(n+1)$.

Let us now examine the SO(3) subgroup of the Lorentz group. It is the group of homographic transformations of the form

$$z \mapsto \frac{az+b}{-\bar{b}z+\bar{a}} . \tag{5.7}$$

By making this transformation on Z and Z', it is readily seen that Perm $[1 + \bar{z}'_i z_j]$ is multiplied by a factor. Therefore, SO(3) preserves the orthogonality property. It follows that D_{j0} is a unitary representation of SO(3).

Geometrically, the action of SO(3) has been described in previous papers [1, 2]. It consists of rotating the sphere S^2. It is also clear that SO(3) preserves antipodality.

Remark 4. If n is even (states are those of integral spins), the representation of SO(3) is *real* since $\mathrm{SO}(3) \subset \mathrm{PO}(n+1, \mathbb{R})$. Note that real constellations only appear in those representations.

Remark 5. Two real constellations of order 2 are harmonic conjugate if and only if their corresponding diameters on S^2 are perpendicular.

6. Clebsch-Gordan Products of Constellations

Given two constellations Z and Z' of order $n=2j$ and $n'=2j'$, respectively, we want to know which constellations are associated with the decomposition of their tensor product. We have already treated the case $n=n'=2$ [5]. Let us recall the results. We know the Clebsch-Gordan series

$$D_1 \times D_1 = D_0 + D_1 + D_2 .$$

This means that by multiplying a vector by a vector, we can get a scalar, a vector and a tensor of rank 2. In the constellation language, we say that by multiplying two constellations Z, Z' both of order 2, we get a constellation or order zero, one of order 2, one of order 4. The constellation of order zero is trivial; the constellation of order 4 is the union of Z and Z'. The constellation of order 2 corresponds to the "vector product" or the "Lie bracket" of Z and Z'. It is the (unique) constellation of order 2 which is h.c. to both Z and Z' (see proof in [5]).

The harmonic conjugation plays an important role in the general case:

Proposition. *Let Z and Z' be two constellations*

$$\begin{aligned} Z &= [z_1, z_2, \ldots, z_n] \quad \textit{order } n \\ Z' &= [z'_1, z'_2, \ldots, z'_{n'}] \quad \textit{order } n' \end{aligned} \tag{6.1}$$

and let us consider the following constellations obtained from Z and Z' by adding $n''-n$ times the star z to Z and $n''-n'$ times the same star z to Z'.

They become

$$\begin{aligned} Z^{\#} &= [z_1, z_2, \ldots, z_n, z, z, \ldots, z] \\ Z'^{\#} &= [z'_1, z'_2, \ldots, z'_{n'}, z, \ldots, z]. \end{aligned} \tag{6.2}$$

Let us make their permanent product. It is a polynomial in z of degree $2n''-n-n'$. It is clear that we have

$$|n\cdot - n'| \leqq 2n'' - n - n'.$$

On the other hand, if $2n''-n-n' > n+n'$, the permanent will be zero whatever is z because there are too many zeroes in it. Then, we have for a nontrivial permanent

$$|n-n'| \leqq 2n''-n-n' \leqq n+n'. \tag{6.3}$$

For a given n'' satisfying those conditions, there exists $2n''-n-n'$ roots of the permanent polynomial. *These roots form a constellation which is the Clebsch-Gordan product of order $2n''-n-n'$ of Z and Z'.*

Proof (see Appendix).

Remark. It could happen that the product of two constellations is undefined. As an example, let us consider the C.-G. series

$$D_1 \times D_1 = D_0 + D_1 + D_2.$$

If the states associated with D_1 are identical, it is clear that on the right hand side there is no state corresponding to D_1 since the vector product of a 3-vector by itself is zero and the null vector is not a representative of a state.

7. Conclusion

In the present paper, we have presented none physical application. Nevertheless we would like to show the physical interest of the constellation description of polarization states when we compare a photon beam with an atom beam.

Photon

(i) A polarization state is a constellation of order *one* (on the Poincaré sphere) [7].

(ii) A beam which arrives on a plane parallel plate, cut in a uniaxial crystal, will be unchanged if its polarization state is invariant under a rotation around a diameter of the Poincaré sphere. There are two such polarization states (they are orthogonal). One corresponds to a polarization along the optical axis, the other to a polarization perpendicular to it.

(iii) In the general case, the constellation has a regular precession motion around the privileged diameter. The total rotation is proportional to the thickness of the plate.

(iv) As a consequence, the polarization will be unchanged if the thickness of the plate is a multiple of the wave length.

(v) If we have a prism instead of a plate, we get two beams (birefringence).

Atom of spin j

(i) A polarization state is a constellation of order 2j.

(ii) A beam which arrives in a *homogeneous* magnetic field **H** will be unchanged if its polarization state is invariant under rotations around **H** (i.e. it is a state of the form $|jm\rangle$ in **H** direction). All these states form an orthogonal basis.

