Commun. math. Phys. 28, 109—122 (1972)

The Algebra Generated by Physical Filters*

Günter Dähn
Mathematisches Institut der Universität Tübingen

Received January 31, 1972

Abstract. This paper investigates mathematical properties of a finite-dimensional real algebra of linear operators which are generated by an orthomodular lattice of filters in the sense of Mielnik [4]. Properties of filter decomposability and a representation theorem for the vector space underlying the algebra mentioned are derived.

I. Introduction

The physical background and the motivation of the subsequent mathematical investigations are the papers by Ludwig [3] whose axiom system was, together with the most important mathematical consequences, restated in [2] in a way more adapt to our mathematical considerations. So, referring to [2] for detailed mathematical notes, we will here only sketch basic mathematical concepts in a contemporary language.

A comprehensive and careful analysis of all current attempts of an axiomatic foundation of physical theories has been given by Mielnik [4] who has subordinated the lattice-representing operators $\boldsymbol{T}_E$ of [2] to the physical concept of filters.

II. Preliminaries

We start from a dual pair (B, B') of two real topological vector spaces. As in [2] B (and hence B') are supposed to be finite-dimensional, say $\dim B = \dim B' = N$.

1. B has an order base K which is convex and closed. The elements of K are denoted by V, the elements of B in general by X.

2. In B there exists a proper positive generating cone B_+ generated by K, i.e.

$$B = B_+ - B_+ , \; B_+ = \bigcup_{\lambda \in \boldsymbol{R}_+} \lambda K .$$

3. B' is partially ordered by

$$Y_1 \leqq Y_2 :\Leftrightarrow \langle V, Y_1 \rangle \leqq \langle V, Y_2 \rangle \quad \text{for every} \quad V \in K .$$

---

* This paper leans on a report presented to and supported by the Deutsche Forschungsgemeinschaft.

4. B' has an order unit $\mathbf{1}$ with $L := \{Y \mid 0 \leqq Y \leqq \mathbf{1}\}$. L is convex and closed. Its elements are denoted by F. $\langle V, \mathbf{1} \rangle = 1$ for every $V \in K$. (In [2] L was denoted by $\hat{L}$.)

5. B' is generated by a proper positive cone B'_+ generated by L, i.e.

$$B' = B'_+ - B'_+, \; B'_+ = \bigcup_{\lambda \in \boldsymbol{R}_+} \lambda L .$$

6. The canonical bilinear functional $\langle \cdot, \cdot \rangle$ over $B \times B'$ is the extension of the (physical motivated) function μ over $K \times L$ restricted by

$$0 \leqq \mu(V, F) \leqq 1 \quad \text{for all} \quad (V, F) \in K \times L$$

separates points in K and L respectively.

7. B is a real Banach space by

$$\|X\| := \sup\{|\mu(X, F)| \mid F \in L\} \quad \text{for each} \quad X \in B .$$

B' is the dual Banach space by

$$\|Y\| := \sup\{|\mu(X, Y)| \mid X \in B \quad \text{and} \quad \|X\| = 1\} \quad \text{for each} \quad Y \in B' .$$

8. The extreme points E of L form an orthomodular lattice G with 0 and $\mathbf{1}$ as zero and unit element, respectively. For every $E \in G$ the orthocomplement $E^\perp$ of E is defined by $E^\perp = \mathbf{1} - E$.

$A(G)$ denotes the set of all atoms P of G, $A(W)$ the set of all atoms of W (see 10.).

9. With the notations

$$K_i(l) := \{V \mid \langle V, F \rangle = i \quad \text{for all} \quad F \in l\}, \; i = 0; 1 \quad \text{and any} \quad l \subseteqq L ,$$

$$L_i(k) := \{F \mid \langle V, F \rangle = i \quad \text{for all} \quad V \in k\}, \; i = 0; 1 \quad \text{and any} \quad k \subseteqq K$$

there exists a lattice isomorphism between G and $U := \{L_0(k) \mid k \subseteqq K\}$ and a dual lattice isomorphism between G and $W := \{K_0(l) \mid l \subseteqq L\}$.

10. W is the set of all extremal sets of K and equals the set of all facets of K ([2], Theorem 2 and corollaries).

11. Throughout this paper $\boldsymbol{N}_n$ denotes the interval $[1; n] \cap \boldsymbol{N}$ for any $n \in \boldsymbol{N}$.

III. Further Properties of the Operators T_E

As in [2] $\mathscr{B}(B')$ denotes the $\boldsymbol{R}$-algebra of all linear (bounded) operators over B'.

$$\mathscr{T} := \{\boldsymbol{T} \mid \boldsymbol{T} \in \mathscr{B}(B') \quad \text{and} \quad \boldsymbol{T}[B'_+] \subseteqq B'_+\}$$

is a proper positive cone in $\mathscr{B}(B')$ ([2]). There we defined

$$\mathscr{T}(G) := \{\boldsymbol{T}_E \mid \boldsymbol{T}_E \in \mathscr{T} \quad \text{and} \quad E \in G\}$$

with T_E uniquely determined by $\langle V, T_E F\rangle = \langle V, F\rangle$ for all $V \in K_1(E)$. T_E was proved to be idempotent and $\mathcal{T}(G)$ was shown to be an orthomodular lattice isomorphic to G.

This section is mainly devoted to the answer of two open questions at the end of [2] (questions 4 and 5). Results similar to those concerning perpendicular projectors on Hilbert space will be obtained.

Theorem 1. *For all* $T_{E_1}, T_{E_2} \in \mathcal{T}(G)$: $T_{E_1 \wedge E_2} = T_{E_1} T_{E_2}$ *iff* $T_{E_1} T_{E_2} = T_{E_2} T_{E_1}$.

Proof. (i) If $T_{E_1 \wedge E_2} = T_{E_1} T_{E_2}$, then $T_{E_1} T_{E_2} = T_{E_2} T_{E_1}$ because of $T_{E_1 \wedge E_2} = T_{E_2 \wedge E_1}$.

(ii) Suppose $T_{E_1} T_{E_2} = T_{E_2} T_{E_1}$. Since T_{E_i} $(i = 1; 2)$ are idempotent, so is $T_{E_1} T_{E_2}$. According to the isomorphism Theorem 16 from [2] there holds $T_{E_1 \wedge E_2} = T_{E_1} \wedge T_{E_2}$; thus, on the one hand, $\langle V, T_{E_1 \wedge E_2} F\rangle = \langle V, F\rangle$ for all $V \in K_1(E_1 \wedge E_2) = K_1(E_1) \cap K_1(E_2)$ and, on the other hand, $\langle V, T_{E_1} T_{E_2} F\rangle = \langle V, T_{E_2} T_{E_1} F\rangle$ for *all* $V \in K$. So there holds especially: $\langle V, T_{E_1} T_{E_2} F\rangle = \langle V, T_{E_2} F\rangle$ for all $V \in K_1(E_1)$ and $\langle V, T_{E_2} T_{E_1} F\rangle = \langle V, T_{E_1} F\rangle$ for all $V \in K_1(E_2)$.

Therefore $\langle V, T_{E_1} T_{E_2} F\rangle = \langle V, F\rangle$ holds for all $V \in K_1(E_1 \wedge E_2)$, i.e. $\langle V, T_{E_1} T_{E_2} F\rangle = \langle V, T_{E_1 \wedge E_2} F\rangle$ for all $V \in K_1(E_1 \wedge E_2)$. This is, according to Theorem 9 in [2], sufficient for $T_{E_1} T_{E_2} F = T_{E_1 \wedge E_2} F$ for all $F \in L$ because of $T_{E_1} T_{E_2} F \leqq E_1 \wedge E_2$ by hypothesis. Hence we obtain $T_{E_1} T_{E_2} = T_{E_1 \wedge E_2}$. ∎

Corollary. *Let* τ *be any finite subset of* $\mathcal{T}(G)$: $\bigwedge_\tau T_E = \prod_\tau T_E$ *iff* τ *consists of pairwise commuting elements.*

Proof. By induction. ∎

In [2] we considered the Sasaki-projection Φ_e defined by $\Phi_e(g) = e \wedge (g \vee e^\perp)$ for all g of an orthomodular lattice and any e therein. This projection Φ_e was compared with the projector T_E. With the compatibility relation "$e_1 \mathscr{C} e_2$ iff $e_1 = (e_1 \wedge e_2) \overset{\perp}{\vee} (e_1 \wedge e_2^\perp)$" there holds due to Nakamura [5] in any orthomodular lattice "$e_1 \mathscr{C} e_2$ iff $\Phi_{e_1} \Phi_{e_2} = \Phi_{e_2} \Phi_{e_1}$".

