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The Enveloping Algebra of a Covariant System

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Abstract. In an earlier work, Doplicher, Kastler and Robinson have examined a mathematical structure consisting of a pair (A, G), where A is a C^*-algebra and G is a locally compact automorphism group of A. We call such a structure a "covariant system". The enveloping von Neumann algebra $\mathscr{A}(A, G)$ of (A, G) is defined as a $*$-algebra of operator valued functions (called options) on the space of covariant representations of (A, G). The system (A, G) is canonically embedded in, and in fact generates, the von Neumann algebra $\mathscr{A}(A, G)$. Further we show there is a natural one-to-one correspondence between the normal $*$-representations of $\mathscr{A}(A, G)$ and the proper covariant representations of (A, G). The relation of $\mathscr{A}(A, G)$ to the covariance C^*-algebra $C^*(A, G)$ is also examined.

§ 1. Introduction

In an earlier work [2], Sergio Doplicher, Daniel Kastler, and Derek Robinson have examined a mathematical structure consisting of a pair (A, G), where A is a C^*-algebra and G is a locally compact group of automorphisms of A. In this paper we shall refer to such a structure as a "covariant system". (The formal definition is given in the next section.) In their paper, Doplicher, Kastler and Robinson examine other algebraic objects associated with a covariant system. Specifically they define and study a Banach $*$-algebra $\mathfrak{A}_1^G$, (which is the analogue of the L_1-group algebra of a group), and its C^*-completion under the minimal regular norm, which they denote $\bar{\mathfrak{A}}^G$ (and which we shall denote $C^*(A, G)$). This latter algebra is referred to by Georges Zeller-Meier, as the crossed product of A by G, [10]. Covariant systems together with their associated crossed product (or covariance C^*-algebra) have received some study in the mathematical literature. (cf. [8, 9, and 10].) The important correspondence of the covariant representation theory of a covariant system (A, G) and the proper $*$-representation theory of its covariance algebra $\mathscr{A}_1^G$ (and hence of its crossed-product $C^*(A, G)$) is presented in § III of [2].

The purpose of this paper is to define and examine other algebraic objects which may be canonically associated with a covariant system. In particular we define a von Neumann algebra $\mathscr{A}(A, G)$, as an algebra of operator valued functions (called options) on the space of covariant representations of (A, G). This algebra is called the enveloping von Neumann algebra of the covariant system (A, G). The covariant system (A, G) is isomorphically and canonically embedded in $\mathscr{A}(A, G)$. We shall thus consider the covariant system to be contained in $\mathscr{A}(A, G)$. Then $A \cup G$ generates $\mathscr{A}(A, G)$. Further the restriction to (A, G), of a normal $*$-representation of $\mathscr{A}(A, G)$, yields a covariant representation of (A, G). In this way, a one-to-one correspondence is determined, between the normal $*$-representations of $\mathscr{A}(A, G)$ and the proper covariant $*$-representations of (A, G).

The crossed product $C^*(A, G)$, considered in [2], is also embedded isomorphically in $\mathscr{A}(A, G)$. The crossed product $C^*(A, G)$ is σ-strong dense in $\mathscr{A}(A, G)$. Indeed the algebra $\mathscr{A}(A, G)$ may then be identified as the von Neumann enveloping algebra of the C^*-algebra $C^*(A, G)$. In the case where $C^*(A, G)$ is separable, the Takesaki duality theory [7] is applicable and $C^*(A, G)$ may be identified as the subset of $\mathscr{A}(A, G)$ of those operator valued functions (options) (defined on the covariant representations of (A, G)), which are continuous relative to an appropriate topology.

In § 2 we give some elementary notions and defintions. In § 3 we develop the theory of the enveloping algebra of a covariant system. Finally in § 4 we exhibit a few other algebraic objects associated with a covariant system (A, G).

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§ 2. Covariant Systems

Definition 2.1. A covariant system is a pair (A, G) where A is a C^*-algebra and G is a locally compact group of automorphisms of A such that, for each x in A, the map $s \to s(x)$ is a continuous function of G into A.

Definition 2.2. A covariant representation of a covariant system (A, G) is a pair (π, U), where π is a $*$-representation of A and U is a strongly continuous unitary representation of G such that both representations act on the same Hilbert space $\mathscr{H}$ and such that

$$U(s)\,\pi(x)\,U(s)^* = \pi(s(x))$$

for all s in G and x in A.

Definition 2.3. A covariant representation (π, U) on the space $\mathscr{H}(\pi, U)$ is said to be proper if the $*$-representation π is proper, i.e., if $\{\pi(x)\psi : \psi \in \mathscr{H}(\pi, U), x \in A\}$ is dense in $\mathscr{H}(\pi, U)$.

In this paper we shall restrict ourselves primarily to proper convariant representations. The next proposition will indicate that this is not a very major restriction. For this we shall first need the following lemma on $*$-representations of C^*-algebras.