(iii) In the general case, the constellation has a regular precession motion around the homogeneous field **H**. The total rotation is proportional to the spread of the magnetic field.

(iv) As a consequence, the polarization will be unchanged if the spread of the field is a multiple of some sonstant.

(v) If the field is inhomogeneous we get $2j+1$ beams (Stern-Gerlach).

In my knowledge, property (iii) of the photon, discovered by Poincaré seems to be unknown. It was natural to emphasize the similarity of this property with the spin precession in a magnetic field.

Appendix

First we will denote by

$$A(z_1, z_2, \ldots, z_n)$$

one of the vectors of $\mathbb{C}^{n+1}$ associated with the constellation $[z_1, z_2, \ldots, z_n]$. In particular for $n=1$ and $z \neq \infty$, a possible choice is

$$A(z) = \begin{vmatrix} 1 \\ z \end{vmatrix}$$

and the permanent product reads

$$\mathrm{Perm}([z],[z']) = z' - z = \varepsilon_{ij} A^i(z) A^j(z'), \tag{A1}$$

where $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$. Therefore the permanent product corresponds to the symplectic product of spinors (ε-contracted product of spinors).

If we want to multiply $A=A(z_1,z_2,\ldots,z_n)$ by $A'=A(z'_1,z'_2,\ldots,z'_n)$, we first consider them as symmetrized products of spinors, i.e.

$$A=A(z_1,z_2,\ldots,z_n)=\begin{vmatrix}1\\z_1\end{vmatrix}\vee\begin{vmatrix}1\\z_2\end{vmatrix}\vee\ldots\vee\begin{vmatrix}1\\z_n\end{vmatrix}\tag{A2}$$

and similarly for A'. Then we take the ε-contracted tensor product of A and A'; if k is the number of contracted indices, we have necessarily

$$0\leqq k\leqq|n-n'|.\tag{A3}$$

Let us denote by $A''=A(z''_1,z''_2,\ldots,z''_{n+n'-2k})$ the result. It is a vector associated with the representation $D_{j+j'-k}$ where $n=2j$, $n'=2j'$. We have

$$\begin{aligned}A''&=A(z''_1,z''_2,\ldots,z''_{n+n'-2k})=A(z_1,z_2,\ldots,z_n)\times A'(z'_1,z'_2,\ldots,z'_{n'})\\&=\sum_{\text{permut}}\operatorname{Perm}(z_{i_1},z_{i_2},\ldots,z_{i_k},z'_{j_1},\ldots,z'_{j_k})A(z_{i_{k+1}},\ldots,z_{i_n},z'_{j_{k+1}},\ldots,z'_{j_{n'}})\end{aligned}\tag{A4}$$

up to an irrelevant factor.

Now, we have for the symmetric function S_ℓ of the constellation product $[z''_1,\ldots,z''_{n+n'-2k}]$

$$S_\ell(z''_1,\ldots,z''_{n+n'-2k})=\sum_{\text{permut}}\operatorname{Perm}(z_{i_1}\ldots z_{ik},z'_{j_1}\ldots z'_{j_k})S_\ell\text{ (other } z \text{ and } z').\tag{A5}$$

Therefore, the z''_i are the roots of a polynomial equation of degree $n''=n+n'-2k$, of the form

$$Z^{n''}-a_1Z^{n''-1}+a_2Z^{n''-2}-\ldots=0,\tag{A6}$$

where the term of degree zero is, up to a sign,

$$\sum_{\text{permut}}\operatorname{Perm}([z_{i_1},\ldots,z_{i_k}],[z'_{j_1},\ldots,z'_{j_k}])z_{i_{k+1}},\ldots,z_{i_n}z'_{j_{k+1}},\ldots,z'_{j_{n'}}.\tag{A7}$$

It is in fact the only term we need. We know that if we translate z_i, z'_i, and z''_i the relation between them is unchanged. This means that the polynomial is only a function of z_i-Z and z'_i-Z. Therefore let us set $z_i=\zeta_i+Z$, $z'_i=\zeta'_i+Z$ in Eq. (A6). Since this equation does no longer depend on Z, we are left with the expression (A7) where z_i and z'_i are replaced by $\zeta_i=z_i-Z$ and $\zeta'_i=z'_i-Z$, respectively. Moreover the permanent product is invariant under translation. Then Eq. (A6) becomes

$$\operatorname{Perm}([z_{i_1},\ldots,z_{i_k}],[z'_{j_1},\ldots,z'_{j_k}])(z_{i_{k+1}}-Z),\ldots,(z'_{j_n}-Z)=0,\tag{A8}$$

which is nothing else than

$$\operatorname{Perm}([z_1,\ldots,z_n,Z,\ldots,Z],[z'_1,\ldots,z'_n,\ldots,Z,Z,\ldots,Z])=0,\tag{A9}$$

which proves our theorem.

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