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Entropy and Normal States

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Abstract. A generalized definition of entropy for any state on a C^* algebra is given and studied. We prove that the entropy characterizes uniquely the normal states.

I. Introduction

For the algebras of the canonical commutation and anticommutation relations the theorem of Dell'Antonio, Doplicher and Ruelle [1] is well-known; it states that normal factor states on the CCR and CAR algebras are characterized by the existence of a number operator on the representation space induced by the state. Physically it means that the states describing a finite number of particles are exactly the normal factor states.

In this work we characterize the normal states (Definition 1) on any C^*-algebra by an other physical quantity, namely the entropy; for the exact formulation see Theorem 1 below. In physical terms, it means that the states of finite entropy are exactly the normal states.

In order to work out the subject we generalize first the motion of entropy of a state on a C^*-algebra (Definition 2). For normal factor states it coincides with the ordinary definition and we prove also that it satisfies properties and inequalities analogous to those satisfied by the usual entropy. Finally we discuss more in detail the entropy definition and give an alternative expression for it (see Definition 3).

II. Normal States and Total Entropy

Definition 1. *Let ω be a state on a C^*-algebra $\mathscr{A}$, ω is called normal if ω is a convex linear combination of pure states on $\mathscr{A}$.*

Remark that, if ω is a normal state, then H_ω, the GNS representation induced by ω, is the direct sum of irreducible representations of $\mathscr{A}$;

it is clear that normal states are not necessarily factor states. Generally in the physics literature a normal state corresponds in our terminology to a normal factor state.

Definition 2. *The entropy $S(\omega)$ of a state ω on $\mathscr{A}$ is given by*

a) *If ω is a normal state, i.e. $\omega = \sum_n \lambda_n \omega_n$, $\sum_n \lambda_n = 1$, $\lambda_n \geqq 0$, and ω_n pure, then*

$$S(\omega) = \inf \sum_n -\lambda_n \log \lambda_n$$

the infimum is taken over all possible decompositions of ω into pure states.

b) *If ω is not normal, then $S(\omega) = \infty$.*

For any state ω on $\mathscr{A}$, denote by Π_ω, $\mathscr{H}_\omega$, Ω_ω respectively the GNS-representation, representation space and cyclic vector induced by ω; let $\psi \in \mathscr{H}_\omega$ then ω_ψ is the state on $\mathscr{A}$, defined by $\omega_\psi(x) = (\psi \mid \Pi_\omega(x)\,\psi)$, $x \in \mathscr{A}$.

Proposition 1. *The entropy $S(\omega)$ of the state ω satisfies:*

i) *If ω is a normal factor state, i.e. there exists an irreducible representation Π on a Hilbert space $\mathscr{H}$ and a density matrix ϱ such that*

$$\omega(x) = \mathrm{Tr}_{\mathscr{H}}\, \varrho \cdot \Pi(x)\,,$$

then

$$S(\omega) = -\mathrm{Tr}_{\mathscr{H}}\, \varrho \log \varrho\,.$$

ii) *If ω_1 and ω_2 are states on $\mathscr{A}$, and λ such that $0 \leqq \lambda \leqq 1$ then*

$$\begin{aligned}\lambda S(\omega_1) + (1+\lambda)\, S(\omega_2) &\leqq S(\lambda\omega_1 + (1-\lambda)\,\omega_2)\\ &\leqq \lambda S(\omega_1) + (1-\lambda) S(\omega_2) - \lambda \log \lambda - (1-\lambda)\log(1-\lambda)\end{aligned}$$

iii) $S(\omega) = 0$ *if and only if ω is a pure state.*

Proof. See Appendix A and B.

Theorem 1. *Let ω be any state on $\mathscr{A}$, then ω is a normal state if and only if the set*

$$P = \{\psi \in \mathscr{H}_\omega \mid \|\psi\| = 1\,,\; S(\omega_\psi) < \infty\}$$

is dense in the unit sphere of $\mathscr{H}_\omega$.

Proof. Suppose first that ω is normal, then $\Pi_\omega = \bigoplus_n \Pi_n$ where Π_n are irreducible representations, $\mathscr{H}_\omega = \bigoplus_n \mathscr{H}_n$. For any finite sequence $(\psi_i)_{i=1,\ldots,p}$ with $\psi_i \in \mathscr{H}_{n_i}$, $\|\psi_i\| = 1$, where all n_i are different for different i, then

$$\omega_{\sum\limits_{i=1}^{p} \lambda_i^{\frac{1}{2}} \psi_i} = \sum_{i=1}^{p} \lambda_i \omega_{\psi_i}$$

for all sets $(\lambda_i)_{i=1,\ldots,p}$, $\lambda_i \geqq 0$, $\sum\limits_{i=1}^{p} \lambda_i = 1$.

