

Coherent States for Arbitrary Lie Group

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Abstract. The concept of coherent states originally closely related to the nilpotent group of Weyl is generalized to arbitrary Lie group. For the simplest Lie groups the system of coherent states is constructed and its features are investigated.

1. Introduction

In a number of fields of quantum theory, and especially in quantum optics and radiophysics, it is convenient to use the system of so called coherent states [1–3].

These states are in close connection with the nilpotent group first considered by Weyl [4].

A question arises: are there exist analogous systems of states for other Lie groups?

The recent paper [5] generalizes the concept of coherent states to some Lie groups. However, the method proposed in this paper cannot be applied to all Lie groups and, in particular, it is inapplicable to compact groups. Besides, with this approach the set of coherent states is noninvariant relative to the action of the group representation operators.

The present paper proposes another method to extend the concept of coherent state¹. This method can be applied to any Lie group and is consistent with the action of the group on the set of coherent states (see Section 2). Sections 3–5 of the paper deal with construction of the system of coherent states and with the investigation of its features for the simplest Lie groups.

¹ Note in this connection that although some states of such type were considered previously, the properties of the system of states as a whole do not appear to have been investigated (except for the Weyl group).

2. General Properties of Coherent States

Let G be an arbitrary Lie group and T be its irreducible unitary representation acting in the Hilbert space \mathcal{H} . A vector of this space is denoted by the symbol $|\psi\rangle$, the scalar product of the vectors $|\psi\rangle$ and $|\varphi\rangle$, linear on $|\psi\rangle$ and antilinear on $|\varphi\rangle$, by the symbol $\langle\varphi|\psi\rangle$, and the projection operator on the vector $|\psi\rangle$ by $|\psi\rangle\langle\psi|$.

Let $|\psi_0\rangle$ be some fixed vector in the space \mathcal{H} . Consider the set of vectors $\{|\psi_g\rangle\}$, where $|\psi_g\rangle = T(g)|\psi_0\rangle$ and g goes over all the group G . It is easy to see that two vectors $|\psi_{g_1}\rangle$ and $|\psi_{g_2}\rangle$ differ from one another only by a phase factor ($|\psi_{g_1}\rangle = e^{i\alpha}|\psi_{g_2}\rangle$, $|e^{i\alpha}| = 1$), or in other words determine the same state only if $T(g_2^{-1} \cdot g_1)|\psi_0\rangle = e^{i\alpha}|\psi_0\rangle$.

Let $H = \{h\}$ be the set of elements of the group G such that $T(h)|\psi_0\rangle = e^{i\alpha(h)}|\psi_0\rangle$. It is evident that H is a subgroup of the group G and we denote it as the stationary group of the state $|\psi_0\rangle$.

From these construction we see that the vectors $|\psi_g\rangle$ for all g which belong to one left coset G on H differ from one another only by a phase factor and that these vectors determine the same state.

Selecting in each coset x one representative $g(x)$ of the group G^2 we get the set of states $\{|\psi_{g(x)}\rangle\}$, or in an abridged form of writing $\{|x\rangle\}$ where $|x\rangle \in \mathcal{H}$, $x \in M = G/H$.

Now we may give the definition of generalized coherent states.

The system of coherent states of the type $(T, |\psi_0\rangle)$ (T is the representation of the group G acting in the some space \mathcal{H} and $|\psi_0\rangle$ is a fixed vector of this space) is called a set of states $\{|\psi_g\rangle\}$, $|\psi_g\rangle = T(g)|\psi_0\rangle$ where g runs over all the group G . Let H be a stationary subgroup of the state $|\psi_0\rangle$. Then the coherent state $|\psi_g\rangle$ is determined by the point $x = x(g)$ of the factor space G/H corresponding to the element g :

$$|\psi_g\rangle = e^{i\alpha}|x\rangle, |\psi_0\rangle = |0\rangle.$$

Let us consider some general properties of the coherent states. It is easy to see that $e^{i\alpha(h_2 h_1)} = e^{i\alpha(h_2)} e^{i\alpha(h_1)}$, i.e. $e^{i\alpha(h)}$ is a one-dimensional unitary representation of the group H .

If this representation is not identity, i.e. if $\alpha(h) \not\equiv 0$, then the factor group A of the group H on its commutant H'^3 is not trivial, i.e. it contains elements different from unity and the character of the group A determines completely the representation of the group H .