Concerning T_E the validity of this equivalence was the open question 4 in [2]. The next theorem answers this question in the affirmative.

Theorem 2. *For all* $E_1, E_2 \in G: E_1 \mathscr{C} E_2$ *iff* $T_{E_1} T_{E_2} = T_{E_2} T_{E_1}$.

Proof. (i) Given $E_1, E_2 \in G$ such that $E_1 \mathscr{C} E_2$ is valid, i.e. $E_1 = (E_1 \wedge E_2) \overset{\perp}{\vee} (E_1 \wedge E_2^\perp)$. Then $T_{E_1} F = T_{(E_1 \wedge E_2) + (E_1 \wedge E_2^\perp)} F \leqq E_1 \wedge E_2 + E_1 \wedge E_2^\perp$, hence $T_{E_2} T_{E_1} F \leqq E_1 \wedge E_2$ for all $F \in L$. Since $T_{E_1 \wedge E_2} T_{E_2} T_{E_1} = T_{E_1 \wedge E_2}$, we conclude from the definition of the T_E-operator $\langle V, T_{E_1 \wedge E_2} T_{E_2} T_{E_1} F\rangle = \langle V, T_{E_2} T_{E_1} F\rangle = \langle V, T_{E_1 \wedge E_2} F\rangle = \langle V, F\rangle$ for all $V \in K_1(E_1 \wedge E_2)$ and any $F \in L$. Then Theorem 9 of [2] gives, because of $T_{E_2} T_{E_1} F \leqq E_1 \wedge E_2$, $T_{E_1 \wedge E_2} F = T_{E_2} T_{E_1} F$ for all $F \in L$, thus $T_{E_1} T_{E_2} = T_{E_2} T_{E_1}$ by Theorem 1.

(ii) Supposing $T_{E_1} T_{E_2} = T_{E_2} T_{E_1}$, we obtain $T_{E_1} T_{E_2} = T_{E_1 \wedge E_2}$ by Theorem 1. Since $E_1 \wedge E_2 \leqq E_1$ orthomodularity of G implies $E_1 = (E_1 \wedge E_2) \overset{\perp}{\vee} (E_1 \wedge (E_1 \wedge E_2)^\perp)$. To prove the assertion means to prove $E_1 \wedge (E_1 \wedge E_2)^\perp = E_1 \wedge E_2^\perp$. This will have been verified when $E_1 \wedge (E_1 \wedge E_2)^\perp \leqq E_1 \wedge E_2^\perp$ has been verified because $E_1 \wedge (E_1 \wedge E_2)^\perp \geqq E_1 \wedge E_2^\perp$ holds always. To this end we observe that $T_{E_1} T_{E_2} = T_{E_1 \wedge E_2}$ implies $T_{E_2} T_{E_1} T_{(E_1 \wedge E_2)^\perp} = \mathbf{0}$. This means $\langle V, T_{E_2} T_{E_1} T_{(E_1 \wedge E_2)^\perp} F \rangle = 0$ for all $V \in K$ and any $F \in L$; thus, in particular $\langle V, T_{E_2} T_{E_1} T_{(E_1 \wedge E_2)^\perp} F \rangle = \langle V, T_{E_1} T_{(E_1 \wedge E_2)^\perp} F \rangle = 0$ for all $V \in K_1(E_2)$ and any $F \in L$. This implies $T_{E_1} T_{(E_1 \wedge E_2)^\perp} F \in L_0 K_1(E_2) = L_0 K_0(E_2^\perp)$ for all $F \in L$, hence $T_{E_1} T_{(E_1 \wedge E_2)^\perp} F \leqq E_2^\perp$ for all $F \in L$.

Consequently there holds especially $T_{E_1} T_{(E_1 \wedge E_2)^\perp} E_1 \wedge (E_1 \wedge E_2)^\perp = E_1 \wedge (E_1 \wedge E_2)^\perp \leqq E_2^\perp$. This yields $E_1 \wedge (E_1 \wedge E_2)^\perp \leqq E_1 \wedge E_2^\perp$ and so $E_1 \mathscr{C} E_2$. ▮

Definition 1 ([6]). (i) Two idempotents $I_1, I_2 \in \mathscr{B}(B')$ are said to be orthogonal iff $I_1 I_2 = I_2 I_1 = \mathbf{0}$.

(ii) An idempotent $I \in \mathscr{B}(B')$ is called *primitive*, *irreducible* or *minimal* iff it cannot be decomposed into a sum of two orthogonal idempotents of $\mathscr{B}(B')$.

Next it will be shown that the orthogonality relation from the preceding definition if restricted to $\mathscr{T}(G)$ is equivalent with the lattice-theoretical one defined in Theorem 16 of [2].

Theorem 3. *For all* $T_{E_1}, T_{E_2} \in \mathscr{T}(G)$: $T_{E_1} \leqq (T_{E_2})^\perp$ *iff* $T_{E_1} T_{E_2} = T_{E_2} T_{E_1} = \mathbf{0}$.

Proof. (i) Suppose $T_{E_1} \leqq (T_{E_2})^\perp = T_{E_2^\perp}$, i.e. $T_{E_1} = T_{E_2^\perp} T_{E_1} = T_{E_1} T_{E_2^\perp}$. By Corollary 1 to Theorem 13 in [2] $T_E T_{E^\perp} = \mathbf{0}$ for all $E \in G$. Therefore $T_{E_2} T_{E_1} = T_{E_2} T_{E_2^\perp} T_{E_1} = \mathbf{0}$ and $T_{E_1} T_{E_2} = T_{E_1} T_{E_2^\perp} T_{E_2} = \mathbf{0}$.

(ii) Supposing $T_{E_1} T_{E_2} = T_{E_2} T_{E_1} = \mathbf{0}$, we have $T_{E_1} T_{E_2} E_2 = T_{E_1} E_2 = 0$, i.e. $\langle V, T_{E_1} E_2 \rangle = \langle V, E_2 \rangle = 0$ for all $V \in K_1(E_1)$, thus $K_1(E_1) \subseteqq K_0(E_2) = K_1(E_2^\perp)$, hence $E_1 \leqq E_2^\perp$. Then the isomorphism Theorem 16 of [2] yields $T_{E_1} \leqq (T_{E_2})^\perp$. ▮

Theorem 17 in [2] expresses that $T_{E_1} \underset{\mathscr{T}}{\leqq} T_{E_2}$ implies $T_{E_1} \underset{\mathscr{T}(G)}{\leqq} T_{E_2}$. The converse of this theorem was formulated as an open question in [2] (question 5) the answer of which shall now be given.

Lemma 1. *For all* $T_E \in \mathscr{T}(G)$: *if* T_E *is orthoadditively decomposable into* $T_E = T_{E_1} + T_{E_2}$, *then* $T_E = T_{E_1 \overset{\perp}{\vee} E_2}$.