As the states ω_{ψ_i} are pure, it follows from Definition 2 that:

$$S\left(\omega_{\sum_{i=1}^{p} \lambda_i \psi_i}\right) \leqq - \sum_{i=1}^{p} \lambda_i \log \lambda_i < \infty .$$

To prove the converse, suppose ω is not normal, then Π_ω is not a direct sum of irreducible representations, hence there exists in the commutant $\Pi_\omega(\mathscr{A})'$ at least one projection E which majorizes no minimal projections. Let ψ be a unit vector of $E\mathscr{H}_\omega$ and $E_\psi \leqq E$ where E_ψ is the orthogonal projection on $\overline{\Pi_\omega(\mathscr{A})\psi}$. Then the induced representation $\Pi_\omega | E_\psi$ is the GNS representation for the vector state ω_ψ. As E_ψ majorizes no minimal projections, $\Pi_\omega | E_\psi$ is not a direct sum of irreducible representations, hence $S(\omega_\psi) = \infty$.

Let $\psi \in \mathscr{H}_\omega$, $\|\psi\| = 1$, then $\psi = \alpha^{\frac{1}{2}} \psi_1 + \sqrt{1-\alpha}\,\psi_2$ where $\psi_1 \in E\mathscr{H}_\omega$, $\|\psi_1\| = 1$, and $\psi_2 \in \mathscr{H}_\omega \ominus E\mathscr{H}_\omega$, $\|\psi_2\| = 1$. As $E \in \Pi_\omega(\mathscr{A})' : \omega_\psi = \alpha\omega_{\psi_1} + (1-\alpha)\,\omega_{\psi_2}$.

From Proposition 1:

$$\alpha S(\omega_{\psi_1}) + (1-\alpha)\, S(\omega_{\psi_2}) \leqq S(\omega_\psi)$$

and for all vectors ψ such that $\alpha \neq 0$, $S(\omega_\psi) = \infty$. This proves that the set P is not dense in the unit sphere of $\mathscr{H}_\omega$. Q.E.D.

III. Discussion of the Entropy Definition

In this section we study in more detail the generalized notion of entropy given in Definition 2 in order to justify the notion of normal states as the states of finite entropy.

We start with the following notation: let $\mathfrak{Z}$ be an abelian von Neumann algebra on a Hilbert space $\mathscr{H}$, Ω a unit vector of $\mathscr{H}$, then denote

$$s_\Omega(\mathfrak{Z}) = \sup_{\mathscr{E} \in \mathscr{C}} h_\Omega(\mathscr{E})$$

where $h_\Omega(\mathscr{E}) = -\sum_{n \in I} (\Omega | E_n \Omega) \log (\Omega | E_n \Omega)$ and $\mathscr{E} = (E_n)_{n \in I}$ is a sequence of two by two orthogonal projections E_n of $\mathfrak{Z}$ such that $\sum_{n \in I} E_n = 1$. Denote by $\mathscr{C}$ the set of such sequences.

Lemma 1. *If there exists a element $\mathscr{E}$ of $\mathscr{C}$ such that all E_n are minimal in $\mathfrak{Z}$ then*

$$s_\Omega(\mathfrak{Z}) = h_\Omega(\mathscr{E}) .$$

Proof. Let G be any projection of $\mathfrak{Z}$. Then

$$G = G\left(\sum_{n \in I} E_n\right) = \sum_{n \in I} G E_n .$$

As all E_n are minimal, $GE_n=0$ or $GE_n=E_n$ for all n. Hence any projection of $\mathfrak{Z}$ is a sum of projections E_n. Take any

$$\mathscr{F}=(F_p)_p\in\mathscr{C}\,;\quad \text{let}\quad I_p=\{n\in I\,|\,E_nF_p=E_n\}\,,$$

then

$$I_p\cap I_q=\emptyset\quad\text{for}\quad p\neq q\quad\text{and}\quad \bigcup_p I_p=I\,.$$

Using the monotonicity of the logarithm we get

$$\begin{aligned} h_\Omega(\mathscr{E}) &= -\sum_p\sum_{n\in I_p}(\Omega\,|\,E_n\Omega)\log(\Omega\,|\,E_n\Omega)\\ &\geqq -\sum_p\sum_{n\in I_p}(\Omega\,|\,E_n\Omega)\log(\Omega\,|\,F_p\Omega)\\ &= -\sum_p(\Omega\,|\,F_p\Omega)\log(\Omega\,|\,F_p\Omega)=h_\Omega(\mathscr{F})\,. \end{aligned}$$

