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On the Fluid-Dynamical Approximation to the Boltzmann Equation at the Level of the Navier-Stokes Equation

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Abstract. The compressible and heat-conductive Navier-Stokes equation obtained as the second approximation of the formal Chapman-Enskog expansion is investigated on its relations to the original nonlinear Boltzmann equation and also to the incompressible Navier-Stokes equation. The solutions of the Boltzmann equation and the incompressible Navier-Stokes equation for small initial data are proved to be asymptotically equivalent (mod decay rate $t^{-5/4}$) as $t \to +\infty$ to that of the compressible Navier-Stokes equation for the corresponding initial data.

1. Introduction

The nonlinear Boltzmann equation for a rarefied simple gas is given in the form

$$F_t + v^j F_{x_j} = \frac{1}{\varepsilon} Q(F, F) \tag{1.1}$$

where $t \geqq 0$: time, $x \in R^3$: physical space, $v \in R^3$: velocity space, ε: mean free path, $F = F(t, x, v)$ is the mass density distribution function and Q represents the quadratic collision operator. Here and in what follows, we use the summation convention when we are not confused. Let us introduce the fluid-dynamical quantities as follows:

mass density:	$\varrho \equiv \int F(t, x, v)\, dv$,
fluid flow velocity:	$u^i \equiv \frac{1}{\varrho} \int v^i F(t, x, v)\, dv$,
momentum:	$m^i \equiv \varrho u^i$,
pressure tensor:	$P^{ij} \equiv \int c^i c^j F(t, x, v)\, dv$,
pressure:	$p \equiv \frac{1}{3} P^{kk}$,
viscous term:	$p^{ij} \equiv P^{ij} - p\delta^{ij}$,
heat flow vector:	$q^i \equiv \frac{1}{2} \int c^i \lvert c \rvert^2 F(t, x, v)\, dv$,

internal energy per unit mass: $e \equiv \frac{1}{\varrho}\int \frac{1}{2}|c|^2 F(t,x,v)\,dv$,

absolute temperature: $\theta \equiv \frac{2}{3R} e$,

total energy: $E \equiv \varrho e + \frac{1}{2}\varrho|u|^2$,

$= \int \frac{1}{2}|v|^2 F(t,x,v)\,dv$,

where $c^i = v^i - u^i$, R: gas constant and δ^{ij}: Kronecker's delta. Then the conservation laws derived from (1.1) are given in the form:

$$
\begin{aligned}
&\varrho_t + (\varrho u^j)_{x_j} = 0, \\
&(\varrho u^i)_t + (\varrho u^i u^j + p\delta^{ij} + p^{ij})_{x_j} = 0, \\
&(\varrho(e + \tfrac{1}{2}|u|^2))_t + (\varrho u^j(e + \tfrac{1}{2}|u|^2) + pu^j + u^k p^{kj} + q^j)_{x_j} = 0,
\end{aligned}
\tag{1.2}
$$

where the equation of state is that of "ideal and polytropic gas", i.e.,

$$p = R\varrho\theta = \tfrac{2}{3}\varrho e.$$

In order to enclose the system (1.2) in the fluid-dynamical variables ϱ, u and θ, the formal Chapman-Enskog expansion

$$F = \sum_{n=0}^{\infty} \varepsilon^n F^{(n)}$$

has been adopted where the functions $F^{(n)}$ can be uniquely computed in turn as the functions of v, (ϱ, u, θ) and their partial derivatives with respect to x (cf. [1, 4]). In fact, the first approximation $F^{(0)}$ is given by the locally Maxwellian, i.e.,

$$F^{(0)} = \frac{\varrho}{(2\pi R\theta)^{3/2}} \exp\left(-\frac{|u-v|^2}{2R\theta}\right), \tag{1.3}$$

for which the system (1.2) comes to be the compressible Euler equation

$$
\begin{aligned}
&\varrho_t + (\varrho u^j)_{x_j} = 0, \\
&(\varrho u^i)_t + (\varrho u^i u^j + p\delta^{ij})_{x_j} = 0, \\
&(\varrho(e + \tfrac{1}{2}|u|^2))_t + (\varrho u^j(e + \tfrac{1}{2}|u|^2) + pu^j)_{x_j} = 0, \\
&p = R\varrho\theta, \ e = \tfrac{3}{2}R\theta.
\end{aligned}
\tag{1.4}
$$

Furthermore, as we show the precise derivation in the later section (Sect. 3), the system (1.2) corresponding to the second approximation $F^{(0)} + \varepsilon F^{(1)}$ is given by the compressible Navier-Stokes equation

$$
\begin{aligned}
&\varrho_t + (\varrho u^j)_{x_j} = 0, \\
&(\varrho u^i)_t + (\varrho u^i u^j + p\delta^{ij})_{x_j} = \varepsilon(\mu(u^i_{x_j} + u^j_{x_i}) - \tfrac{2}{3}\mu u^k_{x_k}\delta^{ij})_{x_j}, \\
&(\varrho(e + \tfrac{1}{2}|u|^2))_t + (\varrho u^j(e + \tfrac{1}{2}|u|^2) + u^j p)_{x_j} \\
&\quad = \varepsilon(\mu u^k(u^k_{x_j} + u^j_{x_k}) - \tfrac{2}{3}\mu u^j u^k_{x_k} + \kappa\theta_{x_j})_{x_j}, \\
&p = R\varrho\theta, \ e = \tfrac{3}{2}R\theta,
\end{aligned}
\tag{1.5}
$$

where $\mu = \mu(\theta)$ and $\kappa = \kappa(\theta)$ represent the coefficient of viscosity and that of heat conduction respectively.

Assuming the cut-off hard potentials in the sense of Grad [5] for the collision, we consider the initial value problem to the nonlinear Boltzmann equation (1.1) in a small neighbourhood of the absolute Maxwellian state

$$M(v) \equiv (2\pi)^{-3/2} \exp\left(-\frac{|v|^2}{2}\right). \tag{1.6}$$

Ukai [15] and Nishida-Imai [13] succeeded to solve the initial value problem globally in time and to show that its solution decays to the absolute Maxwellian state as $t \to +\infty$. Their arguments also include the result not mentioned explicitly that the solutions of the nonlinear and linearized Boltzmann equations are "asymptotically equivalent $\bmod t^{-5/4}$ as $t \to +\infty$" to each other which means, throughout this paper, that the difference of them decays to zero in L^2 at the rate of $(1+t)^{-5/4}$ as $t \to +\infty$. We summarize all these results in Sect. 7.

Recently, we [9, 10] solved the initial value problem to the general compressible Navier-Stokes equations including (1.5) in a small neighbourhood of the constant state $(\bar{\varrho}, 0, \bar{\theta})$ and then we showed that its solution decays to the constant state. In Sect. 4, we summarize these results and also establish the asymptotic equivalence $\bmod t^{-5/4}$ as $t \to +\infty$ between the nonlinear and linearized solutions. Furthermore in Sect. 5, we consider the following incompressible Navier-Stokes equation as an approximation to the compressible Navier-Stokes equation (1.5) around the constant state $(\bar{\varrho}, 0, \bar{\theta})$:

$$\begin{aligned} &v^i_t + v^j v^i_{x_j} - \varepsilon \frac{\mu(\bar{\theta})}{\bar{\varrho}} v^i_{x_j x_j} + \frac{1}{\bar{\varrho}} p_{x_i} = 0 \\ &v^j_{x_j} = 0. \end{aligned} \tag{1.7}$$

The global solutions in time are known in Leray [8]. In the present paper, for appropriately small initial data, we show that the solution of (1.7) is asymptotically equivalent $\bmod t^{-5/4}$ as $t \to +\infty$ to that of the heat equation

$$\begin{aligned} &\tilde{v}^i_t - \varepsilon \frac{\mu(\bar{\theta})}{\bar{\varrho}} \tilde{v}^i_{x_j x_j} = 0, \\ &\tilde{v}^j_{x_j} = 0. \end{aligned} \tag{1.8}$$

On the other hand, in Sect. 6, we show that if the initial data for (1.5) satisfy $(\varrho(0), E(0)) = \text{const}$ and $u(0)^j_{x_j} = 0$, the solution of (1.5) is asymptotically equivalent $\bmod t^{-5/4}$ as $t \to +\infty$ to that of (1.8) and $(\varrho, E) = \text{const}$. Thus, we may assert that the incompressible Navier-Stokes equation (1.7) makes sense as an approximation to the compressible Navier-Stokes equation (1.5) when not only the density but also the total energy can be regarded as identically constant.

The asymptotic problem of the Boltzmann equation as the mean free path ε tends to zero and the relations to the hydrodynamical equations determined by the Chapman-Enskog expansion at the Euler and Navier-Stokes levels have been considered by Grad [6] for the "semilinear" Boltzmann equation locally in time and by McLennan [11], Ellis and Pinsky [2, 3], and Pinsky [14] for the linear Boltzmann equation. As to the full nonlinear Boltzmann equation, Nishida [12] obtained the results at the level of the compressible Euler equation that if the initial deviation from the absolute Maxwellian state is small and analytic in the space variables, the solution of the Boltzmann equation exists in a finite time interval independent of ε and it converges there, as $\varepsilon \to 0$, to the local Maxwellian distribution whose fluid-dynamical quantities satisfy the compressible Euler equation (1.4). In the present paper, we consider the nonlinear Boltzmann equation (1.1) in a small neighbourhood of the absolute Maxwellian state at the level of the compressible Navier-Stokes equation with a fixed ε. Then it is shown in Sect. 8 that the solution of the nonlinear Boltzmann equation (1.1) is asymptotically equivalent $\operatorname{mod} t^{-5/4}$ as $t \to +\infty$ to the solution of the compressible Navier-Stokes equation (1.5) with the corresponding initial data. Here we note that the difference of the solution of (1.1) from the absolute Maxwellian state has decay rate $(1+t)^{-3/4}$ at least for some restricted initial data which are classified later. Thus, the second approximation of the Chapman-Enskog expansion is proved to be valid in a sense.

Finally, an analogous asymptotic problem is considered in Kawashima [7] for one-dimensional Broadwell model of the Boltzmann equation for a simple discrete velocity gas. In comparison with this paper, the interesting fact in [7] is that the solutions of nonlinear and semilinear (not linear) problems are asymptotically equivalent as $t \to +\infty$ to each other.

