Commun. math. Phys. 19, 65–82 (1970)

Estimates of the Unitarity Integral

J. KUPSCH
CERN-Geneva

Received June 18, 1970

Abstract. The elastic unitarity integral is studied for amplitudes which satisfy a Mandelstam representation without subtraction. The double spectral functions are taken to belong to function spaces which allow local, even non-integrable, singularities. The existence of fixed point solutions is derived and the additional restrictions due to inelastic unitarity are discussed.

1. Introduction

The analyticity domain of a two-particle scattering amplitude proposed twelve years ago by Mandelstam [1] has not yet been derived from axiomatic field theory. Also an S matrix theory does not exist which could tell us the analyticity domain of the amplitudes. But a less ambitious consistency problem has been solved; to prove the existence of amplitudes which satisfy Mandelstam analyticity, crossing symmetry for $\pi\,\pi$ scattering, elastic unitarity and the inelastic unitarity bounds [2–6].

The elastic unitarity integral was already written as a system of integral equations for the spectral functions in Mandelstam's original paper [1]. In the case of an unsubtracted amplitude, Atkinson modified these equations to a mapping within a Banach space of Hölder continuous double spectral functions and he showed that its restriction to a subset of this space was a contraction mapping [2]. Hence a fixed point solution which satisfies crossing symmetry and elastic unitarity can be obtained by iteration (under slightly weaker conditions the mere existence of a fixed point can be proved by the Leray-Schauder principle).

This unitarity mapping is not uniquely defined. It depends on an inhomogeneous term which contributes to the double spectral function in the inelastic region $s \geqq 16$, $t \geqq 16$ and which can be chosen arbitrarily within some norm restriction. There is a one-to-one correspondence between this term and the fixed point solution. This arbitrary function might perhaps be determined if one includes all the many particle channels in the unitarity equation, but this is scarcely a solvable problem. However, without too much difficulty one can maintain the unitarity bounds in the inelastic region [as $\operatorname{Im} f_l(s) \geqq |f_l(s)|^2$ in the case of identical particles]. This condition restricts then the choice of the inhomogeneous term.

In Ref. [2], Atkinson characterized a set of these functions such that the fixed point solutions satisfied not only elastic unitarity but also the inelastic unitarity bounds.

These results have been partly extended to amplitudes with subtractions [3–6] (and also the restriction of Ref. [2] to positive double spectral functions is no longer necessary). The proof of convergence of the elastic unitarity iteration goes through without difficulties of principle for any finite number of subtractions [3]. Including the inelastic unitarity bounds, the problem has been solved for one subtraction [4, 6] and also for a class of amplitudes with a finite number of subtractions

$$A(s,t)=F(s,t)+H(s,t) \tag{1.1}$$

where $F(s,t)$ is a once subtracted Mandelstam representation and $H(s,t)$ is holomorphic up to the inelastic thresholds [5] (a mistake in [5] has been corrected in [7]). The class of solutions is still unsatisfactorily small. So we obtain in any case a decreasing total cross-section and the amplitudes are bounded by a polynomial of first degree below the inelastic thresholds, i.e., $|A(s,t)|<\text{const}(1+|t|)$ if $s\leqq 16$. All the papers [3–6] take over the proof of the elastic unitarity from Ref. [2], and hence they only discuss Hölder continuous double spectral functions.

In this article we want to remove the last restriction. We study again the elastic unitarity integral but now for a more general class of double spectral functions, even non-Lebesgue integrable singularities like $\delta(t)$ or $Pf\,\frac{1}{t}$ are allowed. There is some kind of regularity also for our double spectral functions: integration over t (or application on a test function in t) has to lead to Hölder continuous functions in s, at least within the elastic strip $4\leqq s\leqq 16$.

This assumption is connected with our problem in a quite natural way. If we take the imaginary part of the partial waves

$$\operatorname{Im} f_l(s)=\frac{2}{\pi}\,[s(s-4)]^{-\frac{1}{2}}\int dt\, Q_l\left(1+\frac{2t}{s-4}\right)\varrho(s,t)\,, \tag{1.2}$$

$l\geqq l_0\geqq 0$, we know from unitarity that $\operatorname{Im} f_l(s)$ is bounded. Hence, the integration over t smoothes all the singularities of $\varrho(s,t)$. To guarantee in addition a bounded real part

$$\operatorname{Re} f_l(s)\sim\frac{1}{\pi}\int\frac{\operatorname{Im} f_l(s')}{s'-s}\,ds' \tag{1.3}$$

more information about $\varrho(s,t)$ is needed, e.g., Hölder continuity in s after the t integration in (1.2) has been performed.

For our generalization, it is sufficient to study the scattering of identical particles without subtractions in the elastic region, or more precisely, the function $F(s, t)$ in (1.1) is written as an unsubtracted Mandelstam representation. In Section 2, the elastic unitarity condition is formulated as a fixed point problem of a non-linear equation for the double spectral function. (Here we need only the double spectral function within a neighbourhood of the elastic strips in the s and t channels.) This non-linear mapping is discussed on several Banach spaces of functions. These spaces are introduced in Section 3. The estimates of the unitarity integral follow in Sections 4 and 5 (with some technical details in Appendices A and B). As a simple application we prove the existence of fixed point solutions by iteration in Section 6. This iteration may start with a function in a rather general Banach space which allows $\delta(t)$ or $Pf\,\frac{1}{t}$ singularities. But any fixed point solution within this function space exhibits only Lebesgue-integrable singularities.

If we demand in addition the inelastic unitarity bounds, we remain only with such fixed point solutions which are Hölder continuous in s and t within the elastic strips $4 \leqq s \leqq 16$ or $4 \leqq t \leqq 16$ and which possibly yield singularities for $s > 16$ and $t > 16$. There is no general argument which forbids singularities in the intervals $4 \leqq s \leqq 16$ or $4 \leqq t \leqq 16$. But to obtain such singular amplitudes we would need input functions which satisfy inelastic unitarity bounds and show rather strong singularities outside their double spectral region. Our methods, see Ref. [7], are so far not effective enough for this construction.

2. The Unitarity Mapping

We consider the elastic scattering of equal (pseudo) scalar particles of unit mass. The scattering amplitude $A(s, t)$ is decomposed as

$$A(s, t) = F(s, t) + H(s, t) \tag{2.1}$$

where both functions $F(s, t)$ and $H(s, t)$ are symmetric in s, t and $u = 4 - s - t$ and satisfy Mandelstam analyticity. Moreover $F(s, t)$ can be represented as

$$\begin{aligned} F(s, t) = F[\varrho(s, t)] = \frac{1}{\pi^2} \int \frac{\varrho(s', t')}{(s'-s)(t'-t)}\, ds'\, dt' \\ + \text{crossed terms}, \qquad (2.2) \\ \varrho(s, t) = \varrho(t, s)\,. \end{aligned}$$

The function $H(s, t)$ is holomorphic in s up to $s = 16$ and vanishes if $|t| \to \infty$ for values of s in the interval $4 \leqq s \leqq 19$. It is convenient to write

$$\varrho(s, t) = \psi(s, t) + \psi(t, s) \tag{2.3}$$

where $\psi(s, t)$ has no symmetry restrictions.