Lemma 2. *Suppose that $\mathfrak{Z}$ is an abelian von Neumann algebra on a Hilbert space $\mathscr{H}$ which contains no minimal projections. Let Ω be a separating vector for $\mathfrak{Z}$, then:*

a) *for any $\varepsilon>0$ and any projection E of $\mathfrak{Z}$, there exists a projection F in $\mathfrak{Z}$, such that $\|F\Omega\|<\varepsilon$ and $F\leqq E$.*

b) *for any projection $E\in\mathfrak{Z}$ the set*

$$\chi_E=\{\|F\Omega\|\,,\ F\ \textit{projection of}\ \mathfrak{Z},\ F\leqq E\}$$

is dense in the interval $[0,\|E\Omega\|\}$.

Proof. a) Take any $\varepsilon>0$ and E projection in $\mathfrak{Z}$. Suppose there exists a sequence $(H_n)_{n=1,2,\ldots}$ of projections in $\mathfrak{Z}$ such that

$$E\geqq H_1\geqq H_2\geqq\cdots\geqq H_n\geqq\cdots$$
$$\|H_n\Omega\|\geqq\varepsilon\quad\text{for all}\quad n\,.$$

Then $\inf\limits_n H_n\equiv H$ is a projection in $\mathfrak{Z}$ such that $\|H\Omega\|\geqq\varepsilon$([2], (App. 2)). Hence the set of projections $\{G\in\mathfrak{Z}\,|\,G\leqq E,\ \|G\Omega\|\geqq\varepsilon\}$ satisfies the conditions of the lemma of Zorn. Let G_m be the minimal element of this set. It cannot be minimal in $\mathfrak{Z}$, hence there exists a non trivial projection $F\in\mathfrak{Z}$, $F<G_m$ such that $\|F\Omega\|<\varepsilon$.

b) Let E be any projection in $\mathfrak{Z}$; $0,\|E\Omega\|\in\chi_E$. It is sufficient to prove that for any pair $\alpha,\beta:\alpha,\beta\in\chi_E$, $\alpha<\beta$, there exists a $\gamma\in\chi_E$ such that $\alpha<\gamma<\beta$.

Let E_α, $E_\beta\in\mathfrak{Z}$ be such that

$$E_\alpha\leqq E\,,\qquad E_\beta\leqq E\,,\qquad \|E_\alpha\Omega\|=\alpha\,,\qquad \|E_\beta\Omega\|=\beta\,.$$

$\mathfrak{Z}$ being abelian, $E_\beta(1-E_\alpha)$ is a projection majorized by E; $E_\beta(1-E_\alpha)\neq 0$ because otherwise $E_\beta\leqq E_\alpha$ and $\beta\leqq\alpha$.

By a) there exists a projection $F \in \mathfrak{Z}$ such that

$$F < E_\beta(1 - E_\alpha)$$
$$\|F\Omega\|^2 < \beta^2 - \alpha^2$$

and since Ω is separating for $\mathfrak{Z}$:

$$\alpha^2 < \alpha^2 + \|F\Omega\|^2 \leqq \|(E_\alpha + F)\,\Omega\|^2 < \beta^2 .$$

The projection $E_\alpha + F < E$ and the lemma follows. Q.E.D.

Lemma 3. *Let $\mathfrak{Z}$ be an abelian von Neumann algebra on $\mathscr{H}$; Ω a unit vector of $\mathscr{H}$ separating for $\mathfrak{Z}$. Suppose that all elements $\mathscr{E} \in \mathscr{C}$ contain at least one projection E_n which is not minimal, then*

a) *$\mathfrak{Z}$ contains a projection E such that the*

$$\chi_E = \{\|F\Omega\|,\ F \text{ projection in } \mathfrak{Z}, F \leqq E\}$$

is dense in $[0, \|E\Omega\|]$.

b) *There exists an $\alpha \in R$, $0 < \alpha \leqq 1$, and for any integer $n > 1$ a sequence $(F_p)_{p=1,\ldots,n}$ of n pairwise orthogonal projections in $\mathfrak{Z}$, such that*

$$\frac{\alpha}{2n} < \|F_p\Omega\|^2 < \frac{\alpha}{n}, \qquad p = 1, 2, \ldots, n .$$

c) $s_\Omega(\mathfrak{Z}) = \infty$.