2. Some Notations

Letters $x, v \in R^3$ are the space- and velocity-variables and letter $\xi \in R^3$ is the variable for the Fourier-transform in x. $L^p(\cdot)$ ($\cdot = x, v$, or ξ) denotes the Lebesgue space of measurable functions whose p-th powers ($1 \leqq p < +\infty$) are summable in R^3 with the norm $\|f\|_{L^p(\cdot)}$. $H^l(x)$, $l \geqq 0$ denotes the Sobolev space of $L^2(x)$-functions together with the l-th derivatives, $\hat{H}^l(\xi)$ is the Fourier transform of $H^l(x)$ with the norm

$$\|f\|_{H^l(x)} = \|(1+|\xi|)^l \hat{f}(\xi)\|_{L^2(\xi)} = \|\hat{f}\|_{\hat{H}^l(\xi)}.$$

Definition 2.1. $L^2_M(v)$ is the Hilbert space defined by

$$L^2_M(v) = \{f : f M^{1/2} \in L^2(v)\},$$

where M is the absolute Maxwellian $(2\pi)^{-3/2} \exp(-\frac{1}{2}|v|^2)$. The inner product for $f, g \in L^2_M(v)$ is defined by

$$\langle f, g \rangle \equiv \int f \cdot \bar{g} M \, dv.$$

Definition 2.2. $L^2(v; L^2(x))$ [resp. $L^2(v; L^1(x))$] is the Hilbert (resp. Banach) space which consists of $L^2(x)$ [resp. $L^1(x)$] valued L^2_M-functions in $v \in R^3$ with the norm

$$\|f\| \equiv \left(\int \|f(\cdot, v)\|^2_{L^2(x)} M(v) dv\right)^{1/2}$$

$$\left[\text{resp. } \|f\|_{L^{2,1}} \equiv \left(\int \|f(\cdot, v)\|^2_{L^1(x)} M(v) dv\right)^{1/2}\right].$$

Definition 2.3. $B_{m,l}(m, l \geqq 0)$ is the Banach space which consists of $H^l(x)$-valued continuous functions in v with the property

$$(1+|v|)^m M^{1/2} \| f(\cdot, v) \|_{H^l(x)} \to 0 \quad \text{as} \quad |v| \to +\infty .$$

The norm for $f \in B_{m,l}$ is defined by

$$\begin{aligned} \| f \|_{m,l} &= \sup_v (1+|v|)^m M^{1/2} \| f(\cdot, v) \|_{H^l(x)} \\ &= \sup_v (1+|v|)^m M^{1/2} \| \hat{f}(\cdot, v) \|_{\hat{H}^l(\xi)} . \end{aligned}$$

Definition 2.4. Let B be a Banach space. $C^k(0, T; B)$ $(k \geqq 0,\ T>0)$ is the Banach space which consists of B-valued k-times continuously differentiable functions in $t \in [0, T]$. $L^2(0, T; B)$ is the Banach space which consists of B-valued L^2-functions in $t \in [0, T]$.

Definition 2.5. Let $f = (f^1(x), f^2(x), \ldots, f^n(x))$. $D^k f$ $(k \geqq 0)$ is defined by

$$D^k f = \left(\left(\frac{\partial}{\partial x} \right)^\alpha f^i,\ |\alpha| = k, \quad i = 1, 2, \ldots, n \right),$$

which is a vector composed of all k-th partial derivatives with respect to x.

3. Second Fluid-Dynamical Approximation

We consider the Chapman-Enskog expansion

$$F \simeq F^{(0)} + \varepsilon F^{(1)} + \varepsilon^2 F^{(2)} + \ldots \tag{3.1}$$

and the corresponding expansion for the fluid dynamical quantities

$$p^{ij} \simeq p^{(0),ij} + \varepsilon p^{(1),ij} + \ldots, \tag{3.2}$$

$$q^i \simeq q^{(0),i} + \varepsilon q^{(1),i} + \ldots, \tag{3.3}$$

where $\{p^{(m),ij}\}_{m=0}^{\infty}$ and $\{q^{(m),i}\}_{m=0}^{\infty}$ are determined by

$$p^{(m),ij} \equiv \int (c^i c^j - \tfrac{1}{3} |c|^2 \delta^{ij}) F^{(m)} dv , \tag{3.4}$$

$$q^{(m),i} \equiv \tfrac{1}{2} \int c^i |c|^2 F^{(m)} dv . \tag{3.5}$$

It is well known that the first approximation $F^{(0)}$ is the locally Maxwellian (1.3) and $p^{(0),ij} = q^{(0),i} = 0$. In this section, following the procedure in [1], we determine $F^{(1)}$, $p^{(1),ij}$ and $q^{(1),i}$ precisely. To start, we make some preparations. The quadratic collision operator Q in (1.1) can be written in the form (cf. [1, 5])

$$Q(F, G) = \tfrac{1}{2} \int_{S^2 \times R^3} (F' G_1' + F_1' G' - F G_1 - F_1 G) C(\psi, |v_1 - v|) d\omega dv_1 . \tag{3.6}$$

Here v' and v_1' are the velocities after the interaction of the molecules whose velocities were v, v_1 before the interaction and ω represents the unit vector in the direction of the apse line such that

$$\begin{aligned}
&\omega=(\cos\psi, \sin\psi\cos\chi, \sin\psi\sin\chi),\\
&d\omega=\sin\psi\, d\psi\, d\chi,\\
&v'=v+(\omega, v_1-v)\omega,\\
&v_1'=v_1-(\omega, v_1-v)\omega.
\end{aligned} \tag{3.7}$$

Also $F_1=F(t,x,v_1)$, $F'=F(t,x,v')$, $F_1'=F(t,x,v_1')$ and G_1, G', G_1' are defined analogously. Throughout this paper, we assume the cut-off hard potentials (cf. [5]) i.e., the function $C(\psi, |v_1-v|)$ satisfies

$$\begin{aligned}
&0\leqq C(\psi, |v_1-v|)\leqq C_1|\cos\psi|(|v_1-v|+|v_1-v|^{-\delta}),\\
&\int_0^\pi C(\psi, |v_1-v|)\sin\psi\, d\psi \geqq C_2|v_1-v|(1+|v_1-v|)^{-1},
\end{aligned} \tag{3.8}$$

where C_1, C_2, and $\delta<1$ are some positive constants. Two important spacial cases which satisfy (3.8) are the hard sphere for which

$$C(\psi, |v_1-v|)=C_3|v_1-v|\cos\psi \tag{3.9}$$

and the cut-off inverse power forces r^{-s} ($s\geqq 5$), for which

$$\begin{aligned}
&C(\psi, |v_1-v|)=|v_1-v|^\gamma\beta(\psi),\\
&\gamma=\frac{s-5}{s-1},
\end{aligned} \tag{3.10}$$

where C_3 and $\beta(\psi)$ are some positive constant and function of ψ only respectively. Define the summational invariants

$$\{\psi^j\}_{j=0}^4\equiv\left\{1, v^i, \frac{|v|^2-3}{\sqrt{6}}\right\}, \tag{3.11}$$

which satisfy

$$\int\psi^j Q(F,G)dv=0 \quad \text{for} \quad j=0,1,\dots,4. \tag{3.12}$$

Also, introduce the Burnett functions (cf. [2])

$$\begin{aligned}
&\Psi_{11}=(\omega', v)\left(\frac{|v|^2-3}{\sqrt{6}}-\sqrt{\frac{2}{3}}\right),\\
&\Psi_{02}=\frac{|v|^2}{3}-(\omega', v)^2,
\end{aligned} \tag{3.13}$$

where ω' is any fixed unit vector. It is easy to see

$$\langle\Psi_{11}, \Psi_{02}\rangle=\langle\Psi_{11}, \psi^i\rangle=\langle\Psi_{02}, \psi^i\rangle=0 \qquad (i=0,\dots,4). \tag{3.14}$$

For $\Phi \in L_M^2(v)$, define the linear operator L by

$$L(\Phi)=2M^{-1}Q(M,M\Phi). \tag{3.15}$$

By the arguments in [5], the integral equation in $L_M^2(v)$

$$L(\Phi)=\Psi \in L_M^2(v) \tag{3.16}$$

is solvable if and only if

$$\langle \Psi, \psi^i \rangle = 0 \quad \text{for} \quad i=0,1,\dots,4. \tag{3.17}$$

So, from (3.14), $L^{-1}(\Psi_{02})$ and $L^{-1}(\Psi_{11})$ exist in $L_M^2(v)$. Then our assertion in this section is the following.

Proposition 3.1. *$p^{(1),ij}$ and $q^{(1),i}$ are given in the form*

$$\begin{aligned} p^{(1),ij} &= -2\mu(\theta)\left(\frac{u^i_{x_j}+u^j_{x_i}}{2}-\frac{u^k_{x_k}}{3}\delta^{ij}\right), \\ q^{(1),i} &= -\kappa(\theta)\theta_{x_i} \end{aligned} \tag{3.18}$$

such that

$$\begin{aligned} \mu\left(\frac{1}{R}\right) &= -\frac{3}{4}\langle \Psi_{02}, L^{-1}(\Psi_{02})\rangle \quad (>0), \\ \kappa\left(\frac{1}{R}\right) &= -\frac{3R}{2}\langle \Psi_{11}, L^{-1}(\Psi_{11})\rangle \quad (>0). \end{aligned} \tag{3.19}$$

Furthermore, for the special case (3.9) and (3.10), $\mu(\theta)$ and $\kappa(\theta)$ are given explicitly as follows; for the hard sphere

$$\begin{aligned} \mu(\theta) &= (R\theta)^{1/2}\mu\left(\frac{1}{R}\right), \\ \kappa(\theta) &= (R\theta)^{1/2}\kappa\left(\frac{1}{R}\right), \end{aligned} \tag{3.20}$$

and for the cut-off inverse power potentials ($s \geqq 5$)

$$\begin{aligned} \mu(\theta) &= (R\theta)^{\frac{s+3}{2(s-1)}}\mu\left(\frac{1}{R}\right), \\ \kappa(\theta) &= (R\theta)^{\frac{s+3}{2(s-1)}}\kappa\left(\frac{1}{R}\right). \end{aligned} \tag{3.21}$$

Remark. If we take $p^{ij}=\varepsilon p^{(1),ij}$ and $q^j=\varepsilon q^{(1),j}$ in (1.2), then by virtue of this proposition, we immediately obtain the compressible Navier-Stokes equation (1.5) corresponding to the second approximation $F^{(0)}+\varepsilon F^{(1)}$.

Proof. Following [1],

$$F^{(0)}=\frac{\varrho}{(2\pi R\theta)^{3/2}}\exp\left(-\frac{|u-v|^2}{2R\theta}\right),$$

and $F^{(1)}$ is determined by the integral equation

$$2Q(F^{(0)},F^{(1)})=\frac{\partial_0 F^{(0)}}{\partial t}+v^j F^{(0)}_{x_j}, \tag{3.22}$$

where

$$\frac{\partial_0 F^{(0)}}{\partial t}=\frac{\partial F^{(0)}}{\partial\varrho}\frac{\partial_0\varrho}{\partial t}+\frac{\partial F^{(0)}}{\partial u^i}\frac{\partial_0 u^i}{\partial t}+\frac{\partial F^{(0)}}{\partial\theta}\frac{\partial_0\theta}{\partial t}, \tag{3.23}$$

$$\frac{\partial_0\varrho}{\partial t}=-(\varrho u^j)_{x_j}, \tag{3.24}$$

$$\frac{\partial_0 u^i}{\partial t}=-u^j u^i_{x_j}-\frac{1}{\varrho}(R\varrho\theta)_{x_i}, \tag{3.25}$$

$$\frac{\partial_0\theta}{\partial t}=-u^j\theta_{x_j}-\frac{2}{3}\theta u^j_{x_j}. \tag{3.26}$$

Substituting (3.23)–(3.26) into (3.22), we have

$$2Q(F^{(0)},F^{(1)})=F^{(0)}\{A^j(\log\theta)_{x_j}+B^{ij}u^i_{x_j}\}, \tag{3.27}$$

where

$$A^j(v)=\left(\frac{|v-u|^2}{2R\theta}-\frac{5}{2}\right)(v^j-u^j)$$

$$B^{ij}(v)=\left(\frac{(v^i-u^i)(v^j-u^j)}{R\theta}-\frac{|v-u|^2\delta^{ij}}{3R\theta}\right).$$