We then define a mapping $\psi(s,t)\to\psi'(s,t)=\Phi[\psi(s,t)]$ within the set of double spectral functions by the Eq. (2.1)–(2.3) and the unitarity integral

$$\begin{aligned}\operatorname{Im} F[\psi'(s,t)] &= \lambda^2(s)\frac{1}{4\pi}[s(s-4)]^{-\frac{1}{2}}\int_{4-s}^{0} dt'\int_{0}^{2\pi} d\vartheta\, A^*(s,t')\,A(s,t''),\\ t'' &= t+t'+2\frac{tt'}{s-4}-2\sqrt{tt'\left(1+\frac{t}{s-4}\right)\left(1+\frac{t'}{s-4}\right)}\cos\vartheta .\end{aligned} \tag{2.4}$$

Here, we have introduced a cut-off function $\lambda(s)$ with the properties: $\lambda(s)$ is Hölder continuous and

$$\begin{aligned} 0\leqq\lambda(s)\leqq 1 &\quad \text{if} \quad 4\leqq s<\infty,\\ \lambda(s)=1 &\quad \text{if} \quad 4\leqq s<17,\\ \lambda(s)=0 &\quad \text{if} \quad s>19. \end{aligned}$$

The Eq. (2.4) can be transformed in the equivalent relation [1]

$$\psi'(s,t)=\lambda^2(s)\frac{2}{\pi}\int K(s,t,t_1,t_2)\,A_t^*(s,t_1)\,A_t(s,t_2)\,dt_1\,dt_2 \tag{2.5}$$

where we use the absorptive part in the t channel

$$A_t(s,t)=d(s,t)+D(s,t), \tag{2.6}$$

$$d(s,t)=\frac{1}{\pi}\int ds'\left[\frac{1}{s'-s}+\frac{1}{s'-4+s+t}\right]\varrho(s',t), \tag{2.7}$$

$$D(s,t)=\frac{1}{2i}\operatorname{disc}_t H(s,t). \tag{2.8}$$

The kernel $K(s,t,t_1,t_2)$ is defined as

$$K(s,t,t_1,t_2)=\begin{cases} L^{-\frac{1}{2}}(s,t,t_1,t_2) & \text{if} \quad s>4,\, t>16,\\ & \qquad 4<t_1<\alpha(s,t,4),\\ & \qquad 4<t_2<\alpha(s,t,t_1),\\ 0 & \qquad \text{otherwhise} \end{cases} \tag{2.9}$$

with the functions

$$L(s,t,t_1,t_2)=s(s-4)\left[t^2+t_1^2+t_2^2-2(tt_1+tt_2+t_1t_2)-4\frac{tt_1t_2}{s-4}\right] \tag{2.10}$$

and

$$\alpha(s,t,t_1)=t+t_1+2\frac{tt_1}{s-4}-2\sqrt{tt_1\left(1+\frac{t}{s-4}\right)\left(1+\frac{t_1}{s-4}\right)}. \tag{2.11}$$

We assume a support of $\psi(s,t)$ inside the region $4 \leqq s \leqq 19$, $t \geqq 20$. Such a restriction is reproduced by the mapping $\psi(s,t) \to \psi'(s,t) = \Phi[\psi(s,t)]$ [the actual support of (2.5) is even smaller].

For any fixed point solution of this mapping $\psi(s,t) = \Phi[\psi(s,t)]$ the corresponding amplitude $A(s,t)$ satisfies crossing symmetry and elastic unitarity. To derive such solutions one has to study the mapping $\Phi[\psi(s,t)]$ in more detail. The first rigorous proof of existence was given by Atkinson [2]. He restricted the double spectral functions to be elements of a suitable Banach space of Hölder continuous functions and derived all the conditions to apply the Schauder fixed point principle or even the contraction mapping theorem.

The subject of our paper is to study this mapping in more general spaces of spectral functions and to look for fixed point solutions which are not obtained in [2].

We could include a finite number of subtractions for $F(s,t)$ in (2.2) and also take charged particles. But, the generalization we are interested in is independent of the number of subtractions; it can be explained most clearly in the simple case without subtractions.

We would like to mention that nevertheless the whole amplitude $A(s,t)$ may need an arbitrary number of subtractions since $H(s,t)$ is not restricted for $s > 19, t > 19$ and can increase in that region.

3. Banach Spaces of Functions

The spectral functions will be submitted to several restrictions which are most clearly expressed by a norm condition. We therefore define the following Banach spaces of functions:

a) $\mathscr{L}^p(a)$, $1 \leqq p < \infty$, $a \geqq 0$

the space of all measurable functions $f(x)$, $4 \leqq x < \infty$, with the finite norm

$$\|f(x)\| = \left[\int_4^\infty |f(x)\, x^{-a}|^p\, dx\right]^{1/p} \tag{3.1}$$

b) $\mathscr{L}^p(a,b,\mu)$, $1 \leqq p \leqq \infty$, $a \geqq 0$, $-\infty < b < \infty$, $0 < \mu < 1$

the space of all measurable functions $f(x,y)$, $4 \leqq x, y < \infty$, with the finite norm

$$\begin{aligned} \|f(x,y)\| = {} & \sup_{4 \leqq x < \infty} x^{-b} \left[\int_4^\infty |f(x,y)\, y^{-a}|^p\, dy\right]^{1/p} \\ & + \sup_{\substack{4 \leqq x_1, x_2 < \infty \\ |x_1 - x_2| \leqq 1}} x_1^{-b} |x_2 - x_1|^{-\mu} \left[\int_4^\infty |f(x_2,y) - f(x_1,y)|^p\, y^{-ap} dy\right]^{1/p}. \end{aligned} \tag{3.2}$$

The functions $f(x, y)$ of $\mathscr{L}^p(a, b, \mu)$ may be characterized as Hölder continuous functions in x with values $f(x, y) = f_x(y)$ in $\mathscr{L}^p(a)$.

The space $\mathscr{L}^p(a, o, \mu)$ is also denoted as $\mathscr{L}^p(a, \mu)$.