Proof. As a consequence of $E_1 \perp E_2$: $T_E T_{E_1 + E_2} = (T_{E_1} + T_{E_2}) T_{E_1 + E_2} = T_{E_1} T_{E_1 + E_2} + T_{E_2} T_{E_1 + E_2} = T_{E_1} + T_{E_2} = T_E$, thus $T_E \leqq T_{E_1 + E_2}$. Besides, $T_E(E_1 + E_2) = T_{E_1}(E_1 + E_2) + T_{E_2}(E_1 + E_2) = E_1 + E_2 \leqq E$, thus $T_{E_1 + E_2} \leqq T_E$. ▮

Theorem 4. *For all* $E_1, E_2 \in G$: "$T_{E_1} \underset{\mathscr{T}}{\leqq} T_{E_2} \Leftrightarrow T_{E_1} \underset{\mathscr{T}(G)}{\leqq} T_{E_2}$", *iff* G *is Boolean.*

Proof. The implication "$T_{E_1} \underset{\mathscr{T}}{\leqq} T_{E_2} \Rightarrow T_{E_1} \underset{\mathscr{T}(G)}{\leqq} T_{E_2}$" holds by Theorem 17 of [2] in any orthomodular lattice G. Therefore the converse must be shown.

(i) Let G be Boolean. $T_{E_1} \underset{\mathscr{T}(G)}{\leqq} T_{E_2}$ implies $E_1 \leqq E_2$ and so $E_2 = E_1 \curlyvee (E_2 \wedge E_1^{\perp})$. G being Boolean, Theorem 18 of [2] gives $T_{E_1 + (E_2 \wedge E_1^{\perp})} = T_{E_1} + T_{E_2 \wedge E_1^{\perp}} = T_{E_2}$, thus $T_{E_2} - T_{E_1} = T_{E_2 \wedge E_1^{\perp}} \underset{\mathscr{T}}{\geqq} 0$, i.e. $T_{E_2} \underset{\mathscr{T}}{\geqq} T_{E_1}$.

(ii) Let $T_{E_1} \underset{\mathscr{T}(G)}{\leqq} T_{E_2} \Rightarrow T_{E_1} \underset{\mathscr{T}}{\leqq} T_{E_2}$ be valid (for all $E_1 \leqq E_2$). Since $E \leqq \mathbf{1}$ for all $E \in G$, so $T_{\mathbf{1}} - T_E \underset{\mathscr{T}}{\geqq} \mathbf{0}$ and then $\mathbf{0} \underset{\mathscr{T}}{\leqq} T_{\mathbf{1}} - T_E \underset{\mathscr{T}}{\leqq} T_{\mathbf{1}}$. This implies $(T_{\mathbf{1}} - T_E)F \in L$ for all $F \in L$; because of $\langle V, (T_{\mathbf{1}} - T_E)F \rangle = 0$ for all $V \in K_1(E)$, $(T_{\mathbf{1}} - T_E)F \in L_0 K_1(E) = L_0 K_0(E^{\perp})$. Hence $(T_{\mathbf{1}} - T_E)F \leqq E^{\perp}$ for all $F \in L$. Consequently, each $F \in L$ is reduced by any $E \in G$ and from the proof of Theorem 18 in [2] there follows that G is Boolean. ∎

Theorem 5. *If* $T_{E_0} \in \mathscr{T}(G)$ *is orthodecomposable, then the segment* $G(0, E_0)$ *is a reducible lattice.*

Proof. $G(0, E_0)$ is orthomodular with $E_0 \wedge E^{\perp}$ as the orthocomplement for any $E \leqq E_0$. By hypothesis, $T_{E_0} = T_{E_1} + T_{E_2}$. Lemma 1 implies $E_0 = E_1 + E_2 = E_1 + E_0 \wedge E_1^{\perp}$. Then, according to Theorem 4, each $F \in L_{E_0}$ is reduced by E_1 and $E_0 \wedge E_1^{\perp}$, which, therefore, belong to the center of $G(0, E_0)$ thus being reducible. ∎

Corollary. *If* G *is irreducible, then there exists no proper orthodecomposition of* $T_{\mathbf{1}}$.

Remark 1. Given the hypothesis of Theorem 5, there holds for every $E \in G(0, E_0)$:

$$E = (E \wedge E_1) \curlyvee (E \wedge E_0 \wedge E_1^{\perp}) = (E \wedge E_1) \curlyvee (E \wedge E_1^{\perp}),$$

which means that also in G $E \mathscr{C} E_1$ holds for all $E \in G(0, E_0)$. Now we wish to investigate when all segments $G(0, E)$ of the orthomodular lattice G are irreducible, i.e. when no $T_E \in \mathscr{T}(G)$ is orthodecomposable. We first illustrate this situation by the example of Hilbert space from Remark 4 in [2] thereby correcting it:

Let $\mathscr{H}$ be a finite-dimensional real Hilbert space and G the lattice of all perpendicular projectors on $\mathscr{H}$. L is then the set of all self-adjoint operators F with $0 \leqq F \leqq \mathbf{1}$. T_E is given by $T_E F = EFE$ for any $E \in G$ and all $F \in L$. G is modular and we suppose it to be irreducible. Assume the existence of $E \in G$ such that $EFE = (E_1 + E_2)F(E_1 + E_2) = E_1 F E_1$

$+E_2FE_2$ with $E_1 \perp E_2$ E_1FE_2 being positive, we obtain $E_1FE_2 = E_2FE_1 = 0$. Every $F \in L$ has a decomposition $F = \sum_{i \in \boldsymbol{N}_n} \lambda_i^F P_i^F$ with pairwise orthogonal atoms P_i^F and $\lambda_i^F \in \boldsymbol{R}_+^*$. Let $E_1 = \sum_{j \in \boldsymbol{N}_m} P_j^1$, $E_2 = \sum_{k \in \boldsymbol{N}_l} P_k^2$ be atomic orthodecompositions. Then $\sum_{j \in \boldsymbol{N}_m} P_j^1 \sum_{i \in \boldsymbol{N}_n} \lambda_i^F P_i^F \sum_{k \in \boldsymbol{N}_l} P_k^2 = 0$ implies, again by positivity, $P_j^1 P_i P_k^2 = 0$ for all $i \in \boldsymbol{N}_n, j \in \boldsymbol{N}_m$, $k \in \boldsymbol{N}_l$. G being irreducible, the last equation cannot be valid for all $F \in L$: let us consider $x_j \in \operatorname{Im} P_j^1$, $y_k \in \operatorname{Im} P_k^2$, $z_i \in \operatorname{Im} P_i^F$. Since $P_j P_k = 0$, so $x_j \perp y_k$. $P_{x_j} P_{z_i} P_{y_k} h = 0$ for all $h \in \mathscr{H}$ implies $\langle x_j | z_i \rangle \langle z_i | y_k \rangle \langle y_k | h \rangle = 0$ for all $h \in \mathscr{H}$, hence $\langle x_j | z_i \rangle = 0$ or $\langle z_i | y_k \rangle = 0$. This contradicts the fact that in the 2-dimensional subspace generated by x_j, y_k not all vectors z are orthogonal to x_j and y_k, respectively. This statement should be inferred from our general frame.

Theorem 6. *If G is modular and irreducible, no $\boldsymbol{T}_E \in \mathscr{T}(G)$ is non-trivially orthodecomposable in $\mathscr{T}(G)$.*

Proof. Assume the existence of $\boldsymbol{T}_E \neq \boldsymbol{T}_1$ with $\boldsymbol{T}_E = \boldsymbol{T}_{E_1} + \boldsymbol{T}_{E_2}$ and $\boldsymbol{T}_{E_1} \perp \boldsymbol{T}_{E_2}$. By Lemma 1, $\boldsymbol{T}_E = \boldsymbol{T}_{E_1+E_2}$ holds First, we assert the existence of $P \in A(G)$ such that $P < E_1 + E_2$ and $P \nleqq E_i$ for each $i \in \boldsymbol{N}_2$; i.e. $P \wedge E_i = 0$. Let P_1 and P_2 be atoms of E_1 and E_2, respectively. $E_1 \perp E_2$ implies $P_1 \vee P_2 = P_1 + P_2$. G being irreducible and modular, there exists $P \in A(G)$ with $P < P_1 + P_2$ and $P \neq P_i$ for each $i \in \boldsymbol{N}_2$. Orthomodularity of G insures the existence of $Q \in A(G)$ such that $Q \perp P$ and $P + Q = P_1 + P_2$.

