Classical Schwinger Terms

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Abstract. Schwinger terms must be present in equal time ground state Poisson brackets among currents unless the latter are stationary in that state.

A general result in quantum mechanics is the necessary non-vanishing of the equal time commutator of any operator with its time derivative. Only the existence of a unique lowest energy state and the positivity of the Hilbert space metric enter into the derivation which uses the equation of motion

$$\dot{A} \equiv \frac{\partial}{\partial t} A(t) = -i[A(t), H],$$ (1)

to evaluate the vacuum expectation value [1]

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\begin{aligned}
\langle 0|[A(t), \dot{A}]|0\rangle &= -i\langle 0|[A(t), [A(t), H]]|0\rangle \\
&= 2i\langle 0|A(t) H A(t)|0\rangle \\
&= 2i \sum_n E_n |\langle n|A(0)|0\rangle|^2.
\end{aligned}
$$ (2)

The final summation is positive semi-definite since E_n is; it can vanish only if the vacuum is an eigenstate of A. An immediate consequence in local field theory is that the equal time commutator $[j^0(r), j^k(r')]$ of a conserved current $j^\mu(x)$ cannot vanish [1, 2], since its divergence

$$\langle 0|[j^0(r), \ V'_k j^k(r')]|0\rangle = -\langle 0|[j^0(r), \ \partial_0 j^0(r')]|0\rangle$$ (3)

must be non zero (and a non vanishing local current cannot annihilate the vacuum) [3]. A more precise result is that, assuming the equal time

[1] The zero of energy is chosen so that the vacuum lowest energy state has zero energy. The vacuum state is denoted by $|0\rangle$.

commutator exists, then in the general expression

$$-i \langle 0 | [j^0(r), j(0)] | 0 \rangle = \sum_{n=0} c_n \, V(- V^2)^n \delta(r) , \qquad (4)$$

the vanishing of c_0 implies [4] that of the operator j^0, irrespective of conservation, provided only that relativistic invariance holds.

In this note we show that the necessity of Schwinger terms is not a purely quantum phenomenon, but has a somewhat weaker classical counterpart. That Schwinger terms can exist should not be surprising: In the algebra of fields [5] for example, the current is linear in canonical variables. The commutator of Eq. (4) is then just proportional to a canonical commutation relation, and so should survive the transition[2] from commutator to Poisson bracket. However, because there is no classical analogue to vacuum fluctuations it is possible for a local field to "annihilate" the vacuum, hence it is possible for the commutator to vanish. Furthermore, we will see that it is not necessary for c_0 to be non vanishing even if j^μ is not extremal at vacuum; it is only necessary that some c_n be non zero (all must be ≥ 0).

A stable classical ground state (vacuum state) is characterized in terms of the Hamiltonian, by

$$\frac{\partial H}{\partial \chi^a}\bigg|_0 = 0 \qquad (5)$$

and

$$\frac{\delta^2 H}{\delta \chi^a \, \delta \chi^b}\bigg|_0 \equiv \mathscr{H}_{ab} > 0 \qquad (6)$$

where χ^a represents both the canonical variables and their conjugate momenta, $q(x)^n \equiv \chi(x)^{2n-1}$, $p(x)^n \equiv \chi(x)^{2n}$. The coordinate dependence will, in general, be suppressed. The subscript zero on any quantity indicates its value in the ground state and we translate the fields χ so that $\chi = 0$ there. The matrix $(\mathscr{H})_{ab} = \mathscr{H}_{ab}$ is positive definite by the ground state stability assumption.

At vacuum, the equal time Poisson bracket

$$[A, B] \equiv \sum_n \frac{\partial A}{\partial q^n} \frac{\partial B}{\partial p^n} - \frac{\partial A}{\partial p^n} \frac{\partial B}{\partial q^n} \equiv \frac{\partial A}{\partial \chi} \eta \frac{\partial B}{\partial \chi} = - \frac{\delta B}{\delta \chi} \eta \frac{\delta A}{\delta \chi} \quad (7)$$

combined with the role of the Hamiltonian as the generator of time translations

$$\dot{A} \equiv \frac{\partial}{\partial t} A = [A, H] \qquad (8)$$

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[2] We do not consider spinor fields.

yields the result, at vacuum,

$$[A, \dot{A}]|_0 = [A, [A, H]]|_0 = - \left[A, \frac{\delta H}{\delta \chi} \eta \frac{\delta A}{\delta \chi}\right]\bigg|_0$$

$$= \left(\eta \frac{\delta A}{\delta \chi}\right)_0^T \mathscr{H} \left(\eta \frac{\delta A}{\delta \chi}\right)_0 \geq 0 \tag{9}$$

and the equality can hold only if $\dfrac{\delta A}{\delta \chi}\bigg)_0 = 0$ [3]. But this is just the condi-
tion that A be extremal at vacuum, the classical analogue to the vacuum
being an eigenstate of A [4]. However, there is not classical counterpart
to the necessary non vanishing of a local field, e.g., $j^\mu(x)$, applied to the
vacuum; the Schwinger terms exist only if $(\delta j^\mu / \delta \chi) \neq 0$. In particular,
the ordinary bilinear boson currents, whose χ derivatives vanish at
vacuum, have vanishing Schwinger terms.

