Commun. math. Phys. 10, 1—47 (1968)

Boson Fields with the $:\Phi^4:$ Interaction in Three Dimensions*

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Received April 4, 1968

Abstract. The $:\Phi^4:$ interaction for boson fields is considered in three dimensional space time. A space cutoff is included in the interaction term. The main result is that the renormalized Hamiltonian H_{ren} is a densely defined symmetric operator. In addition to the infinite vacuum energy and infinite mass renormalizations, this theory has an infinite wave function renormalization. Consequently the Hilbert space (of physical particles) in which H_{ren} acts is disjoint from the bare particle Fock Hilbert space in which the unrenormalized Hamiltonian is defined.

Contents

* This work was supported in part by the National Science Foundation, NSF GP 7477.

§ 1. Introduction

1.1. Superrenormalizable problems

In Quantum Field Theory, the renormalized Hamiltonian has the form

$$H_{\mathrm{ren}} = H_0 + g\,V + \Sigma\,\alpha_i C_i \tag{1.1.1}$$

where H_0 is a self adjoint operator and V and the C_i are densely defined bilinear forms. The coefficients α_i are constants which depend on g and are generally infinite. We introduce a space cutoff by requiring V to have the form

$$V = \int V(x)\,h(x)\,dx\,,$$

with h (the cutoff) a smooth function of compact support.

If the summation over i in (1.1.1) is finite, the problem is renormalizable; if in addition each α_i is a polynomial in g (with infinite coefficients) plus a finite function of g then the problem is said to be superrenormalizable. An important property of H_{ren} to be established is that it is a positive selfadjoint operator. From the selfadjointness of H_{ren} one can define $e^{-\mathrm{it}H_{\mathrm{ren}}}$, and then

$$\varphi(t) = e^{-\mathrm{it}H_{\mathrm{ren}}}\,\varphi(0)$$

is a solution of the Schrödinger equation

$$i\partial\,\varphi/\partial t = H_{\mathrm{ren}}\,\varphi$$

and gives the dynamics for finite times. As a first step, one could show that H_{ren} is a densely defined symmetric operator or a closable bilinear form; this step is the objective of this paper for the interaction we are studying. For typical interactions in two dimensions, see [2, 6, 8]. Another step is to show that H_{ren}, as a bilinear form, is positive. Since one can approximate H_{ren} by well defined operators (renormalized Hamiltonians with a momentum cutoff for which the α_i's are all finite), it is sufficient to show that each approximating operator is positive or semibounded with a lower bound independent of the approximation. Thus the second step is logically independent of the first. For typical two dimensional problems this second step has been carried out in [3, 4, 8] using two distinct methods. The Friedrichs extension theorem then provides a natural selfadjoint positive extension. Finally it remains to be seen whether the Friedrichs extension is the correct extension. (For example, it might be the only extension.)

1.2 The Domain for H_{ren}

Considering the very singular nature of the perturbation in (1.1.1) when one or more of the coefficients α_i is infinite, one expects that the domains of H_0 and H_{ren} will have only the vector zero in common and furthermore that any vector in the domain of H_{ren} must have a complicated structure closely related to V in order that the infinities will cancel. To construct these vectors we first find an operator T, our dressing transformation, for which

$$H_{\text{ren}} T = T H_0 + \text{error} , \tag{1.2.1}$$

where the error is a densely defined unbounded operator. According to the formal theory, the wave operator

$$W_- = \lim_{t \to -\infty} e^{\text{it} H} e^{-\text{it} H_0}$$

satisfies

$$H_{\text{ren}} W_- = W_- H_0 \tag{1.2.2}$$

and formal series expansions in powers of g are known for W_-, see [1] for example. These expansions appear to diverge but nonetheless they are extremely useful. It seems that if one includes only the important terms or the important part of each term then the series will converge to an invertible operator T which solves (1.2.1). We then define

$$\mathscr{D}(H_{\text{ren}}) = T\mathscr{D}$$

as the domain of H_{ren}, where $\mathscr{D}$ is a suitable dense subspace contained in $\mathscr{D}(H_0)$ and we use (1.2.1) to define H_{ren}.

The infinities (in a model with a space cutoff) are caused entirely by the interaction of particles of large momentum. Thus the important part of W_- is the part corresponding to particles of large momentum, and we can omit from W_- parts of terms corresponding to particles with momentum in some bounded region B_0. A simple but approximate description of T can be given by introducing regions $B_1 \subset B_2 \subset \cdots$, $\lim_j B_j = R^2$ in momentum space and omitting from W_- parts of terms corresponding to the presence of more than j particles with momenta in B_j. Our operator T is close to being the correct wave operator for a world in which

a) all interactions increase or preserve the number of particles

b) there can exist at most j particles with momenta in B_j.

We can describe the definition of T in a more mathematical way by observing that terms in W_- corresponding to l particles with moments in B_j have a size

$$0(\varepsilon_j^l (l!)^{1/2}) .$$

and $\lim_j \varepsilon_j = 0$. Summing over l gives a divergent series and the l^{th} term in the series tends to infinity. If we break off the series at $l = [\varepsilon_j^{-1}]$ then the last terms included in the series and the first terms excluded from the series are both very small. Since the $j + 1^{\text{st}}$ term of W_-^{-1} is defined by a recursion formula involving only the $j - 1$ and j^{th} terms of W_-^{-1}, the error caused by this truncation is $0([\varepsilon_-^{-1}]!)^{-1/2}$ and our truncated series T is an approximate solution of the same recursion formula. The recursion formula is essentially (1.2.2) and the approximate solution gives us (1.2.1). G. Rota has told me that truncations of this nature are standard in the theory of divergent series.

Our method for finding a dressing transformation T which solves (1.2.1) is different from the method used in [2]. The present method seems likely to work for a wider class of interactions.

We remark that the truncations in T complicate the formal or algebraic aspects of the theory. The compensating advantage, of course, is that they make possible the estimates which lead to convergence proofs.

1.3 Infinite Renormalizations

The infinite counter terms in H_{ren} correspond to the infinite vacuum energy and the infinite self energy of the particles. The vacuum energy has terms which are quadratic and cubic in g and the infinite selfenergy is quadratic in g, or in other words the only primitive divergent diagrams are of second or third order. For the Yukawa coupling in three dimensions there are fourth and sixth order divergent diagrams also. In addition to the renormalizations associated with the counter terms, our problem has an infinite wave function renormalization. If we examine T or W_{-}^{-1} or just the first order terms of T or W_{-}^{-1} we find operators which map out of Hilbert space. These operators are essentially tensor product operators. They map a function φ into a function proportional to $q \otimes \varphi$, and q is not in L_2. For such a q, $q \otimes \varphi$ can never be in L_2 and can never belong to our Hilbert space. Moreover q fails to be in L_2 due to an insufficiently rapid decrease for large momenta. Thus according to the philosophy § 1.2, it is just the part of q which must be retained in T which causes the trouble, and so all vectors in the range of T have infinite norm (do not lie in the Fock Hilbert space). However, we will see that $\|T\varphi\|$ can be written as an infinite quantity which does not depend on φ times a finite quantity which does depend on φ. In other words the ratios

$$\|T\varphi\|/\|T\varphi_0\|$$

are well defined and finite even though $\|T\varphi\|$ is not. We use this fact to introduce a new inner product range of T:

$$\langle T\psi, T\varphi\rangle_{\text{ren}} = \langle T\psi, T\varphi\rangle/\|T\varphi_0\|^2 .$$

The resulting Hilbert space, $\mathscr{F}_{\text{ren}}$, is the space on which H_{ren} acts, and we regard T as a transformation from the original Fock Hilbert space $\mathscr{F}$ to $\mathscr{F}_{\text{ren}}$. These definitions agree with standard methods in peturbation theory. $\mathscr{F}$ is interpreted as the space of bare particles and $\mathscr{F}_{\text{ren}}$ is the space of physical particles.

1.4 The Unrenormalized Hamiltonian

We use nonrelativistic notation. Let $\mathscr{F}_n$ be the symmetric tensor product

$$\mathscr{F}_n = L_2(R^2) \otimes_s \ldots \otimes_s L_2(R^2) \subset L_2(R^{2n})$$

with n factors ($\mathscr{F}_0$ is the complex numbers) and let

$$\mathscr{F} = \Sigma \oplus \mathscr{F}_n \tag{1.4.1}$$

be the Fock Hilbert space. We introduce the annihilation and creation operators $a(k)$ and $a^*(k)$, normalized so that (formally)

$$[a(k), a^*(l)] = \delta(k - l)\ . \tag{1.4.2}$$

The interaction term V has the form

$$V = \sum_{j=0}^{4} V_j\ , \tag{1.4.3}$$

where V_j is the part of V which creates j particles,

$$V_j = \int v_j(k_1, \ldots, k_4)\, a^*(k_1) \ldots a^*(k_j)\, a(k_{j+1}) \ldots a(k_4)\, dk \tag{1.4.4}$$

$$v_j(k_1, \ldots, k_4) = \binom{4}{j} \hat{h} \left(\sum_{i=1}^{j} k_i - \sum_{i=j+1}^{4} k_i \right) \prod_{i=1}^{4} \mu_i^{-1/2} \tag{1.4.5}$$

$$\mu_i = \mu(k_i) = (\mu_0^2 + |k_i|^2)^{1/2}\ . \tag{1.4.6}$$

We call v_j the numerical kernel of V_j and we call the integrand of (1.4.4) the operator kernel of V_j. In (1.4.6), μ_0 is the rest mass of the meson; we assume $\mu_0 > 0$. $\hat{h}$ is the Fourier transform of the space cutoff function h; h is assumed to be smooth with compact support and the coupling constant has been absorbed into h. V and each V_j are densely defined bilinear forms, since the numerical kernel v_j is a distribution. $V_0 + V_1$ is also a densely defined operator; this is related to the fact that T consists primarily of creation operators and to the more general fact that annihilators are often more tractable than creators.

To deal rigorously with the subtraction of one infinite quantity from another, we write the infinite quantities as limits of finite quantities, take the difference of the finite quantities and then take the limit of this difference. To find the finite quantities (whose limits are infinite), we introduce an approximate Hamiltonian $H_{\mathrm{ren}\,\sigma}$ with a momentum cutoff depending on a parameter σ. The coeficients α_i in $H_{\mathrm{ren}\,\sigma}$ are finite, depend on σ and generally tend to infinity as $\sigma \to \infty$; we set

$$H_{\mathrm{ren}\,\sigma} = H_0 + V_\sigma + \sum_i \alpha_{i\sigma} C_{i\sigma} \tag{1.4.7}$$

where

$$V_\sigma = \sum V_{j\sigma}$$

and $V_{j\sigma}$ has the numerical kernel

$$v_{j\sigma} = \begin{cases} v_j & \text{if} \quad |k_i| \leqq \sigma, 1 \leqq i \leqq 4 \\ 0 & \text{otherwise} \end{cases} \tag{1.4.8}$$

$C_{i\sigma}$ and $\alpha_{i\sigma}$ will be defined in § 4.1. The free Hamiltonian H_0 is

$$H_0 = \int a^*(k)\, \mu(k)\, a(k)\, dk\,.$$

For an operator or bilinear form W, we use the notation $W^\#$ to denote either W or W^*.