$P \nleqq E_i$ or $Q \nleqq E_i$ for each $i \in \boldsymbol{N}_2$ shall now be shown:

(i) $P \leqq E_1$ leads to the dichotomy

1. $Q \leqq E_1$, which implies $P_2 = P + Q - P_1 \in B'(E_1) \cap B'(E_2)$, a contradiction to $B'(E) = B'(E_1) \oplus B'(E_2)$.

2. $Q \leqq E_2$, which implies, because of the uniqueness of the representation of $P + Q$ by components of $B'(E_i)$, the contradiction $P = P_1$, $Q = P_2$.

So 1. and 2. have the consequence: $P \leqq E_1 \Rightarrow Q \leqq E_i$.

(ii) $P \leqq E_2$ admits only $Q \leqq E_i$ by similar arguments as in (i). The discussion for $Q \leqq E_i$ for each $i \in \boldsymbol{N}_2$ is in a completely analogous way so that finally

$$P \leqq E_i \quad \text{or} \quad Q \leqq E_i \quad \text{for each} \quad i \in \boldsymbol{N}_2 .$$

Without loss of generality let us suppose $P \nleqq E_i$ for each $i \in \boldsymbol{N}_2$; i.e. $P \wedge E_i = 0$. Since $P < E_1 + E_2$, so $\boldsymbol{T}_{E_1} P + \boldsymbol{T}_{E_2} P = \boldsymbol{T}_{E_1+E_2} P = P$ and thus $\boldsymbol{T}_{E_i} P \leqq P$ for each $i \in \boldsymbol{N}_2$. From this there follows $\boldsymbol{T}_{E_i} P \leqq P \wedge E_i = 0$, hence $\boldsymbol{T}_{E_i} P = 0$ with the contradiction $P = 0$. ∎

We can sharpen Theorem 6 by

Theorem 7. *If G is irreducible and modular, then no $\boldsymbol{T}_E \in \mathscr{T}(G)$ is non-trivially orthodecomposable by idempotents of $\mathscr{T} \subset \mathscr{B}(B')$.*

Proof. Assume the existence of $T_E \neq T_1$ with $T_E = T_1 + T_2$, $T_1 T_2 = T_2 T_1 = 0$, $T_i^2 = T_i \in \mathscr{T}$ and $T_i \neq 0$ for each $i \in N_2$.

These assumptions have the immediate consequence $T_i F \leqq E$ for all $F \in L$ and each $i \in N_2$. Moreover, $T_i E < E$ since, otherwise, $T_1 E = E$ for instance would imply $T_2 E = 0$.

Then, because of $0 \leqq T_2 F \leqq E$ for all $F \in L$, $T_2^2 = T_2$ implies $T_2 F = 0$ for all $F \in L$, hence $T_2 = 0$ contrary to $T_i \neq 0$ for each $i \in N_2$. An analogous argument excludes $T_2 E = E$. From $T_i E < E$ for each $i \in N_2$, there follows $K_0(E) \subset K_0(T_i E)$. Determining $E_i \in G$ for each $i \in N_2$ by $K_0(T_i E) = K_0(E_i)$ (see [3]), we get $E_i < E$. $T_i E \leqq E_i < E$ has the consequence $T_i E \leqq T_i E_i \leqq T_i E$, hence $T_i E_i = T_i E$. To show $T_i E_i = E_i$, $T_i E_i < E_i$ must be excluded, for $T_{E_i} E_i \leqq E_i$ holds by construction of E_i. Assume $T_i E_i < E_i$: then, by Lemma 7 in [2], there exists $V_i \in K_1(E_i)$ such that $\langle V_i, T_i E_i \rangle < \langle V_i, E_i \rangle = 1$. Therefore, $K_1(T_i E_i) \subset K_1(E_i)$, and orthomodularity of W gives $K_1(E_i) = K_1(T_i E_i) \curlyvee (K_1(E_i) \wedge K_0(T_i E_i)) = K_1(T_i E_i) \curlyvee (K_1(E_i) \wedge K_0(E_i))$, whence the contradiction $K_1(E_i) \wedge K_0(E_i) \neq \emptyset$.

So $T_i E_i = E_i$ for each $i \in N_2$ holds and thus $E = E_1 + E_2$ and $T_{E_1} \perp T_{E_2}$. Any $F_i \in \operatorname{Im} T_i$ satisfies $F_i < E$, i.e. $F_i = T_i F_i \leqq T_i E = T_i E_i = E_i < E$. Consequently, $F_i \in L_0 K_0(E_i) \subset \operatorname{Im} T_{E_i}$, which implies $T_{E_i} T_i = T_i$. Since $T_E = T_1 + T_2$, so $T_{E_i} = T_{E_i} T_E = T_{E_i} T_1 + T_{E_i} T_2$. In particular, $T_{E_1} = T_1 + T_{E_1} T_2$ and $T_{E_2} = T_{E_2} T_1 + T_2$. Multiplying these equations by T_{E_2} and T_{E_1}, respectively, and using $T_{E_1} \perp T_{E_2}$ give $T_{E_2} T_1 = T_{E_1} T_2 = 0$. So finally, $T_{E_i} = T_i$ for each $i \in N_2$ and thus $T_E = T_{E_1} + T_{E_2}$ contrary to Theorem 6. ∎

We conclude this section by a statement on chains in $\mathscr{T}(G)$.

Theorem 8. *Any chain in $\mathscr{T}(G)$ is linearly independent.*

Proof. It suffices to consider only proper chains in $\mathscr{T}(G)$. They are finite, since B' is finite-dimensional. Let $(T_{E_i})_{i \in N_n}$ be an ascending finite chain with $n \in N$ and suppose $\sum_{i \in N_n} \beta_i T_{E_i} = 0$, $\beta_i \in R$. Orthomodularity of $\mathscr{T}(G)$ implies $T_{E_n} = T_{E_{n-1}} \curlyvee T_{E_n \wedge E_{n-1}^\perp}$ whence, by the chain property, $T_{E_i} \perp T_{E_n \wedge E_{n-1}^\perp}$ for all $i \in N_{n-1}$. Applying Theorem 13 and its Corollary 1 of [2], we obtain

$$T_{E_n \wedge E_{n-1}^\perp} \sum_{i \in N_n} \beta_i T_{E_i} = \beta_n T_{E_n \wedge E_{n-1}^\perp} T_{E_n} = \beta_n T_{E_n \wedge E_{n-1}^\perp} = 0 .$$

From $(T_{E_i})_{i \in N_n}$ being a proper chain there follows $\beta_n = 0$. The same procedure applied to $T_{E_{n-1}}$ verifies the assertion by recursion. ∎

IV. The Algebra $\mathscr{B}(B')$

Remark 2. Theorem 21 in [2] is incorrectly formulated. Its correct version is: "If G is irreducible, then $\mathscr{A}(G) = \mathscr{B}(B')$". This means that, if G is irreducible, there exist no invariant subspaces of B' except (0) and B' for the R-algebra $\mathscr{A}(G)$ generated by $\mathscr{T}(G)$ (see [2]). In this case $\mathscr{A}(G)$ is called *strictly irreducible* [6]. The correction consists in only substituting $\mathscr{A}(G)$ for $\mathscr{A}$ in the Theorems 20 and 21 of [2]. However since we had only outlined the proof of Theorem 21, a complete proof is thought to be necessary. The proof of Theorem 20 shows that the commutant $\mathscr{A}(G)'$ of $\mathscr{A}(G)$ is isomorphic with the *reals* $\boldsymbol{R}$, which is a sharpening of Schur's lemma.