We will now derive from Lorentz convariance, irrespective of con-
servation, the more specific prediction, Eq. (4), that the equal time Poisson
bracket of j^0 with j^k must have a gradient of a delta function term. First,
we will establish that $[j^0(r), j(r')]_0$ is proportional to $\mathscr{H}$, then that $\mathscr{H}$
is proportional to $-\tilde{c}: \nabla \nabla \delta(r - r')$. The latter fact is plausible, since a
general Lorentz covariant Hamiltonian is expected to involve terms
$(\nabla q)^2$ for small excitations around vacuum. [Alternatively, such terms
are expected in order that $\ddot{\chi} = [[\chi, H], H]$ contain the $\nabla^2 \chi$ terms to
combine with $\dot{\chi}$ to form D'Alembertian.]

Consider, at $t = 0$, the Poisson bracket

$$[j^0(r), J^{0k}] = - x^k \partial_0 j^0(r) j^k(r) \tag{10}$$

then take the Poisson bracket of this equation with $j^0(r')$. The double
Poisson bracket on the left is evaluated using the Jacobi Identity and
Lorentz invariance of the vacuum which implies that $[O, J^{\mu\nu}]_0 = 0$
for any quantity O [5]. The result of these operations, independent of whether
j^μ is conserved, is, using $\dot{A} = [A, H]$ and the invariance of the vacuum

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[3] The matrix η possesses an inverse; hence, $\eta \, \delta A/\delta \chi$ cannot vanish unless $\delta A/\delta \chi$
itself does. Note that the classical result requires the ground state to be a local extremum
only whereas the quantum vacuum had to be an absolute minimum. There is a further
analogue, for the second time derivative $[A, \ddot{A}]$ must vanish in both the quantum and
classical cases.

[4] One way to see this is to observe that if $A\bar{\psi} = a\bar{\psi}$, then $\bar{\psi}^+ [A, O]\bar{\psi} = 0$ for all O;
classically, $[A, O] = 0$ implies $\delta A/\delta \chi = 0$, hence we infer the correspondence.

[5] This follows directly from the definition of "Lorentz invariance of the vacuum";
we mean that, in the vacuum no quantity changes under a Lorentz transformation, i.e.,

$$[O, J^{\mu\nu}] = 0 = \frac{\partial O}{\delta \chi} \eta \frac{\delta J^{\mu\nu}}{\delta \chi} \quad \text{or} \quad \frac{\delta J^{\mu\nu}}{\delta \chi} = 0.$$

under time translations,

$$[j^0(\mathbf{r}), j^k(\mathbf{r}')]_0 - [j^0(\mathbf{r}'), j^k(\mathbf{r})]_0 = (\mathbf{r} - \mathbf{r}')^k \left(\eta \frac{\delta j^0(\mathbf{r}')}{\delta \chi}\right)^T H \left(\eta \frac{\delta j^0(\mathbf{r})}{\delta \chi}\right) \qquad (11)$$

If j^0 is the generator of some transformation, then $\delta j^0(\mathbf{r}')/\delta \chi(\mathbf{r}'')$ must have a $\delta(\mathbf{r}' - \mathbf{r}'')$ term; we assume that it does (if it does not, similar arguments will apply to some c_n in the general Poisson bracket or commutator form, Eq. (4), for $n > 0$ [6]. Then since H must be local, it is a series of derivatives of delta functions:

$$H = [h + \boldsymbol{\alpha} \cdot \boldsymbol{V} - \boldsymbol{V} \cdot \tilde{c} \cdot \boldsymbol{V} \ldots \ldots] \delta(\mathbf{r} - \mathbf{r}') \qquad (12)$$

and the factor $(\mathbf{r} - \mathbf{r}')H$ becomes

$$\boldsymbol{\alpha} \, \delta(\mathbf{r} - \mathbf{r}') + 2\tilde{c} \cdot \boldsymbol{V} \delta(\mathbf{r} - \mathbf{r}') \qquad (13)$$