1.5 Products and Their Graphs

To an operator of the form

$$W = \int w(k_1, \ldots, k_l, k'_1, \ldots, k'_m) \prod_i a^*(k) \prod_j a(k'_j)\, dk\, dk' \tag{1.5.1}$$

we associate a graph (or diagram) with l lines (called legs) pointing to the left, m legs pointing to the right, and all legs issuing from a common vertex, see [1]. The graph specifies the number of creators and annihilators in W and W is determined by its graph together with its numerical kernel w. For example the graphs of V_3 and V_4 are given in Fig. 1.

Fig. 1

The product $W_2 W_1$ of two such operators may not have the same form because the creation and annihilation operators may occur in the wrong order, but by use of the commutation relations (1.4.2), $W_2 W_1$ can be written as a sum of terms of the form (1.5.1). The term with no δ function is called the Wick product and is denoted $:W_2 W_1:$. Its numerical kernel is $w_2 \otimes w_1$, or in other words the product of w_2 and w_1 regarded as functions of distinct variables. The term with $j\delta$ functions has a numerical kernel with j contractions and is denoted $W_2 \underset{j}{-\circ-} W_1$; its graph is obtained by connecting j annihilating (right) legs of the graph of W_2 each with a distinct creating (left) leg of the graph of W_1. We write the product

$$V_3 V_4 = :V_3 V_4: + V_3 \underset{1}{-\circ-} V_4$$

in graphs in Fig. 2. We will also encounter products

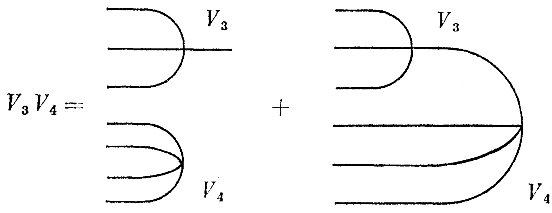


Fig. 2

$$W = W_n W_{n-1} \ldots W_1 \tag{1.5.2}$$

with $n > 2$ factors. If each W_j has the form (1.5.1) then the product can be expressed as a sum of terms of this form, again by use of the commutation relations (1.4.2). Each use of the commutation relations may introduce a δ function and if there are $j\,\delta$ functions in a term then we say that the term has j contractions. If the δ function arises in the commutation of operators $a^{\#}_{i}$ and $a^{\#}_{l}$ associated with the factors W_i and W_l then we say that the i and l factors (or vertices) have been contracted. Group into a single term Y all contributions to the product which have a given number $c(j, l)$ of contractions between each pair of factors W_j and W_l. Y has a graph with n ordered vertices. The j^{th} vertex together with the legs leaving it is identical with the diagram of W_j and if $j < l$ then $c(j, l)$ of the creating legs leaving the j^{th} vertex are joined with distinct annihilating legs leaving the l^{th} vertex. In other words the legs are connected according to the contractions $c(j, l)$ which define our term Y. Y also has a (numerical) integrand y obtained from $w_n \otimes \cdots \otimes w_1$ by equating contracted variables and then summing over all possible contractions which lead to the same graph. If the w_j are symmetric in their annihilating and in their creating variables then each term in the sum defining y is identical and we may multiply any term by the number of ways the contractions may be made. If we multiply y by appropriate factors $a^{\#}(k_1), \ldots$, we obtain the operator integrand. Y is uniquely determined by its graph and its integrand:

$$Y = \int y(k)\, a^{\#}(k_1) \ldots a^{\#}(k_l)\, dk\,. \tag{1.5.3}$$

The graph and the integrand contain more information than Y itself, since they express how Y is obtained as a term in the product (1.5.2). Y may be represented by several different graphs, for example it always has a graph with a single vertex. This single vertex graph of Y is obtained from the n vertex graph defined above by identifying all vertices and contracted (= internal) legs into a single point. We get the kernel of the single vertex into a single point. We get the kernel of the single vertex graph from y by integrating y over all contracted (= internal) variables.

A notable advantage of y and the n vertex graph over the single vertex graph for representing Y is that y may be finite (i.e. a finite valued measurable function) when Y is not. In fact Y is finite (i.e. a densely defined bilinear form) usually in just those cases when y is integrable as a function of its contracted variables. If Z is a second infinite bilinear form with the same graph and a finite integrand z, then we give a well defined meaning to the difference $Y - Z$ by subtracting the finite integrands: $y - z$ is the integrand of $Y - Z$. It is not necessary that Z be a term contributing to a product of n operators but only that z be a finite valued function of the same variables as y.

A subgraph of a graph is a subset of the vertices of the graph together with all legs coming from these vertices. Two subgraphs are called disjoint if they have no common vertices, although in general they may have legs in common. Legs which join two vertices are called internal and the others, which meet a vertex at one end only, are called external. A leg may be internal in the full graph but external with respect to a subgraph. If the graph has an integrand $y = \prod\limits_j w_j$ which is a product of the kernels associated with each of its vertices, then the subgraph has an integrand $y' = \prod_j' w_j$, where we multiply only over the vertices in the subgraph. If G is a graph and H a subgraph then we define the quotient graph G/H to be the graph obtained by identifying all vertices of H and all legs of H which are internal with respect to H. In general the vertices of G/H are not ordered. Let $I(H)$ be the set of variables of y corresponding to legs of G which are in ternal with respect to H. Then $\int\limits_{I(H)} y$, the integral of y over these variables, is the integrand associated with G/H and the pair G/H, $\int\limits_{I(H)} y$ defines the same operator as the pair G, y. The quotient G/G is the single vertex graph of Y constructed above.

Let Ξ be a measurable subset of the variables of Y. We call

$$Y' = \int\limits_{\Xi} y(k)\, a^{\#}(k_1) \ldots a^{\#}(k_l)\, dk \tag{1.5.4}$$

a truncation of Y. If we truncate each of the terms Y contributing to W, then the sum of the truncations is said to be a truncation of W or a truncated product. Thus a truncated product is specified by giving a measurable set for each graph which occurs in the product. Y itself is a truncated product, with all the Ξ's except one equal to the empty set $\emptyset$. $V_{j\sigma}$ is a truncation of V_j. The Wick product $:W_n \ldots W_1:$ is a truncated product as is the attached product

$$W_1 \text{-}\circ\text{-} W_2 = \sum_{j \geqq 1} W_1 \underset{j}{\text{-}\circ\text{-}} W_2 .$$

We also define the connected product, denoted

$$W_1 \underline{\angle} :W_2 \ldots W_n:$$

to consist of all terms in the ordinary product $W_1 :W_2 \ldots W_n$: in which each $W_j (2 \leqq j)$ has at least one leg contracted with W_1, see [1]. Let

$$:e^W: = \sum_{n=0}^{\infty} :W^n:/n! .$$

If W contains only creators or only annihilators then $e^W = :e^W:$. In [1], Friedrichs proved that

$$W_1 :e^W: = :(W_1 \underline{\angle} :e^W:)(:e^W:): \tag{1.5.5}$$

$$= \sum_{n=0}^{\infty} :(W_1 \underline{\angle} :W^n:)(:e^W:):/n! ;$$

for $n = 0$ we define

$$W_1 \underline{\angle} :W^n: = W_1 .$$

The proof of (1.5.5) is by manipulation of power series and the hypothesis is that series in (1.5.5) converge absolutely. We need (1.5.5) with $:e^W:$ on the left replaced by a truncated exponential. We write W as a sum of truncations of W

$$W = \sum W^{(j)}$$

and then form a truncated exponential $:e_T^W:$ in which $W^{(j)}$ occurs to at most the power $n(j)$. Then

$$W_1 :e_T^W:$$

is also the truncation of the right side of (1.5.5) in which $W^{(j)}$ occurs to at most the power $n(j)$.

For an operator W as in (1.5.1) we define a new operator ΓW with the same graph but with the new numerical kernel

$$\gamma w = \left(\sum_{i=1}^{l} \mu(k_i) \right)^{-1} w .$$

One can check that

$$H_0 \Gamma W - :(\Gamma W) H_0: = W$$

and

$$H_0 :e^{\Gamma W}: = :W(:e^{\Gamma W}:): + :(:e^{\Gamma W}:) H_0: \tag{1.5.6}$$

on a suitable domain. We need a trincated version of (1.5.6). If we replace $:e^{\Gamma W}:$ by $:e_T^{\Gamma W}:$ on the left then we must also truncate the right side so that $W^{(j)}$ occurs to a power at most $n(j)$.

It is important to notice that γw decreases more rapidly at infinity than w and consequently ΓW is better behaved than W is. For example ΓV_2 and ΓV_3 are densely defined operators while V_2 and V_3 are only bilinear forms. Similar but different Γ operations were introduced first by FRIEDRICHS [1] and later by the author in [2].

§ 2. Products of the ΓV_j's

2.1 Introduction

The kernels v_j and γv_j of the bilinear forms V_j and ΓV_j decrease at infinity, however the decrease is not sufficiently rapid to place v_j or γv_j in L_2. As a result,arbitrary products

$$(\Gamma V_{j_1})^{\#} (\Gamma V_{j_2})^{\#} \dots \tag{2.1.1}$$

need not be defined. (2.1.1) is a sum of Wick ordered terms, each term corresponding to a unique graph, and some of the terms may be infinite. In general the graph is not connected, and is a union of its connected components. The purpose of this section is to show that *as the number of vertices in each connected component of the graph increases, the decrease of the corresponding kernel at infinity becomes more rapid,* and when each component has three or more vertices then the kernel of that term is in L_2. This improvement of the kernels as the order of the graph increases seems to be characteristic of superrenormalizable theories and is basic to the methods of this paper. We also estimate kernels arising from products (2.1.1) where one or two of the factors are V_j's instead of ΓV_j's. Qualitatively we find the same behavior, namely that some terms in the product are infinite and that the remaining (finite) terms have kernels which decrease more rapidly as the number of factors in each connected component of the product increases. Since the Γ gives the kernel v_j of V_j an extra power of μ^{-1}, the product (2.1.1) with one or two V_j factors has more infinite terms and its finite terms require more complicated estimates than a similar product with no V_j factors.

We will also need to introduce kernels $\delta_{0\,\mathrm{ren}}$, $\delta'_{1\,\mathrm{ren}}$, $\delta''_{1\,\mathrm{ren}}$ and $\delta_{2\,\mathrm{ren}}$ which are functions of 2 variables and are bounded by

$$\delta_* (k_1, k_2) \leqq C_{\beta, N}\, \mu(k_1)^{\beta - 1/2} \mu(k_2)^{-1/2} \mu(k_1 \pm k_2)^{-N} \tag{2.1.2}$$

with $C_{\beta, N}$ a constant, $\beta > 0$, $N = 1, 2, \dots$ Let Δ_* be a bilinear form with kernel δ_* and let $\Gamma\Delta_*$ be a bilinear form with kernel $\gamma\delta_* = \mu(k_1)^{-1}\delta_*$. We now permit (2.1.1) to have an arbitrary number of $(\Gamma V_i)^{\#}$ and $(\Gamma\Delta_*)^{\#}$ factors ($2 \leqq i \leqq 4$) and either V_j^* and V_j ($0 \leqq j \leqq 4$) or Δ_*^* and Δ_* or neither as factors. We require that the factors occur in the following order. To the right are all ΓV_j and $\Gamma\Delta_*$ factors, next comes $\Delta_*^*\Delta_*$ or $V_j^* V_j$ and finally to the left are the $(\Gamma V_j)^*$ and $(\Gamma\Delta_*)^*$ factors.

