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n-Point Functions for the Rectangular Ising Ferromagnet

D. B. Abraham*

Department of Theoretical Chemistry, Oxford University, Oxford OX1 3TG, England

Abstract. A new representation for the n-point functions of the Planar Ising ferromagnet is given. Below the critical temperature the boundary conditions are toroidal; the state is a superposition of the extremal invariant ones, with equal weights.

1. Introduction

This paper presents the final results which are needed to write down the n-point function of the rectangular Ising ferromagnet in an explicit way. As was explained in the first paper [1], this can be done once all matrix elements of spin operators between any eigenvectors of the transfer matrix have been given. In [1] and [2], matrix elements from the vacua to any excited state were considered. The method for completing the problem is quite obvious, but the fact that a Wick theorem still obtains is not; it is also highly significant for the truncation properties of the n-point functions [3]. The results of this series of papers have found application in the rigorous determination of critical indices [4], in heuristic remarks on the equation of state [5] and in the analysis of the density profile between phases [6].

2. Generalised Matrix Elements

Let functions associated with the generalised matrix elements be defined by

$$\begin{aligned}&F((e^{i\beta})_m|(e^{i\alpha})_{m+1,n})\\&\quad=M^{n/2}\exp i\left\{\sum_1^m(\beta_j+\theta(\beta_j))+\sum_{m+1}^n(\alpha_j+\theta(\alpha_j))\right\}\\&\qquad\cdot\langle\Phi_-|G_{\alpha_n}\ldots G_{\alpha_{m+1}}G^+_{-\beta_m}\ldots G^+_{-\beta_1}|\Phi_+\rangle\,.\end{aligned}\tag{2.1}$$

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By using the linear dependence relationship

$$G^{+}_{-\beta}=\sum_{\alpha}\langle\beta|\alpha\rangle(\cos(\theta(\beta)-\theta(\alpha))G^{+}_{-\alpha}+i\sin(\theta(\beta)-\theta(\alpha))G_{\alpha}) \tag{2.2}$$

with

$$\langle\beta|\alpha\rangle=2/M(e^{i(\beta-\alpha)}-1) \tag{2.3}$$

the following recurrence relationship may be derived:

$$\begin{aligned}&F_M((t)_m|(z)_{m+1,n})\\&\quad=\sum_{z_m}\frac{1}{M(z_m/t_m-1)}\left(1-\frac{\Theta(z_m)}{\Theta(t_m)}\right)F_M((t)_{m-1}|(z)_{m,n})\\&\quad+\sum_{j=m+1}^{n}(-1)^{j-m-1}\frac{z_jt_m}{z_jt_m-1}\left(1+\frac{1}{\Theta(z_m)\Theta(t_j)}\right)F_M((t)_m|\Delta_j(z)_{m+1,n}).\end{aligned} \tag{2.4}$$

The first summation is over distinct z_m such that $z_m^M=1$. The relevant object in the limit $M\to\infty$ is given by

$$\begin{aligned}F((t)_m|(z)_{m+1,n})&=\frac{\mathscr{P}}{2\pi i}\int_{C_1}\frac{dz_m}{z_m}\frac{1}{(z_m/t_m-1)}\left(1-\frac{\Theta(z_m)}{\Theta(t_m)}\right)\\F((t)_{m-1}|(z)_{m,n})&\\&+\sum_{j=m+1}^{n}(-1)^{j-m+1}\frac{z_jt_m}{z_jt_m-1}\left(1+\frac{1}{\Theta(t_m)\Theta(z_j)}\right)F((t)_{m-1}|\Delta_j(z)_{m+1,n}).\end{aligned} \tag{2.5}$$

The solution of (2.5) will be developed separately for $T>T_C$ and $T<T_C$ using an inductive ansatz.

By analogy with the introduction of the operator Y_+ in the previous two papers, consider the operator Y_- defined on a dense substance of $L^2(S_1)$ by

$$(Y_-f)(t)=\frac{\mathscr{P}}{\pi i}\int_{C_1}\frac{dz}{z}\frac{1}{z/t-1}\left(1-\frac{\Theta(z)}{\Theta(t)}\right)f(z). \tag{2.6}$$

This may be extended to $\bigotimes_1^n L^2(S_1)$ by

$$(Y_-f)((z)_n)=\frac{\mathscr{P}}{\pi i}\int_{C_1}\frac{dt_1}{t_1}\frac{1}{t/z_1-1}\left(1-\frac{\Theta(t)}{\Theta(z)}\right)f(t,(z)_{2,n}). \tag{2.7}$$

Clearly the norm satisfies $\|Y_-\|\leqq 2$.