² The group G can be considered as a fiber bundle with the base $M = G/H$, and the fiber H . Then the choice of $g(x)$ is the choice of a certain cross section of this fiber bundle.

³ Remind that the commutant H' of the group H consists of the elements h' of the type of $h' = h_1 h_2 h_1^{-1} h_2^{-2}$. The commutant is an invariant subgroup of the group H and the factor group H/H' is the Abelian group.

If $\alpha(h) \equiv 0$, then H is the stationary subgroup of the vector $|\psi_0\rangle$ in the usual sense. In the first (second) case the representation T of the group G being restricted on the subgroup H has to contain the one-dimensional (identity) representation of the group H ⁴.

Note that if the subgroup H is connected then the vector $|\psi_0\rangle$ is the eigenvector of the infinitesimal operators of the representation of the subgroup itself.

Let us consider now the action of the operator $T(g)$ on the state $|\psi_0\rangle = |0\rangle$

$$T(g)|0\rangle = e^{i\alpha(g)}|x(g)\rangle. \quad (1)$$

Here the function $\alpha(g)$ is determined on the whole group G and at $g \in H$ it coincides with the previously considered function $\alpha(h)$ ⁵. Substituting in Eq. (1) g by gh we get

$$\alpha(gh) = \alpha(g) + \alpha(h). \quad (2)$$

Let us now act by the operator $T(g)$ on an arbitrary coherent state $|x\rangle$

$$T(g_1)|x\rangle = e^{-i\alpha(g)}T(g_1)T(g)|0\rangle = e^{i\beta(g_1, g)}|g_1 \cdot x\rangle. \quad (3)$$

Here $\beta(g_1, g) = \alpha(g_1 \cdot g) - \alpha(g)$; $x = x(g)$, $g_1 \cdot x = x_1 \in M$ and the element x_1 is determined by the action of the group G on the homogeneous space $M = G/H$. Note that due to (2) Eq. (3) is correct, i.e. the right-hand side of equality depends not on g but only on the cosets of $x(g)$: $\beta(g_1, g) = \beta(g_1, x)$.

It can be easily seen that the scalar product of two coherent states $|x_1\rangle = |x(g_1)\rangle$ and $|x_2\rangle = |x(g_2)\rangle$ is of the form

$$\langle x_1 | x_2 \rangle = e^{i[\alpha(g_1) - \alpha(g_2)]} \langle 0 | T(g_1^{-1}g_2) | 0 \rangle \quad (4)$$

and due to (2) it is independent of the choice of the representatives g_1 and g_2 . But due to unitarity of the representations $T(g) |\langle x_1 | x_2 \rangle| < 1$ at $x_1 \neq x_2$ and the following equalities take place

$$\langle x_1 | x_2 \rangle = \overline{\langle x_2 | x_1 \rangle}, \quad (5)$$

$$\langle g \cdot x_1 | g \cdot x_2 \rangle = e^{i[\beta(g, x_1) - \beta(g, x_2)]} \langle x_1 | x_2 \rangle. \quad (6)$$

Turning to the problem of completeness, note first of all that the completeness of the system follows immediately from the irreducibility

⁴ In many cases a useful information on possible representations $T(g)$ may be obtained from the reciprocity theorem of Frobenius. Stating that if T_x is the representation of the group G induced by the character $e^{i\alpha}$ of the group H , then the representation T has to be contained in the decomposition of representation T_x into irreducible representations [6].

⁵ Formula (1) defines the mapping $\pi: G \rightarrow \tilde{M}$ where \tilde{M} is the fiber bundle the base of which is $M = G/H$ and the fiber is a circle.

of the representation T . Let it exist the invariant measure dg on the group G . In many cases it induces the invariant measure dx on the homogeneous space $M = G/H$. Supposing the convergence conditions to be fulfilled let us consider the operator

$$B = \int dx |x\rangle \langle x|. \tag{7}$$

From definition of B , the invariance of measure dx and from formula (3) it immediately follows that

$$T(g) B T(g)^{-1} = B. \tag{8}$$

Thus B commutes with all the operators $T(g)$ and so due to irreducibility of the representation T , the operator B is multiple of the identity operator

$$\frac{1}{d} B = I. \tag{9}$$

To find the constant d let us calculate the average value of the operator B in the state $|y\rangle$ ($\langle y|y\rangle = 1$)

$$\langle y|B|y\rangle = \int |\langle y|x\rangle|^2 dx = \int |\langle 0|x\rangle|^2 dx = d. \tag{10}$$