Let Y be a term in a product (2.1.1) with integrand y. We label the internal variables of y as regular or divergent and among the divergent variables we have logarithmically, linearly and quadratically divergent variables. In a connected component of the graph not containing a $V_j^{\#}$ or $\Lambda_*^{\#}$ vertex the variables are regular if

$r1)$ the component is not Λ. (See Fig. 3.)

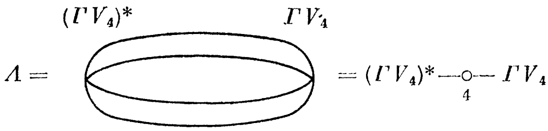


Fig. 3

The variables are logarithmically divergent if

$ln1)$ the component is Λ.

In a connected component containing one $V_j^{\#}$ vertex the variables are regular if both

$r2)$ there are at most 2 legs joining V_j with any other vertex;

$r3)$ the $V_j^{\#}$ component is not in Fig. 4b.

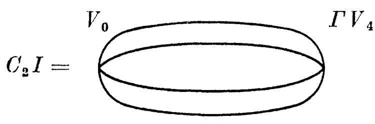


Fig. 4a

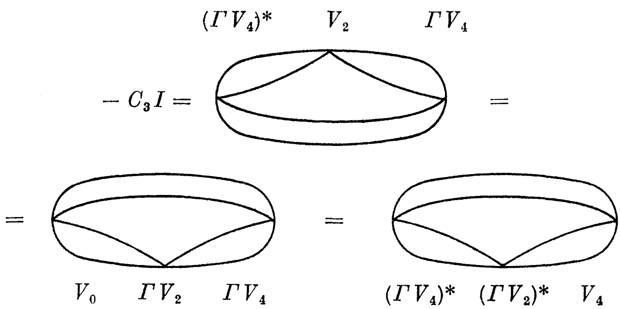


Fig. 4b

The variables joining $V_j^{\#}$ to a $(\Gamma V_l)^{\#}$ vertex are logarithmically divergent if

$ln2)$ there are 3 legs joining V_j with a $(\Gamma V_l)^{\#}$ vertex.

They are linearly divergent if

$l1$) there are 4 legs joining $V_j^{\#}$ with a $(\Gamma V_l)^{\#}$ vertex. (See Fig. 4a.) Each variable of the $V_j^{\#}$ component is logarithmically divergent if

$ln3$) the $V_j^{\#}$ component of the graph occurs in Fig. 4b.

If a component contains one $\Delta_*^{\#}$ vertex, then all variables are regular. If it contains 2 $\Delta_*^{\#}$ vertices, all variables are regular unless: $ln4$) there are 2 legs connecting these 2 $\Delta_*^{\#}$ vertices, in which case these variables are logarithmically divergent. If a connected component of the graph of Y contains V_j^* and V_j vertices then its variables are regular if $r2$) holds and if also

$r4$) there is at most one leg joining the V_j^* with the V_j vertex;

$r5$) the V_j^*, V_j component has external legs, or $(\Gamma\Delta_*)^{\#}$ vertices or $n \neq 4$ vertices.

If there are 2, 3 or 4 legs joining V_j^* and V_j ($ln5$), $l2$), $q1$)) then these variables are logarithmically, linearly or quadratically divergent. If $r2$) fails then we are in the case $ln2$) and these 3 variables are logarithmically divergent. All variables of the component not mentioned in $ln2$) or $ln5$) are logarithmically divergent if

$ln6$) both $l2$) and $r5$) fail.

The divergent graphs for a given model are related to the infinite renormalizations required. The divergent graphs with no $V_j^{\#}$ or $\Delta_*^{\#}$ vertices give infinite wave function renormalizations. If the graph has no external legs then the renormalization is division by an infinite constant:

$$\|\varphi\|_{\text{ren}}^2 = \|\varphi\|^2 e^{-\Delta} .$$

The divergent graphs with one $V_j^{\#}$ vertex are related to the infinite counter terms in the renormalized Hamiltonian. Again the graphs without external legs give infinite constant counter terms (the vacuum energy) and the graphs with external legs give infinite operators. In the model we are considering there are 4 such graphs ($ln2$)), they all have 2 external legs and they give the infinite mass renormalization counter term. The divergent graphs with two $V_j^{\#}$ or two $\Delta_*^{\#}$ vertices do not occur in the renormalization of the S matrix and are caused by the fact that domain of the renormalized Hamiltonian does not contain the simplest Fock space vectors one customarily works with; in fact it seems likely that the free and renormalized Hamiltonians have only the vector zero in their common domain. These domain divergences are cancelled when one works on the correct domain, and it is the role of the dressing transformation T to define this correct domain.

Formal arguments from perturbation theory predict the following picture. If there are no divergent graphs then H_0 and V have a dense common domain. As the perturbation becomes more singular, the domain graphs will be the first to become infinite. If the wave function and counter term graphs are finite then H_0 and $H_0 + V$ are operators but do not have a common dense domain, while $H_0^{1/2}$ and $(H_0 + V)^{1/2}$ do have a common dense domain. If both domain and counter term graphs are infinite then $H_0 + V$ is not an operator but H_{ren} is, and $H_0^{1/2}$ and ${H_{\text{ren}}}^{1/2}$ do not have a common dense domain. Finally if there are infinite graphs of all three types then H_0 and H_{ren} are operators on different Hilbert spaces.

There are a number of divergent graphs and subgraphs which are excluded by the restrictions on the order of the terms in (2.1.1). For example $\Gamma V_2 \underset{2}{-\!\circ\!-} (\Gamma V_2)^*$ is logarithmically infinite but is excluded from (2.1.1). Let n be the number of factors in (2.1.1) or in some subgraph under consideration.

Lemma 2.1.1. *If $n = 3$, if the factors are $V_{j_1}^{\#}$, $(\Gamma V_{j_2})^{\#}$ and $(\Gamma V_{j_3})^{\#}$ in some order and if there are no external legs then the graph of Y occurs in Fig.* 4b.

Proof. Suppose the factors occur in the order above. Then the second and third factors must be ΓV_{j_2} and ΓV_{j_3} because of the order and we can take the first to be V_{j_1} since $V_0 = V_4^*$, etc. We must have $j_1 = 0$ and $j_3 = 4$ to prevent external legs and then $j_2 = 2$ follows for the same reason. Thus we have one of the graphs of Fig. 4b and the other orderings of the factors lead to other graphs in Fig. 4b.

Lemma 2.1.2. *If $n = 3$ with factors V_j^*, V_j and $(\Gamma V_l)^{\#}$ then there are at least 4 external legs.*

Lemma 2.1.3. *If $n = 3$ with factors V_j^*, V_j and $(\Gamma \Delta_*)^{\#}$ then there are at least 2 external legs.*

Lemma 2.1.4. *If $n = 4$ with factors V_j^*, V_j, $(\Gamma \Delta_*)^{\#}$ and $(\Gamma V_j)^{\#}$ then there are at least 2 external legs.*

Proof. $V_j^* \underset{r}{-\!\circ\!-} V_j$ has as many creating legs, $4 - r$, as it has annihilating legs. Thus if there is a third vertex placed one side (Lemmas 2.1.2, 3) or two vertices on either side of $V_j^* \underset{r}{-\!\circ\!-} V_j$ with an unequal number of legs (Lemma 2.1.4), we cannot have all legs contracted. In case the single vertex has 4 legs, it can contract at most $4 - r$ times with $V_j^* \underset{r}{-\!\circ\!-} V_j$ leaving at least

$$4 - r + 4 - (4 - r) = 4$$

external legs.

We note that the number of external legs is always even. Next we analyze condition $ln6$). The factors must be $(\Gamma V_4)^* V_j^* V_j \Gamma V_4$ in that order. If there are r contractions between V_j^* and V_j, we have $V_j^* \underset{r}{-\!\circ\!-} V_j$ as a subgraph. $r = 4$ is excluded by connectedness and $r = 3$ is excluded by the hypothesis that $l2$) fail, but $0 \leqq r \leqq 2$ is possible. We have

$r \leqq j \leqq 4$, but the pairs $r = 0, j = 0$ or 4 are excluded by connectedness. All remaining pairs of r and j are possible. $ln 3$), $ln 5$) and $ln 6$) are the only cases in which variables of V_2 may be divergent.

2.2 Estimates on Products

Consider a term Y in (2.1.1) with graph G and integrand y. Let μ_q and μ_l be the largest of the energies $\mu(k)$ of the quadratically or linearly divergent variables of a given subgraph of type $q1$), $l1$) or $l2$). If there are no such variables, set $\mu_q = 1$, etc. μ_{ln} is defined as the smallest of the energies $\mu(k)$ of the divergent variables of a given subgraph of type $ln1$), . . ., or $ln6$). Let $\prod_e$ and $\prod_r$ denote products taken over the external or regular variables only, while $\prod_d$ is a product over the divergent subgraphs [of any possible type $ln1$), . . ., $q1$)]. Let $I = I(G)$, the set of internal variables of G, and let

$$\mu^\theta = \prod_e \mu^{-2} \prod_r \mu^\varepsilon \prod_d \mu_{ln}^{-\varepsilon'} \mu_l^{-1-\varepsilon'} \mu_q^{-2-\varepsilon'} .$$

Theorem 2.2.1. *There is an* $\varepsilon_0 > 0$ *and a constant* K *such that*

$$\left\| \int_I \mu^\theta \, |y| \right\|_2 \leqq K^n \tag{2.2.1}$$

if $\varepsilon \in (0, \varepsilon_0)$ *and* $\varepsilon' > 0$. *K depends on* ε' *but not on* ε.

Proof. n is the order of the graph. For small n we need only prove that the L_2 norm in (2.2.1) is finite; for $n = 1$ this is clear.

For large n we decompose the graph into a union $G = U_j H_j$ of disjoint subgraphs of bounded size. We prove that the norm $\left\| \int_{I(H_j)} \mu^{\theta_j} |y_j| \right\|_2$ associated with a subgraph is finite if its size n_j is not too small, for example if $n_j \geqq 5$. Since there are only a finite number of possible graphs or subgraphs of bounded size, the set of subgraph norms is finite and hence bounded. We now show that the norm (2.2.1) of the full graph can be estimated by the product of the norms of its subgraphs. Since there are at most n subgraphs, we get K^n as the bound on the norm of the full graph, as required in (2.2.1).

Let $y = y_1 y_2 \ldots$ where y_j is the integrand of H_j and let

$$\mu^\theta \, |y| = \prod_j \mu^{\theta_j} \, |y_j| \, .$$

Then

$$\begin{aligned} \left\| \int_I \mu^\theta \, |y| \right\|_2 &= \left\| \int_{I - U_j I(H_j)} \prod_j \int_{I(H_j)} \mu^{\theta_j} \, |y_j| \right\|_2 \\ &\leqq \int_{I - U_j I(H_j)} \left\| \prod_j \int_{I(H_j)} \mu^{\theta_j} \, |y_j| \right\|_2 \end{aligned} \tag{2.2.2}$$

where $\int\limits_{I-U_j I(H_j)}$ is the integral over the variables joining distinct subgraphs and the last L_2 norm is taken over the external variables only and is a function of the variables in $I - U_j I(H_j)$. Substitute

$$\left\| \prod_j \int\limits_{I(H_j)} \mu^{\theta_j} |y_j| \right\|_2 = \prod_j \left\| \int\limits_{I(H_j)} \mu^{\theta_j} |y_j| \right\|_2$$

in (2.2.2), where the L_2 norms on the right refer to the variables of H_j which are external in G. By the Schwartz inequality in the variables $I - U_j I(H_j)$ we have

$$\left\| \int\limits_I \mu^{\theta} |y| \right\|_2 \leqq \prod_j \left\| \int\limits_{I(H_j)} \mu^{\theta_j} |y_j| \right\|_2 \tag{2.2.3}$$

where the L_2 norms on the right now refer to the variables of H_j which are external in H_j. As a first application of (2.2.3) we choose the H_j to be the connected components of G. Then the theorem is true for G if it is true for each H_j and so without loss of generality we assume that G is connected.