Definition 2 [6]. (i) An algebra $\mathscr{A}$ of linear operators over a vector space $\mathscr{X}$ is said to be *k-fold transitive* on $\mathscr{X}$ iff for any k linearly independent $x_i \in \mathscr{X}(i \in \boldsymbol{N}_k)$ and k $y_i \in \mathscr{X}$ there exists $\boldsymbol{T} \in \mathscr{A}$ such that $\boldsymbol{T}x_i = y_i$ holds.

(ii) $\mathscr{A}$ is called *strictly dense* on $\mathscr{X}$ iff $\mathscr{A}$ is k-fold transitive for any $k \in \boldsymbol{N}$.

Remark 3. (i) 1-fold transitivity is equivalent to irreducibility.

(ii) If $\mathscr{A}$ is 2-fold transitive, then $\mathscr{A}$ is already strictly dense ([6], Lemma (2.4.3)).

Definition 3 [1]. Let M be an $\mathscr{A}$-module.

(i) M is said to be *faithful* iff $aM \neq (0)$ for every $a \in \mathscr{A} \setminus \{0\}$.

(ii) M is said to be *irreducible* iff M and (0) are the only $\mathscr{A}$-submodules and $\mathscr{A}M = \left\{ \sum_{i \in \boldsymbol{N}_n} a_i m_i \,\middle|\, a_i \in \mathscr{A} \text{ and } m_i \in M,\ n \in \boldsymbol{N} \right\} \neq (0)$ holds.

Definition 4 [1]. A ring $\mathscr{A}$ is said to be *primitive* iff there exists a faithful irreducible $\mathscr{A}$-module M.

So we are prepared for the formulation of Jacobson's density theorem [1]: If $\mathscr{A}' = \mathrm{Hom}_{\mathscr{A}}(M, M)$ is the centralizer of a faithful irreducible $\mathscr{A}$-module M, then $\mathscr{A}$ is strictly dense in $\mathrm{Hom}_{\mathscr{A}'}(M, M)$. ($\mathscr{A}'$ is, by Schur's lemma, a field!)

Remark 4. Concerning B' we observe that

1. B' as $\mathscr{A}(G)$-module is faithful for $\mathscr{A}(G) \subseteqq \mathscr{B}(B')$.

2. G irreducible implies B' is irreducible for $\mathscr{A}(G)$, which is, since $\dim B' = N < \infty$, equivalent to the strict irreducibility of B' for $\mathscr{A}(G)$.

3. $\mathscr{A}(G)'$ is isomorphic to $\boldsymbol{R}$, thus $\mathscr{A}(G)'' = \mathscr{B}(B')$. We have only to prove $\mathscr{A}(G) = \mathscr{A}(G)''$ and Theorem 21 in [2] will then be verified in detail:

By the above density theorem $\mathscr{A}(G)$ is strictly dense in $\mathscr{A}(G)'' = \mathscr{B}(B')$. For any $\boldsymbol{T} \in \mathscr{B}(B')$ and any basis $\{Y_i | i \in \boldsymbol{N}_N\}$ of B' we define $\boldsymbol{T}Y_i = \bar{Y}_i$ for each $i \in \boldsymbol{N}_N$. Since $\mathscr{A}(G)$ is dense, there exists $\boldsymbol{A} \in \mathscr{A}(G)$ such that $\boldsymbol{A}\,Y_i = \bar{Y}_i$. Therefore $\boldsymbol{T}$ and $\boldsymbol{A}$ coincide on the basis choosen, hence $\boldsymbol{T} = \boldsymbol{A}$. This proves $\mathscr{B}(B') \subseteqq \mathscr{A}(G)$ and completes the proof of Theorem 21 in [2].

Remark 5. Using Mielnik's terminology [4], we see that the set of all physical filters $\boldsymbol{T}_E$ determines the $\boldsymbol{R}$-algebra of all linear operators of B' which is generated by the set of all physical decision effects E.

Let $\mathscr{L}(\mathscr{S})$ and $\mathscr{R}(\mathscr{S})$ denote the left and right annihilator, respectively, for any $\mathscr{S} \subseteqq \mathscr{B}(B')$. Being the algebra of all linear operators over B', $\mathscr{B}(B')$ satisfies

1. $\mathscr{L}(\mathscr{S})=\mathscr{B}(B')\boldsymbol{I}_1$, 2. $\mathscr{R}(\mathscr{S})=\boldsymbol{I}_2\mathscr{B}(B')$
for all $\mathscr{S}\subseteqq\mathscr{B}(B')$ and $\boldsymbol{I}_1, \boldsymbol{I}_2$ idempotents of $\mathscr{B}(B')$.

1. and 2. are the defining properties of a Baer-ring. By Remark 5 $\mathscr{A}(G)$ is such a ring and there even holds that $\mathscr{B}(B')$, being a Banach algebra, is an annihilator algebra, which can be inferred from a theorem by Rickart [6]:

1. $\mathscr{B}(\mathscr{X})$ being an annihilator Banach algebra is equivalent with the Banach space $\mathscr{X}$ being reflexive. From the textbook [6] we need 4 other theorems:

2. For each minimal right ideal $\mathscr{R}$ of an algebra $\mathscr{A}$ with $\mathscr{R}^2 \neq \{0\}$ there exists an idempotent $e\in\mathscr{A}$ such that $\mathscr{R}=e\mathscr{A}$ and $e\mathscr{A}e$ is a field with unit element e (this e is minimal!).

3. For each minimal idempotent $e\in\mathscr{A}$ $e\mathscr{A}$ is a minimal right ideal, $\mathscr{A}e$ a minimal left ideal.

4. For each minimal idempotent of a Banach algebra there holds

(i) $e\mathscr{A}e$ is isomorphic either to the reals or the complexes or the quaternions.

(ii) If $\mathscr{A}$ is complex, $e\mathscr{A}e=e$.

5. In a semisimple annihilator algebra is (0) the only right ideal which contains no minimal right ideals.

Consequence. By Theorem 20 in [2] $\mathscr{A}(G)=\mathscr{B}(B')$ is simple, hence minimal idempotents exist and 2.–5. give a biunivocal correspondence between the minimal idempotents and the minimal right (and left) ideals (respectively).

We are now prepared for an investigation of all minimal idempotents (and hence all minimal right ideals) in $\mathscr{B}(B')$: Reflexity of B and B' implies the canonical isomorphisms: $B'\otimes_{\boldsymbol{R}}B=\mathscr{B}(B)$; $B\otimes_{\boldsymbol{R}}B'=\mathscr{B}(B')$. So we may define the following linear operators over B and B', respectively:

$$(Y\otimes X)\bar{X}=\langle\bar{X}, Y\rangle X \quad \text{for all} \quad \bar{X}\in B \quad \text{and any} \quad Y\in B', X\in B\,;$$

$$(X\otimes Y)\bar{Y}=\langle X, \bar{Y}\rangle Y \quad \text{for all} \quad \bar{Y}\in B' \quad \text{and any} \quad X\in B,\ Y\in B'\,.$$

Let $X\otimes Y$ be positive, i.e. $(X\otimes Y)[B'_+]\subseteqq B'_+$, then $(X\otimes Y)\bar{Y}\in B'_+$ for all $\bar{Y}\in B'_+$. There are two cases to be distinguished:

(i) $Y\in -B'_+$ implies $\langle X, \bar{Y}\rangle\leqq 0$ for all $\bar{Y}\in B'_+$, thus $X\in -B_+$.

(ii) $Y\in B'_+$ implies $\langle X, \bar{Y}\rangle\geqq 0$ for all $\bar{Y}\in B'_+$ thus $X\in B_+$.
Summerizing we can state

Theorem 9. $\mathscr{T}_1:=\{X\otimes Y \mid X\in B_+$ *and* $Y\in B'_+\}$ *is the set of all positive operators of rank* 1 *in* $\mathscr{B}(B')$.