Hence it is, in particular, seen that a necessary condition for the existence of the operator B is the convergence of the integral (10). In this case, which we call the case of the square-integrable system of coherent states, an important identity holds

$$\frac{1}{d} \int dx |x\rangle \langle x| = I. \tag{11}$$

Making use of this one may expand the arbitrary state in coherent states

$$|\psi\rangle = \frac{1}{d} \int dx c(x) |x\rangle, \quad c(x) = \langle x|\psi\rangle. \tag{12}$$

Here

$$\langle \psi|\psi\rangle = \frac{1}{d} \int dx |c(x)|^2 \tag{13}$$

and the function $c(x)$ is not arbitrary but it must satisfy the condition

$$c(x) = \frac{1}{d} \int \langle x|y\rangle c(y) dy. \tag{14}$$

Thus the kernel $K(x, y) = \frac{1}{d} \langle x|y\rangle$ is the reproducing one

$$K(x, z) = \int dy K(x, y) K(y, z) \tag{15}$$

and the function $\hat{f}(x) = \int K(x, y) f(y) dy$ satisfies Eq. (14) for an arbitrarily chosen function $f(x)$.

It can be also easily seen that between the coherent states there are "linear dependences". Indeed, from (12) it follows that

$$|x\rangle = \frac{1}{d} \int \langle y|x\rangle |y\rangle dy. \quad (16)$$

It means that the system of coherent states is overcomplete, i.e. it contains subsystems of coherent states which are complete systems.

The simplest subsets arise from consideration of the discrete subgroups of the group G . Let Γ be a discrete subgroup of the group G such that the volume V_Γ of the factor space M/Γ is finite. Let us consider the subsystem of the coherent states

$$\{|x_l\rangle\}, \quad x_l = x(\gamma_l), \quad \gamma_l \in \Gamma. \quad (17)$$

A question arises concerning the completeness of such subsystems. It would be interesting to know whether the following statement is valid: at $V_\Gamma > d$ the system of states $\{|x_l\rangle\}$ is not complete but at $V_\Gamma < d$ this system is complete and remains complete even after eliminating any finite number of states. The most interesting case is characterized by $V_\Gamma = d$ (if such a condition can be fulfilled) and it requires a separate, more detailed consideration. Note that for the simplest nilpotent group this problem has been solved in the paper [7].

Let us now illustrate the concept of generalized coherent states using concrete examples.

3. Case of the Special Nilpotent Group⁶

Let us first review some well known facts. The Lie algebra of this group is isomorphic to the Lie algebra produced by the annihilation operators a_1, \dots, a_N , Hermitian-conjugate creation operators a_1^+, \dots, a_N^+ and the identity operator I . The commutation relations between these operators are of the form

$$[a_i, a_j] = [a_i^+, a_j^+] = [a_i, I] = [a_j^+, I] = 0, \quad [a_i, a_j^+] = \delta_{ij} I. \quad (18)$$

A general element of the Lie algebra can be written as

$$tI + i(\bar{\alpha}a - \alpha a^+) \quad (19)$$

where t is a real number, $\alpha = (\alpha_1, \dots, \alpha_N)$, $a = (a_1, \dots, a_N)$ are N -dimensional vectors, α_i and $\bar{\alpha}_i$ are complex conjugate to each other. Here and in the following we use an abbreviated notation for the scalar product of

⁶ This group appears if one writes the Heisenberg commutation relations in the Weyl form [4]. Its properties are considered in detail in the paper [8].

two such vectors:

$$\bar{\alpha}a = \sum_{i=1}^N \bar{\alpha}_i a_i, \quad \alpha a^\dagger = \sum_{i=1}^N \alpha_i a_i^\dagger.$$

The Lie group W_N is obtained from the Lie algebra by means of the exponential mapping. Thus to the element of the algebra (19) corresponds the element g of the group which is denoted by (t, α) . Then the multiplication law in W_N is given by the formula

$$(s, \alpha)(t, \beta) = (s + t + \text{Im}(\alpha\bar{\beta}), \alpha + \beta). \tag{20}$$

The operators of the irreducible unitary representation of the group W_N are of the form

$$T(g) = T(t, \alpha) = e^{it} D(\alpha). \tag{21}$$

From (21) it follows in particular that to the set of elements $G_0 = \{g_k = (2\pi k, 0)\}$ (k is integer) corresponds the identity operator, i.e. the representation under consideration is not a faithful one.