Proof. a) Let $(E_\beta)_{\beta \in I}$ be the set of all minimal projections of $\mathfrak{Z}$, then $E = 1 - \sum_{\beta \in I} E_\beta$ is a non trivial projection of $\mathfrak{Z}$ which majorizes no minimal projections of $\mathfrak{Z}$. Let $\mathscr{H}_E$ be the range of E, and $\mathfrak{Z}_E$ the reduced von Neumann algebra of $\mathfrak{Z}$ on $\mathscr{H}_E$; $\mathfrak{Z}_E$ is abelian and contains no minimal projections; the vector $E\Omega = \Omega_E$ is separating for $\mathfrak{Z}_E$. Applying Lemma 2, the set

$$\{\|F\Omega_E\| \,|\, F \text{ projection of } \mathfrak{Z}_E, F \leqq 1\}$$

is dense in the interval $[0, \|\Omega_E\|]$.

By canonical imbedding of $\mathfrak{Z}_E$ into $\mathfrak{Z}$ one obtains the desired result.

b) Take E as in a), $\alpha = \|E\Omega\|^2$ and choose any integer n. From a) a projection $G_1 < E$ exists such that

$$\frac{\alpha}{2n} < \|G_1\Omega\|^2 < \frac{\alpha}{n}$$

since

$$\|(E - G_1)\Omega\|^2 > \alpha - \frac{\alpha}{n} > 0 .$$

Furthermore $E-G_1$ majorizes no minimal projections of $\mathfrak{Z}$. Applying Lemma 2, the set

$$\{\|F\Omega\|, F \text{ projection in } \mathfrak{Z},\ F \leqq E-G_1\}$$

is dense in the interval $[0, \|(E-G_1)\Omega\|]$. Hence there exists a projection $G_2 \leqq E-G_1$ in $\mathfrak{Z}$ such that

$$0 < \frac{\alpha}{2n} < \|G_2\Omega\|^2 < \frac{\alpha}{n}.$$

Analogously we construct projections G_p such that

$$G_p \leqq E - \sum_{k=1}^{p-1} G_k$$

$$0 < \frac{\alpha}{2n} < \|G_p\Omega\|^2 < \frac{\alpha}{n}.$$

One obtains:

$$\left\|\left(E - \sum_{q=1}^{p} G_p\right)\Omega\right\|^2 > \alpha\left(1 - \frac{p}{n}\right) \geqq \frac{\alpha}{2n}$$

for $p \leqq n - \frac{1}{2}$. The projection G_{n-1} satisfies the right inequality and we may construct G_n. This proves b).

c) Take any integer $n \geqq 3$, $n \in N$. By b) there exists a sequence $\mathscr{F} = (F_p)_{p=1,\ldots,n}$ of pairwise orthogonal projections satisfying

$$\frac{\alpha}{2n} \leqq \|F_p\Omega\|^2 \leqq \frac{\alpha}{n}$$

for some fixed $\alpha : 0 < \alpha \leqq 1$.

For $n \geqq 3$:

$$\|F_p\Omega\|^2 \leqq \frac{\alpha}{3} < \frac{1}{e} \qquad (p = 1, \ldots, n).$$

Using the monotonicity of the function $x \to -x\log x$ in the interval $(0, 1/e]$:

$$-(\Omega \mid F_p\Omega)\log(\Omega \mid F_p\Omega) \geqq -\frac{\alpha}{2n}\log\frac{\alpha}{2n}$$

for $p = 1, \ldots, n$. Hence

$$h_\Omega(\mathscr{F}) \geqq -\frac{\alpha}{2}\log\frac{\alpha}{2n}$$

and

$$\sup_{\mathscr{F} \in \mathscr{C}} h_\Omega(\mathscr{F}) = s_\Omega(\mathfrak{Z}) = \infty. \quad \text{Q.E.D.}$$

Lemma 4. *Let $\mathfrak{Z}$ be an abelian von Neumann algebra on a Hilbert space $\mathscr{H}$, Ω a unit vector of $\mathscr{H}$ separating for $\mathfrak{Z}$. If there exists a sequence*

$\mathscr{E}=(E_n)_n \in \mathscr{C}(\mathfrak{Z})$ *such that all projections* E_n *are minimal in* $\mathfrak{Z}$, *then* $s_\Omega(\mathfrak{Z})=h_\Omega(\mathscr{E})$, *if not*, $s_\Omega(\mathfrak{Z})=\infty$.

Proof. Immediate from Lemma 1 and 3.