Lemma 2. *Every idempotent $\boldsymbol{I} \in \mathscr{B}(B')$ with rank 1 is minimal.*

Proof. Assume $\boldsymbol{I} = \boldsymbol{I}_1 + \boldsymbol{I}_2$ such that $\boldsymbol{I}_1 \boldsymbol{I}_2 = \boldsymbol{I}_2 \boldsymbol{I}_1 = \boldsymbol{0}$ and $I_i \neq \boldsymbol{0}$ for each $i \in \boldsymbol{N}_2$. Then we have $\operatorname{Im} \boldsymbol{I} = \operatorname{Im} \boldsymbol{I}_1 \oplus \operatorname{Im} \boldsymbol{I}_2$, hence $\dim \operatorname{Im} \boldsymbol{I} = \dim \operatorname{Im} \boldsymbol{I}_1 + \dim \operatorname{Im} \boldsymbol{I}_2 \geqq 2$, contrary to $\dim \operatorname{Im} \boldsymbol{I} = 1$.

Theorem 10. *Every positive minimal idempotent $\boldsymbol{I} \in \mathscr{B}(B')$ has the representation*

$$X \otimes Y \quad \text{with} \quad \langle X, Y \rangle = 1, \; X \in B_+ \quad \text{and} \quad Y \in B'_+ \, .$$

Proof. According to [6], p. 65, a right ideal of $\mathscr{B}(B')$ is minimal iff it consists of elements of rank 1. From Theorem 9 and Lemma 2 there follows for such an idempotent $\boldsymbol{I}$ that $\boldsymbol{I} = X \otimes Y$, $X \in B_+$, $Y \in B'_+$ and idempotence of $\boldsymbol{I}$ requires

$$(X \otimes Y)^2 \, \bar{Y} = \langle X, Y \rangle \, \langle X, \bar{Y} \rangle \, Y = \langle X, Y \rangle \, (X \otimes Y) \, \bar{Y} = (X \otimes Y) \, \bar{Y}$$

for all $\bar{Y} \in B'$. This is satisfied iff $\langle X, Y \rangle = 1$. ▮

2. and 5. of Section II admit the representations: every $X \in B_+$ can be written as $X = \alpha V$, $\alpha \in \boldsymbol{R}_+$ and $V \in K$; every $Y \in B'_+$ can be written as $Y = \beta F$, $\beta \in \boldsymbol{R}_+$ and $F \in L$. Therefore, any $\boldsymbol{I}$ from Theorem 10 can be represented as $\boldsymbol{I} = \alpha\beta V \otimes F$ with $\beta \langle V, F \rangle = 1$. If $Y = \beta F$ then β can be choosen so that $K_1(F) \neq \emptyset$ because K is compact. This leads us to

Theorem 11. *Every positive minimal idempotent $\boldsymbol{I} \in \mathscr{B}(B')$ can be represented by $\boldsymbol{I} = \beta V \otimes F$ with $\beta \langle V, F \rangle = 1$ and $K_1(F) \neq \emptyset$.*

Corollary. *$V \otimes F$ is a minimal idempotent iff $V \in K_1(F)$. As an important consequence of Theorem 11 we may verify*

Theorem 12. *Every atom $\boldsymbol{T}_P \in \mathscr{T}(G)$ (i.e. $\boldsymbol{T}_P \in A\mathscr{T}(G)$) is a positive minimal idempotent satisfying*

$$\boldsymbol{T}_P = V_P \otimes P, \; P \in A(G) \quad \text{and} \quad \{V_P\} = K_1(P) \, .$$

Proof. Per definitionem of $\mathscr{T}(G)$ $\boldsymbol{T}_P$ is idempotent and operates by reason of the preface to Theorem 20 in [2] as $\boldsymbol{T}_P Y = \langle V_P, Y \rangle P$ for all $Y \in B'$, thus $\boldsymbol{T}_P = V_P \otimes P$. ▮

From 4. of the quoted theorems in Rickart's textbook [6] there follows in particular that each $\boldsymbol{T}_P \in A\mathscr{T}(G)$ satisfies $\boldsymbol{T}_P \mathscr{B}(B') \boldsymbol{T}_P = \boldsymbol{R} \boldsymbol{T}_P = \boldsymbol{T}_P \boldsymbol{R}$. Of course, we would have been able to calculate this equality directly, ignoring, however, its connexion with irreducibility of idempotents in $\mathscr{B}(B')$.

The next step leads to a linear order isomorphism between B, B' and minimal ideals of $\mathscr{B}(B')$. To this purpose we show

Theorem 13. *For arbitrary but fixed $P \in A(G)$ the sets*

$$\mathscr{R}_P := \{X \otimes P \mid X \in B\} \quad \textit{and} \quad \mathscr{L}_P := \{V_P \otimes Y \mid Y \in B'\}$$

are minimal right and left ideals, respectively, satisfying

$$\mathscr{R}_P = \boldsymbol{T}_P \mathscr{B}(B'), \ \mathscr{L}_P = \mathscr{B}(B') \boldsymbol{T}_P .$$

Proof. The ideal property is obvious. Besides, each element of the ideals has rank 1. We have only to prove $\mathscr{L}_P = \mathscr{B}(B')\boldsymbol{T}_P$, the verification of $\mathscr{R}_P = \boldsymbol{T}_P \mathscr{B}(B')$ is then in a completely analogous way and will be omitted. Let $\{X_i \mid i \in \boldsymbol{N}_N\}$ and $\{Y_j \mid j \in \boldsymbol{N}_N\}$ be bases of B and B', respectively. Then every $\boldsymbol{T} \in \mathscr{B}(B')$ admits the representation $\boldsymbol{T} = \sum_{i,j \in \boldsymbol{N}_N} t_{ij} X_i \otimes Y_j$; thus, for all $Y \in B'$, there holds $\boldsymbol{T}\boldsymbol{T}_P Y = \boldsymbol{T}(V_P \otimes P) Y = \sum_{i,j \in \boldsymbol{N}_N} t_{ij} \langle X_i, P\rangle \cdot (V_P \otimes Y_j) Y$, whence, with $Y := \sum_{i,j \in \boldsymbol{N}_N} t_{ij} \langle X_i, P\rangle Y_j$, $\boldsymbol{T} V_P \otimes P = V_P \otimes Y \in \mathscr{L}_P$, i.e. $\mathscr{B}(B')\boldsymbol{T} \subseteqq \mathscr{L}_P$. Consequently, for every $V_P \otimes Y \in \mathscr{L}_P : (V_P \otimes Y)(V_P \otimes P) = \langle V_P, \cdot\rangle \langle V_P, P\rangle Y = \langle V_P, \cdot \rangle Y = V_P \otimes Y$ since $\langle V_P, P\rangle = 1$. Hence $\mathscr{L}_P \subseteqq \mathscr{B}(B')\boldsymbol{T}_P$. ∎

Corollary. *The correspondences $B \to \mathscr{R}_P$ and $B' \to \mathscr{L}_P$ are linear bijections.*

4. of the Rickart-theorems quoted yields $\mathscr{R}_P \mathscr{L}_P = \boldsymbol{T}_P \mathscr{B}(B') \mathscr{B}(B') \boldsymbol{T}_P = \boldsymbol{T}_P \mathscr{B}(B') \boldsymbol{T}_P = \boldsymbol{R}\,\boldsymbol{T}_P$, which leads because of $(X \otimes P)(V_P \otimes Y) = \langle X, Y\rangle \boldsymbol{T}_P$ to the existence of a bilinear functional Γ over $\mathscr{R}_P \times \mathscr{L}_P$ given by $\Gamma(X \otimes P, V_P \otimes Y) = \langle X, Y\rangle = \mu(X, Y)$. Γ is, therefore, the canonical bilinear functional over $\mathscr{R}_P \times \mathscr{L}_P$, because $\langle \cdot, \cdot \rangle$ has this property on $B \times B'$. This enables us to formulate

Theorem 14. (i) *The bijections from the preceding corollary are order isomorphisms.*

(ii) *The sets $L \subset B'$, $G \subset B'$, $K \subset B$ are represented by*

$$\hat{\mathscr{L}}_P = \{V_P \otimes F \mid F \in L\}, \ \mathscr{G}_P = \{V_P \otimes E \mid E \in G\} \ \textit{and} \ \mathscr{K}_P = \{V_P \otimes P \mid V \in K\},$$

respectively.