Making use of the commutation relations (18) we can easily obtain the multiplication law of the operators

$$D(\alpha) D(\beta) = e^{i\text{Im}(\alpha\bar{\beta})} D(\alpha + \beta). \tag{22}$$

Let us now take some vector $|\psi_0\rangle$ in the representation space. We denote the stationary subgroup of this vector as H . Consider two different cases.

I. Let $|\psi_0\rangle$ be an arbitrary vector of the Hilbert space \mathcal{H} . In this case the subgroup H consists of the elements of type $(t, 0)$ and the factor space $M = W_N/H$ is the N -dimensional complex space \mathbf{C}^N . In correspondence with Section 2 of this paper, the system of generalized coherent states is the set of vectors where

$$|\alpha\rangle = D(\alpha) |\psi_0\rangle. \tag{23}$$

Note that the usual system of coherent states whose properties have been considered in detail for instance in the papers [1–3] corresponds to the choice in (23) of the so called “vacuum” vector $|0\rangle$ ⁷ as initial vector $|\psi_0\rangle$. However, it can be readily shown that the system of generalized coherent states possesses the same properties as the usual system of coherent states. In particular the main identity

$$\frac{1}{\pi^N} \int d^{2N} \alpha |\alpha\rangle \langle \alpha| = I, \quad d^{2N} \alpha = \prod_{i=1}^N d \text{Re} \alpha_i \cdot d \text{Im} \alpha_i \tag{24}$$

remains valid for it.

⁷ The vacuum vector $|0\rangle$ is determined by the equations $a_i|0\rangle = 0$.

II. Let us extend the Hilbert space \mathcal{H} up to the space of generalized functions $\mathcal{H}_{-\infty}$ ⁸ and try to find the eigenvector $|\theta_0\rangle \in \mathcal{H}_{-\infty}$ of the operators $T(h)$, where $h = (t, \alpha_n)$ is the element of the subgroup H , $\alpha_n = \sum_{j=1}^{2N} n_j \omega_j$, n_j are integer numbers. The set of vectors α_n form a lattice L in the space \mathbb{C}^N and we suppose the periods ω_j of this lattice to be really linearly independent. It can be easily seen that the commutant H' of the group H consists of the elements $h' = (t_n, 0)$ where $t_n = 2\pi \sum_{i,j=1}^{2N} B_{ij} n_i n_j$ and the $2N \times 2N$ matrix B has the form $B_{ij} = \frac{1}{\pi} \text{Im}(\omega_i \bar{\omega}_j)$. But according to Section 2 the operator $T(h')$ must be equal to the identity operator, and from this follows that the elements of the matrix B are integral numbers:

$$B_{ij} = \frac{1}{\pi} \text{Im}(\omega_i \bar{\omega}_j) \equiv 0 \pmod{1}. \tag{25}$$

We call such lattice L admissible⁹. The conditions $T(h) |\theta_0\rangle = e^{i\alpha(h)} |\theta_0\rangle$ are in this case equivalent to the equations

$$D(\omega_i) |\theta_0\rangle = e^{i\pi \varepsilon_i} |\theta_0\rangle. \tag{26}$$

The state $|\theta_0\rangle$ is determined by the real numbers ε_i , $i = 1, \dots, 2N$. Acting with the operator $D(\alpha)$ on $|\theta_0\rangle$ we get the system of generalized coherent states

$$|\theta_\alpha\rangle = D(\alpha) |\theta_0\rangle \tag{27}$$

where α runs over the complex torus $M = G/H$. It appears that the states $|\theta_\alpha\rangle$ under the choice of a definite realization of space \mathcal{H} coincide in the essence with the theta functions. Some properties of the theta functions in frame of such approach are considered in the papers [7, 8].

4. Case of the Simplest Compact Group-SU(2) Group

The $SU(2)$ group is the unitary matrix group of the second order with unity determinant. It is locally isomorphic to the $SO(3)$ group – the rotation group of three-dimensional space and it is the most investigated group among all the non-Abelian Lie groups¹⁰. Nevertheless, the coherent states for this group as a special system do not appear to have been considered so far.

⁸ The definition of space $\mathcal{H}_{-\infty}$ and the consideration of some of its properties is given in paper [8].

