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# Note on Trace Inequalities

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**Abstract.** Under conditions which are sufficiently general for physical applications the trace inequalities

$$\operatorname{tr} e^{-(A+B)} \leqq \operatorname{tr} e^{-A} e^{-B}$$

and

$$|\operatorname{tr} e^{-(A+iB)}| \leqq \operatorname{tr} e^{-A}$$

with $A$ and $B$ self adjoint are shown in a rigorous manner.

## Introduction

Recent investigations in statistical mechanics and in particular the treatment of systems of $N$ particles with gravitational and electromagnetic interactions have shown that trace inequalities serve as a prominent tool [1]. The expression for the partition function $z(\beta, \lambda) = \operatorname{tr} e^{-\beta H(\lambda)}$ may serve as an example. Unfortunately the trace operation in an infinitely dimensional space is somewhat more complicated than in its finite dimensional counterpart and there seems to be some confusion about the conditions under which the traces in the general case really exist [2].

While the operators for which a finite trace can be defined – the so-called trace class operators – are well known in the mathematical literature, little can be found concerning trace class operators (or more generally operators with finite $p$-norm) which are of the form $e^{-A}$. All that can be said about the operator $A$ is that it *must* be unbounded and that it *must* have a compact resolvent. It is exactly the unboundedness of the exponent which creates all the difficulties with domain questions etc. So our assumptions in the following may seem rather restrictive; they are, however, general enough for the purpose of applications to physical problems. The usual spectral theorems are not adequate because applications like the analyticity of the partition function lead to non-normal operators.

*Notation.* We denote by $\|A\|_p$ $1 \leqq p < \infty$ the familiar $p$-norm for compact operators on $\left(\text{i.e. } \|A\|_p = \left(\sum_i a_i^p\right)^{1/p}, a_i \text{ the eigenvalues of } (AA^\dagger)^{1/2}\right)$ and by $\|A\|_\infty$ the usual operator norm. $\mathscr{B}_p$, $1 \leqq p < \infty$, is the space of

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operators with finite $p$-norm and $\mathcal{B}_\infty$ the space of compact operators. With $(p)$-lim we denote the limit in the $p$-norm topology and with $s$-lim the limit in the strong operator-topology. We remember: $\|AB\|_r \leqq \|A\|_p \|B\|_q$, $1/r = 1/q + 1/q$, $1 \leqq p, q, r \leqq \infty$.

**Theorem 1.** *Let $A$ and $B$ be self adjoint, positive operators on a Hilbert space $\mathcal{H}$ where $B$ is bounded by $A$ with $A$-bound of $B$ smaller than* 1 *and such that $e^{-A} \in \mathcal{B}_p$, $e^{-B} \in \mathcal{B}_q$, $p^{-1} + q^{-1} = 1$. (Hence, if $\mathcal{H}$ is infinite dimensional $A$ and $B$ are unbounded operators with compact resolvent and $A+B$ is selfadjoint on the domain $D(A+B) = D(A)$.) Then*

$$\operatorname{tr} e^{-(A+B)} \leqq \operatorname{tr} e^{-A} e^{-B}.$$

*Proof.* According to a theorem of Weyl [3] generalized to an infinite dimensional (separable) Hilbert space[1] we have for an arbitrary bounded operator $X$ and non negative integer $m$

$$|\operatorname{tr} X^{2m}| \leqq \operatorname{tr}(X X^\dagger)^m \tag{$*$}$$

where the finiteness of the right hand side implies the existence of the left hand side. Substituting $X = YZ$ with $Y \geqq 0$, $Z \geqq 0$ and $m = 2^n$, $n \geqq 0$, we obtain

$$0 \leqq \operatorname{tr}(YZ)^{2^{n+1}} \leqq \operatorname{tr}(Y^2 Z^2)^{2^n} \leqq \cdots \leqq \operatorname{tr} Y^{2^{n+1}} Z^{2^{n+1}}.$$

Thus for $Y = e^{-A/2^n}$ and $Z = e^{-B/2^n}$ we have

$$\operatorname{tr}(e^{-A/2^n} e^{-B/2^n})^{2^n} \leqq \operatorname{tr} e^{-A} e^{-B}.$$

Due to our assumptions, $e^{-A} e^{-B} \in \mathcal{B}_1$; hence the trace on the left-hand side exists. The left hand side can be rewritten as

$$\operatorname{tr}(e^{-A/2^n} e^{-B/2^n})^{2^n} = \operatorname{tr}[(e^{-A/2^{n+1}} e^{-B/2^{n+1}})(e^{-A/2^{n+1}} e^{-B/2^{n+1}})^\dagger]^{2^n}$$

where the operator on the right hand side is

$$R_n \equiv e^{-A/2^{n+1}} (e^{-B/2} e^{-A/2^n})^{2^n - 1} e^{-B/2^n} e^{-A/2^{n+1}} \geqq 0.$$