Consider first the case $T>T_C$: since $F((z)_{2n})$ is known, when $m=1$ in (2.5) we have

$$\begin{aligned}F(t|(z)_{2,2n})=\sum_{2}^{2n}(-1)^jF(\Delta_j(z)_{2,2n})&\left[\frac{1}{2}(Y_-f_-)(t,z_j)\right.\\&\left.+\frac{z_jt}{z_jt-1}\left(1+\frac{1}{\Theta(z_j)\Theta(t)}\right)\right],\end{aligned} \tag{2.8}$$

where the pair contraction function f_- is expressed in terms of the Wiener-Hopf factorisation (see [1], Appendix B) of $\Theta(z)$ by

$$f_{\pm}(z,t)=\frac{zt}{zt-1}(\Theta_+^{-1}(z)\Theta_-^{-1}(t)\pm\Theta_+^{-1}(t)\Theta_-^{-1}(z)). \tag{2.9}$$

The additional function f_+ will be encountered in the following. Using the properties of the factorisation (see [1], Appendix B) it follows that

$$(Y_-f_-)(t,z_j)=2f_+(t,z_j)-\frac{2z_jt}{z_jt-1}\left(1+\frac{1}{\Theta(z_j)\Theta(t)}\right), \tag{2.10}$$

$$(Y_-f_+)(t,z_j)=2f_-(t,z_j). \tag{2.11}$$

Insertion of (2.10) into (2.8) gives

$$F(t|(z)_{2,2n})=\sum_2^{2n}(-1)^j f_+(t,z_j)F(\Delta_j(z)_{2,2n}). \tag{2.12}$$

This result suggests the inductive ansatz

$$\begin{aligned}F((t)_m|(z)_{m+1,2n})=&\sum_1^{m-1}(-1)^{j-m}f_-(t_m,t_j)F(\Delta_j(t)_{m-1}|(z)_{m+1,2n})\\&+\sum_{m+1}^{2n}(-1)^{j-m+1}f_+(t_m,z_j)F((t)_{m-1}|\Delta_j(z)_{m+1,2n}).\end{aligned} \tag{2.13}$$

In order to test whether this satisfies (2.5), for $m\geqq 2$, (2.10) and (2.11) are needed; then (2.13) is readily verified by induction on m, for any $n\geqq 1$.

If $T<T_C$ and $m=1$ then the expansion (4.18) of Paper I should be used with contraction function and initial condition as follows:

$$f_{\pm}(z,t)=\frac{zt}{zt-1}(\Theta_+^{-1}(z)t^{-1}\Theta_-^{-1}(t)\pm\Theta_+(t)z^{-1}\Theta_-^{-1}(z)), \tag{2.14}$$

$$F(z)=z\Theta_+^{-1}(z)\Theta_+(0)m^*. \tag{2.15}$$

Then from (2.5) it follows that

$$\begin{aligned}F(t|(z)_{2,2n+1})=&-\tfrac{1}{2}(Y_-F)(t)F((z)_{2,2n+1})\\&+\sum_{j=2}^{2n+1}(-1)^j\left\{F(z_j)\tfrac{1}{2}(Y_-F)(t,\Delta_{1j}(z)_{2n+1})\right.\\&\left.+\frac{z_jt}{z_jt-1}\left(1+\frac{1}{\Theta(z_j)\Theta(t)}\right)F(\Delta_j(z)_{2n+1})\right\}.\end{aligned} \tag{2.16}$$

The results needed here are that

$$\begin{aligned}(Y_-f_-)(t,z)=&2f_+(t,z)-\frac{2zt}{zt-1}\left(1+\frac{1}{\Theta(z)\Theta(t)}\right)\\&+2\Theta_+(0)\Theta^{-1}(t)z\Theta_+(z)^{-1},\end{aligned} \tag{2.17}$$

$$(Y_-f_+)(t,z)=2f_-(t,z)-2\Theta_+(0)\Theta^{-1}(t)z\Theta_+(z)^{-1}, \tag{2.18}$$

and

$$(Y_{-}F)(z)=2F(z). \tag{2.19}$$

The terms involving $F(z_j)$ on the right side of (2.16) and (2.17) cancel in (2.15) by appealing to the properties of Pfaffians, giving the result

$$F(t|(z)_{2,2n+1})=\sum_{2}^{2n+1}(-1)^j F(z_j)F(t|\Delta_{1j}(z)_{2n+1}) - F(t)F((z)_{2,2n+1}), \tag{2.20}$$

where $$F(t|\Delta_1(z)_{2n})=\sum_{2}^{2n}(-1)^k f_+(t,z_k)F(\Delta_{1k}(z)_{2n}) \tag{2.21}$$

the final Pfaffian being evaluated according to Paper I of the series. An inductive ansatz analogous to (2.13) can now be made, and established for $T<T_C$; it is (2.36) of Theorem 2.