We give a simplified description of how the subgraphs H_j will be chosen. We require that

$$\int\limits_{I(H_j)} \mu^{\theta_j} |y_j| \; \varepsilon L_2 \,. \tag{2.2.4}$$

Choose a connected subgraph H of minimum size m for which (2.2.4) holds. We will see that $m \leqq 5$ and so $G \sim H$ has at most 12 components. Let H_1 be the subgraph formed by H and all components of $G \sim H$ which do not satisfy (2.2.4). We will see that these components have at most 3 vertices each and so H_1 has at most 41 vertices. Now proceed by induction.

Let k be a regular variable in $I - U_j I_j$ connecting the j^{th} and l^{th} subgraphs. There is a factor $\mu(k)^{\varepsilon}$ to be placed in μ^{θ_j} or in μ^{θ_l}. Because of the assumed order of the factors in Y, $\mu(k)$ will occur in a γ factor $\left(\sum_{i=1}^{j} \mu_i\right)^{-1}$ at one or both of the vertices that its leg joins. There is one exception to this statement, which is when k is the contracted variable in the product $V_j^* \underset{1}{-\!\circ\!-} V_j$. If the vertex in the j^{th} subgraph has $\mu(k)$ in its γ factor then we place $\mu(k)^{2\varepsilon}$ in the product μ^{θ_j}. Otherwise we place $\mu(k)^{-\varepsilon}$ as a factor in μ^{θ_j} and we will deal with the exceptional case when it arises.

Let $\prod\limits_r$ denote a product over regular variables which are internal in the subgraph and let $\prod\limits_{e\pm}$ be a product over regular variables which are external to the subgraph but internal in the full graph and which $(+)$ occur or $(-)$ do not occur in a γ factor of a vertex of the subgraph. Let $\prod\limits_e$ be the product over the variables which are external in the full graph.

Then

$$\mu^{\theta_j} = \prod_e \mu^{-2} \prod_r \mu^{\varepsilon} \prod_{e+} \mu^{2\varepsilon} \prod_{e-} \mu^{-\varepsilon} \prod_d \mu_{\text{ln}}^{-\varepsilon'} \mu_l^{-1-\varepsilon'} \mu_q^{-2-\varepsilon'} . \tag{2.2.5}$$

We call a subgraph (or graph) *decomposable* if it can be represented as a union of disjoint connected proper subsubgraphs such that (2.2.4) holds for each subsubgraph; in making this new decomposition we do not require that the μ^{θ_j} factors of the subsubgraphs be given by (2.2.5), but only that their product give the correct factor μ^θ of the full subgraph, as defined above. If a subgraph is decomposable then (2.2.4) holds for the subgraph.

Let H be a subgraph of G, and define energy factors μ^{θ_1} for H and μ^{θ_2} for the complement $G \sim H$ so that $\mu^\theta = \mu^{\theta_1}\mu^{\theta_2}$. Let y_1 and y_2 be the integrands of H and $G \sim H$. Then

$$\int_{I(H)} \mu^{\theta_1} |y_1| \tag{2.2.6}$$

has the role of kernel times energy factors for the vertex H of G/H. In other words

$$\int_{I(G)} \mu^\theta |y| = \int_{I(G/H)} \mu^{\theta_2} |y_2| \int_{I(H)} \mu^{\theta_1} |y_1| . \tag{2.2.7}$$

Thus if (2.2.4) holds for a quotient of a graph, it holds for the graph also.

We abbreviate

$$k_1 + \cdots + k_j - k_{j+1} - \cdots - k_l$$

by $\sum \pm k_i$ and $\mu(k_i)$ by μ_i. The starting point for our detailed estimates is

$$|v_j(k)| \leqq C_N \prod_i \mu_i^{-1/2} \mu(\textstyle\sum \pm k_i)^{-N} \tag{2.2.8}$$

where C_N is a constant and $N = 1, 2, \ldots$ The $\mu(\sum \pm k_i)^{-N}$ comes from a bound on $\hat{h}(\sum \pm k_i)$; $\hat{h}$ is rapidly decreasing because h is assumed to be smooth. Also

$$|\gamma v_j(k)| \leqq C_N \left(\sum_{i=1}^{j} \mu_i \right)^{-1} \prod_i \mu_i^{-1/2} \mu(\textstyle\sum \pm k_i)^{-N} . \tag{2.2.9}$$

As a direct consequence we have

Lemma 2.2.1. $\gamma v_j \left(\prod_i \mu_i^{\varepsilon_i} \right) \mu_m^{-\varepsilon} \in L_2$ *if* $0 \leqq \varepsilon_i$ *and* $\sum \varepsilon_i < \varepsilon < 1/2$, $j = 3, 4$ or $j = 2$, $m = 3, 4$. *We note that*

$$\mu_2 \leqq \mu_1 + \mu(k_1 \pm k_2)$$

and so

$$\mu_1^{-1} \mu(k_1 \pm k_2)^{-1} \leqq 2\mu(0)^{-1} \mu_2^{-1} ,$$

$$\left.\begin{aligned} \mu_1^{-1} \delta_* &\leqq 2\mu(0)^{-1} \mu_2^{-1} \delta_* \mu(k_1 \pm k_2) \\ \mu_2^{-1} \delta_* &\leqq 2\mu(0)^{-1} \mu_1^{-1} \delta_* \mu(k_1 \pm k_2) . \end{aligned}\right\} \tag{2.2.10}$$

Similarly

$$\mu_1^{-1}\mu(\Sigma \pm k_i)^{-1} \leqq 2\mu(0)^{-1}\mu(k_2 \pm k_3 \pm k_4)^{-1}\,. \tag{2.2.11}$$

Lemma 2.2.2. $\mu_1^{\varepsilon_1}\mu_2^{\varepsilon_2}\gamma\delta_* \in L_2$ if $\varepsilon_1 + \varepsilon_2 < 1$;

$$\mu_1^{\varepsilon_1}\mu_2^{\varepsilon_2}\delta_* \in L_2 \quad \text{if} \quad \varepsilon_1 + \varepsilon_2 < 0\,.$$

In what follows, $B = \int b a^{\#}(k_1) \ldots a^{\#}(k_l)\, dk$ *will be the operator defined by a subgraph* H *and its integrand* b. *We use (2.2.5) to define the energy factor, which we denote* μ^θ.

Lemma 2.2.3. *Suppose* $B = (\Gamma V_{j_1})^{\#} \underset{r}{-\!\!\circ\!\!-} \Gamma V_{j_2}$ *and* $1 \leqq r \leqq 4$. *There is a positive* a, *independent of* ε, *such that*

$$\prod_{e+} \mu^a \int_{I(H)} \mu^\theta |b| \in L_2 \tag{2.2.12}$$

and for $r = 4$ *we may omit either or both* Γ's.

Proof. If $r = 4$ the variables are divergent but the integral is finite by Lemma 2.2.1 because $\mu^\theta = \mu_m^{-\varepsilon'}$ for some m. Let $r = 3$. Since the j_1 in ΓV_{j_1} is at least 2, we must have $\# = *$, or $B = (\Gamma V_{j_1})^* \underset{3}{-\!\!\circ\!\!-} \Gamma V_{j_2}$. Thus $\int_{I(H)} \mu^\theta |b|$ is bounded by

$$\begin{aligned} &C_N \mu_4^{2\varepsilon - 1/2} \mu_5^{2\varepsilon-1/2} \mu(k_4 \pm k_5)^{-N} \\ &\cdot \int \prod_{i=1}^{3} \mu_i^{\varepsilon - 1} \left(\sum_{i=1}^{3} \mu_i\right)^{-2} \mu\left(\sum_{i=1}^{4} \pm k_i\right)^{-N} dk_1\, dk_2\, dk_3 \\ &\leqq C_N \mu_4^{6\varepsilon - 3/2} \mu_5^{2\varepsilon - 1/2} \mu(k_4 \pm k_5)^{-N} \in L_2 \end{aligned} \tag{2.2.13}$$

because

$$\begin{aligned} &\left(\sum_{i=1}^{3} \mu_i\right)^{-1} \mu\left(\sum_{i=1}^{4} \pm k_i\right)^{-1} \\ &\leqq \text{const. } \mu\left(\sum_{i=1}^{3} \pm k_i\right)^{-1} \mu\left(\sum_{i=1}^{4} \pm k_i\right)^{-1} \\ &\leqq \text{const. } \mu_4^{-1} \end{aligned}$$

as in (2.2.11) and similarly we can transfer powers of the energy from k_5 to k_4. We use C_N to denote any constant depending on N and h but independent of ε.

We remark that if we leave out one or both Γ's and $r = 3$ then $B = V_{j_1} \underset{3}{-\!\!\circ\!\!-} \Gamma V_{j_2}$, $(\Gamma V_{j_1})^* \underset{3}{-\!\!\circ\!\!-} V_{j_2}$ or $V_{j_1} \underset{3}{-\!\!\circ\!\!-} V_{j_2}$ and $\int_{I(H)} \mu^\theta |b|$ is bounded by

$$C_N \prod_{e+} \mu^{2\varepsilon} \prod_{e-} \mu^{-\varepsilon}\, \mu_4^{-(\varepsilon'+1)/2}\, \mu_5^{-1/2}\, \mu(k_4 \pm k_5)^{-N}\,. \tag{2.2.14}$$

This is essentially the same bound that we have for δ_* and when $B = :V_{j_1} \underset{3}{-\!\!\circ\!\!-} V_{j_2}:$ (2.2.14) is in L_2. The internal variables are linearly divergent in this case and the factor $\mu_l^{-1-\varepsilon'}$ in μ^θ compensates for the missing Γ.