The functions $f(x, y) \in \mathscr{L}^p(a, b, \mu)$ with $f(4, y) = 0$ for almost all y generate a closed subspace, $\mathscr{L}_0^p(a, b, \mu)$.

c) $\mathscr{E}(a, \mu)$, $-\infty < a < \infty$, $0 < \mu < 1$

the space of Hölder continuous functions $f(x)$, $4 < x < \infty$, with a finite norm

$$\begin{aligned} \|f(x)\|_{\mathscr{E}} = &\sup_{4 \leqq x < \infty} x^{-a} |f(x)| \\ &+ \sup_{\substack{4 \leqq x_1, x_2 < \infty \\ |x_2 - x_1| \leqq 1}} x_1^{-a} |x_2 - x_1|^{-\mu} |f(x_2) - f(x_1)| . \end{aligned} \tag{3.3}$$

d) $\mathscr{F}(a, \mu)$, $-\infty < a < \infty$, $0 < \mu < 1$

the space of Hölder continuous functions $f(x, y)$, $4 \leqq x, y < \infty$ with the finite norm

$$\begin{aligned} \|f(x, y)\|_{\mathscr{F}} = &\sup_{4 \leqq x, y < \infty} (x\, y)^{-a} |f(x, y)| \\ &+ \sup_{\substack{4 \leqq x_1, x_2, y < \infty \\ |x_2 - x_1| \leqq 1}} (x_1 y)^{-a} |x_2 - x_1|^{-\mu} |f(x_2, y) - f(x_1, y)| \\ &+ \sup_{\substack{4 \leqq x, y_1, y_2 < \infty \\ |y_2 - y_1| \leqq 1}} (x\, y_1)^{-a} |y_2 - y_1|^{-\mu} |f(x, y_2) - f(x, y_1)| . \end{aligned} \tag{3.4}$$

The functions $f(x, y) \in \mathscr{F}(a, \mu)$ which vanish if $x = 4$, $f(4, y) = 0$, generate a closed subspace $\mathscr{F}_0(a, \mu)$.

The unitarity mapping can be extended to some classes of generalized functions. In the following we use

e) $\mathscr{E}'(a, \mu)$

the continuous linear functionals $\tau(y)$ on the space $\mathscr{E}(a, \mu)$,

$$f(y) \in \mathscr{E}(a, \mu), \qquad \tau(y) \in \mathscr{E}'(a, \mu)$$

$$|\langle \tau(y) | f(y) \rangle| \leqq C_\tau \|f\|_{\mathscr{E}}$$

f) $\tilde{\mathscr{E}}(a, \mu_1, \mu_2)$, $-\infty < a < \infty$, $0 < \mu_1, \mu_2 < 1$

the continuous linear functionals $T(x, y) \equiv T_x(y)$ on $\mathscr{E}(a, \mu)$ which depend Hölder continuously on a parameter x, $4 \leqq x < \infty$,

$$f(y) \in \mathscr{E}(a, \mu_1), \; T_x(y) \in \tilde{\mathscr{E}}(a, \mu_1, \mu_2),$$

$$\langle T_x(y) | f(y) \rangle = F(x) \in \mathscr{E}(0, \mu_2),$$

$$\|F(x)\|_{\mathscr{E}(0, \mu_2)} \leqq C_T \|f\|_{\mathscr{E}(a, \mu_1)} .$$

We denote by $\tilde{\mathscr{E}}_c(a, \mu_1, \mu_2)$ the subset of all functionals $T(x, y) \in \tilde{\mathscr{E}}(a, \mu_1, \mu_2)$ which have a compact support in x.

g) $\mathscr{F}'(a, \mu)$

the space of the continuous linear functionals on $\mathscr{F}(a, \mu)$.

With the usual definitions of the norm of a mapping the spaces e)–g) are also Banach spaces.

Finally, we want to note some simple relations:

$$\mathscr{L}^1(a) \subset \mathscr{E}'(-a, \mu),$$
$$\mathscr{L}^1(a, \mu) \subset \tilde{\mathscr{E}}(-a, \mu, \mu'),$$
$$\mathscr{F}(a, \mu) \subset \mathscr{L}^1(1+a', a, \mu), \quad a' > a,$$
$$\mathscr{E}(a, \mu) \times \mathscr{E}(a, \mu) \subset \mathscr{F}(a, \mu).$$

If $\tau_1(x)$ and $\tau_2(x) \in \mathscr{E}'(a, \mu)$ then the product $\tau_1(x)\, \tau_2(y)$ defines a continuous functional on $\mathscr{F}(a, 2\mu)$, i.e.,

$$\mathscr{E}'(a, \mu) \times \mathscr{E}'(a, \mu) \subset \mathscr{F}'(a, 2\mu).$$

4. The Reduced Unitarity Mapping

As a first step we consider the mapping $f \times g \to B$

$$B(s, t) = \int K(s, t, t_1, t_2)\, f(t_1)\, g(t_2)\, dt_1\, dt_2 \tag{4.1}$$

where f and g depend only on one variable and K is the kernel (2.9). Since our norms (3.2) are asymmetric in the first and in the second variable we also introduce $\hat{B}(t, s) \equiv B(s, t)$.

In Appendix A the necessary estimates on integrals over $K(s, t, t_1, t_2)$, $K(s', t, t_1, t_2) - K(s, t, t_1, t_2)$ and $K(s, t', t_1, t_2) - K(s, t, t_1, t_2)$ are evaluated to apply the integration theorem of Appendix B. In the following we list some results.

4.1. If $f(t) \in \mathscr{L}^v\left(a + \dfrac{1}{v}\right)$ and $g(t) \in \mathscr{L}^w\left(a + \dfrac{1}{w}\right)$ then $B(s, t)$ and $\hat{B}(t, s)$ are elements of $\mathscr{L}_0^r\left(a + \dfrac{1}{r}, a, \mu\right)$ for the range of the quantities v, w, r, $a, \mu : v \geqq 1$, $w \geqq 1$, $r^{-1} = v^{-1} + w^{-1} + q^{-1} - 2 \geqq 0$ with some q, $1 \leqq q \leqq 2$, $0 < \mu < \dfrac{1}{q} - \dfrac{1}{2}$ and $a \geqq -\dfrac{1}{2} + \mu$.

Furthermore, we obtain the estimates [with the corresponding norms (3.1) and (3.2)]

$$\|B(s, t)\|_{(r)} \leqq C \|f(t)\|_{(v)} \cdot \|g(t)\|_{(w)},$$
$$\|\hat{B}(t, s)\|_{(r)} \leqq C \|f(t)\|_{(v)} \cdot \|g(t)\|_{(w)}.$$

We note particularly two limiting cases. For $v=w=1$ this is a mapping between the spaces

$$\mathscr{L}^1(b)\times\mathscr{L}^1(b)\to\mathscr{L}_0^q\left(b-1+\frac{1}{q},b-1,\mu\right)$$

with $1\leqq q<2$, $0<\mu<\frac{1}{q}-\frac{1}{2}$, $b\geqq\frac{1}{2}+\mu$. If $v^{-1}+w^{-1}<\frac{3}{2}$ the function $B(s,t)$ becomes locally bounded. But it is also continuous, and $\mathscr{L}_0^\infty(a,a,\mu)$ can be read as $\mathscr{F}_0(a,\mu)$. For $\frac{4}{3}<q\leqq 2$, $b\geqq 1-\frac{1}{q}$ we obtain a mapping $\mathscr{L}^q(b)\times\mathscr{L}^q(b)\to\mathscr{F}_0\left(b-\frac{1}{q},\mu\right)$ with $0<\mu<\frac{3}{2}-\frac{2}{q}$.