⁹ Note that the factor space $M = G/H$ is a complex torus and in the case of admissible lattice it can be considered as an Abelian variety.

¹⁰ The properties of this group are considered in detail e.g. in the books [9, 10].

Let us first review some well known facts. The T^j representation of this group is determined by the nonnegative number j , integer or half-integer. The basis vectors $|j, \mu\rangle$ of the space in which the representation acts are labelled by a number μ , that takes the $2j + 1$ integer (if j is integer) or half-integer (if j is half-integer) values from $-j$ up to j . The vectors $|j, \mu\rangle$ satisfy the condition

$$T^j(h) |j, \mu\rangle = e^{i\mu\varphi} |j, \mu\rangle, \quad h = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}. \quad (28)$$

Here h is the element of the rotation subgroup H around the axis x_3 . Note that vector $|j, \mu\rangle$ is the eigenvector of the infinitesimal operator J_3 of the representation T^j corresponding to the subgroup H

$$J_3 |j, \mu\rangle = \mu |j, \mu\rangle. \quad (29)$$

The factor space G/H is the two-dimensional sphere S^2 , the point of this sphere is determined by the unit vector \mathbf{n} , $\mathbf{n}^2 = 1$. Let $g(\mathbf{n})$ be the element of the group G which transforms the vector $\mathbf{n}_0 = (0, 0, 1)$ into the vector $\mathbf{n} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$. As $g(\mathbf{n})$ we may choose, for example, the element $g_\varphi^3 g_\theta^2$ where g_φ^3 corresponds to rotation around the axis x_3 by the angle φ and g_θ^2 to rotation around the axis x_2 by the angle θ . We thus come to the system of coherent states $\{|\mu, \mathbf{n}\rangle\}$:

$$|\mu, \mathbf{n}\rangle = T(g(\mathbf{n})) |\mu\rangle = T(g_\varphi^3) T(g_\theta^2) |\mu\rangle. \quad (30)$$

(Here we have to omit for simplicity index j to abbreviate the notations.)

The system of coherent states can be also determined up to the phase factor $e^{i\alpha(\mathbf{n})}$ using the equation

$$(n_i J_i) |\mu, \mathbf{n}\rangle = \mu |\mu, \mathbf{n}\rangle \quad (31)$$

where J_i is the infinitesimal operator of the representation which corresponds to a rotation around the axis x_i . Here μ should be considered as a fixed parameter and the vector \mathbf{n} as a variable quantity.

Let us consider the properties of this system. The scalar product of two coherent states is generally speaking non-zero and it equals

$$\langle \mu, \mathbf{n}' | \mu, \mathbf{n} \rangle = e^{i\phi(\mathbf{n}', \mathbf{n})} d_{\mu\mu}^j(\theta) = e^{i\phi} \left(\cos \frac{\theta}{2} \right)^{2|\mu|} P_{j-|\mu|}^{(0, 2|\mu|)}(\cos\theta) \quad (32)$$

where $\cos\theta = \mathbf{n}' \mathbf{n}$, $d_{\mu\nu}^j(\theta)$ are standard matrix elements of the $SU(2)$ group [9, 10] and $P_n^{(a, b)}$ are Jacobi polynomials. Hence we find that

$$d = \int |\langle \mu, \mathbf{n}' | \mu, \mathbf{n} \rangle|^2 d\mathbf{n} = \frac{4\pi}{2j + 1} \quad (33)$$

and correspondingly

$$\frac{2j+1}{4\pi} \int d\mathbf{n} |\mu, \mathbf{n}\rangle \langle \mu, \mathbf{n}| = I. \quad (34)$$

Especially simple is the system of coherent states for $\mu = j$. Then, for instance, we get from Eq. (32)

$$|\langle j, \mathbf{n}' | j, \mathbf{n} \rangle|^2 = \left(\frac{1 + \mathbf{n}' \cdot \mathbf{n}}{2} \right)^{2j}. \quad (35)$$

Let us also give the expression for the coherent states in the so-called z -representation [9, 10]. In this case the representation T acts in the space of polynomials of $2j$ degree and the operator of the representation is given by the formula

$$T^j(g) f(z) = (\beta z + \bar{\alpha})^{2j} f\left(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}}\right), \quad g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (36)$$

The basis vectors $|j, \mu\rangle$ in this representation are of the form

$$\langle z | j, \mu \rangle = \sqrt{\frac{(2j)!}{(j+\mu)!(j-\mu)!}} z^{j+\mu}. \quad (37)$$