Let $r = 2$ and let k_1 and k_2 be the contracted variables. The γ factor in γv_{j_2} can be replaced by

$$
\begin{aligned}
&(\mu_1 + \mu_2)^{-1} && (j_2 = 2)\\
&(\mu_1 + \mu_2)^{-1+3\varepsilon}\, \mu_3^{-3\varepsilon} && (j_2 = 3)\\
&(\mu_1 + \mu_2)^{-1+6\varepsilon}\, \mu_3^{-3\varepsilon}\, \mu_4^{-3\varepsilon} && (j_2 = 4)\,.
\end{aligned}
$$

As in (2.2.11) we transfer $(\mu_1 + \mu_2)^{-1+9\varepsilon}$ to the remaining variables k_3 and k_4 of V_{j_2} or the remaining variables k_5 and k_6 of V_{j_1}. If μ_1 and μ_2 occur in the γ factor of γv_{j_1} we similarly transfer $(\mu_1 + \mu_2)^{-1+6\varepsilon}$ to the external variables. Thus $\int_{I(H)} \mu^\theta |b|$ is bounded by

$$
\begin{aligned}
&C_N \prod_{i=3}^{6} \mu_i^{-\varepsilon-1/2}\, \mu(k_3 \pm k_4)^{-1+9\varepsilon}\, \mu(k_5 \pm k_6)^{-1+6\varepsilon} \\
&\cdot \mu\left(\sum_{i=3}^{6} \pm k_i\right)^{-N} \in L_2\,.
\end{aligned}
\tag{2.2.15}
$$

Observe that if we leave out one Γ and $r = 2$ then the remaining Γ must include contracted variables, because of the restrictions on the order of the operators in B, and $\int_{I(H)} \mu^\theta\, |b|$ is bounded by

$$
C_N \prod_{i=3}^{6} \mu_i^{-\varepsilon-1/2}\, \mu(k_3 + k_4)^{-1+9\varepsilon}\, \mu\left(\sum_{i=3}^{6} \pm k_i\right)^{-N}, \tag{2.2.16}
$$

which is close to the bound on ΓV_2. If we leave out both Γ's then the contracted variables are logarithmically divergent and $\int_{I(H)} \mu^\theta\, |b|$ is bounded by

$$
C_N \prod_{i=3}^{6} \mu_i^{-\varepsilon-1/2}\, \mu\left(\sum_{i=3}^{6} \pm k_i\right)^{-N}, \tag{2.2.17}
$$

which is better than the bound that we have for V_j.

The case $r = 1$ is similar.

Suppose $V_j^{\#}$ and $\Delta_*^{\#}$ are not factors of (2.1.1) and let $n \geqq 2$. We decompose the graph of B into subgraphs of 7 types. The first three types consist of a central $(\Gamma\Delta_*)^{\#}$ vertex contracted with $0 \leqq r \leqq 2\,(\Gamma V_j)^{\#}$ vertices and the last four types of subgraphs consist of a central $(\Gamma V_l)^{\#}$ vertex contracted with $1 \leqq s \leqq 4\ (\Gamma V_j)^{\#}$ vertices; the $(\Gamma V_j)^{\#}$ vertices may also be contracted with one another. We choose the subgraphs $H_1, H_2, \ldots$ by induction so that the graph $G_j' = G \sim \bigcup_{i=1}^{j} H_i$ has no components consisting of a single $(\Gamma V_j)^{\#}$ vertex. If $H_1, \ldots, H_j$ have been chosen, we let H_{j+1} be a $(\Gamma\Delta_*)^{\#}$ vertex in G_j' together with all $(\Gamma V_j)^{\#}$ vertices in G_j' which, relative to G_j', are contracted only to that $(\Gamma\Delta_*)^{\#}$. For such a choice of H_{j+1}, there are $r \leqq 2$ such $(\Gamma V_j)^{\#}$ vertices, and for $r = 0$ (2.2.4) follows from Lemma 2.2.2. For $r = 1$ or 2 the subgraph is decomposable, which implies (2.2.4). To prove this, one uses Lemmas 2.2.1 and 2.2.2 and chooses the factors μ^{θ_j} so that the $(\Gamma V_j)^{\#}$ receive extra negative powers μ^{-a} of the energy. In the remaining case there are no $(\Gamma\Delta_*)^{\#}$ vertices in G_j' and we choose H_{j+1} to be a subgraph of one of the four remaining types, $1 \leqq s \leqq 4$, while preserving the induction hypothesis. If there is a G_j' vertex contracted to only one other G_j' vertex α, we take H_{j+1} to be α together with all G_j' vertices contracted only to α. Otherwise we take H_{j+1} to consist of two contracted vertices α, β together with all G_j' vertices contracted only to α and to β. For the fourth type of subgraph ($s = 1$), (2.2.4) follows from Lemma 2.2.3. Now let $2 \leqq s$. If $l = 3$ or 4 at the central, or $(\Gamma V_l)^{\#}$, vertex then the subgraph is decomposable by use of Lemmas 2.2.1 and 2.2.3 because by a suitable choice of the μ^{θ_j}, we can give $s - 1$ vertices some extra negative power μ^{-a} of the energy. Thus we suppose $l = 2$. For the same reason (i.e. the alternative is a decomposable subgraph) we may suppose that both of the variables of the central $(\Gamma V_2)^{\#}$ which are not in its γ factor are contracted to distinct $(\Gamma V_j)^{\#}$ vertices in the subgraph and that these two vertices are not contracted with each other. If $s = 2$ the only remaining possibility is

$$\Gamma V_2 \angle : \Gamma V_{j_1} \Gamma V_{j_2} : \tag{2.2.18}$$

or its adjoint as the operator corresponding to the subgraph. The next lemma shows that (2.2.4) holds in this case and that if $s = 3, 4$ then the subgraph is decomposable.

Lemma 2.2.4. *If B is given by* (2.2.18) *then*

$$\prod_{e+} \mu^a \int_{I(H)} \mu^\theta \, |b| \in L_2$$

for all small $\varepsilon > 0$ and some positive a, independent of ε.

Proof. $\int_{I(H)} \mu^\theta \, |b|$ is bounded by

$$\begin{aligned} C_N \prod_{i=1}^{8} \mu_i^{-\varepsilon-1/2} (\mu_1 + \mu_2)^{-1+6\varepsilon} (\mu_3 + \cdots)^{-1+9\varepsilon} \\ \cdot (\mu_6 + \cdots)^{-1+9\varepsilon} \mu(k_3 \pm k_4 \pm k_5)^{-1+\varepsilon} \\ \cdot \mu(k_6 \pm k_7 \pm k_8)^{-1+\varepsilon} \mu\left(\sum_{i=1}^{8} \pm k_i\right)^{-N} \end{aligned} \tag{2.2.19}$$

where k_1 and k_2 are uncontracted variables of ΓV_2, k_3, k_4, k_5 come from ΓV_{j_1} and k_6, k_7, k_8 come from ΓV_{j_2}. The theorem is proved in the present case.

As a second case suppose that there is one $\Delta_*^{\#}$ factor in Y. (Because of the reduction to terms with one connected component, it is possible to have exactly one $\Delta_*^{\#}$ factor.) Our basic lemma is

Lemma 2.2.5. *Let* $B = \Delta_* \underset{r}{-\circ-} \Gamma V_j$, $r = 1, 2$. Then

$$\prod_{e+} \mu^a \int_{I(H)} \mu^\theta |b| \in L_2$$

for all small $\varepsilon > 0$ *and some positive* a, *independent of* ε, *and we may include an uncontracted variable of* $\Delta_*^\#$ *in the product* $\prod_{e+}$.

Proof. If $r = 1$ then $\int_{I(H)} \mu^\theta |b|$ is bounded by

$$C_{\beta, N}\, \mu_1^{\beta + \varepsilon - 3/2} \prod_{i=2}^{4} \mu_i^{-\varepsilon - 1/2} (\mu_2 + \cdots + \mu_j)^{-1 + 9\varepsilon} \tag{2.2.20}$$

$$\cdot\, \mu \left(\sum_{i=1}^{4} \pm k_i \right)^{-N}$$

where k_1 is the uncontracted variable of Δ_*. We remark that δ_* could be replaced by the function (2.2.14) without affecting (2.2.20). If the Γ were omitted then $\int_{I(H)} \mu^\theta |b|$ would be bounded by

$$C_{\beta, N}\, \mu_1^{\beta + \varepsilon - 3/2} \prod_{i=2}^{4} \mu_i^{-\varepsilon - 1/2}\, \mu \left(\sum_{i=1}^{4} \pm k_i \right)^{-N},$$

which is better than the bound (2.2.9) on v_j.

If $r = 2$ then $\int_{I(H)} \mu^\theta |b|$ is bounded by

$$C_N \mu_1^{-\varepsilon - 1/2}\, \mu_2^{-\varepsilon - 1/2}\, \mu (k_1 \pm k_2)^{-N}, \tag{2.2.21}$$

which is as good as the bound (2.1.2) on δ_*.

Again we could replace δ_* by (2.2.14) and (2.2.21) would be unchanged. If we take $B = V_j \underset{2}{-\circ-} \Gamma \Delta_*$ then (2.2.21) is still a bound for $\int_{I(H)} \mu^\theta |b|$.

If $B = \Delta_*^\# \underset{r}{-\circ-} \Gamma \Delta_*$, $r = 1, 2$, then $\prod_e \mu^a \int_{I(H)} \mu^\theta |b| \in L_2$ where $\prod_e$ is a product over all external variables of b. The statement remains true if we replace the δ_* in $\Delta_*^\#$ by (2.2.14).

Consider the subgraph H of $(\Delta_*^\# \underset{r}{-\circ-} \Gamma \Delta_*)^\#$ or of $(\Delta_*^\# \underset{r}{-\circ-} \Gamma V_l)^\#$ or the full graph $G = H$ if Δ_* is contracted twice to $(\Gamma V_j)^* \underset{3}{-\circ-} \Gamma V_l$. Each component of $G \sim H$ has a single $(\Gamma \Delta_*)^\#$ vertex or a single $(\Gamma V_j)^\#$ vertex with a least one external leg or the component has at least two vertices (and no $V_j^\#$, $\Delta_*^\#$ vertices). Then (2.2.4) holds for H by the above lemma and remarks and (2.2.4) holds for $G \sim H$ by the previous case of the theorem. This proves the theorem in the present case. For later purposes we note that if H' is the subgraph formed by H and one or two $(\Gamma V_j)^\#$ vertices, each contracted at least twice to H, then (2.2.4) holds for H'.

Suppose that Δ_*^* and Δ_* are both factors in (2.1.1). If these operators are contracted twice to each other then $n = 2$ (by connectedness) and we are in the case $ln4$) of logarithmically divergent variables, $\mu^\theta = \mu_i^{-\varepsilon'}$, $i = 1$ or 2, $\int \mu^\theta |y|$ is finite by Lemma 2.2.2, and (2.2.4) follows. If these operators are contracted by one variable let H be the subgraph of $\Delta_*^* \underset{1}{-\circ-} \Delta_*$ and pass to the quotient graph G/H. In the quotient, H is a single vertex with a kernel (2.2.6) bounded by

$$C_{\beta, N}^2 \, \mu_1^{2\beta-\varepsilon-(3/2)} \, \mu_2^{-1/2} \, \mu(k_1 \pm k_2)^{-N} \, .$$

This is better than the bound on the kernel of Δ_* and so (2.2.1) follows from the case of a single Δ_* factor.

We have reduced to the case where $:\Delta_*^* \Delta_*:$ is a factor in (2.1.1). We split the graph into a disjoint union of subgraphs of the type previously considered, Δ_*^* and Δ_* belonging to distinct subgraphs. The theorem is proved in the present case.