4.2. Let us consider the case $\mathscr{L}^1(b)\times\mathscr{L}^1(b)\to\mathscr{L}_0^q(\ldots)$ in more detail. Since $\mathscr{L}^\infty$ is the dual space of $\mathscr{L}^1$ we know that $B_{t_1t_2}(s,t)=K(s,t,t_1,t_2)\cdot(t_1t_2)^b$ is a bounded function of t_1 and t_2 with values $B(s,t)\in\mathscr{L}_0^q$ for almost all t_1 and t_2. In Appendix A the kernel $K(s,t,t_1,t_2)$ is shown to be Hölder continuous in the following sense:

$$|t_1-t_1'|^{-\mu}|K(s,t,t_1,t_2)-K(s,t,t_1',t_2)|=\tilde{K}_{t_1'}(s,t,t_1,t_2)$$

allows estimates of the same kind as $K(s,t,t_1,t_2)$ if only the range of the Hölder indices is restricted. Therefore, the function $B_{t_1t_2}(s,t)$ is also Hölder continuous in t_1 and t_2; and $f(t)$ and $g(t)$ can be continued from $\mathscr{L}^1(b)$ to linear functionals on a space of Hölder continuous functions. More precisely we infer that Eq. (4.1) defines a bounded mapping

$$\mathscr{E}'(-b,\mu_1)\times\mathscr{E}'(-b,\mu_1)\to\mathscr{L}_0^q\left(b-1+\frac{1}{q},b-1,\mu\right)$$

if $1\leqq q<2$, $0<\mu+2\mu_1<\frac{1}{q}-\frac{1}{2}$, $b\geqq\frac{1}{2}+\mu+2\mu_1$. By these conditions μ_1 is restricted to $0<\mu_1<\frac{1}{4}$.

5. The Unitarity Mapping

The integral transform (4.1) is the most subtile part of the unitarity mapping introduced in Section 2. For a complete discussion we have to include an s dependence of the functions f and g in (4.1) and to investigate Hilbert transform in (2.7).

The generalization of Section 4 to a mapping

$$\begin{aligned}&B(s,t)=\int K(s,t,t_1,t_2)\,f(s,t_1)\,g(s,t_2)\,dt_1\,dt_2\,,\\&\hat{B}(t,s)=B(s,t)\end{aligned}\tag{5.1}$$

is straightforward. If $f(s, t)$ and $g(s, t)$ are elements of a space $\mathscr{L}^p(a, 0, \mu)$ or $\tilde{\mathscr{E}}(-a, \mu, \mu')$ we again derive norm conditions of the type (3.2) or (3.4) for $B(s, t)$ and $\hat{B}(t, s)$. The additional s dependence of f and g is easily estimated by

$$\begin{aligned} B(s_2, t) - B(s_1, t) \\ &= \int dt_1\, dt_2 \{(K(s_2, t, t_1, t_2) - K(s_1, t, t_1, t_2))\, f(s_2, t_1)\, g(s_2, t_2) \\ &\quad + K(s_1, t, t_1, t_2)\, (f(s_2, t_1) - f(s_1, t_1))\, g(s_2, t_2) \\ &\quad + K(s_1, t, t_1, t_2)\, f(s_1, t_1)\, (g(s_2, t_2) - g(s_1, t_2))\}\,, \end{aligned} \tag{5.2}$$

and we can reduce the problem to that solved in Section 4.

5.1. The result of subsection 4.1 can now be extended to the following statement:

if $f(s, t) \in \mathscr{L}^v\left(a + \dfrac{1}{v}, \mu\right)$ and $g(s, t) \in \mathscr{L}^w\left(a + \dfrac{1}{w}, \mu\right)$ then $B(s, t)$ and $\hat{B}(t, s)$ are elements of $\mathscr{L}_0^r\left(a + \dfrac{1}{r}, a, \mu\right)$ for $v \geqq 1$, $w \geqq 1$, $r^{-1} = v^{-1} + w^{-1} + q^{-1} - 2 \geqq 0$ with some q, $1 \leqq q < 2$, $0 < \mu < \dfrac{1}{q} - \dfrac{1}{2}$ and $a \geqq -\dfrac{1}{2} + \mu$. The norms of these functions are related by

$$\begin{aligned} \|B(s, t)\|_{(r)} &\leqq C \|f(s, t)\|_{(v)} \cdot \|g(s, t)\|_{(w)}\,, \\ \|\hat{B}(t, s)\|_{(r)} &\leqq C \|f(s, t)\|_{(v)} \cdot \|g(s, t)\|_{(w)}\,. \end{aligned} \tag{5.3}$$

[If $r = \infty$ we can take $\mathscr{F}_0(a, \mu)$ instead of $\mathscr{L}_0^\infty(a, a, \mu)$.]

5.2. As in 4.2 the functions $f(s, t)$ and $g(s, t)$ can be continued to functionals in the variable t. The corresponding result reads:

the Eq. (5.1) defines a continuous mapping

$$\tilde{\mathscr{E}}(-b, \mu_1, \mu) \times \tilde{\mathscr{E}}(-b, \mu_1, \mu) \to \mathscr{L}_0^q\left(b - 1 + \frac{1}{q}, b - 1, \mu\right)$$

if

$$1 \leqq q < 2, 0 < \mu + 2\mu_1 < \frac{1}{q} - \frac{1}{2}, b \geqq \frac{1}{2} + \mu + 2\mu_1\,.$$

5.3. The unitarity mapping of Section 2 also involves the linear transform (2.7) for the absorptive part in the t channel. Let us first assume that the s dependence of the double spectral function is described by $\psi(s, t) \in \mathscr{L}_0^p(a, b, \mu)$, $b < 0$, $0 < \mu < 1$. The mapping

$$d[\psi(s, t)] = \frac{1}{\pi} \int ds' \left[\frac{1}{s' - s} + \frac{1}{s' - 4 + s + t}\right] \psi(s', t) \tag{5.4}$$

can then be estimated as in the case of ordinary Hölder continuous functions. It turns out to be a bounded transform from $\mathscr{L}_0^p(a, b, \mu)$ into $\mathscr{L}^p(a, 0, \mu)$ if $b < 0$ and $0 < \mu < 1$.

The contribution of the crossed term $\hat{\psi}(s, t) = \psi(t, s)$ is evaluated in the same way if also $\hat{\psi}(s, t) \in \mathscr{L}_0^p(a, b, \mu)$. Such a property applies to double spectral functions which are obtained by the elastic iteration (5.1). But the weaker condition $\psi(s, t) \in \mathscr{L}_0^p(a, \mu)$ is sufficient (also for the crossed term) if we know in addition that the support of $\psi(s, t)$ lies within $4 \leqq s \leqq 19$, $t \geqq 20$ (see Section 2). We denote the corresponding subspace of $\mathscr{L}^p(a, \mu)$ by $\mathscr{L}_c^p(a, \mu)$. The supports of $\hat{\psi}(s, t)$ and $\lambda(s)\, d(s, t)$ are then separated and $\psi(s, t) \to \lambda(s)\, d[\hat{\psi}(s, t)]$ is a bounded operator from $\mathscr{L}_c^p(a, \mu)$ into $\mathscr{L}^p(a, \mu)$ if $p \geqq 1$, $0 \leqq a < p^{-1}$ or $p = 1, a = 1$.

