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Nonlinear Realization of Chiral Symmetries and Localizability*

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Abstract. We prove that the nonlinear realization of $SU(n)\times SU(n)$ ($n\geqq 3$) is uniquely determined by the requirement that the Lagrangian, with a minimal number of derivatives of those fields parametrizing the adjoint representation of the diagonal $SU(n)$ subgroup, is localizable in the sense of Jaffe.

I. Introduction

In this paper we generalize a theorem due to Lehmann and Trute [1] which states that the nonlinear realization of the chiral $SU(2)\times SU(2)$ symmetry is uniquely determined by the requirement of localizability. We extend this theorem to $SU(n)\times SU(n)$ ($n\geqq 2$).

Each realization on a manifold of fields of the chiral group $SU(n)\times SU(n)$, which becomes linear if restricted to the diagonal $SU(n)$ subgroup, can be brought by a suitable change of coordinates on the manifold into a "standard form" (Coleman, Wess, and Zumino [2]):

$$g:\xi\to\xi',\ \ \boldsymbol{\Psi}\to\boldsymbol{\Psi}'=\mathscr{D}(e^{\boldsymbol{U}'\boldsymbol{V}})\,\boldsymbol{\Psi} \qquad (1)$$
$$g\in SU(n)\times SU(n)$$

$\mathscr{D}$ is a linear representation of $SU(n)$.

$$g e^{\xi\cdot\boldsymbol{A}}=e^{\xi'\cdot\boldsymbol{A}}e^{\boldsymbol{U}'\cdot\boldsymbol{V}},$$

$$\boldsymbol{A}=(A_1\ \dots\ A_{n^2-1}),\qquad \boldsymbol{V}=(V_1\ \dots\ V_{n^2-1}).$$

With V_i we denote the vectorial and with A_i the axial generators of $SU(n)\times SU(n)$. We choose the V_i and A_i so that they are orthonormal with respect to the Killing form. Each element g_0 in a neighbourhood of the identity of $SU(n)\times SU(n)$ can be uniquely decomposed into a product of the form:

$$g_0=e^{\xi_0\cdot\boldsymbol{A}}e^{\boldsymbol{U}_0\cdot\boldsymbol{V}}.$$

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We shall only consider chiral invariant Lagrangians which are exclusively functions of the (massless, spin zero) fields ξ^i [1]. Let f_{ijk} be the totally antisymmetric $SU(n)$ structure constants. With the notations

$$(t_i)_{jk} = -f_{ijk}, \qquad \boldsymbol{t} = (t_1, \dots t_{n^2-1}),$$
$$D^\mu \boldsymbol{\xi} = (\sinh \boldsymbol{\xi} \cdot \boldsymbol{t} / \boldsymbol{\xi} \cdot \boldsymbol{t})\, \partial_\mu \boldsymbol{\xi}$$

the simplest chiral invariant Lagrangian which emerges from the "standard realization" (1) can be written in the form

$$L = D^\mu \xi\, D_\mu \xi\,. \tag{2}$$

Other chiral invariant expressions of ξ contain more derivatives. We will disregard them in the following.

Any arbitrary nonlinear chiral realization is obtained from the "standard form" (1) by a redefinition of the fields

$$\boldsymbol{\xi} \to \boldsymbol{\xi}' = \boldsymbol{\xi} f_0(\xi) + \sum_{\alpha=1}^{n=2} \boldsymbol{\Psi}_\alpha(\xi) f_\alpha(\xi), \qquad f_0(0) = 1 \tag{3}$$

where the Ψ_α are the different $SU(n)$ vectors which can be formed from the fields ξ^i and the f_α are $SU(n)$ scalar functions of ξ having a power-series expansion around zero in the $(n-1)$ functionally independent $SU(n)$ invariants $y_i (i = 1 \dots, n-1)$

$$f_\alpha = f_\alpha(y_1, \dots y_{n-1})\,.$$

The special form of the substitution is enforced by two conditions; first, that the free Lagrangian has to be reproduced and second, that under $SU(n)$ transformations $\boldsymbol{\xi}'$ should have the same behaviour as $\boldsymbol{\xi}$.

By insertion of (3) into (2) we obtain the most general chiral invariant Lagrangian with a minimal number of derivatives (i.e., which is quadratic in $\partial_\mu \xi$).

$$L_f = : D_f^\mu(\Psi)\, D_{f,\mu}(\Psi) + 2(\partial_\mu \boldsymbol{\Psi} f)\left(\Psi \frac{\partial f}{\partial \xi^i} \partial^\mu \xi^i\right) + \left(\Psi \frac{\partial f}{\partial \xi^i} \partial^\mu \xi^i\right)^2 :$$

$$\Psi = (\xi, \Psi_1 \dots \Psi_{n-2}), \qquad f = (f_0, \dots f_{n-2})$$

$$D_f^\mu(\xi) = \frac{\sinh(\Psi f) \cdot t}{(\Psi f \cdot t)} (\partial^\mu(\Psi) f)\,.$$

After normal ordering indicated by : : L_f is defined as an operator in Fock space.