The matrix elements

$$F^x_M((e^{i\beta})_m|(e^{i\alpha})_{m+1,n})=M^{n/2}\exp i\left\{\sum_{1}^{m}(\beta_j+\theta(\beta_j)) + \sum_{m+1}^{n}(\alpha_j+\theta(\alpha_j))\right\}\langle\Phi_-|G_{\alpha_n}\dots G_{\alpha_{m+1}}\sigma_1^x G^+_{-\beta_m}\dots G^+_{-\beta_1}|\Phi_+\rangle \tag{2.22}$$

are calculated in the appropriate limit as $M\to\infty$ by precisely the same procedure as in Paper II.

For $T>T_C$ we have the equation

$$\begin{aligned}
&F^x((t)_m|(z)_{m+1,2n+1})\\
&=\sum_{j=1}^{m}(-1)^{m-j-1}\Theta(z_j)^{-1}F(\Delta_j(t)_m|(z)_{m+1,2n+1})\\
&\quad+\frac{\mathscr{P}}{2\pi i}\int_{C_1}\frac{dt}{t}\Theta(t)F((t)_m,t|(z)_{m+1,2n+1})\\
&=\sum_{j=1}^{m}(-1)^{m+j-1}\left\{\Theta(t_j)^{-1}+\frac{\mathscr{P}}{2\pi i}\int_{C_1}\frac{dt}{t}\Theta(t)f_-(t,t_j)\right\}\\
&\quad\cdot F(\Delta_j(t)_m|(z)_{m+1,2n+1})\\
&\quad+\sum_{j=m+1}^{n}(-1)^{m+j}\frac{\mathscr{P}}{2\pi i}\int_{C_1}\frac{dt}{t}\Theta(t)f_+(t,z_j)\\
&\quad\cdot F((t)_m|\Delta_j(z)_{m+1,2n+1}).
\end{aligned} \tag{2.23}$$

But we have the results

$$\frac{\mathscr{P}}{2\pi i}\int_{C_1}\frac{dt}{t}\Theta(t)f_+(t,z)=\Theta_-(\infty)\Theta^{-1}(z), \tag{2.24}$$

$$\frac{\mathscr{P}}{2\pi i}\int_{C_1}\frac{dt}{t}\Theta(t)f_-(t,z)=\Theta_-(\infty)\Theta_-^{-1}(z)-\Theta^{-1}(z), \tag{2.25}$$

from which the results given in the Theorem 1 below follow. The analogous results for $T<T_C$ are obtained by conducting the expansion of the Pfaffian in line 1 of (2.23) according to (2.36). Using the results

$$\frac{\mathscr{P}}{2\pi i}\int_{C_1}\frac{dt}{t}\Theta(t)f_+(t,z)=-z\Theta_+^{-1}(z)\Theta_+(0) \tag{2.26}$$

and

$$\frac{\mathscr{P}}{2\pi i}\int_{C_1}\frac{dt}{t}\Theta(t)f_-(t,z)=-\frac{1}{\Theta(z)}+z\Theta_+^{-1}(z)\Theta_+(0) \tag{2.27}$$

together with the normalisation

$$\frac{1}{2\pi i}\oint_{C_1}\frac{dt}{t}\Theta(t)F(t)=m^* \tag{2.28}$$

then gives the appropriate part of Theorem 2. The results are as follows:

Theorem 1. *If* $\mathscr{I}(\Theta)=0(T>T_C)$ *then for* $0\leqq m\leqq 2n+1$

$$F((z)_m|(z)_{m+1,2n+1})=0 \tag{2.29}$$

whereas

$$F((z)_m|(z)_{m+1,2n})=\sum_1^{2n}(-1)^j f(z_1,z_j)F(\Delta_{1j}(z)_m|(z)_{m+1,2n}) \tag{2.30}$$

with

$$f(z_i,z_j)=f_+(z_i,z_j)\,[\text{resp.}\, f_-(z_i,z_j)] \tag{2.31}$$

for $1\leqq i\leqq m$, $m+1\leqq j\leqq 2n$ (resp. $1\leqq i\leqq m$ *and* $1\leqq j\leqq m$ *or* $m+1\leqq i\leqq 2n$, $m+1\leqq j\leqq 2n$).