Hence we have obtained

$$\|\lambda(s)\, d[\psi(s, t) + \psi(t, s)]\| \leqq c_1 \|\psi(s, t)\| \tag{5.5}$$

for the spaces $\mathscr{L}_c^p(a, \mu)$ with $p = 1$, $a = 1$ or $p \geqq 1$, $0 \leqq a < p^{-1}$ and $0 < \mu < 1$.

These results can be generalized to functions

$$\psi(s, t) \in \tilde{\mathscr{E}}(-a, \mu_1, \mu_2)$$

if the support is restricted as above for $\mathscr{L}_c^p(a, \mu)$. We denote these subspaces by $\tilde{\mathscr{E}}_c(-a, \mu_1, \mu_2)$. The mapping

$$\psi(s, t) \to \lambda(s)\, d[\psi(s, t) + \psi(t, s)]$$

is a bounded operator from $\tilde{\mathscr{E}}_c(-a, \mu_1, \mu_2)$ into $\tilde{\mathscr{E}}(-a, \mu_1, \mu_2)$ if $0 \leqq a \leqq 1$ and $0 < \mu_{1,2} < 1$.

6. Applications to Fixed Point Solutions

We first discuss the existence of fixed point solutions. For $\lambda(s)\, D(s, t) \in \mathscr{L}^1(a, \mu)$, $0 < \mu < \frac{1}{2}$, $\frac{1}{2} + \mu \leqq a \leqq 1$, the unitarity mapping is a transform from $\mathscr{L}_c^1(a, \mu)$ into $\mathscr{L}_c^1(a, \mu)$ and the estimates (5.3) and (5.5) allow one to apply the contraction mapping theorem (or the Schauder fixed point principle) as it has been done in Ref. [2]. If the norms of $\lambda(s)\, D(s, t) \in \mathscr{L}^1(a, \mu)$ and $\psi_0(s, t) \in \mathscr{L}_c^1(a, \mu)$ are small enough, the series

$$\psi_0(s, t), \psi_1(s, t), \ldots, \psi_{n+1}(s, t) = \Phi[\psi_n(s, t)], \ldots$$

converges to a fixed point solution $\psi(s, t) \in \mathscr{L}_c^1(a, \mu)$.

We can extend this result to generalized functions $\lambda(s)\, D(s, t) \in \tilde{\mathscr{E}}(-a, \mu_1, \mu_2)$ and $\psi_0(s, t) \in \tilde{\mathscr{E}}_c(-a, \mu_1, \mu_2)$, $0 < \mu_1 < \frac{1}{4}$, $0 < \mu_2 < 1$. From Section 5.1 we know that the iterated functions $\psi_n(s, t)$, $n = 1, 2, \ldots$, are all elements of $\mathscr{L}^1(a, \mu)$, $\mu = \min(\frac{1}{2} - 2\mu_1, \mu_2)$. The convergence of this series follows as above if $\lambda(s)\, D(s, t)$ and $\psi_0(s, t)$ are small.

The proof of the existence of fixed point solutions using the spaces $\mathscr{L}^1(a, \mu)$ is easier than Atkinson's method [2]. Compared to Ref. [2] it has also the advantage that functions which decrease only like $(\log t)^{-1-\varepsilon}$ are included in $\mathscr{L}^1(1, \mu)$. (This generalization is also possible by a modification of the norm in Ref. [2], see Ref. [6].) But to satisfy the inelastic unitarity bounds

$$\operatorname{Im} f_l(s) \geqq |f_l(s)|^2, \qquad l=0, 2, \ldots, s>16 \tag{6.1}$$

we need some additional work. The calculations of Section 5 imply $\psi(s,t)$ and $\hat{\psi}(s,t) \in \mathscr{L}^q\left(a-1+\frac{1}{q}, a-1, \mu\right)$, $1 \leqq q < 2$, $0<\mu<\frac{1}{q}-\frac{1}{2}$. These norm conditions are not sufficient to derive the necessary estimates of the contribution of $\psi(s,t)+\psi(t,s)$ to the partial waves.

But, before we discuss the problem of inelastic unitarity we explore the singularity structure of fixed point solutions of the elastic unitarity integral. The space $\tilde{\mathscr{E}}(-a, \mu_1, \mu_2)$, $0<\mu_1<\frac{1}{4}$, allows singularities of the type $\delta(t)$ or $Pf\,\frac{1}{t}$. The unitarity mapping smoothes these local singularities and they do not show up in the fixed point solution. From Section 5 we infer the following statements for any fixed point solution $\psi(s,t) \in \tilde{\mathscr{E}}(-1, \mu_1, \mu)$, $0<2\mu_1+\mu<\frac{1}{2}$ whether it is obtained by iteration or not.

a) If $\lambda(s)\, D(s,t) \in \tilde{\mathscr{E}}(-1, \mu_1, \mu)$ then $\psi(s,t)$ is an element of $\mathscr{L}^q\left(\frac{1}{q}, \mu\right)$ with $\frac{1}{2}+2\mu_1+\mu<q^{-1} \leqq 1$.

b) If $\lambda(s)\, D(s,t)$ has only $\mathscr{L}^p$ singularities, $1 \leqq p \leqq \frac{4}{3}$, then $\psi(s,t)$ is locally $\mathscr{L}^r$ integrable for all r with $2p^{-1}-\frac{3}{2}<r^{-1} \leqq 2p^{-1}-1$.

c) If $\lambda(s)\, D(s,t) \in \mathscr{L}^q\left(\frac{1}{q}, \mu\right)$, $\frac{4}{3}<q \leqq 2$, $0<\mu<\frac{3}{2}-\frac{2}{q}$ then $\psi(s,t)$ is an element of $\mathscr{F}(0, \mu)$.

To obtain δ like singularities for the fixed point solution, if they exist at all, the inhomogeneous term $\lambda(s)\, D(s,t)$ has to be taken out of a more general function space than $\tilde{\mathscr{E}}(-a, \mu_1, \mu)$, $0<\mu_1<\frac{1}{4}$.

At this point we have to remember the inelastic unitarity condition. Using the proofs [2, 4, 5] as an advice on how to obtain the bounds (6.1) the inhomogeneous term $H(s,t)$ has to satisfy (among other conditions):

a) $D(s,t)$ is a positive measure in t, which depends analytically on s, in the strip $0<s<16$;

b) the inelastic bounds (6.1) apply also to $H(s,t)$ for energies above $s=17$ [i.e., for those energies with $\lambda(s) \neq 1$].

In the region $4<s<16$ and $t>17$ the property *b)* restricts the measure $D(s,t)$ to an $\mathscr{L}^p$ function, the values of p depend on s, we can take any p with $1<p<\sqrt{\frac{16}{s}}$, [8]. For $s<9$ we reach values $p \geqq \frac{4}{3}$ and the contribution of this part of $D(s,t)$ (i.e., $s<9, t>17$) to $\psi(s,t)$ is then Hölder continuous in s and t. From the other regions the fixed point solution may however get singular contributions.