Remark 6. According to Theorem 10 any minimal idempotent $X_0 \otimes Y_0$ with $\langle X_0, Y_0 \rangle = 1$ can be substituted for $\boldsymbol{T}_P$ in the preceding theorem, for it leads to equivalent representations.

Open remains the question of what algebras posses a bilinear functional Γ satisfying all postulates of μ (and its extension) and of what subsets of these algebras represent $L \subset B'$, $G \subset B'$ and $K \subset B$, i.e. satisfy all the axioms postulated of L and K. Theorem 14 suggests to attempt a characterization of the dual pair (B, B') by the algebra $\mathscr{B}(B')$. The following considerations prepare this task.

Lemma 3. *In* $(\mathscr{B}(B'), \|\cdot\|_{b'N})$ *as a Banach space there exists an additional norm defined by* $\|\boldsymbol{T}\|_L := \sup\{\|\boldsymbol{T}F\| \mid F \in L\}$, b'^N *denoting the* N*-dimensional unit ball of* B'.

Proof. Verification of the norm axioms: 1) $\|\boldsymbol{T}\|_L \geqq 0$ for all $\boldsymbol{T} \in \mathscr{B}(B')$. 2) $\|\boldsymbol{T}\|_L = 0$ implies $\|\boldsymbol{T}F\| = 0$ for all $F \in L$, hence $\boldsymbol{T}F = 0$ for all $F \in L$. Since L is generating in B', so $\boldsymbol{T} = \boldsymbol{0}$. Immediately from the above definition there follow

3. $\|\beta \boldsymbol{T}\|_L = |\beta|\, \|\boldsymbol{T}\|_L$ for all $\beta \in \boldsymbol{R}$ and

4. $\|\boldsymbol{T}_1 + \boldsymbol{T}_2\|_L \leqq \|\boldsymbol{T}_1\|_L + \|\boldsymbol{T}_2\|_L$. ∎

Remark 7. Concerning this norm $\mathscr{B}(B')$ is as a finite-dimensional vector space complete and the two norms $\|\cdot\|_{b'N}$ and $\|\cdot\|_L$ determine the same topology in $\mathscr{B}(B')$, hence they are equivalent. Therefore, all $\boldsymbol{T} \in \mathscr{B}(B')$ are already distinguished sufficiently sharp by the effects $F \in L$. This is the physical meaning of the norm $\|\cdot\|_L$.

We can even strengthen Lemma 3 by

Theorem 15. $(\mathscr{B}(B'), \|\cdot\|_L)$ *is (just as* $(\mathscr{B}(B'), \|\cdot\|_{b'N})$*) a Banach algebra.*

Proof. Since $\|\boldsymbol{T}F\| \leqq \beta_L \|F\|$ with $\beta_L \in \boldsymbol{R}_+^*$ for all $F \in L$, we have in addition to the norm properties in Lemma 3,

$$\|\boldsymbol{T}_1 \boldsymbol{T}_2\|_L = \sup\{\|\boldsymbol{T}_1 \boldsymbol{T}_2 F\| \mid F \in L\} \leqq \|\boldsymbol{T}_1\|_L \|\boldsymbol{T}_2 F\| \leqq \|\boldsymbol{T}_1\|_L \|\boldsymbol{T}_2\|_L .$$

Moreover, there holds $\|\boldsymbol{T}_1\|_L = \|\boldsymbol{id}_{B'}\|_L = \sup\{\|F\| \mid F \in L\} = 1$. ∎

We intend now to tackle the problem of how to make $(\mathscr{B}(B'), \|\cdot\|_L)$ a $*$-algebra with $\mathscr{T}(G)$ as a subset of the set of all elements remaining fixed under the involution $*$. Remark 5 gives an appropriate hint:

$\mathscr{A}(G)$ is the smallest $\boldsymbol{R}$-algebra containing $\mathscr{T}(G)$. Thus it contains the $\boldsymbol{R}$-algebra of all finite linear combinations of all finite products of elements of $\mathscr{T}(G)$. By Remark 5 it equals $\mathscr{B}(B')$. Consequently, each $\boldsymbol{T} \in \mathscr{B}(B')$ has a representation by finitely many $\boldsymbol{T}_E \in \mathscr{T}(G)$ because of $\dim \mathscr{B}(B') = N^2 < \infty$. Without loss of generality these $\boldsymbol{T}_E$ may be selected to be linearly independent:

$$\boldsymbol{T} = \sum_{i_1 \in \boldsymbol{N}_l} t(i_1)\boldsymbol{T}_{E_{i_1}} + \sum_{\substack{i_1, i_2 \\ i_1 \neq i_2}} t(i_1 i_2)\boldsymbol{T}_{E_{i_1}} \boldsymbol{T}_{E_{i_2}} + \cdots$$
$$\cdots + \sum_{\substack{i_1, \ldots, i_m \\ i_1 \neq i_2 \neq \cdots \neq i_m}} t(i_1, \ldots, i_m)\, \boldsymbol{T}_{E_{i_1}} \boldsymbol{T}_{E_{i_2}} \ldots \boldsymbol{T}_{E_{i_m}};$$

$m \in \boldsymbol{N}$, $i_k \in \boldsymbol{N}_l$ for all $k \in \boldsymbol{N}_m$, where $l \leqq \dim \mathscr{B}(B') = N^2$.

We define for every $m \in \boldsymbol{N}$:

$$\mathscr{T}(G)^m := \{\boldsymbol{T}_{E_{i_1}} \boldsymbol{T}_{E_{i_2}} \ldots \boldsymbol{T}_{E_{i_m}} \mid \boldsymbol{T}_{E_{i_k}} \in \mathscr{T}(G) \quad \text{and} \quad k \in \boldsymbol{N}_m\}.$$

Then we regard on $\bigcup_{m\in N} \mathscr{T}(G)^m$ the relation * defined by

$$(T_{E_{i_1}} T_{E_{i_2}} \dots T_{E_{i_{m-1}}} T_{E_{i_m}})^* = T_{E_{i_m}} T_{E_{i_{m-1}}} \dots T_{E_{i_2}} T_{E_{i_1}} .$$

This confronts us with the question when the defining property for * to be a mapping holds: "For any $T_1,\ T_2 \in \mathscr{T}(G)^m : T_1 = T_2 \Rightarrow T_1^* = T_2^*$. Since $T_E = T_E T_E$ implies $T_E^* = T_E$ for each $T_E \in \mathscr{T}(G)$ the defining property of * is obviously valid for all $T_E \in \mathscr{T}(G)$ being a subset of the set of all fixed elements under *. Let us consider $\mathscr{T}(G)^2$ and assume $T_{E_1} T_{E_2} = T_{E_3} T_{E_4}$. The question is whether "$T_{E_1} T_{E_2} = T_{E_3} T_{E_4}$ implies $T_{E_2} T_{E_1} = T_{E_4} T_{E_3}$" holds.

Theorem 16. *For all* $i \in N_4$ *and* $T_{E_i} \in \mathscr{T}(G) : T_{E_1} T_{E_2} = T_{E_3} T_{E_4}$ *and* $E_1 \mathscr{C} E_2$ *implies* $T_{E_2} T_{E_1} = T_{E_4} T_{E_3}$.