Lemma 2.4. *Let A be a C^*-algebra with approximate identity $\{e_\lambda\}$. Let π be a $*$-representation of A and let E_π denote the projection on $\mathscr{H}(\pi)$ with range equal to the essential space of π. Then the net of operators $\pi(e_\lambda)$ converges strongly to E_π, in $\mathscr{L}(\mathscr{H}(\pi))$.*

Proof. Here, as usual, $\mathscr{L}(\mathscr{H}(\pi))$ denotes the algebra of all bounded linear operators on the representation space $\mathscr{H}(\pi)$ of π.

Since $\{e_\lambda\}$ is an approximate identity in A, we have that e_λ is a net in A such that $\|e_\lambda\| \leqq 1$ for all λ and $e_\lambda x$ converges to x, for every x in A. Since π is a $*$-representation which is norm continuous, we have $\pi(e_\lambda x) = \pi(e_\lambda)\pi(x)$ converges to $\pi(x)$ in the norm topology of $\mathscr{L}(\mathscr{H}(\pi))$, for every x in A. Thus $\pi(e_\lambda)\varphi$ converges to φ for every vector φ of the form $\varphi = \pi(x)\psi$, with $x \in A$ and $\psi \in \mathscr{H}(\pi)$. Hence, by a standard $\varepsilon - \delta$ argument, we have that $\pi(e_\lambda)\varphi$ converges to φ for every φ is the closure of the linear space $\{\pi(x)\psi : x \in A, \psi \in \mathscr{H}(\pi)\}$, i.e., for every φ in the range of E_π. Now let φ be any vector of $\mathscr{H}(\pi)$. Then

$$\begin{aligned}
&\|(\pi(e_\lambda) - E_\pi)\varphi\| \\
&\quad = \|\pi(e_\lambda)\varphi - E_\pi\varphi\| \\
&\quad = \|\pi(e_\lambda)(E_\pi\varphi + (I - E_\pi)\varphi) - E_\pi\varphi\| \\
&\quad = \|\pi(e_\lambda)(E_\pi\varphi) - E_\pi\varphi\| \to 0 .
\end{aligned}$$

Thus $\pi(e_\lambda)$ converges strongly to E_π.

Proposition 2.5. *If (π, U) is a covariant representation of (A, G), then the essential space of π is an invariant subspace of the unitary representation U, and hence the restriction of (π, U), to the essential space of π, is a proper covariant representation.*

Proof. Let $\{e_\lambda\}$ be an approximate identity in A and let E_π denote the projection of $\mathscr{H}(\pi)$ onto the essential space of π. Since each s in G is an automorphism of A, we have that $\{s(e_\lambda)\}$ is an approximate identity, for each $s \in G$. Thus by our previous lemma, $\pi(s(e_\lambda))$ converges strongly to E_π, for each s in G. Further, for each s, $U(s)\pi(e_\lambda)U(s)^*$ converges strongly to $U(s)E_\pi U(s)^*$. But, for each s and each λ

$$U(s)\pi(e_\lambda)U(s)^* = \pi(s(e_\lambda)) .$$

Hence, for each s in G,

$$U(s)\, E_\pi U(s)^* = E_\pi$$

and thus the range of E_π is an invariant subspace of U.

Definition 2.6. Let $\mathscr{A}(\pi, U)$ denote the von Neumann algebra generated by the range of (π, U). The covariant representation (π, U) is said to be cyclic, with cyclic vector ψ, if the linear space $\mathscr{A}(\pi, U)\,\psi$ is dense in $\mathscr{H}(\pi, U)$.

In [2], Doplicher, Kastler, and Robinson define the notion of cyclic covariant representation slightly differently. They define a covariant representation to be cyclic with cyclic vector ψ if the set of vectors of $\mathscr{H}(\pi, U)$ obtained by applying to ψ arbitrary products of operators of the form $\pi(x)$, x in A, and $U(s)$, s in G, generates, linearly, a dense subset of $\mathscr{H}$. It is an easy exercise to verify that this definition is equivalent to the definition given above.

Notice that this definition of cyclic representation has the curious property that a cyclic representation need not be proper. (We leave the reader to puzzle about that.) However Proposition 2.5 tells us that every covariant representation may be written as a direct sum of a proper covariant representation and a covariant representation (π_0, U_0) where π_0 is a zero representation of A, i.e., $\pi_0(x)$ is the zero operator on $\mathscr{H}(\pi_0, U_0)$, for all x in A. Thus a slight variation of the usual proof for C^*-algebras yields the following proposition (cf. Proposition 2.2.7 of [3]).

Proposition 2.7. *Every covariant representation (π, U) of a covariant system (A, G) may be expressed as a direct sum*

$$\pi = \pi_0 \oplus \sum_{\lambda\in\Lambda} \oplus \pi_\lambda, \qquad U = U_0 \oplus \sum_{\lambda\in\Lambda} \oplus U_\lambda$$

where (π_λ, U_λ) is a proper cyclic covariant representation for each λ in Λ, and π_0 is a zero representation of A (of course (π_0, U_0) appears in the decomposition if and only if (π, U) is improper).