In this case from (30) and (36) it immediately follows

$$\langle z | \mu, \mathbf{n} \rangle = \sqrt{\frac{(2j)!}{(j+\mu)!(j-\mu)!}} (\beta z + \bar{\alpha})^{j-\mu} (\alpha z - \bar{\beta})^{j+\mu} \quad (38)$$

where

$$\alpha = \cos \frac{\theta}{2} e^{i\varphi/2}, \quad \beta = \sin \frac{\theta}{2} e^{i\varphi/2}.$$

Let us map the sphere S^2 onto the plane of the complex variable ζ using the stereographic projection

$$\zeta = \operatorname{ctg} \frac{\theta}{2} e^{i\varphi}. \quad (39)$$

Expression (38) takes now the form

$$\langle z | \mu, \mathbf{n} \rangle = e^{i\phi} \langle z | \mu, \zeta \rangle \quad \text{where} \quad e^{i\phi} = (-1)^{j+\mu} e^{-i\mu\varphi} \quad (40)$$

$$\langle z | \mu, \zeta \rangle = \sqrt{\frac{(2j)!}{(j+\mu)!(j-\mu)!}} (1 + |\zeta|^2)^{-j} (z + \bar{\zeta})^{j-\mu} (1 - \zeta z)^{j+\mu}. \quad (41)$$

Acting on the function $|\mu, \zeta\rangle$ by the operator $T(g)$ we get $T(g)|\mu, \zeta\rangle = e^{i\phi}|\mu, g \cdot \zeta\rangle$, where

$$e^{i\phi} = \left(\frac{\bar{\beta}\zeta + \bar{\alpha}}{\beta\bar{\zeta} + \alpha} \right)^\mu, \quad g \cdot \zeta = \frac{\alpha\zeta - \beta}{\bar{\beta}\zeta + \bar{\alpha}}. \tag{42}$$

Let us now give the formula for the scalar product of coherent states in this representation

$$\langle j, \zeta' | j, \zeta \rangle = (1 + |\zeta'|^2)^{-j} (1 + |\zeta|^2)^{-j} (1 + \bar{\zeta}'\zeta)^{2j} \tag{43}$$

Note that just as the usual coherent states appear naturally in the problem of an oscillator which is under the action of an external time-dependent force [1, 2] the states $|\mu, \mathbf{n}\rangle$ appear naturally when one considers the problem of the spin motion in a time-dependent magnetic field. In this case the variation of the state over time is determined by the Schrödinger equation

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = -\mathbf{a}\mathbf{J}|\psi(t)\rangle \tag{44}$$

where $\mathbf{a} = \mu\mathbf{H}$, μ is the magnetic moment of the particle, $\mathbf{H}(t)$ is the magnetic field, $\mathbf{J} = (J_1, J_2, J_3)$, J_i is the operator of infinitesimally small rotations around the axis x_i .

It can be easily seen that if at initial time we have the coherent state $|\psi(0)\rangle = |\mathbf{n}_0\rangle$, then at any following time this state remains coherent, i.e.

$$|\psi(t)\rangle = e^{i\alpha(t)}|\mathbf{n}(t)\rangle \tag{45}$$

where the vector $\mathbf{n}(t)$ is determined by the classical equation of motion

$$\dot{\mathbf{n}}(t) = -[\mathbf{a}(t), \mathbf{n}(t)]. \tag{46}$$

The coherent states may be also used to describe the density matrix ϱ of a particle with spin¹¹. Namely, the density matrix ϱ is completely determined either by the function $P(\mathbf{n})$ or $Q(\mathbf{n})$, according to the formulae

$$\varrho = \int d\mathbf{n} P(\mathbf{n}) |\mathbf{n}\rangle \langle \mathbf{n}|, \tag{47}$$

$$Q(\mathbf{n}) = \langle \mathbf{n} | \varrho | \mathbf{n} \rangle. \tag{48}$$

Note one more useful identity

$$|\mathbf{n}\rangle \langle \mathbf{n}| = \frac{2j+1}{16\pi^2} \int c(\mathbf{n}, g^{-1}) T(g) dg, \tag{49}$$

$$c(\mathbf{n}, g) = \langle \mathbf{n} | T(g) | \mathbf{n} \rangle \tag{50}$$

which follows from the orthogonality of the matrix elements $T_{\mu\nu}^j(g)$.