Here

$$f_\pm(z,t)=\frac{zt}{zt-1}(\Theta_+^{-1}(z)\Theta_-^{-1}(t)\pm\Theta_+^{-1}(t)\Theta_-^{-1}(z)). \tag{2.32}$$

on the other hand

$$F^x((z)_m|(z)_{m+1,2n})=0 \tag{2.33}$$

whereas

$$\begin{aligned}F^x((z)_m|z_{m+1,2n+1})=&\sum_{j=1}^{2n+1}\Theta_-(\infty)(-1)^j\Theta_-^{-1}(z_j)\\&\cdot F(\Delta_j(z)_m|(z)_{m+1,2n+1}),\end{aligned} \tag{2.34}$$

where the generalised Pfaffian is given by (2.30) *and* (2.31).

Theorem 2. *If* $\mathscr{I}(\Theta)=-1(T<T_C)$ *then for* $0\leqq m\leqq 2n$

$$F((z)_m|(z)_{m+1,2n})=0 \tag{2.35}$$

whereas

$$F((z)_m|(z)_{m+1,2n+1})=\sum_{j=1}^{2n+1}(-1)^j F(z_j)F(\Delta_j(z)_m|(z)_{m+1,2n+1}) \tag{2.36}$$

the second matrix element factor on the right hand side being given by (2.22) *and* (2.23) *of the previous theorem, with*

$$f_\pm(z,t)=\frac{zt}{zt-1}\Theta_+^{-1}(z)t^{-1}\Theta_-^{-1}(t)\pm\Theta_+^{-1}(t)z^{-1}\Theta_-^{-1}(z), \tag{2.37}$$

$$F(z)=\Theta_+(0)m^*z\Theta_+^{-1}(z). \tag{2.38}$$

On the other hand

$$F^x((z)_m|(z)_{m+1,2n+1})=0 \tag{2.39}$$

whereas

$$F^x((z)_m|(z)_{m+1,2n})=F((z)_m|(z)_{m+1,2n}) \tag{2.40}$$

the right hand side being given by (2.22) *and* (2.23), *with the pair contraction function* $f_\pm$ *given by*

$$f_\pm(z,t)=\frac{zt}{zt-1}(\Theta_+(z)^{-1}t^{-1}\Theta_-(t)^{-1}\pm\Theta_+(t)^{-1}z^{-1}\Theta_-(z)^{-1}). \tag{2.41}$$

Remarks. 1. The matrix elements are written in terms of Pfaffians which are generalised further to include *symmetric* contractions. It should be noted that there is still antisymmetry under permutations of the $\{t\}$ or the $\{z\}$ separately, as there should be. It is quite surprising that a Wick theorem result holds in this case also.

2. The case $T<T_C$, $m=n=2$ was used in the theory of the interface between phases for the rectangular Ising ferromagnet [6].

3. Representation of the n-Point Function

The following formula was developed in Paper I [1] of this series. The notation $(r)_n=(r_1,\dots,r_n)$ will be used for the location of the n particles, with $r_j\in\mathbb{Z}^2$. The relative coordinates are $x_k=(r_{k+1}-r_k)\cdot i$ and $y_k=(r_{k+1}-r_k)\cdot j$ where i and j are unit vectors for the lattice $\mathbb{Z}^2$ and i is the transfer direction. The points are ordered so that $x_k\geqq 0$, $k=1,\dots,n-1$. The n-point function is

$$\begin{aligned}\langle\sigma(r)_n\rangle=\lim_{M\to\infty}\sum_{j_1\dots j_{n+1}}&\exp-\sum_{k=1}^{n-1}(\gamma(j_k)x_k-i\omega(j_k)y_k)\\ &\cdot\langle\Phi_+|\sigma_1^x|\Phi_{j_1}\rangle\prod_1^{n-2}\langle\Phi_{j_l}|\sigma_1^x|\Phi_{j_{l+1}}\rangle\\ &\cdot\langle\Phi_{j_{n-1}}|\sigma_1^x|\Phi_+\rangle.\end{aligned} \tag{3.1}$$