It is clear that all the usual definitions such as invariant subspace, irreducible representation, and unitary equivalence have their obvious analogues for covariant representations. In particular the usual properties of quasi-equivalence holds (cf. § 5.3 of [3]) and we may define two covariant representations (π, U) and (π', U') to be quasi-equivalent (denoted $(\pi, U) \sim (\pi', U')$) if there is an isomorphism φ of $\mathscr{A}(\pi, U)$ onto $\mathscr{A}(\pi', U')$ such that $\varphi(\pi(x)) = \pi'(x)$ for all x in A and $\varphi(U(s)) = U'(s)$ for all s in G. Thus one notes that if $(\pi, U) \sim (\pi', U')$, then $\pi \sim \pi'$ and $U \sim U'$. A similar fact holds, of course, for unitary equivalence. In the reverse direction, however, one can apparently conclude nothing about the equivalence of covariant representations, from the equivalence of their component parts.

We conclude these elementary remarks with the following observation. (Notice that if either π or U is irreducible, then (π, U) is irreducible.)

Proposition 2.8. *Let (A, G) be a covariant system and let π be an irreducible $*$-representation of A. Then there exists at most one, up to multiplication by a one dimensional representation, unitary representation U of G on $\mathscr{H}(\pi)$ such that (π, U) is a covariant representation of (A, G).*

Proof. Consider two unitary representations, U_1 and U_2, of G on $\mathscr{H}(\pi)$ and suppose both (π, U_1) and (π, U_2) are covariant representations. Thus for each x in A and s in G we have

$$\begin{aligned}\pi(x) = \pi(s^{-1}(sx)) &= U_2(s^{-1})\,\pi(s(x))\,U_2(s^{-1})^* \\ &= U_2(s^{-1})\,U_1(s)\,\pi(x)\,U_1(s)^*\,U_2(s^{-1})^*\end{aligned}$$

and hence $\pi(x)\,U_2(s)^*\,U_1(s) = U_2(s)^*\,U_1(s)\,\pi(x)$. Thus $U_2(s)^*\,U_1(s)$ is a unitary operator in the commutator $\mathscr{A}(\pi)'$ of π, which is a complex multiple $\alpha(s)$ of identity operator, since π is irreducible. Thus $U_1(s) = \alpha(s)\,U_2(s)$ for every s in G. Since $\alpha(s)\,I = U_2(s)\,U_1(s)$, $s \to \alpha(s)$ is a one dimensional unitary representation.

§ 3. The Enveloping von Neumann Algebra

Definition 3.1. Fix a Hilbert space $\mathscr{H}_0$ of sufficiently high dimension that every cyclic covariant representation of (A, G) is unitary equivalent to a covariant representation acting on a closed subspace of $\mathscr{H}_0$. The concrete covariant dual is the set $\mathscr{R}(A, G)$ of all proper covariant representations (π, U) of (A, G) such that $\mathscr{H}(\pi, U)$ is a closed subspace of $\mathscr{H}_0$.

It is an easy, but non-trivial, observation that if A and G are both separable, then $\mathscr{H}_0$ may be chosen of dimension $\aleph_0$.

Definition 3.2. We next define the covariance von Neumann algebra for a covariant system (A, G). Let $\mathscr{A}(A, G)$ denote the set of maps S defined on the concrete covariant representation space of (A, G) satisfying the following axioms.

(i) For each covariant representation (π, U) in $\mathscr{R}(A, G)$, $S(\pi, U)$ is a bounded linear operator on $\mathscr{H}(\pi, U)$, the representation space of (π, U).

(ii) Sup $\{\|S(\pi, U)\| : (\pi, U) \in \mathscr{R}(A, G)\} < +\infty$.

(iii) If $(\pi, U) \in \mathscr{R}(A, G)$ and $(v, V) \in \mathscr{R}(A, G)$ and $\mathscr{H}(\pi, U) \perp \mathscr{H}(v, V)$ so that

$$(\pi \oplus v, U \oplus V) \in \mathscr{R}(A, G)$$

then

$$S(\pi \oplus v, U \oplus V) = S(\pi, U) \oplus S(v, V).$$

(iv) If (π, U) and (v, V) are elements of $\mathscr{R}(A, G)$ and if u is a linear isometry of $\mathscr{H}(\pi, U)$ onto $\mathscr{H}(v, V)$ such that

$$\pi = u^* v u \quad \text{and} \quad U = u^* V u$$

then

$$S(\pi, U) = u^* S(v, V)\, u.$$

Any operator valued function on $\mathscr{R}(A, G)$ satisfying these properties will be called an option on $\mathscr{R}(A, G)$.