Set $V^i=(R\theta)^{-1/2}(v^i-u^i)$. Then $F^{(0)}$ is written by the absolute Maxwellian $M(v)$ as

$$F^{(0)}(v)=\varrho(R\theta)^{-3/2}M(V) \tag{3.28}$$

so that

$$\begin{aligned}&2M(V)^{-1}Q(M(V),F^{(1)}((R\theta)^{1/2}V+u))\\&\quad=(R\theta)^{1/2}\tilde{A}^j(V)(\log\theta)_{x_j}+\tilde{B}^{ij}(V)u^i_{x_j},\end{aligned} \tag{3.29}$$

where

$$\tilde{A}^j(V)=\left(\frac{|V|^2}{2}-\frac{5}{2}\right)V^j,$$

$$\tilde{B}^{ij}(V)=V^iV^j-\frac{|V|^2}{3}\delta^{ij}.$$

Define the linear operator L_θ from $L^2_M(V)$ to $L^2_M(V)$ by

$$L_\theta(\Phi)(V) = 2M(V)^{-1} \int_{S^2\times R^3} (M'\Phi'_1 + M'_1\Phi' - M\Phi_1 - M_1\Phi) \cdot C(\psi, (R\theta)^{1/2}|V_1 - V|)\,d\omega\, dV_1, \tag{3.30}$$

where we note that $L_{1/R} = L$. Then noting that

$$\langle \tilde{A}^j, \psi^k\rangle = 0,$$
$$\langle \tilde{B}^{ij}, \psi^k\rangle = 0 \qquad (0 \leqq k \leqq 4),$$

we have

$$F^{(1)}(u + (R\theta)^{1/2}V) = M(V)\{(R\theta)^{-1}(\log\theta)_{x_j} L_\theta^{-1}(\tilde{A}^j) + (R\theta)^{-3/2} u^i_{x_j} L_\theta^{-1}(\tilde{B}^{ij})\}. \tag{3.31}$$

Hence (3.4) and (3.5) give

$$\begin{aligned} p^{(1),ij} &= (R\theta)^{5/2} \int V^i V^j F^{(1)}(u + (R\theta)^{1/2}V)\,dV, \\ q^{(1),j} &= (R\theta)^3 \int \frac{|V|^2}{2} V^j F^{(1)}(u + (R\theta)^{1/2}V)\,dV. \end{aligned} \tag{3.32}$$

Substituting (3.31) into the above, we arrive at after computation

$$\begin{aligned} p^{(1),ij} &= \frac{R\theta}{5} \int M\tilde{B}^{ij} L_\theta^{-1}(\tilde{B}^{ij})\,dV \times \left(\frac{u^i_{x_j} + u^j_{x_i}}{2} - \frac{(u^k_{x_k})}{3}\delta^{ij}\right), \\ q^{(1),j} &= \frac{R^2\theta}{3} \int M\tilde{A}^j L_\theta^{-1}(\tilde{A}^j)\,dV \times \theta_{x_j}. \end{aligned} \tag{3.33}$$

Here we use the fact that $L_\theta^{-1}(\tilde{A}^i)$ and $L_\theta^{-1}(\tilde{B}^{ij})$ are given in the form (cf. [1])

$$\begin{aligned} L_\theta^{-1}(\tilde{A}^i) &= a^\theta(|V|)\tilde{A}^i, \\ L_\theta^{-1}(\tilde{B}^{ij}) &= b^\theta(|V|)\tilde{B}^{ij}, \end{aligned} \tag{3.34}$$

where a^θ and b^θ are some functions depending only on $|V|$. Thus we have (3.18) by setting

$$\begin{aligned} \mu(\theta) &\equiv -\frac{R\theta}{10}\left\langle \left(v^i v^j - \frac{|v|^2}{3}\delta^{ij}\right), L_\theta^{-1}\left(v^i v^j - \frac{|v|^2}{3}\delta^{ij}\right)\right\rangle, \\ \kappa(\theta) &\equiv -\frac{R^2\theta}{3}\left\langle \left(\frac{|v|^2}{2} - \frac{5}{2}\right)v^j, L_\theta^{-1}\left\{\left(\frac{|v|^2}{2} - \frac{5}{2}\right)v^j\right\}\right\rangle. \end{aligned} \tag{3.35}$$

Then, using the fact that the right hand side of (3.19) is independent of ω' and (3.34) again, we have

$$\begin{aligned} \mu\left(\frac{1}{R}\right) &= -\frac{1}{10}\left\langle v^i v^j - \frac{|v|^2}{3}\delta^{ij}, L^{-1}\left(v^i v^j - \frac{|v|^2}{3}\delta^{ij}\right)\right\rangle \\ &= -\tfrac{3}{4}\langle \Psi_{02}, L^{-1}(\Psi_{02})\rangle, \\ \kappa\left(\frac{1}{R}\right) &= -\frac{3R}{2}\langle \Psi_{11}, L^{-1}(\Psi_{11})\rangle, \end{aligned}$$

that imply (3.19). Finally (3.20) and (3.21) are the consequence of

$$L_\theta=(R\theta)^{1/2}L \qquad \text{for hard sphere},$$

$$L_\theta=(R\theta)^{\frac{s-5}{2(s-1)}}L \quad \text{for power force},$$

which are proved easily by (3.9), (3.10) and (3.30). This completes the proof of Proposition 3.1.

4. Solutions of the Compressible Navier-Stokes Equation

We consider the initial value problem to the compressible Navier-Stokes equation (1.5) with a fixed ε, so that we may set $\varepsilon=1$ without loss of generality. Writing (1.5) in the variables ϱ, u and θ, we consider

$$\begin{aligned}
&\varrho_t+(\varrho u^j)_{x_j}=0\,,\\
&u^i_t+u^j u^i_{x_j}+\frac{1}{\varrho}(R\varrho\theta)_{x_i}=\frac{1}{\varrho}\Big(\mu(\theta)(u^i_{x_j}+u^j_{x_i})-\frac{2}{3}\mu(\theta)u^k_{x_k}\delta^{ij}\Big)_{x_j}\,,\\
&\theta_t+u^j\theta_{x_j}+\frac{2}{3}\theta u^j_{x_j}=\frac{2}{3R\varrho}(\kappa(\theta)\theta_{x_j})_{x_j}+\frac{2\Psi}{3R\varrho}\,,
\end{aligned} \tag{4.1}$$

with the initial data

$$(\varrho(0,x),u(0,x),\theta(0,x))=(\varrho_0(x),u_0(x),\theta_0(x))\,, \tag{4.2}$$

where

$$\Psi=\frac{\mu}{2}(u^i_{x_j}+u^j_{x_i})(u^i_{x_j}+u^j_{x_i})-\frac{2}{3}\mu(u^k_{x_k})^2\,.$$

Applying our arguments in [10] to (4.1)–(4.2), we obtain the solution in a small neighbourhood of the constant state $(1,0,R^{-1})$ which corresponds to the absolute Maxwellian (1.6).

Theorem 4.1. *Suppose the initial data*

$$(\varrho_0-1,u_0,\theta_0-R^{-1})\in H^3(x)\cap L^1(x)$$

and set

$$M_0\equiv\|\varrho_0-1,u_0,\theta_0-R^{-1}\|_{H^3(x)}+\|\varrho_0-1,u_0,\theta_0-R^{-1}\|_{L^1(x)}\,.$$

Then there exists a positive constant ε_0 such that if $M_0<\varepsilon_0$, the initial value problem (4.1)–(4.2) has a unique solution globally in time such as

$$\varrho-1\in C^0(0,+\infty;H^3(x))\cap C^1(0,+\infty;H^2(x))\cap L^2(0,+\infty;H^3(x))\,,$$

$$(u,\theta-R^{-1})\in C^0(0,+\infty;H^3(x))\cap C^1(0,+\infty;H^1(x))\cap L^2(0,+\infty;H^4(x))\,.$$

Furthermore the solution is classical for $t>0$ and has the estimates

$$\sup_t\Big\{\|(\varrho-1,u,\theta-R^{-1})(t)\|^2_{H^3(x)}+\int_0^t\|(\varrho-1)(\tau)\|^2_{H^3(x)}$$
$$+\|(u,\theta-R^{-1})(\tau)\|^2_{H^4(x)}\,d\tau\Big\}\leqq\text{const}\,M_0^2\,, \tag{4.3}$$
$$\|(\varrho-1,u,\theta-R^{-1})(t)\|_{H^2(x)}\leqq\text{const}(1+t)^{-3/4}M_0\,.$$

In order to study the further asymptotic property, we rewrite (4.1)–(4.2) in the variables ϱ, m and E as

$$\begin{aligned}
&\varrho_t+m^j_{x_j}=0\,,\\
&m^i_t+\left\{\frac{m^im^j}{\varrho}+\frac{2}{3}\left(E-\frac{|m|^2}{2\varrho}\right)\delta^{ij}\right\}_{x_j}\\
&\quad=\left[\mu\left\{\left(\frac{m^i}{\varrho}\right)_{x_j}+\left(\frac{m^j}{\varrho}\right)_{x_i}\right\}-\frac{2}{3}\mu\left(\frac{m^k}{\varrho}\right)_{x_k}\delta^{ij}\right]_{x_j},\\
&E_t+\left(\frac{5Em^j}{3\varrho}-\frac{|m|^2m^j}{3\varrho^2}\right)_{x_j}\\
&\quad=\left[\frac{\mu m^k}{\varrho}\left\{\left(\frac{m^k}{\varrho}\right)_{x_j}+\left(\frac{m^j}{\varrho}\right)_{x_k}\right\}-\frac{2}{3}\frac{\mu m^j}{\varrho}\left(\frac{m^k}{\varrho}\right)_{x_k}+\frac{2\kappa}{3R}\left(\frac{E}{\varrho}-\frac{|m|^2}{2\varrho^2}\right)_{x_j}\right]_{x_j},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
(\varrho(0),m(0),E(0))&=\left(\varrho_0,\varrho_0u_0,\varrho_0\left(\frac{3R}{2}\theta_0+\frac{|u_0|^2}{2}\right)\right)\\
&\equiv(\varrho_0,m_0,E_0)\,,
\end{aligned} \tag{4.5}$$

where

$$\mu=\mu\left(\frac{2}{3R}\left(\frac{E}{\varrho}-\frac{|m|^2}{2\varrho^2}\right)\right),$$
$$\kappa=\kappa\left(\frac{2}{3R}\left(\frac{E}{\varrho}-\frac{|m|^2}{2\varrho^2}\right)\right).$$