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[1] For the case of physical interest $[SU(2) \times SU(2),\ SU(3) \times SU(3)]$ this means a restriction to meson selfinteractions.

II. Uniqueness of the Localizable Chiral Invariant Lagrangian

It is our aim to prove that the Wightman two-point function $\langle 0|L_f L_f|0\rangle$ is localizable[2] in the sense of Jaffe [3] if and only if $f\equiv(1,0,\ldots,0)$. The localizability of L_f with $f\equiv(1,0,\ldots,0)$ can be checked by a trivial majorization argument.

The arguments for the necessity of this condition are simplified by the following observation:

Let $\xi^{i'}$, $\{i'\}\subset\{1\ldots n^2-1\}$ be any subset of the (n^2-1) fields ξ^i.

$$L'=L_f/\xi^{i'}\equiv 0\,,$$
$$L_f=L'+\Delta L\,.$$

Because of the identity $\langle 0|L'\Delta L|0\rangle=0$ and the positivity of the spectral-functions of $\langle 0|L'L'|0\rangle$ and $\langle 0|\Delta L\Delta L|0\rangle$ it is obvious that the localizability of L_f implies that of L'.

We choose a special basis of generators A_i (which are orthonormal with respect to the Killing form) in such a way that A_1, A_2 and A_3 form a $SU(2)$ algebra and A_3 together with $A_4\ldots A_{n+1}$ define a basis of a Cartan subalgebra of $SU(n)$[3]. We shall draw all our conclusions from the assumed localizability of the reduced Lagrangian

$$\tilde{L}=L_f/\xi^{n+2}=\cdots\xi^{n^2-1}\equiv 0\,.$$

The first three $SU(n)$ components of Ψ become proportional to $(\xi^1,\xi^2,\xi^3)=:\boldsymbol{\eta}$ as this is the only possible remaining $SU(2)$ vector

$$(\Psi^1,\Psi^2,\Psi^3)=\boldsymbol{\eta}(f_0+\Sigma f_\alpha P_\alpha(\xi^4,\ldots\xi^{n+1}))=:\boldsymbol{\eta}g\,. \tag{4}$$

P_α denotes a polynomial in $\xi_4\ldots\xi_{n+1}$. The other components of Ψ lying in the direction of the Cartan subalgebra elements $A_4\ldots A_{n+1}$ the "covariant" derivative $D_f^\mu(\xi)$ takes a form characteristic for the case of $SU(2)\times SU(2)$

$$D_f^\mu(\xi)_{\xi^{n+2}=\cdots\xi^{n2-1}\equiv 0}=((\partial^\mu\Psi)f)_{\xi^{n+2}=\cdots\xi^{n2-1}\equiv 0}$$
$$+\tfrac{1}{3!}(\partial^\mu\eta\otimes\eta)\otimes\eta g^3+\cdots\,.$$

[2] Glaser and Epstein [4] have shown that the localizability of the twopoint function is sufficient to render Green's functions localizable in all orders of perturbation theory. We call L_f localizable if the two-point function is localizable.

[3] The fields ξ^i are transformed under the action of the diagonal $SU(n)$ by the adjoint representation. Thus, in this context we can identify the basis elements A_i of the representation space with the generators of the diagonal subgroup (up to an isomorphism).

With $\otimes$ we denote the normal 3-dimensional vector product. Evaluating the vector products and using Lehmann and Trute's technique [1]

$$\Box_x g(z)|_{x=y} = \frac{(\partial_\mu \eta \otimes \eta)^2}{z^3} g' + \frac{(\partial_\mu \eta \eta)^2}{z^2} g'' \,|\, g' = \frac{\partial g}{\partial z}, \; z^2 = \eta^2$$

$$\eta = \eta(x), \; \xi^4 = \xi^4(y) \ldots \xi^{n+1} = \xi^{n+1}(y)$$

$\tilde{L}$ can be represented in the form $\tilde{L} = \tilde{L}_1 + \tilde{L}_2$

$$\tilde{L}_1 = (\partial_\mu \eta \otimes \eta)^2 / z^2 \left(\frac{\sin^2(gz)}{z^2} - 1 - \frac{h'}{z} \right).$$

Because of the invariance under the $SU(2)$ subgroup ξ^1, ξ^2 and ξ^3 occur in g only in the form ${\xi^1}^2 + {\xi^2}^2 + {\xi^3}^2 = z^2$.

$$h'' = 2 g g' z + g'^2 z^2 + g^2 + \sum_{i=1}^{n^2-1} \left(\frac{\partial}{\partial z} \Psi^i f \right)^2,$$

$$\tilde{L}_2 = \partial^x_\mu \eta \, \partial^\mu_x \eta + \partial^x_\mu k^\mu + H + \Box_x h|_{x=y}.$$

The explicit form of the terms k^μ and H is of no interest for our purpose. We only note that they do not contain derivatives $\partial_\mu \eta$. The antisymmetric structure of the vector product $\partial_\mu \eta \otimes \eta$ implies (cf. [1]): $\langle \tilde{L}_1 \tilde{L}_2 \rangle = 0$. On account of the positivity of the spectral functions of $\langle \tilde{L}_1, \tilde{L}_1 \rangle$ and $\langle \tilde{L}_2, \tilde{L}_2 \rangle \tilde{L}_1$ and $\tilde{L}_2$ must be separately localizable.