$\mathscr{A}(A, G)$ is given a $*$-algebra structure by defining addition, multiplication and the adjoint operation pointwise. Thus, for example, if S, T are options, then $S + T$ denotes the element of $\mathscr{A}(A, G)$ defined by

$$(S + T)(\pi, U) = S(\pi, U) + T(\pi, U)$$

for all (π, U) in $\mathscr{R}(A, G)$.

The weak topology of $\mathscr{A}(A, G)$ is defined to be the smallest topology such that the functions $S \to (S((\pi, U)\, \psi, \varphi)$ are continuous, for all (π, U) in $\mathscr{R}(A, G)$ and all ψ, φ in $\mathscr{H}(\pi, U)$.

Similarly, the strong topology of $\mathscr{A}(A, G)$ is defined to be the topology determined by the family of semi-norms

$$S \to \|S(\pi, U)\, \psi\|$$

where $(\pi, U) \in \mathscr{R}(A, G)$ and $\psi \in \mathscr{H}(\pi, U)$.

The conjugate space X^* of a complex Banach space X is the Banach space of all bounded linear functionals on X. Richard Arens [1] has described a natural multiplication in the second conjugate space X^{**} of any Banach algebra X.

Theorem 3.3. *$\mathscr{A}(A, G)$ is a von Neumann algebra. The weak and strong topologies defined above are the intrinsic σ-weak and σ-strong operator topologies, respectively. Further $\mathscr{A}(A, G)$ is canonically isomorphic to the second conjugate space of the covariance C^*-algebra, $C^*(A, G)$, endowed with Arens multiplication.*

Proof. This is basically an adaptation of a theory already developed for locally compact groups and for C^*-algebras (cf. [4], [6], § 1 of [7] and § 12 of [3]). More explicitly $\mathscr{A}(A, G)$ may be represented as a von Neumann algebra as follows. Let (v, V) denote the universal covariant representation, which we define as the direct sum of all the representations in the concrete covariant dual $\mathscr{R}(A, G)$ of (A, G). Let $\mathscr{A}(v, V)$ denote the von Neumann algebra on $\mathscr{H}(v, V)$ generated by the range of (v, V). Then the map Φ, defined by

$$\Phi(S) = \sum_{(\pi, U) \in \mathscr{R}(A, G)} \oplus S(\pi, U)$$

is an isomorphism of $\mathscr{A}(A, G)$ onto $\mathscr{A}(v, V)$. Further the reasoning of Remarks 2.6, 2.7 and Proposition 2.8 of [6] applies in this context to establish the other facts listed in the theorem.

Remark 3.4. By using Theorem 3 of [2] we may associate the concrete covariant dual $\mathscr{R}(A, G)$ with the space $\mathrm{Rep}(C^*(A, G), \mathscr{H}_0)$ of all $*$-representations of $C^*(A, G)$ on $\mathscr{H}_0$ defined by Takesaki (cf. § 1 of [7]). Indeed each proper covariant representation (π, U) of (A, G) may be integrated to yield a proper $*$-representation T of $C^*(A, G)$ in the same space $\mathscr{H}(\pi, U)$. This in turn corresponds to the element of $\mathrm{Rep}(C^*(A, G), \mathscr{H}_0)$ whose essential space is $\mathscr{H}(\pi, U)$ and whose restriction to $\mathscr{H}(\pi, U)$ is T. This sets up a one-to-one mapping of $\mathscr{R}(A, G)$ into $\mathrm{Rep}(C^*(A, G), \mathscr{H}_0)$ which is almost onto, except that Takesaki considers the zero representation to be an element of $\mathrm{Rep}(C^*(A, G), \mathscr{H}_0)$, which does not correspond to an element of $\mathscr{R}(A, G)$. However this difference is unimportant for the first section of Takesaki's paper. Thus $\mathscr{A}(A, G)$ is precisely the von Neumann enveloping algebra of the covariance C^*-algebra, $C^*(A, G)$ described in § 1 of [7] and § 12 of [3]. Thus $C^*(A, G)$ is canonically and isomorphically embedded as a σ-strongly dense subalgebra of $\mathscr{A}(A, G)$. The fact that $\mathscr{A}(A, G)$ is pretty roomy is shown by the fact that the covariant system (A, G) is also embedded in $\mathscr{A}(A, G)$, and its image also generates $\mathscr{A}(A, G)$.

Theorem 3.5. *For each s in G, define the option $\hat{s}$ on $\mathscr{R}(A, G)$ by $\hat{s}(\pi, U) = U(s)$ for all (π, U) in $\mathscr{R}(A, G)$. Similarly for each x in A, define the option $\hat{x}$ in $\mathscr{R}(A, G)$ by $\hat{x}(\pi, U) = \pi(x)$, for all (π, U) in $\mathscr{R}(A, G)$.*

Then $s \to \hat{s}$ is a group isomorphism and σ-strong homeomorphism of G onto a group of unitary operators in $\mathscr{A}(A, G)$. Similarly $x \to \hat{x}$ is a $*$-isomorphism and isometry of A into $\mathscr{A}(A, G)$.