Suppose one V_j is a factor in (2.1.1). The singular case $Y = V_0 \underset{4}{-\circ-} \Gamma V_4$ was already considered. Let H be the graph of $\left(V_j \underset{3}{-\circ-} \Gamma V_l\right)^{\#}$ or $\left(V_j \underset{2}{-\circ-} \Gamma \Delta_*\right)^{\#}$. If H is a subgraph of G then we pass to the quotient G/H. The integrand of H is estimated in (2.2.14) or (2.2.21) and the theorem is proved as in the case of one $\Delta_*^{\#}$ factor. Thus we suppose that V_j is contracted at most twice to any $(\Gamma V_l)^{\#}$ vertex and at most once to any $(\Gamma \Delta_*)^{\#}$ vertex. We also assume that V_j is contracted to at least two $(\Gamma V_l)^{\#}$ vertices because otherwise V_j will have at least two legs that are external in G or contracted to $(\Gamma \Delta_*)^{\#}$ vertices and if we take H to be the subgraph formed by V_j and adjacent $(\Gamma \Delta_*)^{\#}$ vertices, then the corresponding kernel (2.2.6) is bounded by

$$C_{\beta, N} \, (\mu_1 \mu_2)^{-\varepsilon-1/2} \, (\mu_3 \mu_4)^{\beta+3\varepsilon-5/2} \, \mu\left(\sum_{i=1}^{4} \pm k_i\right)^{-N} \tag{2.2.22}$$

and the theorem follows.

Let $n = 3$ with no external legs. Then we are in case $ln3$), with the help of Lemma 2.1.1 and the above reductions. We have $\int_I \mu^\theta |y|$ bounded by

$$C_N \int |\gamma v_4| \prod_{i=1}^{4} \mu_i^{-1/2} \, \mu(k_1 + k_2)^{-1} \, \mu\left(\sum_{i=1}^{4} k_i\right)^{-N} A \, dk$$

$$A = \begin{cases} \mu_j^{-\varepsilon'} \, \mu(k_1 + k_2)^{\varepsilon'/2}, \ 1 \leqq j \leqq 4, \text{ or} \\ \mu(k_1 + k_2)^{-\varepsilon'/2} \end{cases}$$

and (2.2.4) follows.

For general n we decompose the graph into a disjoint union of subgraphs. There is a subgraph H consisting of V_j contracted to a $(\Gamma V_{l_1})^{\#}$ and a $(\Gamma V_{l_2})^{\#}$ vertex, and H has at least 2 legs which are external in H.

Let H' be the subgraph formed by H and all $(\Gamma V_l)^{\#}$ vertices contracted four times to H. The parts of the theorem already proved give us (2.2.4) for $G \sim H'$. One can compute that the kernel (2.2.6) corresponding to H is bounded by one of the following:

$$C_N(\mu_1\mu_2)^{-\varepsilon-1/2}\,\mu(k_1 \pm k_2)^{-N} \tag{2.2.23}$$

$$C_N \prod_{e+} \mu^{3\varepsilon} \prod_{i=1}^{4} \mu_i^{-\varepsilon-1/2} A\,\mu\left(\sum_{i=1}^{4} \pm k_i\right)^{-N} \tag{2.2.24a}$$

$$A = \begin{cases} \mu(k_2 \pm k_3)^{3\varepsilon-1}\,\mu_4^{3\varepsilon-1} & \text{or} \\ \mu(k_1 \pm k_2)^{3\varepsilon-1}\,\mu(k_3 \pm k_4)^{3\varepsilon-1} \end{cases} \tag{2.2.24b}$$

$$C_N \prod_{e+} \mu^{3\varepsilon} \prod_{i=1}^{6} \mu_i^{-\varepsilon-1/2} A\,\mu\left(\sum_{i=1}^{6} \pm k_i\right)^{-N} \tag{2.2.25a}$$

$$A = \begin{cases} \mu(k_3 \pm k_4)^{3\varepsilon-2}\,\mu(k_5 \pm k_6)^{\varepsilon-1} & \text{or} \\ (\mu_2 + \cdots)^{-1}\,\mu(k_2 \pm k_3 \pm k_4)^{\varepsilon-1}\,\mu(k_5 \pm k_6)^{3\varepsilon-1} \end{cases} \tag{2.2.25b}$$

$$C_N \prod_{e+} \mu^{3\varepsilon} \prod_{i=1}^{8} \mu_i^{-\varepsilon-1/2} A\,\mu\left(\sum_{i=1}^{8} \pm k_i\right)^{-N} \tag{2.2.26a}$$

$$\begin{aligned} A = {} & (\mu_3 + \cdots)^{-1}\,\mu(k_3 \pm k_4 \pm k_5)^{\varepsilon-1}\,(\mu_6 + \cdots)^{-1} \\ & \cdot \mu(k_6 \pm k_7 \pm k_8)^{\varepsilon-1}\,. \end{aligned} \tag{2.2.26b}$$

The uncontracted variables of $(\Gamma V_{l_1})^{\#}$ occur in one of the factors $\mu(k_m \pm \cdots)^{-a}$ of A and the variables which do not occur in A come from V_j. One can check (2.2.4) for H' and the theorem is proved in the present case. For later purposes we note that (2.2.4) holds for any subgraph H'' formed by H and some of the $(\Gamma V_l)^{\#}$ vertices adjacent to H.

As our final case, we suppose that V_j^* and V_j are factors of (2.1.1). The divergent graph $Y = V_0 \underset{4}{-\circ-} V_4$ was considered in Lemma 2.2.3. The expression $V_j^* \underset{3}{-\circ-} V_j$ is also divergent and the corresponding integrand was bounded in (2.2.14). If H is the subgraph of $V_j^* \underset{3}{-\circ-} V_j$ then we pass to the quotient graph G/H and proceed as in the case of a single $\Delta_*^{\#}$ factor in (2.1.1). If $V_j^* \underset{2}{-\circ-} V_j$ occurs as a subgraph H then we also go to the quotient G/H and use (2.2.17) to bound the kernel of H; this case is then the same as a single $V_j^{\#}$ in (2.1.1). There are 3 types of divergent graphs containing a single V_j: $ln2$), $l1$) and $ln3$). The first two concern 2 vertex subgraphs of the quotient G/H, or 3 vertex subgraphs of G with V_j^*, V_j and $(\Gamma V_l)^{\#}$ vertices and at most 2 external legs. By Lemma 2.1.2, such a subgraph cannot occur in G and so $ln2$), $l1$) are impossible in G/H. The remaining case, $ln3$) in G/H, corresponds to a subcase of $ln6$) in G and both graphs have a logarithmic divergence. Thus the estimates for G/H imply (2.2.4) for G. We now suppose that V_j^* and V_j are contracted with each other at most once.

Next we consider the graph $ln6$). If V_j is contracted three times to $(\Gamma V_4)^{\#}$ then V_j^* is also contracted three times to the other $(\Gamma V_4)^{\#}$, and (2.2.1) follows from (2.2.14) (with $\varepsilon = 0$). If V_j is contracted twice to each $(\Gamma V_4)^{\#}$ then so is V_j^* and we have the bound

$$C_N \prod_{i=1}^{4} \mu_i^{-1/2} (\mu_1 + \mu_2)^{-1+\varepsilon'/4} \mu(\Sigma \pm k_i)^{-N}$$

on the kernel of $B = V_j^{\#} \underset{2}{-\circ-} \Gamma V_4$ or $(\Gamma V_4)^* \underset{2}{-\circ-} V_j^{\#}$. For one of these B's the factor μ^θ is $\mu_j^{-\varepsilon'}$ with k_j an external variable of B, and (2.2.1) is bounded by

$$C_N \int \prod_{i=1}^{4} \mu_i^{-1} (\mu_1 + \mu_2)^{-1+\varepsilon'/4} (\mu_3 + \mu_4)^{-1+\varepsilon'/4} \mu_j^{-\varepsilon'} \mu(\Sigma \pm k_i)^{-N} \, dk \, ,$$

which is finite. In the remaining case V_j is contracted twice to one of the $(\Gamma V_4)^{\#}$ and once to the other $(\Gamma V_4)^{\#}$ and the B above are still subgraphs of G; if the factor $\mu^\theta = \mu_j^{-\varepsilon'}$ occurs in an external variable of B then (2.2.1) is bounded by

$$C_N \int \prod_{i=1}^{4} \mu_i^{-1} (\mu_1 + \mu_2)^{-1+\varepsilon'/4} (\mu_2 + \mu_3)^{-1+\varepsilon'/4} \mu_j^{-\varepsilon'} \mu(\Sigma \pm k_i)^{-N} \, dk \, ,$$

which is finite. If the factor μ^θ occurs in an internal variable of one of the B's then (2.2.1) is bounded by the finite quantity

$$C_N \int |v_j| \prod_{i=1}^{4} \mu_i^{-1/2} \mu_1^{-\varepsilon'} (\mu_1 + \mu_2)^{-1+\varepsilon'/4} \mu_3^{-1+\varepsilon'/4} \mu(\Sigma \pm k_i)^{-N} \, dk \, .$$

We suppose that the $V_j^{\#}$ are not both contracted three times to single vertices since otherwise we use (2.2.14) and pass to a quotient graph. For the quotient, (2.2.1) follows from the case of two $\Delta_*^{\#}$.

The kernel (2.2.6) corresponding to

$$B = \left(V_j^* \underset{1}{-\circ-} V_j\right) \underset{3}{-\circ-} \Gamma V_l$$

is bounded by

$$C_N \prod_{i=1}^{4} \mu_i^{-\varepsilon - 1/2} A \mu \left(\sum_{i=1}^{4} \pm k_i \right)^{-N}$$

$$A = \begin{cases} \mu(k_1 \pm k_2)^{-1+8\varepsilon} & \text{or} \\ \mu_i^{-1+4\varepsilon} & \end{cases}$$

and in the second case, k_2, k_3 and k_4 belong to a single $V_j^{\#}$. This is better than the bound on V_j. If H is the corresponding subgraph then we proceed in G/H as in the case of one V_j. The graph $l1$) in G/H corresponds to $ln6$) in G and has been estimated. The graphs $ln2$) and $ln3$) in G/H satisfy (2.2.4) because of the extra decrease at infinity implied by the factor A. In the case $ln2$) and $A = \mu_1^{-1+4\varepsilon}$ we make use of the fact that one of the $V_j^{\#}$ is contracted three times with ΓV_l and so the other $V_j^{\#}$ is not contracted three times with a single vertex. Thus we suppose this H is not a subgraph of G. By similar reasoning we can suppose that the graphs of

$$(V_j^* \underset{1}{-\circ-} V_j) \underset{2}{-\circ-} \Gamma\Delta_*$$

$$:V_j^* V_j: \underset{4}{-\circ-} \Gamma V_4$$

are not subgraphs of G.

If a $V_j^{\#}$ is contracted twice with a $(\Gamma\Delta_*)^{\#}$ or three times with a $(\Gamma V_l)^{\#}$ vertex then the subgraph H formed by these two vertices has a kernel bounded by (2.2.21) or (2.2.14) and by the bound on δ_*. The quotient G/H is then in the case of one $V_j^{\#}$ vertex and one vertex, H, of $\Delta_*^{\#}$ type and by our above restrictions on G, these two vertices are not contracted to each other. The proof of (2.2.1) is essentially a combination of the individual cases of one $V_j^{\#}$ and of one $\Delta_*^{\#}$ previously considered.

We assert that in the remaining cases we can find disjoint subgraphs H and H^* as follows. $H^{\#}$ contains $V_j^{\#}$ and $r = 0, 1, 2$ vertices contracted with $V_j^{\#}$ and at least $2 - r$ legs of $V_j^{\#}$ are external in G. Then the kernel (2.2.6) of $H^{\#}$ is estimated by (2.2.22)–(2.2.26). Let H_1 be the subgraph formed by $H \cup H^*$ together with all $(\Gamma V_l)^{\#}$ vertices totally contracted to $H \cup H^*$. As before ,(2.2.4) is valid for $G \sim H_1$. We write $H_1 = H' \cup H^{*\prime}$ where $H^{\#\prime}$ contains $H^{\#}$ and some of the $(\Gamma V_l)^{\#}$ vertices contracted to $H^{\#}$ two or more times. (2.2.4) has already been proved for $H^{\#\prime}$ and so the theorem follows from the assertion.