Furthermore H must be a symmetric matrix, hence $\boldsymbol{\alpha}$ must be antisymmetric and, comparing with Eq. (4), we find

$$[j^0(\mathbf{r}), j^k(\mathbf{r}')] = (\eta \, \delta j^0/\delta \chi)^T \tilde{c} \cdot \boldsymbol{V}(\eta \, \delta j^0/\delta \chi)$$
$$= c_0 \, \boldsymbol{V}^k \, \delta(\mathbf{r} - \mathbf{r}'), \quad c_0 \geqq 0 \qquad (14)$$

where c_0 can vanish only if $\delta j^0/\delta \chi$ does or if $\tilde{c} = 0$. We now show that $\tilde{c}$ cannot be zero in a Lorentz covariant theory. The structure relations of the Poincaré group are

$$[J^{0k}, P^l] = -\delta^{kl} P^0 \simeq -\delta^{kl} \tfrac{1}{2} \chi \mathcal{H} \chi \qquad (15\,\mathrm{a})$$

and

$$[J^{0k}, P^0] = -P^k = \tfrac{1}{2} \sum_n \int d^3r \, p^n(\mathbf{r}) \, \boldsymbol{V}^k q^n(\mathbf{r}) = \tfrac{1}{2} \chi \eta \, \boldsymbol{V}^k \chi \qquad (15\,\mathrm{b})$$

The Hamiltonian P^0 has been evaluated near vacuum and the momentum P^k is exact. The fields χ must transform as some representation of the Lorentz group,

$$[J^{0k}, \chi(x)] = (x^0 \, \partial^k - x^k \, \partial^0) \, \chi(x) + S^{0k} \chi(x) \qquad (16)$$

For $t = 0$ and near vacuum, this equation becomes

$$[J^{0k}, \chi(x)] = (x^k(\eta \, \mathcal{H}) + S^{0k}) \, \chi(x). \qquad (17)$$

Now, assume that $\mathcal{H}$ has no second or higher derivatives of a delta function, i.e., that $\mathcal{H} = h + \boldsymbol{\alpha} \cdot \boldsymbol{V}$; but the $\boldsymbol{\alpha} \cdot \boldsymbol{V}$ term is not positive definite [7], hence it must vanish and we can rewrite the structure relations, Eqs. (15),

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[6] In the classical, as in the quantum, theory all the c_n's in Eq. (4) are nonnegative.

[7] This conclusion is, of course, not valid for anticommuting variables; there the matrix $\mathcal{H}$ must be antisymmetric rather than symmetric, forcing instead $\chi \mathcal{H} \chi \sim \chi \boldsymbol{\alpha} \cdot \boldsymbol{V} \chi$ with *no* $\boldsymbol{V}^2$ terms.

near vacuum, as

$$[J^{0k}, P^l] = \chi\eta\, V^l(x^k\eta h + S^{0k})\chi = -\tfrac{1}{2}\delta^{kl}\chi h\chi \tag{18a}$$

and

$$[J^{0k}, P^0] = \chi h(x^k\eta h + S^{0k})\chi = -\tfrac{1}{2}\chi\eta\, V^k\chi. \tag{18b}$$

Only the symmetric part of the matrix between the χ's contributes, hence the first equation reads

$$\tfrac{1}{2}h = \tfrac{1}{2}[V^l x^k h - h x^k\, V^l - V^l\eta S^{0k} - S^{0k^T}\eta\, V^l] \tag{19}$$

or, since, by locality, $[S^{0k}, V^l] = 0$,

$$\eta\, S^{0k} + S^{0k^T}\eta = 0 \tag{20}$$

and the second equation reads

$$\begin{aligned}
\tfrac{1}{2}\eta\, V^k &= -\tfrac{1}{2}h S^{0k} - \tfrac{1}{2}S^{0k^T}h \\
&= -\tfrac{1}{2}[h S^{0k} + \eta S^{0k}\eta h] \\
&= \tfrac{1}{2}\eta[\eta h, S^{0k}]
\end{aligned}$$

or

$$V^k = [\eta h, S^{0k}]. \tag{21}$$

The matrix ηh is the direct product of a finite dimensional matrix and the unit matrix $\langle r|1|r'\rangle = \delta(r - r')$, thus, only the finite dimensional factor can produce a non vanishing commutator and the trace over the finite dimensional subspace must vanish on the right and does not on the left. Hence, Eq. (21) is impossible; the only assumption (aside from Lorentz covariance) which went into this was the vanishing of $\tilde{c}$ and all higher derivative terms, and we have proven that they cannot vanish. Thus the Poisson bracket, Eq. (14), is established with a nonvanishing c_0 (or $c_n > 0$).

In summary, we have shown that classical boson currents containing terms linear in the canonical variable must have Schwinger terms in their ground state Poisson brackets, just as do their commutator counterparts.

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