Further $\widehat{s(x)} = \hat{s}\hat{x}\hat{s}^*$ for all s in G, x in A and $\{\hat{x}, \hat{s} : x \in A, s \in G\}$ generates $\mathscr{A}(A, G)$.

Proof. It is an easy matter to verify that $\hat{x}$ and $\hat{s}$ are options in $\mathscr{R}(A, G)$ for all x in A and s in G, and that the maps $x \to \hat{x}$, $y \to \hat{y}$ are homomorphisms.

Notice that if (v, V) is the universal representation, then $\{v(x), V(s) : x \in A, s \in G\}$ generates $\mathscr{A}(v, V)$. But under the isomorphism Φ of $\mathscr{A}(A, G)$ onto $\mathscr{A}(v, V)$ we have $\Phi(\hat{x}) = v(x)$ for x in A and $\Phi(\hat{s}) = V(s)$ for s in G. Thus $\{\hat{x}, \hat{s} : x \in A, s \in G\}$ generates $\mathscr{A}(A, G)$.

Further, for any (π, U) in $\mathscr{R}(A, G)$ we have

$$\begin{aligned}\widehat{s(x)}\,(\pi, U) &= \pi(s(x))\\ &= U(s)\,\pi(x)\,U(s)^*\\ &= \hat{s}(\pi, U)\,\hat{x}(\pi, U)\,\hat{s}^*(\pi, U)\\ &= \hat{s}\hat{x}\hat{s}^*(\pi, U)\,.\end{aligned}$$

Thus $\widehat{s(x)} = \hat{s}\hat{x}\hat{s}^*$.

To see that the maps are indeed isomorphisms requires the consideration of a faithful representation of (A, G). The fact that (A, G) admits a faithful representation is well-known. Indeed if π_0 is a faithful representation of A we may construct the corresponding induced representation of (A, G) from the identity representation of the trivial subgroup $\{e\}$ of G, according to [8]. This is equivalent to the following explicit construction.

Definition 3.6. Let π_0 be any non-zero representation of A. Then the left π_0-regular covariant representation (π, L) of (A, G) is defined as follows: Let μ denote left invariant Haar measure on G. Then $\mathscr{H}(\pi, L)$ is $L^2(G, \mathscr{H}(\pi_0))$, the space of all μ-square integrable functions of G into $\mathscr{H}(\pi_0)$. We define (π, L) on $L^2(G, \mathscr{H}(\pi_0))$ by

$$(L(s)\,\xi)\,(t) = \xi(s^{-1}t)$$

for all s, t in G and ξ in $L^2(G, \mathscr{H}(\pi_0))$, and

$$(\pi(x)\,\xi)\,(t) = \pi_0(t^{-1}x)\,\xi(t)$$

for all x in A, t in G and ξ in $L^2(G, \mathscr{H}(\pi_0))$.

The unitary representation L in the above definition is a faithful representation of G. In fact, L is unitary equivalent to αL_0 where $\alpha = \dim \mathscr{H}(\pi_0)$ and L_0 is the left regular representation of G. Indeed the choice of an orthonormal basis for $\mathscr{H}(\pi_0)$ enables one to define an isometry of $L^2(G, \mathscr{H}(\pi_0))$ with the direct sum of α copies of $L^2(G)$ and on each of these subspaces, L acts like the left regular representation. Hence if π_0 is a faithful representation of A, then (π, L) is a faithful covariant representation of (A, G). (By this we mean only that each of the two component representations are faithful. Indeed if π_0 is proper and A admits an identity i, then clearly $\pi(i) = L(e)$ and hence (π, L) maps these two points into the identity operator.)

By Proposition 2.7, the existence of a faithful representation implies there exist sufficiently many proper cyclic covariant representations of (A, G) to distinguish the points of both A and G, i.e., the $*$-representations which appear distinguish the points of A and the unitary representations which appear distinguish the points of G. Thus two distinct points of A determine distinct options on $\mathscr{R}(A, G)$, and hence the map $x \to \hat{x}$ is an injective map of A into $\mathscr{A}(A, G)$. But any isomorphism of a C^*-algebra into another C^*-algebra is necessarily an isometry (cf. Propositions 1.3.7 and 1.8.1 of [3]). Similarly the map $s \to \hat{s}$ is an injective mapping of G into $\mathscr{A}(A, G)$.

Thus it remains to show that $s \to \hat{s}$ is a σ-strong homeomorphism. Recall the σ-strong topology of $\mathscr{A}(A, G)$ is given by the family of semi-norms $S \to \|S(\pi, U)\psi\|$ such that $(\pi, U) \in \mathscr{R}(A, G)$ and $\psi \in \mathscr{H}(\pi, U)$. If $s_\lambda \to s$ is a convergent net in G, then for any (π, U) in $\mathscr{R}(A, G)$ and ψ in $\mathscr{H}(\pi, U)$, we have

$$\|(\hat{s}_\lambda - \hat{s})(\pi, U)\psi\| = \|[U(s_\lambda) - U(s)]\psi\|$$

which converges to zero because U is assumed to be strongly continuous. Thus $\hat{s}_\lambda \to \hat{s}$ σ-strongly in $\mathscr{A}(A, G)$.