Suppose $B = V_j^* \underset{1}{-\circ-} V_j$ is a factor in Y. Since B annihilates and creates three particles, we have B contracted with at least four vertices when no legs of B are external in G. In fact any vertex is contracted at most twice with B and two vertices are needed to contract the three annihilators of B, two more are needed to contract the three creating legs of B. There are at most two vertices in G which are contracted twice to B; if a vertex is contracted twice to V_j then it must be in H and then the remaining vertex in H is unique and H and H^* can be chosen as asserted. If two vertices are each contracted once to V_j and once to V_j^* then one goes in H and the other in H^* and H and H^* can be chosen; they can also be chosen in all remaining cases (if there are no external legs). For each missing vertex (of the four contracted to B above) there will be an external leg, and so the assertion is proved for this B.

Suppose $B = :V_j * V_j:$ is a factor in Y. Then B annihilates and creates four particles and at most three of these can be contracted to any one vertex. Thus if B has no external legs, it must be contracted to at least four vertices and H and H^* can be constructed as above. This proves the theorem (all cases).

§ 3. The Dressing Transformation

3.1 Introduction

The infinite wave function renormalization is due entirely to terms from V_4 and is caused by the fact that ΓV_4 is not an operator on $\mathscr{F}$. (ΓV_4 contains no annihilators and the kernel γv_4 is not in L_2.) Our dressing transformation T is built up from the bilinear forms V_4, V_3 and V_2, but for simplicity replace V_3 and V_2 by zero; then T would be a truncated approximation to the exponential $\exp(-\Gamma V_4)$. To compute the norm

$$\|T\varphi\|^2 = (\varphi, T^* T\varphi)$$

we expand the product

$$(\Gamma V_4)^* \, \Gamma V_4 \tag{3.1.1}$$

as a sum of five Wick ordered terms; each term has j contractions, $0 \leqq j \leqq 4$, and all terms except the last one, with 4 contractions, are (finite) densely defined bilinear forms. The exceptional term, illustrated in Fig. 3, is a multiple ΛI of the identity, with

$$\Lambda = 4!\,\|\gamma v_4\|^2 \tag{3.1.2}$$

infinite. Thus

$$(\Gamma V_4)^* \, \Gamma V_4 - \Lambda I \tag{3.1.3}$$

is a (finite) bilinear form. If we expanded

$$\exp(-\Gamma V_4 - \Lambda I/2)^* \exp(-\Gamma V_4 - \Lambda I/2) \tag{3.1.4}$$

in a formal power series, we would find that each term is a finite bilinear form, after cancellation of infinities, as in (3.1.3). The series appears not to converge, for reasons described in § 1.2. However using our truncated exponential T, we find that

$$(T e^{-\Lambda I/2})^* \, (T e^{-\Lambda I/2})$$

is a convergent series, each term of which is a bilinear form. Thus $e^{\Lambda/2}$ is the infinite part of the norm $\|T\varphi\|$ and is clearly independent of φ, while

$$\|T\varphi\| \, e^{-\Lambda/2} = \|T\varphi\|_{\text{ren}}$$

is finite and defines a Hilbert space norm on the range of T. In the $:\Phi^4:$ interaction in four dimensions, a more complicated wave function renormalization is required because the infinite part of $\|T\varphi\|$ depends on φ.

3.2 The Definition of T

Let the domain $\mathscr{D} = \mathscr{D}(T)$ be the set of all vectors $\varphi = \varphi_0, \varphi_1, \ldots$ in $\mathscr{F}$ with a finite number of particles ($\varphi_n = 0$ for large n) and bounded momentum $\left(\varphi_n(k) = 0 \text{ for large } \sum_{i=1}^{n} |k_i|\right)$. Let

$$Q = \Delta_{2\,\text{ren}} + V_3 + V_2' - (V_3 + V_2) -\!\circ\!- \Gamma V_4 + V_2 \angle (\Gamma V_4)^2/2\,. \tag{3.2.1}$$

In this formula $\Delta_{2\,\mathrm{ren}}$ has the form

$$\Delta_{2\,\mathrm{ren}} = \int \delta_{2\,\mathrm{ren}}(k_1, k_2)\, a^*(k_1)\, a^*(k_2)\, dk_1\, dk_2 \tag{3.2.2}$$

and $\delta_{2\,\mathrm{ren}}$ will be specified later and will satisfy (2.1.2). Also

$$V_2' = \int_{2(|k_3| + |k_4|) < |k_1| + |k_2|} v_2(k)\, a^*(k_1)\, a^*(k_2)\, a(k_3)\, a(k_4)\, dk\,. \tag{3.2.3}$$

As a formal, or untruncated series, we take

$$T^{\sim} = T_1 T_2$$

where

$$T_1 = \exp(-\Gamma V_4)$$

and

$$T_2 = \sum_{n=0}^{\infty} T_2^{(n)}$$

$$T_2^{(n)} = -\Gamma(Q T_2^{(n-1)})\,, \quad n \geqq 1$$

$$T_2^{(0)} = I\,.$$

Then Q and T_2 are formal solutions of the equations

$$(H_0 + V_2 + V_3 + V_4 + \Delta_{2\,\mathrm{ren}})\, T_1 = T_1(H_0 + Q + V_2 - V_2') \tag{3.2.4}$$

$$(H_0 + Q)\, T_2 = :T_2 H_0:\,. \tag{3.2.5}$$

We write

$$V_4 = \sum_{j=0}^{\infty} V_4{}^{(j)} \tag{3.2.6}$$

$$Q = \sum_{j=0}^{\infty} Q^{(j)} \tag{3.2.7}$$

where in $V_4{}^{(j)}$ the momentum k of largest magnitude is bounded as follows:

$$|k| \in \begin{cases} [2^j, 2^{j+1}), & j \geqq 1 \\ [0, 2), & j = 0\,. \end{cases} \tag{3.2.8}$$

$Q^{(j)}$ is defined by imposing the same restriction (3.2.8) on the momentum k of largest magnitude created by V_2, V_2', V_3 or $\Delta_{2\,\mathrm{ren}}$ (i.e. V_4 momenta are not considered).

We need two truncations to obtain T. $T^{\sim}$ is a power series in V_4 and because of (3.2.6), it is a power series in $V_4^{(j)}$. We retain in T only those terms which have a degree at most j in $V_4^{(j)}$. $T^{\sim}$ is also a power series in Q or in the $Q^{(j)}$. Furthermore the Q occur in a definite order (from right to left, in order of multiplication), and so the $Q^{(j)}$ also occur in a definite order. For each sequence $j_1, \ldots, j_n$ we have a unique contribution to T; in this term the Q to the extreme right (the first Q) is replaced by $Q^{(j_1)}$, etc. We retain in T only those terms for which the corresponding sequence $j_1, \ldots, j_n$ satisfies the conditions $1 \leqq j_1$ and

$$\left(\sum_{i=1}^{p-1} j_i\right)^{3/4} \leqq j_p, \quad 2 \leqq p \leqq n. \tag{3.2.9}$$

Roughly speaking, the sequence (3.2.9) are characterized by a strongly increasing property, and this accords with our general philosophly that the terms of T refer to sequences of events in which particles of progressively larger momentum are created. T is defined to be the sum of all terms in $T^{\sim}$ which are retained after the two truncations described above.

To determine the rate of growth of j_p with p implied by (3.2.9), we first note that $1 \leqq j_p$ and so $0(p^{3/4}) \leqq j_p$. Thus

$$0(p^{7/4}) \leqq \sum_{i=1}^{p-1} j_i$$

and $0(p^{21/16}) \leqq j_p$ for large p. Integrating again we find

$$0(p^{37/16}) \leqq \sum_{i=1}^{p-1} j_i$$

and $0(p^{27/16}) \leqq j_p$ for large p. Another integration yields $0(p^{15/8}) \leqq j_p$ and finally

$$p^2 \leqq j_p \tag{3.2.10}$$

for large p. Further integration would yield $p^{3-\varepsilon}$ as a lower bound on the growth of j_p.

We also define a cutoff dressing transformation T_σ. To define T_σ, replace V_4 by $V_{4\sigma}$, V_3 by $V_{3\sigma}$ etc. in the definition of T. $\Delta_{2\,\mathrm{ren}\,\sigma}$ will be defined by a kernel $\delta_{2\,\mathrm{ren}\,\sigma}(k_1, k_2)$ to be given later. The cutoff kernel will converge pointwise to $\delta_{2\,\mathrm{ren}}$ as $\sigma \to \infty$ and will be bounded as in (2.1.2) with the constant independent of σ.

We observe that T and T_σ are invertible because they have the form $I + A + B$ where A increases the number of particles and B preserves the number of particles but increases their total (free) energy by at least μ_0.

3.3 The Renormalized Inner Product

In this section we define the renormalized inner product on the range of T; we must prove that our definition (by means of a limiting process) makes sense and that it yields finite values. The inner product is then easily seen to be semidefinite and bilinear, because these properties are preserved by taking limits. The inner product is also definite, but this fact requires a proof. T is a densely defined unbounded linear transformation from $\mathscr{F}$ to $\mathscr{F}_{\mathrm{ren}}$, the Hilbert space completion of the range of T.

We remark that the annihilation and creation operators act in a natural fashion on $\mathscr{F}_{\mathrm{ren}}$ and that this representation appears to be inequivalent to the Fock representation. It would be of interest to study systematically the representations which can be constructed by these methods.

Q is a sum of six terms and each has a associated graph; the first three terms have only one vertex in the graph. The next two terms, V_3 -○- ΓV_4 and V_2 -○- ΓV_4, have two vertices in their graph; the vertices are ordered with the V_3 or V_2 vertex coming last (= left). The remaining term has three vertices with the V_2 vertex last, and the two identical ΓV_4 vertices are not ordered, relative to each other. The bilinear form T is also a sum of terms and each is associated with a unique graph. A graph will have $n = 0, 1, \ldots$ partially ordered vertices. The vertices coming from a single Q in the T_2 part of T inherit their order from Q, and otherwise the order is determined by the order of multiplication in $T^{\sim}$, but the identical T_1 vertices are not ordered, relative to each other. We note that the graph uniquely determines the corresponding term of T, once V and our definition of T is given. In the same way we can write $T^* T$ as a sum of terms and each term has a graph with partially ordered vertices and is uniquely determined by its graph. Some of the terms in $T^* T$ are infinite. In fact a term is infinite if and only if its graph has one or more connected components equal to Λ. (See Fig. 3.)

The terms of $T_\sigma^* T_\sigma$ are all finite and associated with the same graphs that occur for $T^* T$, but of course the operators corresponding to the same graph of $T_\sigma^* T_\sigma$ or of $T^* T$ are not the same because the integrands are not the same.