The index j of each state $|\Phi_j\rangle$ is given by a set of wavenumbers $(\omega)_{m_j}$ with $m_j\geqq 0$ ($m_j=0$ corresponds to $|\Phi_+\rangle$) with $\omega\in[0,2\pi]$. The summations become integrations in the thermodynamic limit, as can be seen by considering Section 5 of

[1], giving the result

$$\langle\sigma((r)_n)\rangle = \sum_{m_1,\dots,m_{n-1}=0}^{\infty} \int_0^{2\pi}\dots\int d(\omega_1)_{m_1}\dots d(\omega_{n-1})_{m_{n-1}} \prod_1^{n-1} \frac{1}{(2\pi)^{m_j} m_j!}$$
$$\cdot F_x((e^{i\omega_1})_{m_1}) \prod_1^{n-2} F_x((e^{i\omega_j})_{m_j} | (e^{i\omega_{j+1}})_{m_{j+1}}) F_x((e^{i\omega_{n-1}})_{m_{n-1}})$$
$$\cdot \exp \sum_{l=1}^{n-1} \sum_{k=1}^{m_l} (-\gamma(\omega_{kl})x_l + iy_l\omega_{kl}\operatorname{sgn} l)\,, \tag{3.2}$$

where the notation

$$(\omega_j)_n = (\omega_{1j}, \dots, \omega_{nj}) \tag{3.3}$$

will be used.

Just as for the 2-point function, there is an illuminating graphical representation of these results. For an n-point function consider vertex sets $\mathscr{R}_j$, $j=1,\dots,n-1$. The k^{th}-vertex within $\mathscr{R}_j$ is labelled ω_{kj}, $k=1,\dots,m_j$ with $|\mathscr{R}_j| = m_j$, $\mathscr{R}_j$ is the set of wavenumbers describing $|\Phi_j\rangle$. For pictorial purposes it is convenient to arrange each $\mathscr{R}_j$ horizontally and then order the $\mathscr{R}_j$ vertically.

The union of the $\mathscr{R}_j$ will now be taken as the vertex set V for a graph $\mathscr{G} = \{V, E\}$; the contractions $f_\pm$ which occur in Theorems 1 and 2 of the previous section will be assigned as edge weights on $\mathscr{G}$. Evidently there will be f_- edges within rows $\mathscr{R}_j$ but f_+ edges between $\mathscr{R}_j$ and $\mathscr{R}_{j+1}$ for $j=1,\dots,n-1$. First we rationalise the contraction functions so that the edge weights become real. By analogy with [2] we introduce the functions

$$e_\pm^{>}(\omega_1, \omega_2) = (\sinh\gamma(\omega_1) \pm \sinh\gamma(\omega_2))/2\sin((\omega_1+\omega_2)/2)\,, \tag{3.4}$$

$$e_\pm^{<}(\omega_1, \omega_2) = (p(\omega_1)q(\omega_2) \pm p(\omega_2)q(\omega_1))/2\sin((\omega_1+\omega_2)/2)\,. \tag{3.5a}$$

with

$$\begin{aligned} p(\omega) &= (-2\cos\omega + A + 1/A)^{1/2} \\ q(\omega) &= (-2\cos\omega + B + 1/B)^{1/2} \end{aligned} \tag{3.5b}$$

for the rectangular Ising model. The integration weight for each vertex is now

$$d\mu(\omega) = d\omega/2\pi \sinh\gamma(\omega) \tag{3.6}$$

and the factors of i arising from the replacement of f by e can readily be shown to cancel.

Reference to Theorems 1 and 2 shows that the graphs in the two cases will be different. The case $T < T_C$ is the simpler: our considerations here will apply only to periodic boundary conditions, for which $\langle\sigma(r)_n\rangle = 0$ whenever n is odd (This is obviously not so with $+$ boundary conditions: take $n=1$). Allowed graphs $\mathscr{G}$ are unions of disjoint closed cycles $\mathscr{C}_l$. Each $\mathscr{C}_l$ has an even number of edges, weighted by $e_-^{<}(\cdot,\cdot)$ if both vertex labels come from the same $\mathscr{R}_j$. Within the vertical ordering $e_+^{<}(\cdot,\cdot)$ can only connect elements of $\mathscr{R}_j$ and $\mathscr{R}_k$ if $j = k \pm 1$. Closure of any $\mathscr{C}_l$ requires that the number of e_+ weighted edges be even. The final problem here concerns the sign factors in the expansions over permutations. This is given by

Lemma 1. *Any closed cycle has a permutation factor of* (-1).