But, unfortunately we are not able to prove the existence of such solutions which satisfy the inelastic unitarity bounds and are not Hölder continuous in the elastic strip $4 < s < 16$. To illustrate these difficulties we first consider the case where also $H(s, t)$ is given by an unsubtracted Mandelstam representation [2]

$$H(s,t) = \frac{1}{\pi^2} \int \frac{\omega(s',t')}{(s'-s)(t'-t)} ds'\, dt' + \text{crossed terms}$$

$$\omega(s,t) \in \mathscr{L}^1(a, a-1, \mu),\ \tfrac{1}{2} + \mu < a < 1\,.$$

Then we can neglect the cut-off function $\lambda(s)$, i.e., $\lambda(s) = 1$ for $4 \leqq s < \infty$. The proof of the convergence of the iteration $\psi_n(s, t)$ goes through if we start with a smooth $\psi_0(s, t)$. (The fixed point solutions do not depend on this choice.) For the mapping (5.1) we need $D(s, t) \in \mathscr{L}^1(a, \mu)$ which is satisfied. The transform (2.7) is defined for $\psi_n(s, t) + \psi_n(t, s)$ since $\psi_n(s, t)$ and $\hat{\psi}_n(s, t)$ are elements of $\mathscr{L}^1(a, a-1, \mu)$.

A condition like $\omega(s, t) \in \mathscr{L}^1(a, a-1, \mu)$ was necessary to obtain $D(s, t) \in \mathscr{L}^1(a, \mu)$. But then crossing symmetry, $\omega(s, t) = \omega(t, s)$ implies that

$$D(s,t) = \frac{1}{\pi} \int ds' \left[\frac{1}{s'-s} + \frac{1}{s'-4+s+t}\right] \omega(s', t)$$

is Hölder continuous in s and t outside the support of $\omega(s, t)$. Hence, in this simplest case we always obtain Hölder continuous solutions for $s < 16$.

If we use a cut-off function $\lambda(s)$ we are not faced with this problem due to crossing symmetry for energies s with $\lambda(s) = 0$. But in that region the inhomogeneous term has to satisfy the inelastic unitarity bounds. So far all such amplitudes which we can write down (see Ref. [7]), are locally $\mathscr{L}^2$ integrable outside their double spectral region, i.e., $D(s, t)$ is $\mathscr{L}^2$ integrable for $s < 16$. This leads to Hölder continuous fixed point solutions $\psi(s, t)$ if $4 < s < 16$.

We have seen how the elastic unitarity integral can be estimated in function spaces which allow local singularities, and we have proved the existence of fixed point solutions in these spaces. But, if in addition inelastic unitarity bounds are required our present methods only lead to solutions which are Hölder continuous in the elastic strips $4 < s < 16$, $t > 4$ and $4 < t < 16$, $s > 4$. In the region $s > 16$, $t > 16$ local singularities are still possible.

Acknowledgements. I would like to thank Professors D. Atkinson and A. Martin for discussions and reading the manuscript.

Appendix A

In this Appendix we derive estimates on the unitarity kernel $K(s, t, t_1, t_2)$.

The function (2.10)

$$L(s, t, t_1, t_2) = s(s-4)\,\varphi(t, t_1, t_2) - 4stt_1t_2 \tag{A.1}$$

with

$$\varphi(t, t_1, t_2) = t^2 + t_1^2 + t_2^2 - 2(tt_1 + tt_2 + t_1t_2) \tag{A.2}$$

may be written as

$$L(s, t, t_1, t_2) = s(s-4)\,[\alpha(s, t, t_1) - t_2]\,[\beta(s, t, t_1) - t_2] \tag{A.3}$$

or

$$L(s, t, t_1, t_2) = s(s-4)\,[t - \alpha(s, t_1, t_2)]\,[t - \beta(s, t_1, t_2)] \tag{A.4}$$

using the expression (2.11) and

$$\beta(s, t, t_1) = t + t_1 + 2\frac{tt_1}{s-4} + 2\sqrt{tt_1\left(1+\frac{t}{s-4}\right)\left(1+\frac{t_1}{s-4}\right)}\,. \tag{A.5}$$

If $t_1 > 4$ and $t_2 > 4$ we can describe the region where $L(s, t, t_1, t_2)$ is positive in the following ways

$$s > 4,\ t > 16,\ 4 < t_1 < \alpha(s, t, 4),\ 4 < t_2 < \alpha(s, t, t_1)$$

[see (2.9)], or

$$t_1, t_2 > 4,\ t > t_1 + t_2 + 2\sqrt{t_1 t_2},\ \ s > 4 + \frac{4\,tt_1t_2}{\varphi(t, t_1, t_2)}$$

or

$$t_1, t_2 > 4,\ s > 4,\ t > \beta(s, t_1, t_2) \qquad (>16)\,.$$

The domain in the variables (s, t, t_1, t_2) characterized by these equivalent conditions is called $\mathfrak{D}$. We list some estimates which are valid within $\mathfrak{D}$:

$$\begin{gathered}
4 < \alpha(s, t, t_1) < \frac{t}{4t_1}(s-4)\,,\\
16 < t_1 t_2 < \frac{1}{4}\,t(s-4)\,,\\
4\frac{tt_1}{s-4} < \beta(s, t, t_1) - \alpha(s, t, t_1)\,,\\
4\frac{tt_1}{s-4} < \beta(s, t, t_1) < 4t\left(1+\frac{t}{s-4}\right),\\
0 < \varphi(t, t_1, t_2) < t^2\,.
\end{gathered} \tag{A.6}$$

These inequalities give sufficient information about the boundaries of the region $\mathfrak{D}$. The unitarity kernel $K(s,t,t_1,t_2)$ was defined in (2.9) as

$$K(s,t,t_1,t_2)=\begin{cases} L^{-\frac{1}{2}}(s,t,t_1,t_2) & \text{if} \quad (s,t,t_1,t_2)\in\mathfrak{D}\,,\\ 0 & \text{if} \quad (s,t,t_1,t_2)\notin\mathfrak{D}\,. \end{cases}$$

For the following we need estimates of the difference $K(s',t',t_1,t_2)-K(s,t,t_1,t_2)$. We notice that $L(s,t,t_1,t_2)$ is a quadratic form in each of its variables, it is therefore sufficient to study $Q(x)=(x-a)(x-b)$ with $a\geqq b\geqq 0$ and

$$Q_+^{\lambda}(x)=\begin{cases} Q^{\lambda}(x) & \text{if} \quad x>a\,,\\ 0 & \text{if} \quad x<a\,. \end{cases}$$

For $Q_+^{-1}(x)-Q_+^{-1}(y)$ we calculate

$$0<Q_+^{-1}(x)-Q_+^{-1}(y)\leqq Q_+^{-1}(x)\left|\frac{(x-y)(x+y)}{Q(y)}\right|^{\mu}\leqq Q_+^{-1-\mu}(x)\left|(x-y)\frac{2x^2}{y}\right|^{\mu}$$

with $0\leqq\mu\leqq 1$, if $a<x<y$, and

$$|Q_+^{-1}(x)-Q_+^{-1}(y)|\leqq Q_+^{-1}(y)\leqq|(y-x)\,y|^{\mu}Q_+^{-1-\mu}(y)$$

with $0\leqq\mu\leqq 1$, if $x<y$, $x<a$.