Corresponding to $(1,0,R^{-1})$, we consider (ϱ,m,E) around the constant state $(1,0,\frac{3}{2})$. Getting together the linearized parts of (4.4) at $(1,0,\frac{3}{2})$ in left-hand side, we rewrite (4.4)–(4.5) again as

$$\begin{aligned}
&\varrho_t+m^j_{x_j}=0\,,\\
&m^i_t+\tfrac{2}{3}E_{x_i}-\bar\mu m^i_{x_jx_j}-\tfrac{1}{3}\bar\mu m^j_{x_ix_j}=f^{ij}_{x_j}\,,\\
&E_t+\frac{5}{2}m^j_{x_j}+\frac{\bar\kappa}{R}\varrho_{x_jx_j}-\frac{2\bar\kappa}{3R}E_{x_jx_j}=g^j_{x_j}\,,
\end{aligned} \tag{4.6}$$

$$(\varrho(0),m(0),E(0))=(\varrho_0,m_0,E_0)\,, \tag{4.7}$$

where $\bar{\mu}=\mu(R^{-1})$, $\bar{\kappa}=\kappa(R^{-1})$,

$$\begin{aligned} f^{ij}=&-\frac{1}{\varrho}\left(m^i m^j-\frac{|m|^2}{3}\delta^{ij}\right)+(\mu-\bar{\mu})\{m^i_{x_j}+m^j_{x_i}\}\\ &-\frac{2}{3}(\mu-\bar{\mu})m^k_{x_k}\delta^{ij}-\mu\left\{\left(\frac{(\varrho-1)m^i}{\varrho}\right)_{x_j}+\left(\frac{(\varrho-1)m^j}{\varrho}\right)_{x_i}\right\}\\ &+\frac{2}{3}\mu\left(\frac{(\varrho-1)m^k}{\varrho}\right)_{x_k}\delta^{ij}\\ =&-\varrho(u^i u^j-\tfrac{1}{3}|u|^2\delta^{ij})+(\mu-\bar{\mu})(u^i_{x_j}+u^j_{x_i})\\ &-\tfrac{2}{3}(\mu-\bar{\mu})u^k_{x_k}\delta^{ij}-\bar{\mu}\{((\varrho-1)u^i)_{x_j}+((\varrho-1)u^j)_{x_i}\}\\ &+\frac{2\bar{\mu}}{3}((\varrho-1)u^k)_{x_k}\delta^{ij}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} g^j=&-\left\{\frac{5}{3\varrho}\left(E-\frac{3}{2}\varrho\right)m^j-\frac{|m|^2 m^j}{3\varrho^2}\right\}\\ &+\frac{\mu m^k}{\varrho}\left\{\left(\frac{m^k}{\varrho}\right)_{x_j}+\left(\frac{m^j}{\varrho}\right)_{x_k}\right\}-\frac{2\mu m^j}{3\varrho}\left(\frac{m^k}{\varrho}\right)_{x_k}\\ &+\frac{2}{3R}(\kappa-\bar{\kappa})\left(E-\frac{3}{2}\varrho\right)_{x_j}\\ &-\frac{2\kappa}{3R}\left\{\frac{(\varrho-1)}{\varrho}\left(E-\frac{3}{2}\varrho\right)+\frac{|m|^2}{2\varrho^2}\right\}_{x_j}\\ =&-\varrho\left(\frac{5R}{2}(\theta-R^{-1})u^j+\frac{|u|^2u^j}{2}\right)\\ &+\mu u^k(u^k_{x_j}+u^j_{x_k})-\frac{2\mu}{3}u^j u^k_{x_k}\\ &+\frac{2}{3R}(\kappa-\bar{\kappa})\left(\frac{3R}{2}(\theta-R^{-1})+\frac{1}{2}|u|^2\right)_{x_j}\\ &-\frac{2\bar{\kappa}}{3R}\left\{(\varrho-1)\left(\frac{3R}{2}(\theta-R^{-1})+\frac{1}{2}|u|^2\right)\right\}_{x_j}-\frac{2\kappa}{3R}u^k u^k_{x_j}. \end{aligned} \tag{4.9}$$

Setting

$$\begin{aligned} N\equiv&\,{}^t(n^0,n^i,n^4)\\ \equiv&\,{}^t(\varrho-1,m^i,\sqrt{\tfrac{2}{3}}E-\sqrt{\tfrac{3}{2}}\varrho), \end{aligned} \tag{4.10}$$

$$F^j\equiv{}^t(0,f^{ij},\sqrt{\tfrac{2}{3}}g^j), \tag{4.11}$$

$$A\equiv\begin{pmatrix} 0 & -i\partial_{x_k} & 0\\ -i\partial_{x_j} & \bar{\mu}\Delta\delta^{jk}+\dfrac{\bar{\mu}}{3}\partial^2_{x_jx_k} & -\sqrt{\tfrac{2}{3}}\partial_{x_j}\\ 0 & -\sqrt{\tfrac{2}{3}}\partial_{x_k} & \dfrac{2\bar{\kappa}}{3R}\Delta \end{pmatrix}, \tag{4.12}$$

(4.6)–(4.7) is written in the form

$$N_t = AN + F^j_{x_j}$$
$$N(0) = N_0 \equiv {}^t(\varrho_0 - 1, m^i_0, \sqrt{\tfrac{2}{3}}\, E_0 - \sqrt{\tfrac{3}{2}}\, \varrho_0) \tag{4.13}$$

or the Fourier transform of (4.13)

$$\hat{N}_t = A(\xi)\hat{N} + i\xi_j \hat{F}^j,$$
$$\hat{N}(0) = \hat{N}_0, \tag{4.14}$$

where

$$A(\xi) = -\begin{pmatrix} 0 & i\xi_k & 0 \\ i\xi_j & \bar{\mu}|\xi|^2\delta^{jk} + \dfrac{\bar{\mu}}{3}\xi_j\xi_k & i\sqrt{\tfrac{2}{3}}\,\xi_j \\ 0 & i\sqrt{\tfrac{2}{3}}\,\xi_k & \dfrac{2\bar{\kappa}}{3R}|\xi|^2 \end{pmatrix}. \tag{4.15}$$

Let us review the results of spectral analysis for $A(\xi)$ that was precisely investigated in [10]. The characteristic equation for $A(\xi)$ is

$$\det(\lambda I - A(\xi)) = (\lambda + \bar{\mu}|\xi|^2)^2 f(\lambda) = 0, \tag{4.16}$$

where

$$f(\lambda) = \lambda^3 + \left(\frac{4}{3}\bar{\mu} + \frac{2\bar{\kappa}}{3R}\right)|\xi|^2\lambda^2 + \left(\frac{8}{9R}\bar{\mu}\bar{\kappa}|\xi|^4 + \frac{5}{3}|\xi|^2\right)\lambda + \frac{2\bar{\kappa}}{3R}|\xi|^4.$$

Set $\lambda_3(\xi) = -\bar{\mu}|\xi|^2$. Denote the roots of $f(\lambda) = 0$ by $\{\lambda_j(\xi)\}_{j=0}^2$ and the spectral resolution for $e^{tA(\xi)}$ by

$$e^{tA(\xi)} = \sum_{j=0}^{3} e^{t\lambda_j(\xi)} P_j(\xi), \tag{4.17}$$

where $P_j(\xi)$ is the corresponding projection matrix.

Lemma 4.2. i) $\lambda_j(0) = 0$ *and* $\mathrm{Re}\,\lambda_j(\xi) < 0$ *for any* $|\xi| > 0$ $(0 \leqq j \leqq 3)$.

ii) *Rank*$(\lambda_3(\xi)I - A(\xi)) = 3$ *for any* $|\xi| > 0$ *except at most one point of* $|\xi|$.

iii) *There exists a positive constant* r_1 *such that for any* $|\xi| \leqq r_1$, $\lambda_j(\xi)$ *has the Taylor series expansion*

$$\lambda_j(\xi) = \sum_{n=1}^{\infty} \lambda_j^{(n)}(i|\xi|)^n, \quad (0 \leqq j \leqq 3) \tag{4.18}$$

and more concretely

$$\lambda_0(\xi) = -\frac{2\bar{\kappa}}{5R}|\xi|^2 + 0(|\xi|^3),$$

$$\lambda_1(\xi) = i\sqrt{\frac{5}{3}}|\xi| - \frac{2}{3}\left(\bar{\mu} + \frac{\bar{\kappa}}{5R}\right)|\xi|^2 + 0(|\xi|^3),$$
$$\lambda_2(\xi) = \overline{\lambda_1(\xi)},$$
$$\lambda_3(\xi) = -\bar{\mu}|\xi|^2.$$

iv) *There exist positive constants* β_0 *and* β_1 *such that for any* $|\xi|\leqq r_1$, $-\beta_0|\xi|^2 \leqq \mathrm{Re}\lambda_j(\xi)\leqq -\beta_1|\xi|^2$ $(0\leqq j\leqq 3)$.

v) *There exists a positive constant* β_2 *such that for any* $|\xi|>r_1$, $\mathrm{Re}\lambda_j(\xi)<-\beta_2$ $(0\leqq j\leqq 3)$.

vi) *The representation* (4.17) *makes sense for any* $|\xi|\geqq 0$ *and for* $|\xi|>r_1$,

$$\|e^{tA(\xi)}\|\leqq C(1+t)^3 e^{-\beta_2 t},$$

where $\|\cdot\|$ *represents matrix norm, and for* $|\xi|\leqq r_1$, $P_j(\xi)$ *has the Taylor series expansion*

$$P_j(\xi)=\sum_{n=0}^{\infty}(i|\xi|)^n P_j^{(n)}(\omega), \qquad \omega=\xi/|\xi|,$$

where $\{P_j^{(0)}(\omega)\}_{j=0}^{3}$ *are orthogonal projections and are given by*

$$P_0^{(0)}(\omega)=\begin{pmatrix} \frac{2}{5} & 0 & -\frac{2}{5}\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 \\ -\frac{2}{5}\sqrt{\frac{3}{2}} & 0 & \frac{3}{5} \end{pmatrix},$$

$$P_1^{(0)}(\omega)=\begin{pmatrix} \frac{3}{10} & -\frac{1}{2}\sqrt{\frac{3}{5}}\,\omega_j & \frac{1}{5}\sqrt{\frac{3}{2}} \\ -\frac{1}{2}\sqrt{\frac{3}{5}}\,\omega_i & \frac{1}{2}\omega_i\omega_j & -\frac{1}{2}\sqrt{\frac{2}{5}}\,\omega_i \\ \frac{1}{5}\sqrt{\frac{3}{2}} & -\frac{1}{2}\sqrt{\frac{2}{5}}\,\omega_j & \frac{1}{5} \end{pmatrix},$$

$$P_2^{(0)}(\omega)=\begin{pmatrix} \frac{3}{10} & \frac{1}{2}\sqrt{\frac{3}{5}}\,\omega_j & \frac{1}{5}\sqrt{\frac{3}{2}} \\ \frac{1}{2}\sqrt{\frac{3}{5}}\,\omega_i & \frac{1}{2}\omega_i\omega_j & \frac{1}{2}\sqrt{\frac{2}{5}}\,\omega_i \\ \frac{1}{5}\sqrt{\frac{3}{2}} & \frac{1}{2}\sqrt{\frac{2}{5}}\,\omega_j & \frac{1}{5} \end{pmatrix},$$

$$P_3^{(0)}(\omega)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^{ij}-\omega_i\omega_j & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Lemma 4.3. *Define* $e^{tA}G$ *for* $G\in L^2(x)$ *by*

$$e^{tA}G\equiv(2\pi)^{-3/2}\int e^{ix\cdot\xi}e^{tA(\xi)}\hat{G}(\xi)d\xi.$$

Then for $G\in L^2(x)\cap L^1(x)$,

$$\|e^{tA}G\|_{L^2(x)}\leqq c(1+t)^{-3/4}(\|G\|_{L^2(x)}+\|G\|_{L^1(x)}),$$

and for $G\in H^1(x)\cap L^1(x)$,

$$\|D^1(e^{tA}G)\|_{L^2(x)}\leqq c(1+t)^{-5/4}(\|G\|_{H^1(x)}+\|G\|_{L^1(x)}).$$

By virtue of Lemmas 4.2 and 4.3, we can show the asymptotic equivalence $\mathrm{mod}\,t^{-5/4}$ as $t\to+\infty$ between the nonlinear solution $N(t)$ and the solution of the linearized equation which is defined by

$$\begin{aligned} W(t)&\equiv{}^t(w^0,w^i,w^4)\\ &\equiv e^{tA}N_0, \end{aligned} \tag{4.19}$$

i.e., $W(t)$ is the solution to the initial value problem

$$\begin{aligned} &W_t = AW, \\ &W(0) = N_0 . \end{aligned} \tag{4.20}$$

For this purpose, we prepare

Lemma 4.4. *Let* (ϱ, u, θ) *be the solution of* (4.1)–(4.2) *constructed in Theorem 4.1 and* F^j *be defined by* (4.11). *Then*

$$\begin{aligned} \sum_{j=1}^{3} (\|F^j\|_{H^1(x)} + \|F^j\|_{L^1(x)}) &\leqq c\|(\varrho-1, u, \theta - R^{-1})(t)\|^2_{H^2(x)} \\ &\leqq cM_0^2(1+t)^{-3/2} . \end{aligned}$$

For the proof, we may apply the estimates of composite function and the decay estimates (4.3) to (4.11) (cf. [10]). Thus, it is ready to state the main result in this section.