¹¹ The description of the oscillator density matrix using the usual coherent states can be found in the papers [1–3].

Note that if in (47) $P(\mathbf{n})$ is expanded in a series of spherical functions

$$P(\mathbf{n}) = \sum_{l,m} C_{lm} Y_{lm}(\mathbf{n}) \quad (51)$$

one gets the expansion of the density matrix

$$\varrho = \sum_{l,m} C_{lm} \hat{P}_{lm} \quad (52)$$

in the operators

$$\hat{P}_{lm} = \int d\mathbf{n} Y_{lm}(\mathbf{n}) |\mathbf{n}\rangle \langle \mathbf{n}|. \quad (53)$$

Calculating the integral entering (53) we find

$$\langle v' | \hat{P}_{lm} | v \rangle = \frac{\sqrt{4\pi(2l+1)}}{2j+1} (j, v'; l, m | j, v) (j, \mu; l, 0 | j, \mu) \quad (54)$$

where $(j, v'; l, m | j, v)$ is the Clebsch-Gordan coefficient.

In conclusion note that the formulae obtained in this section carry over to the case arbitrary compact Lie group. In order to do this it is only necessary to replace the group H by the Cartan subgroup and to take into consideration that $2j+1$ is the dimension of the representation $T(g)$ and 4π is the volume of the factor space $M = G/H$.

5. Case of the Simplest Noncompact Group- $SU(1, 1)$ Group

The $SU(1, 1)$ group is the group of unimodular matrices that leave the form $|z_1|^2 - |z_2|^2$ invariant. The element g of this group has the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1. \quad (55)$$

The $SU(1, 1)$ group is isomorphic to the $Sp(2, R)$ group (to the group of real symplectic second order matrices) and it is locally isomorphic to the $SO(2, 1)$ group, the group of "rotations" of the three-dimensional pseudoeuclidian space. It has several series of unitary irreducible representations and, in particular, two discrete series T^+ and T^- . It is sufficient to consider only one of these, e.g. T^+ , because all the results are automatically carried over to the other case.

The representation of the series T^+ is characterized by the positive integer or half-integer numbers k . It may be realized in the space of the

function \mathcal{F}_k that are analytic in the unit disk $|\zeta| < 1$ and satisfy the condition

$$\|f\|^2 = \int d\mu_k(z) |f(z)|^2 < \infty \tag{56}$$

where

$$d\mu_k(z) = \frac{2k-1}{\pi} (1-|z|^2)^{2k-2} d^2z. \tag{57}$$

It is not difficult to see that if $f(z) = \sum_{n=0}^{\infty} c_n z^n$ then

$$\|f\|^2 = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(n+2k)} |c_n|^2. \tag{58}$$

The space \mathcal{F}_k becomes a Hilbert space if the scalar product of two vectors is defined in it according to the formula

$$\langle f|g \rangle = \int d\mu_k(z) \bar{f}(z) g(z). \tag{59}$$

Now it can be easily checked that the functions

$$|n\rangle = f_n(z) = \sqrt{\frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)}} z^n \tag{60}$$

form an orthonormal basis in the space \mathcal{F}_k ¹².

Let us define the action of the operators $T(g)$ in the space \mathcal{F}_k

$$T^k(g) f(z) = (\beta z + \bar{\alpha})^{-2k} f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right). \tag{61}$$

It can be shown [10–12] that the operators $T^k(g)$ determine the unitary irreducible representation of the $SU(1, 1)$ group.

Let us choose in \mathcal{F}_k the vector of the lowest weight $|0\rangle$ as the fixed vector $|\psi_0\rangle$ (the corresponding function $f_0(z) \equiv 1$). Acting on it by the operators $T^k(g)$ we get the system of states

$$|g\rangle = T(g) |0\rangle = (\beta z + \bar{\alpha})^{-2k}. \tag{62}$$

This expression can be easily transformed to the form

$$|g\rangle = e^{i\phi} |\zeta\rangle; \quad |\zeta\rangle = (1-|\zeta|^2)^k (1-\zeta z)^{-2k}, \quad |\zeta| < 1. \tag{63}$$

The set $\{|\zeta\rangle\}$ is just the system of coherent states.

¹² Note that if k tends to infinity and z to zero so that $kz = \text{const}$ then as the functions $f_n(z)$, and all other quantities go over into corresponding quantities for a special nilpotent group (see Section 3).