Conversely, if $\hat{s}_\lambda \to \hat{s}$ in $\mathscr{A}(A, G)$, then $\hat{s}_\lambda(\pi, U)$ converges strongly to $\hat{s}(\pi, U)$ for every (π, U) in $\mathscr{R}(A, G)$, i.e., $U(s_\lambda)$ converges strongly to $U(s)$, for every unitary representation U for which there exists a π for which $(\pi, U) \in \mathscr{R}(A, G)$. Let (π, L) denote the π_0-left regular representation of (A, G), where π_0 is some proper $*$-representation of A. Since (π, L) is the direct sum of proper cyclic representations of (A, G), we have that $L(s_\lambda)$ converges strongly to $L(s)$. Let L_0 denote the left regular representation of G. Since $L \cong \alpha L_0$, where $\alpha = \dim \pi_0$, it follows that $L_0(s_\lambda)$ converges strongly to $L_0(s)$. But the left regular representation is a weak homeomorphism (cf. Lemma 2.2 of [6]). Since the strong and weak operator topologies are identical on the unitary group on a Hilbert space, L_0 is a strong homeomorphism. Thus s_λ converges to s in G. This completes the proof of Theorem 3.5.

Since both (A, G) and $C^*(A, G)$ are canonically embedded in $\mathscr{A}(A, G)$, we now examine the important correspondence of the representation theory of these different objects (cf. Section III of [2]). By a normal $*$-representation of a von Neumann algebra $\mathscr{A}$ we shall mean a σ-strongly continuous $*$-representation which maps the identity element of $\mathscr{A}$ into the identity operator. Since both $A \cup G$ and $C^*(A, G)$ generate $\mathscr{A}(A, G)$ as a Neumann algebra, a normal $*$-representation of (A, G) is uniquely determined by its values, either on (A, G) or $C^*(A, G)$.

Theorem 3.7. *Every proper covariant $*$-representation of a covariant system (A, G) has a unique extension to a normal $*$-representation of the enveloping algebra $\mathscr{A}(A, G)$ of (A, G). Further the restriction of any normal $*$-representation of $\mathscr{A}(A, G)$ to (A, G) is a proper covariant $*$-representation of (A, G). Corresponding representations generate the same von Neumann algebra. Hence the correspondence preserves most properties of the representation theory, such as unitary equivalence, quasi-equivalence, irreducibility, type I, etc.*

Proof. Let (π, U) be a proper covariant $*$-representation of (A, G). The corresponding extension $N_{\pi,U}$ of $\mathscr{A}(A, G)$ may be defined for (π, U) in $\mathscr{R}(A, G)$ by

$$N_{\pi,U}(S) = S(\pi, U)$$

for every option S in $\mathscr{A}(A, G)$. The verification that $N_{\pi,U}$ is a normal $*$-representation of $\mathscr{A}(A, G)$ is now straightforward. The extension for proper covariant $*$-representations, which are not contained in $\mathscr{R}(A, G)$ is achieved by using Proposition 2.7. Again this extension is necessarily unique since $A \cup G$ generates $\mathscr{A}(A, G)$.

Now let N be a normal $*$-representation of $\mathscr{A}(A, G)$. Lemma 2.4 may now be applied to all the proper representations in $\mathscr{R}(A, G)$ to conclude that the approximate identity in A converges σ-strongly to the identity element of $\mathscr{A}(A, G)$. Thus the restriction of N to A yields a proper $*$-representation π_N of A. Since the original topology of G is equivalent to the σ-strong topology of G, when embedded in $\mathscr{A}(A, G)$, the restriction of N to G yields a strongly continuous unitary representation U_N of G. Finally since N is a $*$-representation we have, using Theorem 3.5, that $\pi_N(s(x)) = N(s(x)) = N(s x s^*) = N(s)\, N(x)\, N(s)^* = U_N(s)\, \pi_N(x)\, U_N(s)^*$. Thus the restriction (π_N, U_N) of N to (A, G), yields a proper covariant representation of (A, G).

Remark 3.8. This correspondence of the proper covariant representation theory of (A, G) and the normal $*$-representation theory of $\mathscr{A}(A, G)$ commutes with the correspondence of the proper covariant representation theory of (A, G) and the proper $*$-representation theory of $C^*(A, G)$ established by Doplicher, Kastler, and Robinson in [2]. Indeed the correspondence of the $*$-representation theory of a C^*-algebra with its enveloping algebra is well-known (cf. Proposition 12.1.5 of [3]).