Writing $\tilde{L}_2$ in the form

$$\tilde{L}_2 = \frac{(\partial_\mu \eta \otimes \eta)^2}{z^3} h' + \tilde{\tilde{L}}_2 \tag{5}$$

we deduce the following identities

$$\tilde{L}_{2|\xi^1 = \xi^2 \equiv 0} = \tilde{\tilde{L}}_2|_{\xi^1 = \xi^2 \equiv 0} =: L_a, \tag{6}$$

$$L_a + \partial_\mu \xi^3 \, \partial^\mu \xi^3 z^2 g^2 = L_f|_{\xi_1 = \xi_2 = \xi_{n+2} \cdots = \xi_{n2-1} \equiv 0} =: L_G. \tag{7}$$

L_a and L_G have been shown to be necessarily localizable. The same can now be proved for the term $\partial_\mu \xi^3 \, \partial^\mu \xi^3 g^2 z^2$: Suppose that $\partial_\mu \xi^3 \, \partial^\mu \xi^3 g^2$ is nonlocalizable. The term $\langle \partial_\mu \xi^3 \, \partial^\mu \xi^3 g^2 z^2, \; \partial_\mu \xi^3 \, \partial^\mu \xi^3 g^2 z^2 \rangle$ in $\langle L_a + \partial_\mu \xi^3 \, \partial^\mu \xi^3 g^2 z^2, \; L_a + \partial_\mu \xi^3 \, \partial^\mu \xi^3 g^2 z^2 \rangle$ gives the dominant contribution for high energies, since the remaining terms involve localizable operators. Consequently L_G would be nonlocalizable contrary to the assumptions.

Introducing in $\tilde{\tilde{L}}_2$ again the fields ξ^1 and ξ^2 we don't loose localizability, since the relevant terms are functions of z^2. With the same argument as above one shows that $\left(\frac{(\partial_\mu \xi \otimes \xi)^2}{z^3} h'\right)$ and therewith $\frac{(\partial_\mu \xi \otimes \xi)^2}{z^2}\left(\frac{\sin^2 zg}{z^2} - 1\right)$ have to be localizable.

In $L_I = \frac{(\partial_\mu \xi \otimes \xi)^2}{z^2}\left(\frac{\sin^2 zg}{z^2} - 1\right)$ we replace the fields $\xi_4 \dots \xi_{n+1}$ by arbitrary complex numbers $v_1 \dots v_{n-2}$. $L_I \to \tilde{L}_I(z; v_1 \dots v_{n-2})$. $\tilde{L}_I$ is localizable together with L_I. This can be concluded by a simple estimate of the two point function, which uses the fact that in passing from L_I to $\tilde{L}_I$ the combinatorial factors arising from contractions of the fields $\xi^4 \dots \xi^{n+1}$ drop out. Thus we arrive at a necessary condition for g: $\sin^2 zg(z, v_1 \dots v_{n-1})$ has to be an entire function in z of order <2 for arbitrary complex $v_1 \dots v_{n-1}$ [4].

The same analytic property for $g^2(z, v_1 \dots v_{n-2})$ is deduced from the localizability of $\partial_\mu \xi^3 \partial^\mu \xi^3 g^2 z^2$. Moreover the increase bound for $\sin^2(gz)$ requires g to be at most a square root of a linear function in z. But as g should have a power series expansion around zero in z^2 g can only be identical to one.

The polynomials P_i in (4) (now polynomials in $v_1 \dots v_{n-2}$) are functionally independent with respect to $SU(n)$ scalar coefficients. Therefore, to fullfill the foregoing condition, f has to be of the form $(1, 0, \dots, 0)$.

Up to this point we have shown that $f_\alpha(y_1, \dots, y_{n-1})$, $(y_i = y_i \cdot (z, v_1 \dots v_{n-2}))$ as functions of z have to be constants for arbitrary complex $v_1 \dots v_{n-2}$. Exploiting this arbitrariness and letting z vary we can cover a compact region of the space $(y_1 \dots y_{n-1})$. f_α must constant in this region and consequently every where. This is what we wanted to prove.

Another proof of the above stated theorem was suggested by Ashmore [5]. This author overlooks the fact, that the adjoint representation of $SU(n)$ has $(n-1)$ functionally independent invariants. This is the essential complication in passing from $SU(2) \times SU(2)$ to $SU(n) \times SU(n)$ $(n \geqq 3)$.

I would like to thank Professor H. Lehmann for suggesting these investigations and for his helpful discussions.

[4] The requirement both sufficient and (apart from some subtleties unessential for our purpose) necessary for a Lagrangian $L(z)$ to be localizable is, that L is an entire function of order <2 in z. This is also true if there is an additional factor $(\partial_\mu \eta \otimes \eta)^2$ in the Lagrangian.

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