The corresponding results for $Q_+^{-\frac{1}{2}}(x)-Q_+^{-\frac{1}{2}}(y)$ follow immediately from $|A^{\frac{1}{2}}-B^{\frac{1}{2}}|\leqq|A-B|^{\frac{1}{2}}$, $A\geqq 0$, $B\geqq 0$. The unitarity kernel can now be estimated by

$$s'\geqq s,\ t'\geqq t,\ 0\leqq\mu\leqq\tfrac{1}{2}\,,$$

$$\begin{aligned} &0\leqq K(s,t,t_1,t_2)-K(s',t',t_1,t_2)\\ &\leqq|K(s,t,t_1,t_2)|^{1+2\mu}\left\{\left|2t^2s^2\,\frac{s'-s}{s'}\right|^{\mu}+\left|2s(s-4)\,t^2\,\frac{t'-t}{t'}\right|^{\mu}\right\} \end{aligned}$$

if $(s,t,t_1,t_2)\in\mathfrak{D}$,

$$|K(s,t,t_1,t_2)-K(s,t',t_1,t_2)|\leqq|K(s,t',t_1,t_2)|^{1+2\mu}|s(s-4)\,t'(t'-t)|^{\mu}$$

if $(s,t,t_1,t_2)\notin\mathfrak{D}$, $(s,t',t_1,t_2)\in\mathfrak{D}$, (A.7)

$$|K(s,t,t_1,t_2)-K(s',t,t_1,t_2)|\leqq|K(s',t,t_1,t_2)|^{1+2\mu}|t^2s'(s'-s)|^{\mu}$$

if $(s,t,t_1,t_2)\notin\mathfrak{D}$, $(s',t,t_1,t_2)\in\mathfrak{D}$.

These results are also contained in Atkinson's paper [2]. But we use somewhat simpler techniques and need (A.5)–(A.7) anyhow for further reference.

For different values of t_1 (or t_2) we derive in the same way

$$0 < K(s,t,t_1',t_2) - K(s,t,t_1,t_2) \leqq |K(s,t,t_1',t_2)|^{1+2\mu} |4\,st(s+t)(t_1'-t_1)|^{\mu}$$

$$\text{if } (s,t,t_1',t_2) \in \mathfrak{D},\ t_1' > t_1\,,$$

$$|K(s,t,t_1',t_2) - K(s,t,t_1,t_2)| \leqq |K(s,t,t_1,t_2)^{1+2\mu} |4\,st(s+t)(t_1'-t_1)|^{\mu}$$

if $(s,t,t_1',t_2) \notin \mathfrak{D}$, $(s,t,t_1,t_2) \in \mathfrak{D}$, (A.8)

$$0 \leqq \mu \leqq \tfrac{1}{2}\,.$$

For application in Section 4 we calculate integrals over $K(s,t,t_1,t_2,)$. Let q be a real number, $1 \leqq q < 2$, then

$$\int |t_2^a K(s,t,t_1,t_2)|^q dt_2$$

can be evaluated using the representation (A.3) for $L(s,t,t_1,t_2)$ and the bounds (A.6). Estimates for s or t integrations are most easily derived from (A.1) or (A.4). We write the results as follows:

$$\int |(t_1t_2)^a K(s,t,t_1,t_2)|^q dt_{1,2} \leqq C\cdot s^{-\frac{q}{2}}(s-4)^{aq+1-\frac{q}{2}} t^{aq+1-q} t_{2,1}^{-1}, \quad a \geqq \frac{1}{2} - \frac{1}{q}\,,$$

$$\int |s^{-a} K(s,t,t_1,t_2)|^q ds \leqq C\cdot t^{aq-1} (t_1t_2)^{1-aq-q}, \quad a \geqq \frac{1}{q} - \frac{1}{2}\,, \tag{A.9}$$

$$\int |t^{-a} K(s,t,t_1,t_2)|^q dt \leqq C\cdot s^{-\frac{q}{2}}(s-4)^{aq-1+\frac{q}{2}} (t_1t_2)^{1-aq-q}, \quad a \geqq \frac{1}{q} - \frac{1}{2}\,,$$

$1 \leqq q < 2$.

It is convenient to introduce the q dependent kernel

$$R(s,t,t_1,t_2) = (st)^{-a+1-\frac{2}{q}} (t_1t_2)^a K(s,t,t_1,t_2), \qquad 1 \leqq q < 2,\ a \geqq \frac{1}{2} - \frac{1}{q} \tag{A.10}$$

then (A.9) can be formulated in a more symmetric way

$$\int |R(s,t,t_1,t_2)|^q\, dt_{1,2} \leqq C(stt_{2,1})^{-1}\,,$$

$$\int |R(s,t,t_1,t_2)|^q\, ds \leqq C(tt_1t_2)^{-1}\,, \tag{A.11}$$

$$\int |R(s,t,t_1,t_2)|^q\, dt \leqq C(st_1t_2)^{-1}$$

where the right-hand side of the first and the third inequality can be multiplied by a threshold factor

$$\left(\frac{s-4}{s}\right)^{aq-\frac{q}{2}+1}.$$

The unitarity mapping (4.1) is then transformed to

$$B(s,t)=(s\,t)^{a-1+\frac{2}{q}}\int R(s,t,t_1,t_2)\,t_1^{-a}f(t_1)\,t_2^{-a}g(t_2)\,dt_1\,dt_2\,. \quad \text{(A.12)}$$

Using (A.11) this integral can be estimated by the theorem of Appendix B.

Four our applications in Sec. 4 we also need bounds for $B(s',t)-B(s,t)$ and $B(s,t')-B(s,t)$ which easily follow from the inequalities (A.7) as

$$|B(s',t)-B(s,t)|\leqq\left|2\,\frac{s'-s}{s'}\right|^{\mu}(G(s,t)+G(s',t)) \quad \text{(A.13)}$$

with

$$G(s,t)=(s\,t)^{2\mu}\int|K(s,t,t_1,t_2)|^{1+2\mu}|f(t_1)\,g(t_2)|\,dt_1\,dt_2\,. \quad \text{(A.14)}$$

This expression can also be written in the form of (A.12),

$$G(s,t)=(s\,t)^{a-1+\frac{2}{q}}\int\tilde{R}(s,t,t_1,t_2)\,t_1^{-a}|f(t_1)|\,t_2^{-a}|g(t_2)|\,dt_1\,dt_2$$

with

$$\tilde{R}(s,t,t_1,t_2)=(s\,t)^{-a+1-\frac{2}{q}+2\mu}|K(s,t,t_1,t_2))|^{1+2\mu}(t_1\,t_2)^a,$$

$$0\leqq\mu<\frac{1}{q}-\frac{1}{2},\qquad a\geqq\frac{1}{2}-\frac{1}{q}+\mu\,.$$

This kernel $\tilde{R}$ satisfies again the estimates (A.11) (with another constant C and without the threshold factor). So the problem is reduced to an integral transform of the type (A.12).