Theorem 3.9. *The lattice of all quasi-equivalence classes of proper covariant representations of a covariant system is lattice isomorphic to the lattice of all projections in the center of* $\mathscr{A}(A, G)$. *In particular the quasi-dual* $\widetilde{(A, G)}$ *of* (A, G), *consisting of all quasi-equivalence classes of factor covariant representations of* (A, G), *admits a canonical one-to-one correspondence with the minimal central projections of* $\mathscr{A}(A, G)$.

Proof. The correspondence is generated as follows. Suppose $\mathscr{A}(A, G)$. is concretely and faithfully represented as a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ (cf. the proof of Theorem 3.3, for example). Then each central projection E of $\mathscr{A}(A, G)$ gives rise to an induction

$$\mathscr{A}(A, G) \to \mathscr{A}(A, G)_E$$

(cf. A-15, page 335 of [3].) Which is a normal $*$-representation of $\mathscr{A}(A, G)$. Thus its restriction to (A, G) is a proper covariant representation. That this generates the asserted lattice isomorphism follows from standard arguments outlined in [5] and Remark 2.9 of [6]. In general, the lattice of quasi-equivalence classes of normal $*$-representations of a von Neumann algebra $\mathscr{A}$ is lattice isomorphic to its lattice of central projections.

Remark 3.10. In the case where $C^*(A, G)$ is a separable C^*-algebra, the duality theory of M. Takesaki allows one to identify the elements of $C^*(A, G)$, within $\mathscr{A}(A, G)$, as those options which are continuous with respect to an appropriate topology on $\mathscr{R}(A, G)$. By Remark 3.4 our concrete covariant dual, $\mathscr{R}(A, G)$, may be identified (except for one exceptional point) with the concrete $*$-representation space Rep$(C^*(A, G), \mathscr{H}_0)$ considered in [7]. Takesaki considers the zero representation of $C^*(A, G)$ to be an element of Rep$(C^*(A, G), \mathscr{H}_0)$ and this does not correspond to any proper covariant representation of (A, G). The natural topology for Rep$(C^*(A, G), \mathscr{H}_0)$ is defined in [7]. In a remark at the end of his paper, Takesaki points out that he does not require this exceptional point in his concrete dual if the C^*-algebra has a unit. Thus we may transfer his topology to $\mathscr{R}(A, G)$ directly, in the case where G is discrete and A has a unit. In the general case, however, we must resort to the subterfuge of adding an additional point (the zero covariant representation acting on the zero dimensional space $\{0\}$, if you like) to $\mathscr{R}(A, G)$ and extending the domain of definition of every option on $\mathscr{R}(A, G)$ to this (slightly) larger space $\mathscr{R}_0(A, G)$, by defining every option to be the zero operator at this exceptional point. This does not change the previous theory of $\mathscr{A}(A, G)$ in any way. We may then proceed to transfer Takesaki's topology to $\mathscr{R}_0(A, G)$ and hence identify $C^*(A, G)$, within $\mathscr{A}(A, G)$, as those options which are continuous in the following sense. An option S on $\mathscr{R}_0(A, G)$ is said to be $*$-strong continuous if, for each vector ψ in $\mathscr{H}_0$ (cf. Definition 3.1) this maps

$$(\pi, U) \to \|S(\pi, U)\, E(\pi, U)\, \psi\|$$

and

$$(\pi, U) \to \|S(\pi, U)^*\, E(\pi, U)\, \psi\|$$

are continuous on $\mathscr{R}_0(A, G)$, where $E(\pi, U)$ denotes the projection of $\mathscr{H}_0$ onto the representation space of (π, U).

It would be most interesting to find appropriate axioms for the options which would enable one to identify the covariant system within $\mathscr{A}(A, G)$. This could be called the duality question for covariant systems.

§ 4. Other Structures Associated with (A, G)

In this section we examine the relationship to $\mathscr{A}(A, G)$, of the other algebraic objects associated with the covariant system, namely $C^*(G)$, the C^*-group algebra of G (cf. § 13.9 of [3]), $\mathscr{A}(G)$, the enveloping von

Neumann algebra of G (cf. [4]) and $\mathscr{A}(A)$, the enveloping von Neumann algebra of A (cf. § 12 of [3]).

We first note that there is a natural map of $C^*(G)$ into $\mathscr{A}(A, G)$ defined as follows. If $g \in C^*(G)$, define the option $\hat{g}$ on $\mathscr{R}(A, G)$ by

$$\hat{g}(\pi, U) = U'(g)$$

where U' denotes the canonical (i.e., integrated) proper $*$-representation of $C^*(G)$ associated with U (cf. § 13.9.3 of [3]). We leave the verification that $g \to \hat{g}$ is a $*$-algebra homomorphism of $C^*(G)$ into $\mathscr{A}(A, G)$. Unfortunately we do not know whether this is an embedding, i.e., whether the map $g \to \hat{g}$ is injective.