Definition 3. *The entropy* $S'(\omega)$ *of a state* ω *on* $\mathscr{A}$ *is given by*

$$S'(\omega)=\inf_{\mathfrak{Z}} s_{\Omega_\omega}(\mathfrak{Z})$$

where the infimum is taken over all maximal abelian von Neumann algebras $\mathfrak{Z}$ *of the commutant* $\Pi_\omega(\mathscr{A})'$.

Each sequence $\mathscr{E}=(E_n)_n$ of minimal projections in $\Pi_\omega(\mathscr{A})'$ such that $\sum\limits_n E_n=1$ generates a maximal abelian von Neumann algebra $\mathfrak{Z}$ of $\Pi_\omega(\mathscr{A})'$ and a decomposition of ω in pure states: let $\Omega_n=\dfrac{E_n\Omega}{\|E_n\Omega\|}$ then

$$\omega=\sum_n(\Omega\,|\,E_n\Omega)\,\omega_{\Omega_n}\,.$$

If there exists such a sequence then the state ω is normal and from the definitions $S'(\omega)\geqq S(\omega)$. If there exists no such a sequence then by Lemma 4: $S'(\omega)=\infty$. Hence in general

$$S'(\omega)\geqq S(\omega)\,. \tag{$\star$}$$

Lemma 5. *Let* ω *be a normal factor state on* $\mathscr{A}$, *then there exists a countable sequence* $\mathscr{E}=(E_n)_n$ *of minimal projections* $E_n\in\Pi_\omega(\mathscr{A})'$ *such that*

$$S(\omega)=h_{\Omega_\omega}(\mathscr{E})\,.$$

In particular, $S(\omega)=S'(\omega)$.

Proof. By Proposition 1: $S(\omega)=-\operatorname{Tr}_{\mathscr{H}_0}\varrho\log\varrho$ (with the obvious notations). Let $\varrho=\sum\limits_n\lambda_n E_{\psi_n}$ be the spectral decomposition of ϱ, $(\psi_n)_n$ is an orthonormal basis of $\mathscr{H}_0$; then $\Pi_\omega=\bigoplus\limits_n\Pi_0\ \mathscr{H}_\omega=\oplus\mathscr{H}_0, \Omega_\omega=\sum\limits_n\sqrt{\lambda_n}\psi_n$ is the GNS-triplet induced by ω. Let E_n be the projection on the n^{th}-term $\mathscr{H}_0$ of the direct sum $\mathscr{H}_\omega=\bigoplus\limits_n\mathscr{H}_0$, then $\mathscr{E}=(E_n)_n$ is a sequence of minimal projections in $\Pi_\omega(\mathscr{A})'$ such that $h_\Omega(\mathscr{E})=-\operatorname{Tr}_{\mathscr{H}_0}\varrho\log\varrho$ hence $S(\omega)=h_\Omega(\mathscr{E})$.
Q.E.D.

Proposition 2. *For any state* ω *on a* C^**-algebra* $\mathscr{A}$: $S(\omega)=S'(\omega)$.

Proof. If $S(\omega)=\infty$, the equality follows from ($\star$). It is sufficient to consider the case that ω is normal. In this case, there exists a unique decomposition $\omega=\sum\limits_p\lambda_p\omega_p$ of ω in disjoint factor states [3] 5.4.9, p. 109. Remark that $\Pi_\omega=\bigoplus\limits_p\Pi_p, \Omega_\omega=\sum\limits_p\sqrt{\lambda_p}\Omega_p$ where $F_p\Omega=\sqrt{\lambda_p}\Omega_p$, $\Pi_p=\Pi_\omega\,|\,F_p$,

F_p is a central projection of $\Pi_\omega(\mathscr{A})'$. For any sequence $\mathscr{E}=(E_n)_{n\in I}$ of minimal projections $E_n\in\Pi_\omega(\mathscr{A})'$, there exists a partition $(I_p)_p$ of the index set I such that