Theorem 4.5. *Let* (ϱ, u, θ) *be the solution for* (4.1)–(4.2) *constructed in Theorem 4.1 and let* $N(t)$ *and* $W(t)$ *be defined by* (4.10) *and* (4.19) *respectively. Then*

$$\|N(t) - W(t)\|_{L^2(x)} \leqq cM_0^2(1+t)^{-5/4} .$$

Proof. By (4.13),

$$\begin{aligned} N(t) &= e^{tA}N(0) + \int_0^t e^{(t-s)A}F^j_{x_j}(s)ds \\ &= W(t) + \int_0^t e^{(t-s)A}F^j_{x_j}(s)ds . \end{aligned}$$

Hence from Lemmas 4.3 and 4.4,

$$\begin{aligned} \|N(t) - W(t)\|_{L^2(x)} &\leqq \int_0^t \|e^{(t-s)A}F^j_{x_j}(s)\|_{L^2(x)}\, ds \\ &\leqq c\sum_j \int_0^t (1+t-s)^{-5/4}(\|F^j(s)\|_{H^1(x)} + \|F^j(s)\|_{L^1(x)})ds \\ &\leqq cM_0^2 \int_0^t (1+t-s)^{-5/4}(1+s)^{-3/2}ds \\ &\leqq cM_0^2(1+t)^{-5/4} . \end{aligned}$$

This completes the proof of Theorem 4.5.

Finally, for the arguments in the later sections, let us study the condition that the solution $W(t)$ [or $N(t)$] has the decay rate $t^{-3/4}$ at least.

Lemma 4.6. *Define* $W'(t)$ *from* $W(t)$ *in the Fourier transform by*

$$\hat{W}'(t,\xi) \equiv \sum_{j=0}^{3} e^{t\lambda_j(\xi)}P_j^{(0)}(\omega)\hat{N}_0(\xi) .$$

Then

$$\|W(t) - W'(t)\|_{L^2(x)} \leqq c(1+t)^{-5/4}(\|N_0\|_{L^2(x)} + \|N_0\|_{L^1(x)}) .$$

Lemma 4.7. *Suppose*

$$\int N_0(x)dx \neq 0$$

and set

$$M_1 \equiv \left|\int N_0(x)dx\right| > 0 .$$

Then there exists a positive constant δ such that

$$\|W(t)\|_{L^2(x)} \geqq \delta M_1(1+t)^{-3/4} - c(1+t)^{-5/4}(\|N_0\|_{L^2(x)} + \|N_0\|_{L^1(x)}).$$

Proof of Lemma 4.6

$$\begin{aligned}
&\|W(t)-W'(t)\|_{L^2(x)} \\
&\quad = \|\hat{W}(t)-\hat{W}'(t)\|_{L^2(\xi)} \\
&\quad = \left\| \sum_{j=0}^{3} e^{t\lambda_j(\xi)}(P_j(\xi)-P_j^{(0)}(\omega))\hat{N}_0(\xi) \right\|_{L^2(\xi)} \\
&\quad \leqq c(1+t)^3 e^{-\beta_2 t}\|N_0\|_{L^2(x)} + c\left(\int_{|\xi| \leqq r_1} |\xi|^2 e^{-2t\beta_1|\xi|^2} d\xi \right)^{1/2} \|N_0\|_{L^1(x)} \\
&\quad \leqq c(1+t)^{-5/4}(\|N_0\|_{L^2(x)} + \|N_0\|_{L^1(x)}).
\end{aligned}$$

Proof of Lemma 4.7. Using Lemma 4.6, we may estimate $W'(t)$ because

$$\|W(t)\|_{L^2(x)} \geqq \|W'(t)\|_{L^2(x)} - c\|W(t)-W'(t)\|_{L^2(x)}. \tag{4.21}$$

Noting that

$$\begin{aligned}
&{}^tP_j^{(0)}(\omega) = P_j^{(0)}(\omega) = (P_j^{(0)}(\omega))^2 , \\
&P_j^{(0)}(\omega)P_k^{(0)}(\omega) = 0 \quad \text{for} \quad j \neq k ,
\end{aligned}$$

we have

$$\begin{aligned}
|\hat{W}'(t,\xi)|^2 &= {}^t\overline{\hat{W}'(t,\xi)}\,\hat{W}'(t,\xi) \\
&= \sum_{j,k=0}^{3} {}^t\overline{(e^{t\lambda_j}P_j^{(0)}\hat{N}_0)}(e^{t\lambda_k}P_k^{(0)}\hat{N}_0) \\
&= \sum_{j,k} e^{t(\bar{\lambda}_j+\lambda_k)}\,{}^t\bar{\hat{N}}_0\,{}^tP_j^{(0)}P_k^{(0)}\hat{N}_0 \\
&= \sum_{j=0}^{3} e^{2t\mathrm{Re}\lambda_j(\xi)}|P_j^{(0)}(\omega)\hat{N}_0(\xi)|^2 .
\end{aligned}$$

Hence it follows from Lemma 4.2 that for $|\xi| \leqq r_1$,

$$\begin{aligned}
|\hat{W}'(t,\xi)|^2 &\geqq e^{-2\beta_0|\xi|^2 t}\sum_j |P_j^{(0)}(\omega)\hat{N}_0(\xi)|^2 \\
&= e^{-2\beta_0|\xi|^2 t}|\hat{N}_0(\xi)|^2 .
\end{aligned} \tag{4.22}$$

By the assumption that $N_0 \in L^1(x)$ and

$$\hat{N}_0(0) = \int N_0(x)dx \neq 0 ,$$

we can assume

$$|\hat{N}_0(\xi)| \geqq \tfrac{1}{2}|\hat{N}_0(0)| \quad \text{for} \quad |\xi| \leqq r_1 \tag{4.23}$$

because we can regard r_1 as sufficiently small. Therefore (4.22) and (4.23) give

$$\begin{aligned}\int_{|\xi|\leqq r_1} |\hat{W}'(t,\xi)|^2\, d\xi &\geqq \tfrac{1}{4}|\hat{N}_0(0)|^2 \int_{|\xi|\leqq r_1} e^{-2\beta_0|\xi|^2 t}\, d\xi \\ &= cM_1^2(1+t)^{-3/2}.\end{aligned} \tag{4.24}$$

On the other hand, it is easy to see

$$\int_{|\xi|>r_1} |\hat{W}'(t,\xi)|^2\, d\xi \leqq ce^{-2\beta_2 t}\|N_0\|^2_{L^2(x)}. \tag{4.25}$$

Thus (4.21), (4.24), and (4.25) give the desired estimate

$$\|W(t)\|_{L^2(x)} \geqq \delta M_1(1+t)^{-3/4} - c(1+t)^{-5/4}(\|N_0\|_{L^2(x)} + \|N_0\|_{L^1(x)}).$$

This completes the proof of Lemma 4.7.

5. Solutions of the Incompressible Navier-Stokes Equation

We consider the initial value problem to the incompressible Navier-Stokes equation

$$\left.\begin{aligned} &v^i_t - \frac{\tilde{\mu}}{\tilde{\varrho}} v^i_{x_j x_j} + v^j v^i_{x_j} + \frac{1}{\tilde{\varrho}} p_{x_i} = 0 \qquad (i=1,2,3), \\ &v^i_{x_i} = 0, \end{aligned}\right. \tag{5.1}$$

$$v(0,x) = v_0(x)\,(v^i_{0x_i} = 0), \tag{5.2}$$

where $\tilde{\varrho}$ and $\tilde{\mu}$ represent the constant density and coefficient of viscosity respectively. By the arguments in Leray [8], we have

Theorem 5.1. *Suppose $v_0(x) \in H^1(x)$ and $v^j_{0x_j} = 0$. Then there exists a positive constant ε_1 such that if $\|v_0\|_{H^1(x)} < \varepsilon_1$, the initial value problem (5.1)–(5.2) has a unique solution $v \in C^0(0, +\infty; H^1(x))$ which is classical (C^∞) for $t > 0$.*

Let us study the decay estimates of the solution. Fourier transform of (5.1) gives

$$\left.\begin{aligned} &\hat{v}^j_t + \frac{\tilde{\mu}}{\tilde{\varrho}}|\xi|^2 \hat{v}^j + i\xi_k \widehat{(v^j v^k)} + \frac{i\xi_j}{\tilde{\varrho}}\hat{p} = 0, \\ &i\xi_j \hat{v}^j = 0, \end{aligned}\right. \tag{5.3}$$

which also implies

$$\hat{p}(\xi) = -\tilde{\varrho}\frac{\xi_j \xi_k}{|\xi|^2}\widehat{(v^j v^k)}(\xi). \tag{5.4}$$

Therefore the problem (5.1)–(5.2) is reduced to the integral equation

$$\begin{aligned}\hat{v}^j(t,\xi) &= e^{-\frac{\tilde{\mu}}{\tilde{\varrho}}|\xi|^2 t}\hat{v}^j_0(\xi) \\ &\quad + \int_0^t e^{-\frac{\tilde{\mu}}{\tilde{\varrho}}|\xi|^2(t-s)} i\xi_k \left\{\frac{\xi_j \xi_l}{|\xi|^2}\widehat{(v^l v^k)}(s,\xi) - \widehat{(v^j v^k)}(s,\xi)\right\} ds.\end{aligned} \tag{5.5}$$

Here we note that if we define $v'(t,x)$ in the Fourier transform by

$$\hat{v}'(t,\xi) \equiv e^{-\frac{\tilde{\mu}}{\tilde{\varrho}}|\xi|^2 t}\hat{v}_0(\xi), \tag{5.6}$$

$v'(t,x)$ is the solution of the initial value problem to the heat equation

$$\begin{aligned} &v_t'^i = \frac{\tilde{\mu}}{\tilde{\varrho}} v_{x_j x_j}'^i \qquad (i=1,2,3), \\ &v'(0,x) = v_0(x). \end{aligned} \tag{5.7}$$

By the analogous arguments as in [10, 13, 15] or Sect. 4, we have easily

Theorem 5.2. *Suppose $v_0(x) \in H^2(x) \cap L^1(x)$ and $v_{0x_j}^j = 0$. Then there exists a positive constant ε_2 such that if $\|v_0\|_{H^2(x)} + \|v_0\|_{L^1(x)} < \varepsilon_2$, the initial value problem* (5.1)–(5.2) *has a unique solution $v \in C^0(0, +\infty; H^2(x)) \cap C^1(0, +\infty; L^2(x))$ which is classical for $t > 0$ and also has the decay estimate*

$$\|v(t)\|_{H^2(x)} \leqq c(1+t)^{-3/4}(\|v_0\|_{H^2(x)} + \|v_0\|_{L^1(x)}).$$

Furthermore it holds for v' defined by (5.6) that

$$\|v(t) - v'(t)\|_{L^2(x)} \leqq c(1+t)^{-5/4}.$$

6. Compressible and Incompressible Navier-Stokes Equations

In this section we establish the asymptotic equivalence mod $t^{-5/4}$ as $t \to +\infty$ between the solution of the compressible equation and that of the incompressible one. In order to compare their solutions, we can assume without loss of generality

$$\begin{aligned} &\tilde{\varrho} = \bar{\varrho} \equiv 1,\ \tilde{\mu} = \bar{\mu} \equiv \mu(R^{-1}), \\ &\varrho_0(x) = 1, u_0(x) = v_0(x) \quad \text{and} \quad u_{0x_j}^j = v_{0x_j}^j = 0. \end{aligned} \tag{6.1}$$