Proposition 4.1. *The canonical map $g \to \hat{g}$ of $C^*(G)$ into $\mathscr{A}(A, G)$ is injective if and only if there exists a covariant representation (π, U) of (A, G) such that U' is a faithful $*$-representation of $C^*(G)$, where U' denotes the canonical proper $*$-representation of $C^*(G)$ associated with U.*

Proof. If such a (π, U) exists, then clearly $\hat{g}_1 = \hat{g}_2$ implies $U'(g_1) = U'(g_2)$ and hence $g_1 = g_2$, i.e., the canonical map of $C^*(G)$ into $\mathscr{A}(A, G)$ is injective.

On the other hand, if $g \to \hat{g}$ is faithful, then there exist enough proper covariant representations of (A, G) to distinguish the elements in $C^*(G)$. The direct sum of all these representations is then the required covariant representation of (A, G).

Corollary 4.2. *If G is a discrete amenable group, then $C^*(G)$ is canonically embedded, isometrically, in $\mathscr{A}(A, G)$.*

Proof. According to corollary 6 of [9], under these circumstances, the representation L_0' of $C^*(G)$ corresponding to the left-regular representation L_0 of G, is faithful on $C^*(G)$.

Let π_0 be any proper representation of A. Then the left π_0-regular covariant representation (π, L) of (A, G) is a proper covariant representation such that L is unitary equivalent to αL_0, where α is the dimension of $\mathscr{H}(\pi_0)$ (cf. the proof of Theorem 3.5). Hence the $*$-representation L' of $C^*(G)$ associated with L, is faithful on $C^*(G)$.

We conclude with an examination of the enveloping von Neumann algebras $\mathscr{A}(A)$ and $\mathscr{A}(G)$, of A and G respectively. We define the covariant enveloping algebra $\mathscr{A}_c(A)$ of A, to be the strong closure of A, in $\mathscr{A}(A, G)$. Similarly the covariant enveloping algebra $\mathscr{A}_c(G)$ of G is defined to be the von Neumann subalgebra of $\mathscr{A}(A, G)$ generated by G.

There is a canonical normal $*$-homomorphism of $\mathscr{A}(A)$ onto $\mathscr{A}_c(A)$, defined as follows. The embedding of A into $\mathscr{A}(A, G)$ (Theorem 3.5) is a $*$-representation of A and hence (Proposition 12.15 of [3]) admits a unique extension to a normal $*$-representation of $\mathscr{A}(A)$. Since A is σ-strongly dense in $\mathscr{A}(A)$, this σ-strong continuous homomorphism maps $\mathscr{A}(A)$ onto the σ-strong closure of A, i.e., $\mathscr{A}_c(A)$.

Similarly there is a canonical normal $*$-homomorphism of $\mathscr{A}(G)$ onto $\mathscr{A}_c(G)$. Indeed the embedding (Theorem 3.5) of G into $\mathscr{A}(A, G)$ is a strongly continuous unitary representation of G and hence (§ 1 of [6]) has a unique extension to a normal $*$-homomorphism of $\mathscr{A}(G)$ into $\mathscr{A}(A, G)$. Since G generates $\mathscr{A}(G)$ (§ 1 of [6]), the range of this homomorphism is necessarily $\mathscr{A}_c(G)$.

By analogy with the definition of a locally compact automorphism group of a C^*-algebra (i.e., the notion of a covariant system), one may define the notion of a locally compact automorphism group of a von Neumann algebra. The idea is that in a C^*-algebra the norm topology is the relevant topology, while for a von Neumann algebra, the strong operator topology is the relevant topology.

Definition 4.3. If G is a locally compact group of automorphisms of a von Neumann algebra $\mathscr{A}$, we say G is a locally compact automorphism group of $\mathscr{A}$ if, for each $x \in \mathscr{A}$, the map $s \to s(x)$ is a strongly continuous map of G into $\mathscr{A}$.

Proposition 4.4. *If (A, G) is a covariant system, then G is a locally compact automorphism group of $\mathscr{A}_c(A)$.*

Proof. Since $\mathscr{A}_c(A)$ and G are both contained in $\mathscr{A}(A, G)$ we define the action of G on $\mathscr{A}_c(A)$ by $s(x) = sxs^*$ for all s in G. Since G is a unitary group in $\mathscr{A}(A, G)$, this defines G as a group of automorphisms on $\mathscr{A}_c(G)$. Since A is invariant under G, and A is strongly dense in $\mathscr{A}_c(A)$, each automorphism maps $\mathscr{A}_c(A)$ onto $\mathscr{A}_c(A)$. Since on the unitary operators the weak and strong operator topology are equivalent, $s \to s^*$ is strongly continuous. Further if x is a fixed element of $\mathscr{A}_c(G)$ then $s \to xs^*$ is continuous. Finally multiplication on bounded sets is strongly continuous and hence $s \to sxs^* = s(x)$ is strongly continuous.

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