Note that the group of matrices of type $h = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$ is the stationary subgroup H of the vector $|0\rangle$ and the factor space G/H is the unit disk. We see that according to Section 2, the coherent state $|\zeta\rangle$ is completely determined by the point ζ of the factor-space G/H . Note that this space could be also realized as an upper sheet of the hyperboloid $n_0^2 - n_1^2 - n_2^2 = 1$.

Expanding the state $|\zeta\rangle$ in states $|n\rangle$ we get

$$|\zeta\rangle = (1 - |\zeta|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)}} \zeta^n |n\rangle. \quad (64)$$

Hence for the scalar product of two coherent states the following formula obtains

$$\langle \zeta' | \zeta \rangle = (1 - |\zeta'|^2)^k (1 - |\zeta|^2)^k (1 - \bar{\zeta}'\zeta)^{-2k}. \quad (65)$$

Correspondingly

$$d = \int d\mu(\zeta) |\langle 0 | \zeta \rangle|^2 = \frac{\pi}{2k-1} \quad (66)$$

and the condition of completeness takes the form

$$\frac{2k-1}{\pi} \int d\mu(\zeta) |\zeta\rangle \langle \zeta| = I. \quad (67)$$

Here $d\mu(\zeta) = \frac{d^2\zeta}{(1-|\zeta|^2)^2}$ is the invariant measure on the disk $|\zeta| < 1$.

Let us now consider an arbitrary normalized vector $|\psi\rangle$ belonging to the Hilbert space \mathcal{H} . To this vector the function $\langle \zeta | \psi \rangle$ may be taken into correspondence and if $|\psi\rangle = \sum c_n |n\rangle$, then

$$\langle \zeta | \psi \rangle = (1 - |\zeta|^2)^k \psi(\bar{\zeta}) \quad (68)$$

where

$$\psi(\zeta) = \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)}} c_n \zeta^n. \quad (69)$$

Note that in this case

$$\|\psi\|^2 = \langle \psi | \psi \rangle = \int d\mu_k(\zeta) |\psi(\zeta)|^2, \quad (70)$$

i.e. $\psi(\zeta) \in \mathcal{F}_k$. The formula (69) establishes the isomorphism between the spaces \mathcal{H} and \mathcal{F}_k .

Moreover, from the inequality $|\langle \zeta | \psi \rangle|^2 \leq \|\psi\|^2$ follows a restriction on the growth of function $\psi(\zeta)$

$$|\psi(\zeta)|^2 \leq (1 - |\zeta|^2)^{-2k} \|\psi\|^2. \quad (71)$$

From (71) we immediately obtain that strong convergence of the sequence $|\psi_n\rangle$ implies pointwise convergence $\psi_n(\zeta)$ uniform on any compact subset of the plane ζ .

A characteristic feature of spaces of the type \mathcal{F}_k are the so called "reproducing kernels" which play the role of usual δ -functions. Such kernel can be found in the usual way. Namely

$$\delta_{z'}(z) = \sum_{n=0}^{\infty} \overline{f_n(z')} f_n(z) = (1 - \bar{z}'z)^{-2k}. \quad (72)$$

At fixed z' $\delta_{z'}(z)$ is a function of z and its norm is equal to

$$\|\delta_{z'}\|^2 = (1 - |z'|^2)^{-2k}. \quad (73)$$

It can be easily checked by direct calculations that $\delta_{z'}(z)$ is the analog of δ -function, i.e. the equality

$$\langle \delta_z | f \rangle = \int d\mu(z') \bar{\delta}_z(z') f(z') \equiv f(z), \quad (f(z) \in \mathcal{F}_k) \quad (74)$$

holds.

Up to now we considered the representations of the $SU(1, 1)$ group. One can however consider also the representations of its universal covering group, namely the group $\overline{SU(1, 1)}$ ¹³ which as is well known, covers the group $SU(1, 1)$ an infinite number of times. It can be easily seen that this results only in replacing the number k , that was previously restricted to non-negative integer or half-integer values, by an arbitrary non-negative number.

Note, that analogous results obtain also for other semi-simple Lie groups, having a discrete series of representations.

In this paper we have briefly considered the simplest systems of coherent states. It would be interesting to consider other systems of such states, in particular, the systems related to the continuous spectrum.

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¹³ Note that the group $\overline{SU(1, 1)}$ is a dynamical symmetry group in some model many body problems [13].

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