$\sum_{n\in I_p} E_n = F_p$. Then

$$\begin{aligned} h_\Omega(\mathscr{E}) &= -\sum_p \sum_{n\in I_p} (\Omega' E_n \Omega)\log(\Omega\,|\,E_n\Omega) \\ &= -\sum_p \sum_{n\in I_p} \lambda_p(\Omega_p\,|\,E_n\Omega_p)\log\lambda_p(\Omega_p\,|\,E_n\Omega_p) \\ &= -\sum_p \sum_{n\in I_p} \lambda_p(\Omega_p|E_n\Omega_p)\log\lambda_p - \sum_p\sum_{n\in I_p}\lambda_p(\Omega_p|E_n\Omega_p)\log(\Omega_p|E_n\Omega_p) \\ &= -\sum_p \lambda_p\log\lambda_p + \sum_p \lambda_p h_{\Omega_p}(\mathscr{E}_p) \end{aligned}$$

where $\mathscr{E}_p=(E_nF_p)_{n\in I_p}$. It is clear that any choice of $\mathscr{E}$ corresponds to a choice of sequences $\mathscr{E}_p$ and vice-versa. Hence

$$S'(\omega)=\sum_p \lambda_p(S'(\omega_p)-\log\lambda_p)\,.$$

By Lemma 5:

$$S'(\omega)=\sum_p \lambda_p(S(\omega_p)-\log\lambda_p)\,.$$

By Proposition B.1 (Appendix B):

$$S'(\omega)=S(\omega)\,. \quad \text{Q.E.D.}$$

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Appendix A

Here we prove Proposition 1 (i): i.e. the equivalence of our definition of entropy with the ordinary one for normal factor states.

Let ϱ be a positive trace-class operator on a Hilbert space $\mathscr{H}$, and $\mathscr{H}'$ an infinite separable Hilbert space, $\varrho=\sum_n \lambda_n E_{\varphi_n}$ the spectral decomposition of ϱ; $(\varphi_n)_n$ $((\psi_n)_n)$ an orthonormal basis of $\mathscr{H}(\mathscr{H}')$;

$$\Omega=\sum_n \lambda_n^{\frac{1}{2}}\,\varphi_n\otimes\psi_n\in\mathscr{H}\otimes\mathscr{H}'\,.$$

Then for all bounded operators A on $\mathscr{H}$, denoted by $\mathscr{B}(\mathscr{H})$

$$\operatorname{Tr}\varrho A=(\Omega\,|\,A\otimes 1\,\Omega)\,. \tag{A.1}$$

Define the projection E_n on $\mathscr{H}\otimes\mathscr{H}'$ by

$$E_n=1\otimes E_{\psi_n}\,. \tag{A.2}$$

The E_n are minimal in $1 \otimes \mathscr{B}(\mathscr{H}')$ and $\sum_n E_n = 1$, $(\Omega | E_n \Omega) = \lambda_n$, and hence

$$- \operatorname{Tr} \varrho \log \varrho = - \sum_n (\Omega | E_n \Omega) \log(\Omega | E_n \Omega) . \tag{A.3}$$

Let $(F_p)_p$ be any sequence of pairwise orthogonal minimal projections in $1 \otimes \mathscr{B}(\mathscr{H}')$ such that $\sum_p F_p = 1$. Then F_p is of the form $F_p = 1 \otimes H_p$ where H_p is minimal projection on $\mathscr{H}'$; let $(\chi_p)_p$ be the orthonormal basis corresponding to $(H_p)_p$ then

$$(\Omega | F_p \Omega) = \sum_n \lambda_n (\psi_n | H_p \psi_n) = \sum_n \lambda_n |(\psi_n | \chi_p)|^2 .$$

By the convexity of the function $-x \log x$ for $x > 0$:

$$- \sum_p (\Omega | F_p \Omega) \log(\Omega | F_p \Omega) \geqq - \sum_n \lambda_n \log \lambda_n = - \operatorname{Tr} \varrho \log \varrho . \tag{A.4}$$

Now we prove a proposition, which seems to be known:

Proposition A.1. *If ϱ_1 and ϱ_2 are positive trace-class operators on a Hilbert space $\mathscr{H}$, then:*

$$- \operatorname{Tr}(\varrho_1 + \varrho_2) \log(\varrho_1 + \varrho_2) \leqq - \operatorname{Tr} \varrho_1 \log \varrho_1 - \operatorname{Tr} \varrho_2 \log \varrho_2 .$$

Proof. Take a Hilbert space $\mathscr{H}'$ as above, let Ω_1 and Ω_2 be the vectors of $\mathscr{H} \otimes \mathscr{H}'$ such that (see A.1)

$$\begin{aligned} \operatorname{Tr} \varrho_1 A &= (\Omega_1 | A \otimes \mathbf{1}\, \Omega_1) \\ \operatorname{Tr} \varrho_2 A &= (\Omega_2 | A \otimes \mathbf{1}\, \Omega_2) , \qquad A \in \mathscr{B}(\mathscr{H}) . \end{aligned}$$