Proof. This is analogous to that in [2]. The only difference is that a product of $2n-1$ permutations has to be handled because a cycle is permitted to intersect all $\mathcal{R}_j$, $j=1,\ldots,2n-1$.

When $T>T_C$, open chains occur. $\mathcal{R}_1$ and $\mathcal{R}_{2n-1}$ have each one chain end arising from the first and last matrix elements respectively in (3.2). There is one in each $\mathcal{R}_j \cup \mathcal{R}_{j+1}$ for $j=1,\ldots,2n-2$ arising from the intermediate matrix elements. Each chain end has an edge emanating from it; the degree of all remaining vertices is two. Thus any allowed graph is a disjoint union of n chains and any number of closed cycles. These are weighted in accordance with the rules for $T>T_C$, mutatis mutandis.

The permutation sign of a given chain is given by the following lemma:

Lemma 2. *An open chain which has ends in* $\mathcal{R}_j$ *and* $\mathcal{R}_k$ *has an even (resp. odd) number of edges if* $(j-k)$ *is even (resp. odd). The permutation sign is* $(-1)^{(j-k)}$.

Proof. This is an elementary extension of that in [2].

The final information required to specify the graphical representation is the vertex weight function for a vertex label ω_{jk} in row $\mathcal{R}_k$. This weight, denoted $v_k(\omega_{jk})$ is given by

$$v_k(\omega_{jk}) = \exp(-|x_k|\gamma(\omega_{jk}) + iy_k\omega_{jk}\,\mathrm{Sgn}\,k)\,. \tag{3.7}$$

The appearance of $\mathrm{Sgn}\,k$ in (3.7) is a consequence of the choice of wavenumbers in (2.1) and (2.22). All spins are translated in the direction perpendicular to transfer to the standard position 1 in accordance with the procedures of [1].

The sums over appropriate weighted graphs for $T>T_C$ and $T<T_C$ are denoted $\varrho^{\pm}((x_{n-1},(y)_{n-1})$. The vacuum scalar products required as boundary conditions for the Pfaffian expansion of Theorems 1 and 2 are given in [2]. One obtains:

$T>T_C$:

$$\langle\sigma(r)_{2n}\rangle = (\hat{m}(K_1,K_2)/\cosh K_1^*)^{2n}\varrho^+((x)_{2n-1},(y)_{2n-1})\,,$$

where

$$\hat{m}(K_1,K_2) = (1-(\sinh 2K_1 \sinh 2K_2)^2)^{1/8}$$

and

$$e^{-2K_1^*} = \tanh K_1\,.$$

$T<T_C$:

$$\langle\sigma(r)_{2n}\rangle = (m^*(K_1,K_2))^{2n}\varrho^-((x)_{2n-1},(y)_{2n-1})\,,$$

where $m^*(K_1,K_2)$ is the spontaneous magnetisation, given first by Onsager [7]:

$$m^*(K_1,K_2) = (1-(\sinh 2K_1, \sinh K_2)^{-2})^{1/8}\,.$$

Remarks. 1. A conjecture has been given on the scaling limit of the truncated n-point functions [5] which suggests that the equation of state of the ising ferromagnet has an asymptotic form

$$m(h,t) \sim t^{1/8} f(ht^{-15/8}) \tag{2.42}$$

where $t = (T - T_C)/T_C$ and h is the applied field. But the precise meaning of the symbol $\sim$ is yet to be given, as well as the properties and form of f.

2. Duneau et al. [3] have stressed the relationship between spanning tree decay properties of truncated n-point functions and analyticity. It appears difficult to establish such results rigorously. Study 4-point functions indicates that the occurrence of the contraction $f_+(\cdot, \cdot)$ is involved in an *essential way* in the truncation.

3. The results for $T < T_C$ have been obtained with toroidal boundary conditions. Messager and Miracle-Sole [8] have shown that below the critical temperature there are just two extremal invariant states $\omega_\pm$; the state considered here is just $(\omega_+ + \omega_-)/2$. The results for ω_+, and hence for *any* invariant equilibrium state will be given in another paper, using some of the methods of the last-named article in [7].

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Note Added in Proof

R. Z. Bariev [Phys. Lett. **64**A, 169 (1977)] has derived n-point functions independently.