Proof. From $T_{E_1} T_{E_2} = T_{E_3} T_{E_4}$ there follows $T_{E_1} T_{E_2}(E_1 \wedge E_2) = E_1 \wedge E_2 = T_{E_3} T_{E_4}(E_1 \wedge E_2) \leqq E_1 \wedge E_3$ and $T_{E_1} T_{E_2}(E_3 \wedge E_4) = T_{E_3} T_{E_4}(E_3 \wedge E_4) = E_3 \wedge E_4 \leqq E_1 \wedge E_3$. $E_1 \mathscr{C} E_2$ is, by Theorem 2, equivalent with $T_{E_1} T_{E_2} = T_{E_2} T_{E_1}$. Applying then Theorem 1, we obtain $T_{E_1 \wedge E_2}(E_3 \wedge E_4) = T_{E_3}\ T_{E_4}(E_3 \wedge E_4) = E_3 \wedge E_4 \leqq E_1 \wedge E_2$. $T_{E_1 \wedge E_2} E_4^\perp = T_{E_3}\ T_{E_4}\ E_4^\perp = 0$ implies $\langle V, T_{E_1 \wedge E_2} E_4^\perp \rangle = \langle V, E_4^\perp \rangle = 0$ for every $V \in K_1(E_1 \wedge E_2)$, hence $K_1(E_1 \wedge E_2)\ K_0(E_4^\perp) = K_1(E_4)$, thus $E_1 \wedge E_2 \leqq E_4$. $T_{E_3^\perp}\ T_{E_1 \wedge E_2} = T_{E_3^\perp} T_{E_3} T_{E_4} = \mathbf{0}$ implies $T_{E_3^\perp} E_1 \wedge E_2 = 0$, i.e. $0 = \langle V, T_{E_3^\perp} E_1 \wedge E_2 \rangle = \langle V, E_1 \wedge E_2 \rangle$ for every $V \in K_1(E_3^\perp)$, hence $K_1(E_3^\perp) = K_0(E_3) \subseteqq K_0(E_1 \wedge E_2)$, thus $E_1 \wedge E_2 \leqq E_3$. This completes the proof of $E_1 \wedge E_2 = E_3 \wedge E_4$ from which $T_{E_1 \wedge E_2} = T_{E_3 \wedge E_4} = T_{E_3} T_{E_4}$ results and thus $T_{E_3} T_{E_4} = T_{E_4} T_{E_3}$. ∎

Physically speaking, the above implication was only verified for commensurable decision effects. Generally, however, the above implication is valid for $(A\mathscr{T}(G))^2$:

Theorem 17. *For all* $T_{P_1}, T_{P_2}, T_{Q_1}, T_{Q_2} \in A\mathscr{T}(G)$: $T_{P_1} T_{P_2} = T_{Q_1} T_{Q_2}$ *implies* $T_{P_2} T_{P_1} = T_{Q_2} T_{Q_1}$.

Proof. (i) If $T_{P_1} T_{P_2} = T_{Q_1} T_{Q_2} = \mathbf{0}$, then $P_1 \perp P_2$ and $Q_1 \perp Q_2$ and the implication is true.

(ii) If $T_{P_1} T_{P_2} \neq \mathbf{0}$, then $T_{P_1} T_{P_2} = \langle V_{P_1}, P_2 \rangle\, V_{P_2} \otimes P_1 = \langle V_{Q_1}, Q_2 \rangle\, V_{Q_2} \otimes Q_1$ with $\langle V_{P_1}, P_2 \rangle \neq 0 \neq \langle V_{Q_1}, Q_2 \rangle$ implies $P_1 = Q_1$. Moreover, for every atom $P \leqq P_2^\perp$ we obtain $T_{Q_2} T_P = \mathbf{0}$, hence $P \leqq Q_2^\perp$. For every atom $Q \leqq Q_2^\perp$ we obtain $T_{P_2} T_Q = \mathbf{0}$, hence $Q \leqq P_2^\perp$. So, $\bigvee_{Q \leqq Q_2^\perp} Q = Q_2^\perp \leqq P_2^\perp = \bigvee_{P \leqq P_2^\perp} P \leqq Q_2^\perp$, whence $P_2 = Q_2$. ∎

Suppose * is a mapping, then * is obviously involutory on $\bigcup_{m\in N} \mathscr{T}(G)^m$ and the question arises whether * can be linearly extended to $\mathscr{B}(B')$ to give an algebra involution. A necessary condition for * to have such an extension is:

"For every $T \in \bigcup_{m\in N} \mathscr{T}(G)^m : T^* = \mathbf{0} \Rightarrow T = \mathbf{0}$."

Obviously, this implication is true if $E_{i_j} \perp E_{i_{j-1}}$ for at least one $j \in N_m \setminus \{1\}$. Let us consider $T_{E_{i_1}} \ldots T_{E_{i_m}} \in \mathscr{T}(G)^m$ such that $(T_{E_{i_1}} \ldots T_{E_{i_m}})^* = T_{E_{i_m}} \ldots T_{E_{i_1}} = 0$. This implies $\langle V, T_{E_{i_m}} T_{E_{i_{m-1}}} \ldots T_{E_{i_1}} F \rangle = 0$ for every $F \in L$ and every $V \in K$; particularly, $\langle V, T_{E_{i_m}} T_{E_{i_{m-1}}} \ldots T_{E_{i_1}} F \rangle = \langle V, T_{E_{i_{m-1}}} \ldots T_{E_{i_1}} F \rangle = 0$ for every $V \in K_1(E_{i_m}) = K_0(E_{i_m}^\perp)$. Since $T_{E_{i_{m-1}}} \ldots T_{E_{i_1}} F \in L_0 K_0(E_{i_{m-1}})$, so $T_{E_{i_{m-1}}} \ldots T_{E_{i_1}} F \in L_0 K_0(E_{i_{m-1}}) \cap L_0 K_0(E_{i_m}^\perp) = L_0 K_0(E_{i_{m-1}} \wedge E_{i_m}^\perp)$. We failed to prove the desired implication generally though it holds in an important special case.

Theorem 18. *For every* $T \in \bigcup_{m \in N} A \mathscr{T}(G)^m : T^* = 0 \Rightarrow T = 0$.

Proof. By the above consideration we get the following dichotomy:

(i) $P_{i_{m-1}} \wedge P_{i_m}^\perp = P_{i_{m-1}}$, hence $P_{i_{m-1}} \leqq P_{i_m}^\perp$ and the implication is true.

(ii) $P_{i_{m-1}} \wedge P_{i_m}^\perp = 0$ leads to $T_{P_{i_{m-1}}} \ldots T_{P_{i_1}} = 0$. To $T_{P_{i_{m-1}}} \ldots T_{P_{i_1}} = 0$ the same procedure applies and we arrive at least at one $j \in N_m \setminus \{1\}$ such that $P_{i_j} \perp P_{i_{j-1}}$ which states the validity of the implication. (i) and (ii) express the existence of at least two orthogonal atoms being neighbouring factors of the product in question. ∎

There remains the open question whether for instance modularity of G guarantees the existence of an involution * on $\mathscr{B}(B')$ which makes $\mathscr{B}(B')$ a C^*-algebra with $\mathscr{T}(G)$ in the set of all fixed elements under *. The converse of this question (the open question 3 in [2]) will be answered in a subsequent paper in the affirmative.

References

1. Behrens, E. A.: Algebren, Bd. 97/97a. Mannheim: Bibliogr. Institut 1965.
2. Dähn, G.: Attempt of an axiomatic foundation of quantum mechanics and more general theories. IV. Commun. math. Phys. **9**, 192—211 (1968).
3. Ludwig, G.: Attempt of an axiomatic foundation of quantum mechanics and more general theories. III. Commun. math. Phys. **9**, 1—12 (1968).
4. Mielnik, B.: Theory of filters. Commun. math. Phys. **15**, 1—46 (1969).
5. Nakamura, M.: The permutability in a certain orthocomplemented lattice. Kodai Math. Sem. Rep. **9**, 158—160 (1957).
6. Rickart, C. E.: General theory of Banach algebras. Princeton: Van Nostrand 1960.

Günter Dähn
Mathematisches Institut der Universität
D-7400 Tübingen
Brunnenstraße 27
Federal Republic of Germany