Let $(E_n^1)_n$ and $(E_n^2)_n$ be the sequences of minimal projections in $\mathbf{1} \otimes \mathscr{B}(\mathscr{H}')$ such that (see A.3):

$$\begin{aligned} - \operatorname{Tr} \varrho_1 \log \varrho_1 &= - \sum_n (\Omega_1 | E_n^1 \Omega_1) \log(\Omega_1 | E_n^1 \Omega_1) \\ - \operatorname{Tr} \varrho_2 \log \varrho_2 &= - \sum_n (\Omega_2 | E_n^2 \Omega_2) \log(\Omega_2 | E_n^2 \Omega_2) \end{aligned}$$

Form $\mathscr{H}'' = \mathscr{H}' \oplus \mathscr{H}'$, then $\mathscr{H} \otimes \mathscr{H}'' = \mathscr{H} \otimes \mathscr{H}' \oplus \mathscr{H} \otimes \mathscr{H}'$ and form $\Omega = \Omega_1 \otimes \Omega_2 \in \mathscr{H} \otimes \mathscr{H}''$. Then

$$\operatorname{Tr}(\varrho_1 + \varrho_2) A = (\Omega | A \otimes \mathbf{1}\, \Omega) \quad A \in \mathscr{B}(\mathscr{H}) .$$

The projections E_n^1 and E_n^2 are minimal in $\mathbf{1} \otimes \mathscr{B}(\mathscr{H}')$ and $\sum_n (E_n^1 + E_n^2) = \mathbf{1}_{\mathscr{H}''}$. By (A.4)

$$\begin{aligned} - \sum_n (\Omega | E_n^1 \Omega) \log(\Omega | E_n^1 \Omega) - \sum_n (\Omega | E_n^2 \Omega) \log(\Omega | E_n^2 \Omega) \\ \geqq - \operatorname{Tr}(\varrho_1 + \varrho_2) \log(\varrho_1 + \varrho_2) . \end{aligned}$$

But $(\Omega|E_n^1\Omega)=(\Omega_1|E_n^1\Omega_1)$, $(\Omega|E_n^2\Omega)=(\Omega_2|E_n^2\Omega_2)$. Hence the result.

Q.E.D.

Proposition A.2. *Let ω be a normal factor state on $\mathcal{A}$, then*

$$S(\omega) = -\operatorname{Tr}_{\mathcal{H}} \varrho \log \varrho .$$

Proof. As ω is normal, $\omega=\sum_n \mu_n \omega_n$ where ω_n are pure states; ω_n induces the GNS-triplet $(\Pi_n, \mathcal{H}_{n.}, \Omega_n)$; as ω is a factor state, all Π_n are equivalent, $\Pi \equiv \Pi_n$ for all n. Let $(E_n)_n$ be the set of one dimensional projections on $\mathcal{H}$ such that

$$\omega_n(A) = \operatorname{Tr}_{\mathcal{H}} E_n \Pi(A), \qquad A \in \mathcal{A}$$

then

$$\omega(A) = \operatorname{Tr} \varrho \Pi(A)$$

where $\varrho=\sum_n \mu_n E_n$ is the unique density matrix induced by ω. From Proposition A.1:

$$-\operatorname{Tr}\varrho \log \varrho \leqq -\operatorname{Tr}(\mu_n E_n)\log(\mu_n E_n) = -\Sigma \mu_n \log \mu_n .$$

This is true for any decomposition of ω, hence:

$$-\operatorname{Tr}\varrho \log\varrho \leqq S(\omega) .$$

To prove the contrary inequality, let $\varrho=\sum_n \lambda_n E_{\psi_n}$ be the spectral decomposition of the density matrix ϱ, then also $\omega=\sum_n \lambda_n \omega_{\psi_n}$ and $S(\omega) \leqq -\sum_n \lambda_n \log \lambda_n = -\operatorname{Tr}\varrho \log\varrho$, hence the result. Q.E.D.

Appendix B

We prove Proposition 1 ii), iii), i.e. we prove that our generalized definition of entropy satisfies the same kind of inequalities as the ordinary definition of entropy for a factor state.

Proposition B.1. *Let $(\omega_n)_n$ be a countable sequence of normal states on $\mathcal{A}$, $\omega=\sum_n \lambda_n\omega_n$ a convex combination of the $\omega_n \left(0 \leqq \lambda_n \leqq 1, \sum_n \lambda_n = 1\right)$ then*

$$S(\omega) \leqq \sum_n \lambda_n [S(\omega_n) - \log \lambda_n] .$$

The equality sign holds if the states ω_n induce factor representations two by two disjoint.