In addition, we need to assume

$$E_0(x) \equiv \frac{3R}{2}\theta_0(x) + \frac{1}{2}|u_0(x)|^2 = \frac{3}{2}, \tag{6.2}$$

which determines the initial data of the absolute temperature θ. Then N_0 in (4.13) is given by

$$N_0 = {}^t(0, u_0(x), 0), \tag{6.3}$$

for which $W'(t)$ in Lemma 4.6 is given after simple calculation by

$$\begin{aligned} \hat{W}'(t,\xi) &= \sum_{j=0}^{3} e^{t\lambda_j(\xi)} P_j^{(0)}(\omega)\, {}^t(0, \hat{u}_0^i(\xi), 0) \\ &= e^{t\lambda_3(\xi)} P_3^{(0)}(\omega)\, {}^t(0, \hat{u}_0^i(\xi), 0) \\ &= {}^t(0, e^{-\bar{\mu}|\xi|^2 t}\hat{u}_0(\xi), 0) \\ &= {}^t(0, e^{-\bar{\mu}|\xi|^2 t}\hat{v}_0(\xi), 0). \end{aligned}$$

Therefore $W'(t,x)$ is ${}^t(0, v'(t,x), 0)$ exactly, where v' is the solution of the heat Eq. (5.7) with $\tilde{\varrho}=1$, $\tilde{\mu}=\bar{\mu}$ and $v'(0)=v_0=u_0$. Thus we have

Theorem 6.1. *Suppose* u_0, v_0, *and* $\theta_0 - R^{-1} \in H^3(x) \cap L^1(x)$ *and also suppose* (6.1) *and* (6.2). *Let* (ϱ, u, θ) *and* v *be the solutions of* (4.1)–(4.2) *and* (5.1)–(5.2) *constructed in Theorems 4.1 and 5.2 respectively. Then*

$$\|(1, v^i(t), \tfrac{3}{2}) - (\varrho, \varrho u^i, E)(t)\|_{L^2(x)} \leqq c(1+t)^{-5/4}.$$

Proof. It follows easily from Lemma 4.6, Theorems 4.5 and 5.2 that

$$\begin{aligned}
&\|(1, v^i(t), \tfrac{3}{2}) - (\varrho, \varrho u^i, E)(t)\|_{L^2(x)} \\
&\quad = \|(0, v^i(t), 0) - (\varrho - 1, m^i, E - \tfrac{3}{2})(t)\|_{L^2(x)} \\
&\quad \leqq c\|{}^t(0, v^i(t), 0) - N(t)\|_{L^2(x)} \\
&\quad \leqq c\|v(t) - v'(t)\|_{L^2(x)} + c\|{}^t(0, v'(t), 0) - W'(t)\|_{L^2(x)} \\
&\qquad + c\|W'(t) - W(t)\|_{L^2(x)} + c\|W(t) - N(t)\|_{L^2(x)} \\
&\quad \leqq c(1+t)^{-5/4}.
\end{aligned}$$

This completes the proof of Theorem 6.1.

Remark 1. By Lemma 4.7, if we suppose

$$\int v_0\, dx = \int u_0\, dx \neq 0,$$

$v(t)$ has the decay rate $t^{-3/4}$ at least.

Remark 2. Theorem 6.1 indicates that the incompressible Navier-Stokes equation is an approximation to the compressible one when not only the density but also the total energy can be regarded as identically constant.

7. Solutions of the Boltzmann Equation

Under the same assumptions in Sect. 3, we consider the initial value problem to the nonlinear Boltzmann equation

$$\begin{aligned}
&F_t + v^j F_{x_j} = Q(F, F), \\
&F(0, x, v) = F_0(x, v),
\end{aligned} \tag{7.1}$$

where we set $\varepsilon = 1$. In order to linearize (7.1), setting

$$F(t, x, v) = M(v)(1 + f(t, x, v)), \tag{7.2}$$

where M is the absolute Maxwellian, (7.1) is written in the form

$$\begin{aligned}
&f_t + v^j f_{x_j} - Lf = \Gamma(f, f) \\
&f(0, x, v) = M^{-1}(v) F_0(x, v) - 1 \equiv f_0(x, v)
\end{aligned} \tag{7.3}$$

or in the Fourier transform of (7.3)

$$\begin{aligned}
&\hat{f}_t + (iv^j \xi_j - L)\hat{f} = \hat{\Gamma}(f, f) \\
&\hat{f}(0, \xi, v) = \hat{f}_0(\xi, v),
\end{aligned} \tag{7.4}$$

where Lf and $\Gamma(f,f)$ are defined by

$$\begin{aligned} &Lf = 2M^{-1}Q(M, Mf),\\ &\Gamma(f,f) = M^{-1}Q(Mf, Mf). \end{aligned} \tag{7.5}$$

Let $f'(t,x,v)$ be the solution of the linearized problem

$$\begin{aligned} &f'_t + v^j f'_{x_j} - Lf' = 0\\ &f'(0,x,v) = f_0(x,v). \end{aligned} \tag{7.6}$$

First, we review the arguments in Ukai [15] and Nishida-Imai [13].

Theorem 7.1. *Suppose the initial data satisfies*

$$f_0 \equiv M^{-1}(F_0 - M) \in B_{3,3} \cap L^2(v; L^1(x)).$$

Then there exists a positive constant ε_3 *such that if*

$$\|f_0\|_{3,3} + \|f_0\|_{L^{2,1}} < \varepsilon_3,$$

the initial value problem (7.3), *i.e.* (7.1), *has a unique solution* f *globally in time which satisfies*

$$f \in C^0(0, +\infty; B_{3,3}) \cap C^1(0, +\infty; B_{2,2}),$$

$$\|f(t)\|_{3,3} \leqq c(1+t)^{-3/4},$$

and for the linearized solution f' *defined by* (7.6),

$$\|(f - f')(t)\|_{3,3} \leqq c(1+t)^{-5/4}.$$

Next, we review the fluid-dynamical eigenvalues and eigenfunctions $(\alpha(\xi), e(\xi, v))$ in $L^2_M(v)$ defined by

$$(L - iv^j\xi_j)e = \alpha e \tag{7.7}$$

which were precisely investigated in [2, 3].

Proposition 7.2 (Ellis and Pinsky). *There exist five linearly independent solutions* $(\alpha(\xi), e(\xi, v))$ *of* (7.7) *with* $\alpha(0) = 0$; *these can be represented in the form*

$$\begin{aligned} &\alpha_j(\xi) = \sum_{n=1}^{\infty} \alpha_j^{(n)}(i|\xi|)^n\\ &e_j(\xi, v) = \sum_{n=0}^{\infty} e_j^{(n)}(\omega, v)(i|\xi|)^n \qquad (0 \leqq j \leqq 4), \end{aligned} \tag{7.8}$$

where the series are asymptotic for $|\xi| \to 0$. *Here* $\{e_j\}_{j=0}^4$ *satisfy*

$$\bar{e}_j(\xi, v) = e_j(-\xi, v)$$

and are normalized as

$$\langle e_j(\xi), e_k(-\xi)\rangle = \delta^{jk} \qquad (0 \leqq j, k \leqq 4).$$

More concretely, they can be represented in the form

$$\begin{aligned}
\alpha_0(\xi)&=\tfrac{3}{5}\langle\Psi_{11},\mathrm{L}^{-1}\Psi_{11}\rangle|\xi|^2+0(|\xi|^3),\\
\alpha_1(\xi)&=i\sqrt{\tfrac{5}{3}}|\xi|+(\tfrac{1}{2}\langle\Psi_{02},L^{-1}\Psi_{02}\rangle\\
&\qquad+\tfrac{1}{5}\langle\Psi_{11},L^{-1}\Psi_{11}\rangle)|\xi|^2+0(|\xi|^3),\\
\alpha_2(\xi)&=\overline{\alpha_1(\xi)}\\
\alpha_3(\xi)&=\tfrac{3}{4}\langle\Psi_{02},L^{-1}\Psi_{02}\rangle|\xi|^2+0(|\xi|^3),\\
\alpha_4(\xi)&=\tfrac{3}{4}\langle\Psi_{02},L^{-1}\Psi_{02}\rangle|\xi|^2+0(|\xi|^3),
\end{aligned}\tag{7.9}$$

$$\begin{aligned}
e_0^{(0)}(\omega,v)&=-\sqrt{\tfrac{2}{5}}\,\psi^0+\sqrt{\tfrac{3}{5}}\,\psi^4,\\
e_1^{(0)}(\omega,v)&=\sqrt{\tfrac{3}{10}}\,\psi^0-\sqrt{\tfrac{1}{2}}\,\omega_j\psi^j+\sqrt{\tfrac{1}{5}}\,\psi^4,\\
e_2^{(0)}(\omega,v)&=\sqrt{\tfrac{3}{10}}\,\psi^0+\sqrt{\tfrac{1}{2}}\,\omega_j\psi^j+\sqrt{\tfrac{1}{5}}\,\psi^4,\\
e_3^{(0)}(\omega,v)&=C_4^j\psi^j,\\
e_4^{(0)}(\omega,v)&=C_5^j\psi^j,
\end{aligned}\tag{7.10}$$

where C_4^j and C_5^j represent some unit vectors such as $C_4^jC_5^j=0$ and $C_4^iC_4^j+C_5^iC_5^j=\delta^{ij}-\omega_i\omega_j$.