The first, third and fourth factor in this expression converge towards the identity in the strong operator topology as $n \to \infty$, while for the second term we have

$$(e^{-B/2^n} e^{-A/2^n})^{2^n - 1} = \left( e^{-B \frac{2^n-1}{2^n} / 2^n - 1} e^{-A \frac{2^n-1}{2^n} / 2^n - 1} \right)^{2^n - 1} \to e^{-(A+B)}$$

according to a theorem of Trotter [4]. Hence $R_n \to e^{-(A+B)} \geqq 0$ and $R_n \geqq 0 \forall n$. Since the trace norm is weakly semi-continuous from below and coincides with the trace for positive operators we finally have

$$\operatorname{tr} e^{-(A+B)} \leqq \liminf \operatorname{tr} R_n = \liminf \operatorname{tr}(e^{-A/2^n} e^{-B/2^n})^{2^n} \leqq \operatorname{tr} e^{-A} e^{-B}.$$

[1] Relation $(*)$ holds for arbitrary $n \times n$ matrices $X_n$. It also holds for $X \in \mathcal{B}_{2m}: X = (2m) - \lim X_n$. $X^{2m} = (1) - \lim X_n^{2m}$ and $(X X^\dagger)^m = (1) - \lim (X_n X_n^\dagger)^m$ and since the trace operation is continuous in the trace norm topology relation $(*)$ also holds in the limit.

**Theorem 2.** *Let $A$ be a self adjoint operator with $e^{-A}$ trace class and $B$ a self adjoint operator with $A$-bound smaller than one. Then $e^{-(A+iB)} \in \mathscr{B}_1$ and $|\operatorname{tr} e^{-(A+iB)}| \leqq \|e^{-(A+iB)}\|_1 \leqq \operatorname{tr} e^{-A}$.*

*Proof.* $A+iB$ is closable because the adjoint $(A+iB)^\dagger \supseteq A-iB$ is densely defined. A familiar argument shows that the spectrum of $A+iB$ is restricted to the right half-plane such that $e^{-(A+iB)}$ can be defined: We consider the quadratic form $t[x] := \langle x|(A+iB)x\rangle$, $x \in D(A)$; Theorem (VI, 1.38) of [5] shows that $\langle x|Bx\rangle$ is bounded by $\langle x|Ax\rangle$ with bound less than one, Theorem (VI, 1.33) of [5] now implies that $t$ is sectorial (i.e. $\operatorname{im} t$ is contained in a sector symmetric to the real axis with half-angle less than $\pi/2$) and finally theorem (VI, 2.1) of [5] shows that $A+iB$ can be extended to a closed $m$-sectorial operator (i.e. its associated form is sectorial and its resolvent set contains a left half-plane). This extension is a generalization of the well known Friedrichs extension and is defined uniquely by the condition that the spectrum be contained in the same right half plane as the spectrum of $A$. The relation between the growth order of a semi-group and the spectrum $\sigma$ of its generator [6] (i.e. $\|e^{-tA}\| \leqq Ce^{\omega t}$, $\omega = \inf \operatorname{Re} a$, $a \in \sigma(A)$) shows that the extension of $A+iB$ is just the generator of the semi-group defined by the Trotter formula:

$$\|(e^{-tA/2^n} e^{-itB/2^n})^{2^n}\| \leqq \|e^{-tA/2^n}\|^{2^n} = \|e^{-tA}\| \leqq e^{-\omega t}$$

implies that all Trotter approximants and thus the strong limit must have a growth order smaller or equal than $\omega$. Applying again $(*)$ of the proof of Theorem 1 with $X = e^{-A/2^n} e^{-iB/2^n}$ and $m = 2^{n-1}$ we obtain

$$|\operatorname{tr}(e^{-A/2^n} e^{-iB/2^n})^{2^n}| \leqq \operatorname{tr} e^{-A}.$$

Furthermore

$$\|e^{-A/2^n} e^{-iB/2^n}\|_{2^n}^{2^n} = \|e^{-A/2^n}\|_{2^n}^{2^n} = \sum_i (e^{-a_i/2^n})^{2^n} = \operatorname{tr} e^{-A} \quad \forall n.$$

(we denote the eigenvalues of $A$ by $a_i$) because $e^{-iB/2^n}$ is unitary. Therefore

$$|\operatorname{tr}(e^{-A/2^n} e^{-iB/2^n})^{2^n}| \leqq \|(e^{-A/2^n} e^{-iB/2^n})^{2^n}\|_1 \leqq \|e^{-A/2^n} e^{-iB/2^n}\|_{2^n}^{2^n} = \operatorname{tr} e^{-A} \quad \forall n.$$

Trotters formula together with the lower semi-continuity[2] of the trace norm finally yields

$$|\operatorname{tr} e^{-(A+iB)}| \leqq \|e^{-(A+iB)}\|_1 \leqq \liminf \|(e^{-A/2^n} e^{-A/2^n} e^{-iB/2^n})^{2^n}\|_1 \leqq \operatorname{tr} e^{-A}.$$

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[2] $\|A\|_1 = \sup \sum_n |\langle \phi_n | A | \psi_n \rangle|$ where the sup extends over all orthonormal systems $\{\phi_n\}$ and $\{\psi_n\}$, [7].

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*Note.* After completion of this work our attention was drawn to the manuscript by Ruskai (Commun. math. Phys. **26**, 280–289 (1972)) where Theorem 1 is proven in a more general form for the trace on any semifinite von Neumann-algebra.

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