Proof. Let $\omega_n = \sum_p \lambda_p^n \omega_p^n$ be a decomposition of ω_n in pure states, then $\omega = \sum_n \sum_p \lambda_n \lambda_p^n \omega_p^n$ is a decomposition of ω in pure states, hence ω is normal and

$$S(\omega) \leqq -\sum_{n,p} \lambda_n \lambda_p^n \log \lambda_n \lambda_p^n = -\sum_n \lambda_n \left(\sum_p \lambda_p^n \log \lambda_p^n \right) - \sum_n \lambda_n \log \lambda_n .$$

Since the decomposition of the ω_n is arbitrary

$$S(\omega) \leqq \sum_n \lambda_n [S(\omega_n) - \log \lambda_n] .$$

If the states ω_n induce factor representations two by two disjoint, then $\omega = \sum_n \lambda_n \omega_n$ is a unique decomposition in this sense. It follows that all decompositions of ω in pure states are obtained by all decompositions of the ω_n in pure states, and the equality holds. Q.E.D.

Proposition B.2. *Let* ω_1 *and* ω_2 *be normal states, and* $\omega = \lambda \omega_1 + (1-\lambda)\omega_2$ *where* $0 \leqq \lambda \leqq 1$, *then*

$$\lambda S(\omega_1) + (1-\lambda) S(\omega_2) \leqq S(\omega) .$$

Proof. Let $\omega_i = \sum_{p \in I_i} \lambda_p^i \omega_p^i$, $i = 1, 2$, be the decomposition of ω_1 and ω_2 into disjoint factor states;

K is the set of indices $p \in I_1$ such that the representations $\Pi_{\omega_p^1}$ are disjoint from all subrepresentations of ω_2;

L is the set of indices $p \in I_1$ such that $\Pi_{\omega_p^1}$ is quasi-equivalent with some subrepresentation $\Pi_{\omega_{n_p}^2}$ induced by ω_2;

M is the set of indices $p \in I_2$ such that $\Pi_{\omega_p^2}$ is disjoint from all subrepresentations induced by ω_2.

Then

$$\omega = \sum_{p \in K} \lambda \lambda_p^1 \omega_p^1 + \sum_{p \in M} (1-\lambda) \lambda_p^2 \omega_p^2 + \sum_{p \in L} (\lambda \lambda_p^1 + (1-\lambda) \lambda_{n_p}^2)\, \omega_{p,n_p}$$

where

$$\omega_{p,n} = \frac{1}{\lambda \lambda_p^1 + (1-\lambda)\lambda_{n_p}^2} [\lambda \lambda_p^1 \omega_p^1 + (1-\lambda)\lambda_{n_p}^2 \lambda \omega_{n_p}^2] , \qquad p \in L$$

is the unique decomposition of ω in disjoint factor states.

From Proposition B.1

$$\begin{aligned} S(\omega) = {} & \sum_{p \in K} \lambda \lambda_p^1 [S(\omega_p^1) - \log \lambda \lambda_p^1] + \sum_{p \in M} (1-\lambda) \lambda_p^2 [S(\omega_p^2) - \log (1-\lambda) \lambda_p^2] \\ & + \sum_{p \in L} (\lambda \lambda_p^1 + (1-\lambda)\lambda_{n_p}^2) [S(\omega_{p,n_p}) - \log(\lambda \lambda_p^1 + (1-\lambda)\lambda_{n_p}^2)] . \end{aligned}$$

As the states ω_p^1 and $\omega_{n_p}^2$ are quasi-equivalent factor states (see [4] p. 27)

$$S(\omega_{p,n_p}) \geqq \frac{1}{\lambda \lambda_p^1 + (1-\lambda)\lambda_{n_p}^2} [\lambda \lambda_p^1 S(\omega_p^1) + (1-\lambda)\lambda_{n_p}^2 S(\omega_{n_p}^2)]$$

and

$$S(\omega) \geqq \lambda \sum_{p \in I_1} \lambda_p^1 [S(\omega_p^1) - \log \lambda_p^1] + (1-\lambda) \sum_{p \in I_2} \lambda_p^2 [S(\omega_p^2) - \log \lambda_p^2] .$$

Using proposition B.1

$$S(\omega) \geqq \lambda S(\omega_1) + (1-\lambda) S(\omega_2) . \quad \text{Q.E.D.}$$

Proposition B.3. *$S(\omega) = 0$ if and only if ω is a pure state.*

Proof. Follows immediately from the definition of $S(\omega)$ and from Proposition B.1. Q.E.D.

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