Remark. In Proposition 3.1, we showed

$$\begin{aligned}
\bar{\mu}&=\mu\left(\frac{1}{R}\right)=-\frac{3}{4}\langle\Psi_{02},L^{-1}\Psi_{02}\rangle\\
\bar{\kappa}&=\kappa\left(\frac{1}{R}\right)=-\frac{3R}{2}\langle\Psi_{11},L^{-1}\Psi_{11}\rangle.
\end{aligned}\tag{7.11}$$

Substituting (7.11) into (7.9), it follows that

$$\begin{aligned}
\alpha_0(\xi)&=-\frac{2\bar{\kappa}}{5R}|\xi|^2+0(|\xi|^3),\\
\alpha_1(\xi)&=i\sqrt{\frac{5}{3}}|\xi|-\frac{2}{3}\left(\bar{\mu}+\frac{\bar{\kappa}}{5R}\right)|\xi|^2+0(|\xi|^3),\\
\alpha_2(\xi)&=\overline{\alpha_1(\xi)},\\
\alpha_3(\xi)&=-\bar{\mu}|\xi|^2+0(|\xi|^3),\\
\alpha_4(\xi)&=-\bar{\mu}|\xi|^2+0(|\xi|^3).
\end{aligned}\tag{7.9'}$$

Hence, comparing (7.9′) with iii) in Lemma 4.2, we have

$$\alpha_j^{(1)}=\lambda_j^{(1)}\quad\text{and}\quad\alpha_j^{(2)}=\lambda_j^{(2)}\qquad(0\leqq j\leqq 3).\tag{7.12}$$

Finally in this section, we derive "Navier-Stokes part" f_N from the linearized solution f'. The Fourier transform of f' is written explicitely in the form (cf. [13, 15])

$$\hat{f}'(t,\xi,v)=\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}e^{pt}(pI-L+iv^j\xi_j)^{-1}\hat{f}_0(\xi,v)\,dp\tag{7.13}$$

for $|\xi| \geqq r$ and

$$\hat{f}'(t,\xi,v) = \sum_{j=0}^{4} e^{t\alpha_j(\xi)} \langle \hat{f}_0(\xi,\cdot), e_j(-\xi,\cdot)\rangle e_j(\xi,v) + \frac{1}{2\pi i} \int_{-\beta-i\infty}^{-\beta+i\infty} e^{pt}(pI - L + iv^j\xi_j)^{-1} \hat{f}_0(\xi,v)\,dp \tag{7.14}$$

for $|\xi| < r$ where r and β are some positive constants. Let us define f_N in the Fourier transform by

$$\hat{f}_N(t,\xi,v) \equiv \sum_{j=0}^{4} e^{t\alpha'_j(\xi)} \langle \hat{f}_0(\xi,\cdot), e_j^{(0)}(\omega,\cdot)\rangle e_j^{(0)}(\omega,v), \tag{7.15}$$

where $\alpha'_j(\xi) \equiv i\alpha_j^{(1)}|\xi| - \alpha_j^{(2)}|\xi|^2$. Using Proposition 7.2, we have easily

Lemma 7.3. *Suppose* $f_0 \in B_{3,3} \cap L^2(v; L^1(x))$. *Then*

$$\|(f' - f_N)(t)\| \leqq c(1+t)^{-5/4}.$$

Now define $W_N = {}^t(w_N^0, w_N^i, w_N^4)$ by

$$w_N^i(t,x) \equiv \langle f_N(t,x,\cdot), \psi^i(\cdot)\rangle \qquad (0 \leqq i \leqq 4) \tag{7.16}$$

for which we simply write

$$W_N = \langle f_N, \psi\rangle. \tag{7.17}$$

Then we have

Lemma 7.4. i) W_N *defined by* (7.17) *is the solution of the initial value problem*

$$\begin{aligned} &(W_N)_t = A_N W_N \\ &W_N(0) = \langle f_0, \psi\rangle, \end{aligned} \tag{7.18}$$

where the symbol $A_N(\xi)$ *of* A_N *is given by*

$$A_N(\xi) = -\begin{pmatrix} \frac{2}{5}\left(\bar{\mu} + \frac{3\bar{\kappa}}{5R}\right)|\xi|^2 & i\xi_k & \frac{2}{5}\sqrt{\frac{2}{3}}\left(\bar{\mu} - \frac{2\bar{\kappa}}{5R}\right)|\xi|^2 \\ i\xi_j & \bar{\mu}|\xi|^2\delta^{jk} - \frac{1}{3}\left(\bar{\mu} - \frac{2\bar{\kappa}}{5R}\right)\xi_j\xi_k & i\sqrt{\frac{2}{3}}\xi_j \\ \frac{2}{5}\sqrt{\frac{2}{3}}\left(\bar{\mu} - \frac{2\bar{\kappa}}{5R}\right)|\xi|^2 & i\sqrt{\frac{2}{3}}\xi_k & \frac{4}{15}\left(\bar{\mu} + \frac{11\bar{\kappa}}{10R}\right)|\xi|^2 \end{pmatrix}.$$

ii) $e^{tA_N(\xi)}$ *is given in the form*

$$e^{tA_N(\xi)} = \sum_{j=0}^{3} e^{t\alpha'_j(\xi)} P_j^{(0)}(\omega), \tag{7.20}$$

where $\{P_j^{(0)}(\omega)\}_{j=0}^{3}$ *are as in Lemma 4.2.*

Proof. Defining the matrix $B=\{B^{ij}\}$ $(0\leqq i,j\leqq 4)$ by

$$\{B^{ij}\}=\begin{pmatrix} -\sqrt{\frac{2}{5}} & 0 & 0 & 0 & \sqrt{\frac{3}{5}} \\ \sqrt{\frac{3}{10}} & -\frac{\omega_1}{\sqrt{2}} & -\frac{\omega_2}{\sqrt{2}} & -\frac{\omega_3}{\sqrt{2}} & \sqrt{\frac{1}{5}} \\ \sqrt{\frac{3}{10}} & \frac{\omega_1}{\sqrt{2}} & \frac{\omega_2}{\sqrt{2}} & \frac{\omega_3}{\sqrt{2}} & \sqrt{\frac{1}{5}} \\ 0 & C_4^1 & C_4^2 & C_4^3 & 0 \\ 0 & C_5^1 & C_5^2 & C_5^3 & 0 \end{pmatrix},$$

we can write (7.10) as

$$e_i^{(0)}(\omega,v)=\sum_{j=0}^{4} B^{ij}\psi^j \qquad (0\leqq i\leqq 4). \tag{7.21}$$

Substituting (7.21) into (7.15),

$$\begin{aligned} \hat{f}_N &= \sum_{j=0}^{4} e^{t\alpha_j'}\left\langle \hat{f}_0, \sum_{l=0}^{4} B^{jl}\psi^l\right\rangle\left(\sum_{k=0}^{4} B^{jk}\psi^k\right) \\ &= \sum_{j,k,l=0}^{4} e^{t\alpha_j'}B^{jl}B^{jk}\psi^k\hat{w}_N^l(0) \end{aligned} \tag{7.22}$$

for which

$$\begin{aligned} \hat{w}_N^m &= \langle \hat{f}_N, \psi^m\rangle \\ &= \sum_{j,k,l=0}^{4} e^{t\alpha_j'}B^{jl}B^{jk}\langle\psi^k,\psi^m\rangle\,\hat{w}_N^l(0) \\ &= \sum_{j,l=0}^{4} e^{t\alpha_j'}B^{jl}B^{jm}\hat{w}_N^l(0) \\ &= \sum_{j=0}^{3}\sum_{l=0}^{4} e^{t\alpha_j'}\{P_j^{(0)}(\omega)\}^{l,m}\hat{w}_N^l(0) \end{aligned} \tag{7.23}$$

which implies (7.20). Noting that

$$\sum_{n=0}^{4} B^{ln}B^{mn}=\delta^{lm},$$

we have

$$\begin{aligned} (\hat{w}_N^m)_t &= \sum_{j,l=0}^{4} \alpha_j' e^{t\alpha_j'}B^{jl}B^{jm}\hat{w}_N^l(0) \\ &= \sum_{j,l=0}^{4} e^{t\alpha_j'}B^{jl}\hat{w}_N^l(0)\left(\sum_{k=0}^{4}\alpha_k' B^{km}\delta^{jk}\right) \\ &= \sum_{j,l=0}^{4} e^{t\alpha_j'}B^{jl}\hat{w}_N^l(0)\left(\sum_{k,n=0}^{4}\alpha_k' B^{km}B^{jn}B^{kn}\right) \\ &= \sum_{n=0}^{4}\left(\sum_{k=0}^{4}\alpha_k' B^{km}B^{kn}\right)\hat{w}_N^n \\ &= \sum_{n=0}^{4}\{A_N(\xi)\}^{m,n}\hat{w}_N^n. \end{aligned}$$

Thus (7.18) is proved. This completes the proof of Lemma 7.4.

Here we note the difference between the systems (7.18) for W_N and (4.20) for W. The system (4.20) is not parabolic (so to speak, incompletely parabolic system), but the system (7.18) is parabolic. That is

Lemma 7.5. *For any vector $V \in R^5$, there exists a positive constant δ such that*

$$ {}^tV(-A_N^{(2)}(\xi))V \geqq \delta|\xi|^2|V|^2, $$

where $A_N^{(2)}(\xi)$ is the principal part of $A_N(\xi)$ defined by

$$ A_N^{(2)}(\xi) = -\begin{pmatrix} \frac{2}{5}\left(\bar{\mu}+\frac{3\bar{\kappa}}{5R}\right)|\xi|^2 & 0 & \frac{2}{5}\sqrt{\frac{2}{3}}\left(\bar{\mu}-\frac{2\bar{\kappa}}{5R}\right)|\xi|^2 \\ 0 & \bar{\mu}|\xi|^2\delta^{ij}-\frac{1}{3}\left(\bar{\mu}-\frac{2\bar{\kappa}}{5R}\right)\xi_i\xi_j & 0 \\ \frac{2}{5}\sqrt{\frac{2}{3}}\left(\bar{\mu}-\frac{2\bar{\kappa}}{5R}\right)|\xi|^2 & 0 & \frac{4}{15}\left(\bar{\mu}+\frac{11\bar{\kappa}}{10R}\right)|\xi|^2 \end{pmatrix}. $$

Proof. Set $V={}^t(a_0, a_i, a_4)$. Then

$$ \begin{aligned} &{}^tV(-A_N^{(2)}(\xi))V \\ &= \frac{2}{5}\left(\bar{\mu}+\frac{3\bar{\kappa}}{5R}\right)|\xi|^2a_0^2 + \frac{4}{5}\sqrt{\frac{2}{3}}\left(\bar{\mu}-\frac{2\bar{\kappa}}{5R}\right)|\xi|^2a_0a_4 \\ &\quad + \frac{4}{15}\left(\bar{\mu}+\frac{11\bar{\kappa}}{10R}\right)|\xi|^2a_4^2 + \sum_{i,j=1}^{3}\left\{\bar{\mu}|\xi|^2\delta^{ij}-\frac{1}{3}\left(\bar{\mu}-\frac{2\bar{\kappa}}{5R}\right)\xi_i\xi_j\right\}a_ia_j \\ &= \frac{2}{5}|\xi|^2\left\{\left(\bar{\mu}+\frac{3\bar{\kappa}}{5R}\right)a_0^2 + 2\sqrt{\frac{2}{3}}\left(\bar{\mu}-\frac{2\bar{\kappa}}{5R}\right)a_0a_4 + \frac{2}{3}\left(\bar{\mu}+\frac{11\bar{\kappa}}{10R}\right)a_4^2\right\} \\ &\quad + |\xi|^2\sum_{i,j}\left\{\bar{\mu}\delta^{ij}-\frac{1}{3}\left(\bar{\mu}-\frac{2\bar{\kappa}}{3R}\right)\omega_i\omega_j\right\}a_ia_j \\ &\equiv \frac{2}{5}|\xi|^2f(a_0,a_4)+|\xi|^2g(a_1,a_2,a_3)\,; \end{aligned} \tag{7.24} $$

$$ \begin{aligned} f(a_0,a_4) &= \left(\bar{\mu}+\frac{3\bar{\kappa}}{5R}\right)\left\{a_0+\frac{2}{3}\left(\frac{\bar{\mu}-\frac{2\bar{\kappa}}{5R}}{\bar{\mu}+\frac{3\bar{\kappa}}{5R}}\right)a_4\right\}^2 + \frac{\frac{5\bar{\kappa}}{3R}\left(1+\frac{\bar{\kappa}}{5R}\right)}{\bar{\mu}+\frac{3\bar{\kappa}}{5R}}a_4^2 \\ &= \frac{2}{3}\left(\bar{\mu}+\frac{11\bar{\kappa}}{10R}\right)\left\{a_4+\frac{3}{2}\left(\frac{\bar{\mu}-\frac{2\bar{\kappa}}{5R}}{\bar{\mu}+\frac{11\bar{\kappa}}{10R}}\right)a_0\right\}^2 + \frac{\frac{5\bar{\kappa}}{2R}\left(1+\frac{\bar{\kappa}}{5R}\right)}{\bar{\mu}+\frac{11\bar{\kappa}}{10R}}a_0^2 \\ &\geqq C_1(|a_0|^2+|a_4|^2), \end{aligned} \tag{7.25} $$

where C_1 is some positive constant,

$$\begin{aligned}
g(a_1, a_2, a_3) &= \sum_{i,j} \left(\bar{\mu}\delta^{ij} - \frac{1}{3}\left(\bar{\mu} - \frac{2\bar{\kappa}}{5R}\right)\omega_i\omega_j \right) a_i a_j \\
&= \bar{\mu} \sum_{i=1}^{3} |a_i|^2 - \frac{1}{3}\left(\bar{\mu} - \frac{2\bar{\kappa}}{5R}\right)\left(\sum_{i=1}^{3} \omega_i a_i\right)^2 \\
&= \bar{\mu}\left(\sum_{i=1}^{3} |a_i|^2 - \frac{1}{3}\left(\sum_{i=1}^{3} \omega_i a_i\right)^2\right) + \frac{2\bar{\kappa}}{15R}\left(\sum_{i=1}^{3} \omega_i a_i\right)^2 \\
&\geqq \frac{2}{3}\bar{\mu} \sum_{i=1}^{3} |a_i|^2 .
\end{aligned} \tag{7.26}$$

Hence (7.24)–(7.26) immediately imply

$$ {}^tV(-A_N^{(2)}(\xi))V \geqq \delta|\xi|^2|V|^2 $$

for some positive constant δ. This completes the proof of Lemma 7.5.