Finally we would like to mention that $f(t)$ and $g(t)$ can also be taken as generalized functions. If $f(t)$ is a linear functional on the space of Hölder continuous functions with index μ_1 the mapping (4.1) corresponds to a transform of integrable functions but the kernel substituted by

$$|t_1-t_1'|^{-\mu_1}|K(s,t,t_1,t_2)-K(s,t,t_1',t_2)|\,.$$

In (A.8) this expression is estimated by

$$|8\,s^2t^2|^{\mu_1}\max_{x=t_1,t_1'}|K(s,t,x,t_2)|^{1+2\mu_1}$$

So we have again to handle integrals like (A.14). If $f(t)$ and $g(t)$ are elements of $\mathscr{E}'(-a,\mu_1)$ the difference $B(s',t)-B(s,t)$ is majorized by an expression like (A.13) where the range of the Hölder indices is restricted to $0<\mu+2\mu_1<\dfrac{1}{q}-\dfrac{1}{2}$ and $a\geqq\dfrac{1}{2}-\dfrac{1}{q}+\mu+2\mu_1$.

Appendix B

In this Appendix we derive a theorem on integral transforms of the type (4.1). We consider complex valued Lebesgue measurable functions defined on the real axis. The function spaces $\mathscr{L}^p[\sigma]$, $1 \leqq p \leqq \infty$, are defined as usual [9] with a measure $d\sigma(x)$ and the norms

$$\|f\|_{p,\sigma} = [\int |f(x)|^p \, d\sigma(x)]^{\frac{1}{p}}, \qquad 1 \leqq p < \infty,$$

and

$$\|f\|_{\infty,\sigma} = \operatorname{ess.\,sup}_x |f(x)|$$

In the case of the Lebesgue measure $d\sigma(x) = dx$ we drop the index σ.

Theorem. *Given the functions $K(x_1, x_2, x_3)$, $f(x)$ and $g(x)$ with the following restrictions*

a) the integral kernel $K(x_1, x_2, x_3)$ is estimated by

$$[\int |K(x_1, x_2, x_3)|^q \, dx_i]^{\frac{1}{q}} \leqq \varphi(x_j)\, \varphi(x_k)$$

for $i = 1, 2, 3$, $(i, j, k) = \mathrm{perm}(1, 2, 3)$ and a value of q, $1 \leqq q < \infty$;

b) the functions $f(x)$ and $g(x)$ satisfy

$$f(x)\,[\varphi(x)]^{1-q+\frac{q}{v}} \in \mathscr{L}^v, \qquad g(x)\,[\varphi(x)]^{1-q+\frac{q}{w}} \in \mathscr{L}^w$$

for some real numbers $v \geqq 1$ and $w \geqq 1$, $v^{-1} + w^{-1} \geqq 2 - q^{-1}$, then the bilinear form

$$F(x_3) = \int K(x_1, x_2, x_3)\, f(x_1)\, g(x_2)\, dx_1\, dx_2$$

is defined for almost all values of x_3 and we obtain

$$F(x)\,[\varphi(x)]^{-1+\frac{q}{r}} \in \mathscr{L}^r, \qquad r^{-1} = v^{-1} + w^{-1} + q^{-1} - 2$$

and

$$\|F \cdot \varphi^{-1+\frac{q}{r}}\|_r \leqq \|f \cdot \varphi^{1-q+\frac{q}{v}}\|_v \cdot \|g \cdot \varphi^{1-q+\frac{q}{w}}\|_w .$$

Proof. The Hölder inequality for the x_1 or the x_2 integration gives

$$|F(x_3)| \leqq \begin{cases} \|f\|_p \cdot \|g \cdot \varphi\|_1 \cdot \varphi(x_3) \\ \|f \cdot \varphi\|_1 \cdot \|g\|_p \cdot \varphi(x_3) \end{cases} \text{with} \quad p^{-1} + q^{-1} = 1 . \tag{B.1}$$

In addition we calculate

$$[\int |F(x_3)|^q \, dx_3]^{\frac{1}{q}} .$$

If $q = 1$ this can be done by an interchange of the order of integration,

$$\int |F(x)|\, dx \leqq \|f \cdot \varphi\|_1 \, \|g \cdot \varphi\|_1 .$$

If $1<q<\infty$ we multiply $F(x)$ by a function $h(x) \in \mathscr{L}^p$, the dual space of $\mathscr{L}^q$, and the Hölder inequality yields

$$\int |F(x)\, h(x)|\, dx \leqq \|h\|_p \cdot \|f\varphi\|_1 \|g\varphi\|_1 .$$

Hence we obtain in both cases

$$\|F\|_q \leqq \|f\varphi\|_1 \|g\varphi\|_1, \qquad 1 \leqq q < \infty \tag{B.2}$$

Now we define the functions

$$\tilde{f}(x) = \begin{cases} [\varphi(x)]^{1-q} f(x) & \text{if} \quad \varphi(x) > 0 \\ 0 & \text{if} \quad \varphi(x) = 0 \end{cases}$$

and

$$\bar{F}(x) = \begin{cases} [\varphi(x)]^{-1} F(x) & \text{if} \quad \varphi(x) > 0 \\ 0 & \text{if} \quad \varphi(x) = 0 \end{cases}$$

and introduce the spaces $\mathscr{L}^r[\sigma]$, $1 \leqq r \leqq \infty$, with the measure $d\sigma(x) = [\varphi(x)]^q dx$. The results (B.1) and (B.2) can then be written as

$$\|\bar{F}\|_{\infty,\sigma} \leqq \begin{cases} \|\tilde{f}\|_{p,\sigma} \|\tilde{g}\|_{1,\sigma} \\ \|\tilde{f}\|_{1,\sigma} \|\tilde{g}\|_{p,\sigma} \end{cases}$$

and

$$\|\bar{F}\|_{q,\sigma} \leqq \|\tilde{f}\|_{1,\sigma} \|\tilde{g}\|_{1,\sigma} .$$

The Riesz convexity theorem [9] allows to generalize these inequalities to

$$\|\bar{F}\|_{r,\sigma} \leqq \|\tilde{f}\|_{v,\sigma} \|\tilde{g}\|_{w,\sigma}$$

with $v, w \geqq 1$ and $r^{-1} = v^{-1} + w^{-1} + q^{-1} - 2 \geqq 0$. But this is exactly our theorem if we use the original functions f, g and F.

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J. Kupsch
Theory Division
CERN
CH-1211 Genf 23