8. Boltzmann and Compressible Navier-Stokes Equations

In the last section, we establish the asymptotic equivalence $\operatorname{mod} t^{-5/4}$ as $t \to +\infty$ between the solution of the nonlinear Boltzmann equation (7.1) and the solution of the compressible Navier-Stokes equation (4.1). To distinguish the former from the latter, especially in this section, we write

$$\begin{aligned}
\varrho_F(t,x) &\equiv \int F(t,x,v)\,dv\,, \\
m_F^i(t,x) &\equiv \int v^i F(t,x,v)\,dv\,, \\
E_F(t,x) &\equiv \int \tfrac{1}{2}|v|^2 F(t,x,v)\,dv\,,
\end{aligned} \tag{8.1}$$

$$ f^N(t,x,v) \equiv \sum_{i=0}^{4} n^i(t,x)\psi^i(v)\,, \tag{8.2}$$

where $N(t)$ was defined by (4.10), i.e.,

$$\begin{aligned}
N(t,x) &= {}^t(n^0, n^i, n^4) \\
&= {}^t(\varrho - 1, m^i, \sqrt{\tfrac{2}{3}}\,E - \sqrt{\tfrac{3}{2}}\,\varrho)\,.
\end{aligned}$$

Then it holds for $F = M(1+f)$ that

$$\begin{aligned}
&{}^t(\varrho_F - 1, m_F^i, \sqrt{\tfrac{2}{3}}\,E_F - \sqrt{\tfrac{3}{2}}\,\varrho_F) \\
&\quad = {}^t(\langle f, \psi^0\rangle, \langle f, \psi^i\rangle, \langle f, \psi^4\rangle) \\
&\quad = \langle f, \psi\rangle\,.
\end{aligned} \tag{8.3}$$

In fact, (8.3) follows from

$$\begin{aligned}\varrho_F - 1 &= \int F - M\,dv\\ &= \int Mf\,dv\\ &= \langle f, \psi^0\rangle,\\ m_F^i &= \int v^i F\,dv\\ &= \int Mv^i + Mfv^i\,dv\\ &= \langle f, \psi^i\rangle,\\ \sqrt{\tfrac{2}{3}}E_F - \sqrt{\tfrac{3}{2}}\varrho_F &= \int (\tfrac{1}{2}\sqrt{\tfrac{2}{3}}|v|^2 - \sqrt{\tfrac{3}{2}})M + (\tfrac{1}{2}\sqrt{\tfrac{2}{3}}|v|^2 - \sqrt{\tfrac{3}{2}})fM\,dv\\ &= \langle f, \psi^4\rangle.\end{aligned}$$

In order to compare the problems (7.1) and (4.1)–(4.2), we assume that the initial data satisfy

$$\begin{aligned}&\varrho_{F_0}(x) = \varrho_0(x),\\ &m_{F_0}^i(x) = \varrho_0(x)u_0^i(x) \equiv m_0^i(x),\\ &E_{F_0}(x) = \frac{3R}{2}\varrho_0(x)\theta_0(x) + \frac{1}{2}\varrho_0(x)|u_0(x)|^2 \equiv E_0(x).\end{aligned} \tag{8.4}$$

Then we note from (8.3) and (8.4) that

$$\begin{aligned}N_0 &\equiv {}^t(\varrho_0 - 1, m_0, \sqrt{\tfrac{2}{3}}E_0 - \sqrt{\tfrac{3}{2}}\varrho_0)\\ &= {}^t(\varrho_{F_0} - 1, m_{F_0}, \sqrt{\tfrac{2}{3}}E_{F_0} - \sqrt{\tfrac{3}{2}}\varrho_{F_0})\\ &= \langle f_0, \psi\rangle.\end{aligned} \tag{8.5}$$

Now it is ready to state our main result in this paper.

Theorem 8.1. *Suppose that the initial data* F_0 *and* $(\varrho_0, u_0, \theta_0)$ *satisfy* (8.4) *and*

$$f_0 \equiv M^{-1}(F_0 - M) \in B_{3,3} \cap L^2(v; L^1(x)). \tag{8.6}$$

Then there exists a positive constant ε_4 *such that if*

$$\|f_0\|_{3,3} + \|f_0\|_{L^{2,1}} < \varepsilon_4,$$

both the initial value problems (4.1)–(4.2) *and* (7.1) *have the global solutions in time (Theorems 4.1 and 7.1) which satisfy*

$$\|(\varrho_F - \varrho, m_F - m, E_F - E)(t)\|_{L^2(x)} \leqq c(1+t)^{-5/4}, \tag{8.7}$$

$$\|(f - f^N)(t)\| \leqq c(1+t)^{-5/4}, \tag{8.8}$$

where (ϱ_F, m_F, E_F) *and* f^N *are given by* (8.1) *and* (8.2) *respectively and* $F = M(1+f)$.

Remark. By Lemma 4.7, if we suppose

$$\int \langle f_0, \psi\rangle\,dx \neq 0,$$

$\langle f, \psi\rangle(t)$ has the decay rate $t^{-3/4}$ essentially.

Proof. First, we note that (8.6) implies

$$(\varrho_0-1, u_0, \theta_0-R^{-1})\in H^3(x)\cap L^1(x).$$

In fact, $\varrho_0-1\in H^3(x)$ follows from

$$\begin{aligned}\|\varrho_0-1\|^2_{H^3(x)} &= \|\varrho_{F_0-M}\|^2_{H^3(x)}\\ &= \sum_{k=0}^{3}\int|D^k\int(F_0-M)dv|^2\,dx\\ &\leqq \sum_{k=0}^{3}\iint|D^k f_0 M^{1/2}|^2\,dx\,dv\\ &\leqq c\left\{\sup_v(1+|v|)^3 M^{1/2}\|f_0(\cdot,v)\|_{H^3(x)}\right\}^2\\ &\leqq c\|f_0\|^2_{3,3}\end{aligned}$$

and the others are obtained in the same way. Hence, if ε_4 is sufficiently small, Theorems 4.1 and 7.1 guarantee the global existence in time. Let us show only (8.7) because (8.8) is proved in the same way. It follows that

$$\begin{aligned}&\|(\varrho_F-\varrho, m_F-m, E_F-E)\|_{L^2(x)}\\ &\quad\leqq c\|(\varrho_F-1, m_F, \sqrt{\tfrac{2}{3}}E_F-\sqrt{\tfrac{3}{2}}\varrho_F)-(\varrho-1, m, \sqrt{\tfrac{2}{3}}E-\sqrt{\tfrac{3}{2}}\varrho)\|_{L^2(x)}\\ &\quad= c\|\langle f,\psi\rangle-N\|_{L^2(x)}\\ &\quad\leqq c\|\langle f-f',\psi\rangle\|_{L^2(x)}+c\|\langle f'-f_N,\psi\rangle\|_{L^2(x)}\\ &\qquad+c\|W_N-W'\|_{L^2(x)}+c\|W'-W\|_{L^2(x)}+c\|W-N\|_{L^2(x)},\end{aligned}\tag{8.9}$$

where f, f', f_N, N, W and W' are given by (7.3), (7.6), (7.15), (4.13), (4.20) and Lemma 4.6 respectively. Since we can easily see

$$\begin{aligned}\|\langle g,\psi\rangle\|_{L^2(x)} &\leqq c\|g\|\\ &\leqq c\|g\|_{3,3}\quad\text{for}\quad g\in B_{3,3},\end{aligned}$$

Theorem 7.1 and Lemma 7.3 give

$$\|\langle f-f',\psi\rangle(t)\|_{L^2(x)}+\|\langle f'-f_N,\psi\rangle(t)\|_{L^2(x)}\leqq c(1+t)^{-5/4},\tag{8.10}$$

and Theorem 4.5 and Lemma 4.6 give

$$\|W(t)-N(t)\|_{L^2(x)}+\|W(t)-W'(t)\|_{L^2(x)}\leqq c(1+t)^{-5/4}.\tag{8.11}$$

Therefore if we can show

$$\|W_N(t)-W'(t)\|_{L^2(x)}\leqq c(1+t)^{-5/4},\tag{8.12}$$

then (8.9)–(8.12) complete the proof. Noting (8.5), W_N and W' were represented as

$$\begin{aligned}W_N(t) &= e^{tA_N}W_N(0)=e^{tA_N}\langle f_0,\psi\rangle\\ &= \sum_{j=0}^{3} e^{t\alpha_j(\xi)}P_j^{(0)}(\omega)N_0,\end{aligned}$$

$$W'(t)=\sum_{j=0}^{3} e^{t\lambda_j(\xi)}P_j^{(0)}(\omega)N_0.$$

Hence, it follows from Lemma 4.2 and Remark of Proposition 7.2 that

$$\begin{aligned}
&\| W_N(t) - W'(t) \|^2_{L^2(x)} \\
&\quad = \| \hat{W}_N(t) - \hat{W}'(t) \|^2_{L^2(\xi)} \\
&\quad \leqq \left(\int_{|\xi| \leqq r_1} + \int_{|\xi| > r_1} \right) \left(\left| \sum_{j=0}^{3} (e^{t\alpha'_j(\xi)} - e^{t\lambda_j(\xi)}) P_j^{(0)}(\omega) \hat{N}_0 \right|^2 \right) d\xi \\
&\quad \leqq c(1+t)^6 e^{-2\beta_2 t} \| \hat{N}_0 \|^2_{L^2(\xi)} + \sup_{\xi} |\hat{N}_0(\xi)|^2 \int_{|\xi| \leqq r_1} ct^2 |\xi|^6 e^{-2\beta_1 |\xi|^2 t} d\xi \\
&\quad \leqq c(1+t)^{-5/2} (\| N_0 \|^2_{L^1(x)} + \| N^0 \|^2_{L^2(x)}).
\end{aligned}$$

Thus we have

$$\| W_N(t) - W'(t) \|_{L^2(x)} \leqq c(1+t)^{-5/4}$$

which also completes the proof of Theorem 8.1.

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