In addition to the products $T^* T$, we will need to consider $T^* V V T$ and $T^* V T$, for example. In order to cancel the infinities and obtain finite quantities, it is convenient to consider different parts of these products separately. Thus we let W be the bilinear form (1.5.1) and we let P be a truncation of the product $W T$. This means that for each graph of the product $W T$ we give some measurable set Ξ and we restrict the integration to Ξ. Such a truncated product P is too general to work with and so we suppose that Ξ depends only on the variables of the T_2 part of T and of the W component of the graph of T. We thus write $\Xi = \Xi' \cdot R^l$ where R^l is a Euclidean space and the variables mentioned above span a different Euclidean space, R^j, and $\Xi' \subset R^j$. As a further restriction on P we suppose that the set Ξ' is determined by the subgraph formed by vertices in T_2 and in the W component of the graph. Thus if these subgraphs are identical for two distinct graphs of P, then the sets Ξ' are assumed to be identical also.

We also suppose that a cutoff operator W_σ is defined by means of some given cutoff kernel w_σ; we require that $w_\sigma \to w$ pointwise and that

$$w^* = \sup_\sigma |w_\sigma|$$

be finite. Further conditions will be imposed on w_* later. The integrands r_σ associated with a term R_σ converge pointwise to a limit r. To obtain bounded convergence we replace $v_{j\sigma}$ by $|v_j|$, w_σ by w_* and $\delta_{2\,\mathrm{ren}\,\sigma}$ by the right side of (2.1.2). The resulting integrand b is called the *majorant* of r_σ; obviously

$$\sup_\sigma |r_\sigma| \leqq b\,.$$

If defined, B is the operator with integrand b.

Let P_σ be defined as the same truncation of the product $W_\sigma T_\sigma$. In other words, P_σ is defined by integrating the integrands of $W_\sigma T_\sigma$ over the same measurable sets Ξ used to define P.

For any bilinear form R defined by a graph and an integrand r, we let $|R|$ be the bilinear form with the same graph but with the integrand $|r|$. We have

$$|(\psi, R\,\varphi)| \leqq (|\psi|, |R|\,|\varphi|) \leqq (|\psi|, B\,|\varphi|)\,. \tag{3.3.1}$$

We let $R_{0\sigma}$ be a term from

$$T_\sigma^* T_\sigma, \qquad T_\sigma^* P_\sigma \quad \text{or} \quad P_\sigma^* P_\sigma \tag{3.3.2}$$

whose graph has no Λ components and we let $R_{j\sigma}$ be the term from (3.3.2) whose graph differs from the graph of $R_{0\sigma}$ by the inclusion of $j\Lambda$ components. $R_{0\sigma}$ is called a reduced term and its graph is called a reduced graph; the reduced graph of $R_{j\sigma}$ is the graph of $R_{0\sigma}$. We assert that

$$|(\psi, R_{j\sigma}\,\varphi)| \leqq \Lambda(\sigma)^j (j!)^{-1} (|\psi|, |R_{0\sigma}|\,|\varphi|)\,, \tag{3.3.3}$$

$$\Lambda(\sigma) = 4!\,\|\gamma\, v_{4\sigma}\|_2^2\,. \tag{3.3.4}$$

The proof of (3.3.3) is primarily combinatorial. Suppose that the graph of $R_{0\sigma}$ has n T_1^* vertices and m T_1 vertices, not counting vertices in the W^* or W components of T^* or of T. Then the graph of $R_{j\sigma}$ has $n + j$ and $m + j$ such vertices respectively. The integrand of $R_{0\sigma}$ acquires $(n!\,m!)^{-1}$ from the factorials in T_1^* and T_1 and the integrand of $R_{j\sigma}$ similarly acquires $((n + j)!\,(m + j)!)^{-1}$. In fact suppose the W component of T has l T_1 vertices. These vertices can be selected in $\binom{m+l}{l}$ ways from the $m + l$ T_1 vertices of T. Since the expontial contributes $(m + l)!^{-1}$ as a factor and since $(m + l)!^{-1}\binom{m+l}{l} = (m!\,l!)^{-1}$, the integrand $r_{0\sigma}$ acquires $m!^{-1}$ as asserted, the extra $l!^{-1}$ being absorbed into the integrand since it occurs in both $R_{0\sigma}$ and $R_{j\sigma}$. There are

$$\binom{n+j}{j}\binom{m+j}{j} j!\,4!^j$$

ways to contract $2j$ vertices into $j\Lambda$ components and each of the remaining legs must be contracted in a manner determined by the graph of $R_{0\sigma}$.

Thus

$$4!^j((n+j)!\,(m+j)!)^{-1}\binom{n+j}{j}\binom{m+j}{j}j! = 4!^j(j!\,n!\,m!)^{-1}$$

occurs as a factor in $R_{j\sigma}$ and if we replace $\Lambda(\sigma)$ by its integral definition (3.3.4), then $R_{j\sigma}$ and $\Lambda(\sigma)^j j!^{-1} R_{0\sigma}$ are bilinear forms with the same integrand, but integrated over different regions determined by the truncations. The truncation which defines P depends only on the variables in T_2 or in the W component of T and the measurable subset Ξ' of these variables determining the truncation is the same for $R_{0\sigma}$ and $R_{j\sigma}$ by hypothesis. Thus this truncation has the same effect on $R_{j\sigma}$ that it has on $\Lambda(\sigma)^j j!^{-1} R_{0\sigma}$. The truncations involved in the definition of T^* and T give $R_{j\sigma}$ a smaller region of integration. Hence (3.3.3) follows.

We sum over all graphs in (3.3.3) and obtain

$$e^{-\Lambda(\sigma)}\,|(\varphi, T_\sigma^* T_\sigma \psi)| \leqq (|\varphi|, \textstyle\sum B\,|\psi|) \tag{3.3.5}$$

where the summation extends over all reduced terms of the product $T_\sigma^* T_\sigma$. There is a similar estimate for the other operators in (3.3.2).

It will be necessary to consider a second type of truncation of the product $W T$. Let Θ be a measurable subset of R^8. Let

$$\Lambda(\sigma, \Theta) = 4!\,\|\gamma v_{4\sigma\Theta}\|_2^2 \tag{3.3.6}$$

$$v_{4\sigma\Theta}(k) = \begin{cases} v_{4\sigma}(k) & \text{if} \quad k \in \Theta \\ 0 & \text{otherwise}\,. \end{cases}$$

We substitute

$$v_{4\sigma\Theta} + v_{4\sigma\sim\Theta}$$

for $v_{4\sigma}$ in the definition of T and expand. To a graph with m ΓV_4 vertices there correspond 2^m terms and we give each term a graph by the simple expedient of labeling each of these m vertices as either a $\Gamma V_{4\Theta}$ vertex or a $\Gamma V_{4\sim\Theta}$ vertex. (We have hereby changed our definition of a graph.)

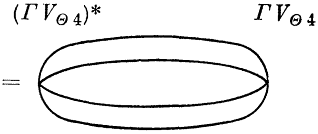
$\Lambda(\Theta) =$ 

Fig. 5

In defining our truncation P, we now permit $\Xi = \Xi' \cdot R^l$ to depend on the variables of the $v_{4\sigma\Theta}$ vertices as well as on the variables of T_2 and of the W component of the graph. As before we require that the set Ξ' be determined by the subgraph formed by the vertices above. In the product $T^* P$ or $P^* P$ the graphs do not have Λ components, but they may have $\Lambda(\Theta)$ or $\Lambda(\sim\Theta)$ components, see Fig. 5.

A reduced term R_σ is now a term whose graph has no $\Lambda(\sim\Theta)$ components and $\sum_{j=0}^{\infty} R_{j\sigma}$ is now the sum of all terms from $T_\sigma^* P_\sigma$ or $P_\sigma^* P_\sigma$ whose reduced graph (graph with $\Lambda(\sigma, \sim\Theta)$ components removed) equals the graph of $R_{0\sigma}$. As before we prove

$$e^{-\Lambda(\sigma,\sim\Theta)} \sum_{j=0}^{\infty} |(\varphi, R_{j\sigma}\psi)| \leqq (|\varphi|, B|\psi|) \tag{3.3.7}$$

$$e^{-\Lambda(\sigma,\sim\Theta)} |(\varphi, T_\sigma^* P_\sigma \psi)| \leqq (|\varphi|, \sum B\,|\psi|) \tag{3.3.8}$$

$$e^{-\Lambda(\sigma,\sim\Theta)} |(\varphi, P_\sigma^* P_\sigma \psi)| \leqq (|\varphi|, \sum B\,|\psi|) \tag{3.3.9}$$

where the summation on the right extends over the majorants B of all reduced terms of the product $T_\sigma^* P_\sigma$ or $P_\sigma^* P_\sigma$.

We now state the main results of this section.

Theorem 3.3.1. For φ and ψ in $\mathscr{D}$, the limit

$$\lim_{\sigma\to\infty} (T_\sigma\varphi, T_\sigma\psi)\, e^{-\Lambda(\sigma)} = (T\varphi, T\psi)_{\mathrm{ren}} \tag{3.3.10}$$

exists.

We define

$$\|T\varphi\|_{\mathrm{ren}}^2 = (T\varphi, T\varphi)_{\mathrm{ren}} = \lim \|T_\sigma\varphi\|^2\, e^{-\Lambda(\sigma)}\,. \tag{3.3.11}$$

Theorem 3.3.2. The inner product $(,)_{\mathrm{ren}}$ is positive definite on the range of T.

Let $\mathscr{F}_{\mathrm{ren}}$ be the Hilbert space formed by completing the range of T in the norm $\|\ \|_{\mathrm{ren}}$.

Theorem 3.3.3. Let P be a truncated product as above. Suppose that for all φ and ψ in $\mathscr{D}$ the limit

$$\lim_{\sigma\to\infty} (T_\sigma\varphi, P_\sigma\psi)\, e^{-\Lambda(\sigma)} = (T\psi, P\psi)_{\mathrm{ren}} \tag{3.3.12}$$

exists and that

$$\limsup_{\sigma} \|P_\sigma\psi\|^2\, e^{-\Lambda(\sigma)} \leqq (|\psi|, \sum B\,|\psi|) < \infty\,. \tag{3.3.13}$$

Then (3.3.12) defines $P\psi$ as an element of $\mathscr{F}_{\mathrm{ren}}$,

$$\|P\psi\|_{\mathrm{ren}}^2 \leqq \limsup_{\sigma} \|P_\sigma\psi\|^2\, e^{-\Lambda(\sigma)}$$

and PT^{-1} is a densely defined operator on $\mathscr{F}_{\mathrm{ren}}$ with domain $T\mathscr{D}$.

The third theorem follows from the first two, together with the Riesz representation theorem. The limits (3.3.10) and (3.3.12) are established using the bounded convergence theorem. If the truncation depends on a region Θ in the variables of the graph then we require (for the next two lemmas only) that Θ be bounded and we take σ larger than the magnitude of any variable in Θ. Consider a fixed reduced graph G and let $r_\sigma = r_{0\sigma}$ be the integrand of this graph. Let $r_{j\sigma}, j = 1, 2, \ldots$ be the integrands of the other terms having the same reduced graph. Let

$$\int_{\Lambda} r_{j\sigma} \tag{3.3.14}$$

be the result of integrating $r_{j\sigma}$ over the variables in the Λ components of its graph $\left(r_{0\sigma} = \int_{\Lambda} r_{0\sigma}\right)$. Then $r_{0\sigma}$ and $\int_{\Lambda} r_{j\sigma}$ are functions of the same variables. Let

$$r_{G\sigma} = \left(\sum_{j=0}^{\infty} \int_{\Lambda} r_{j\sigma}\right) e^{-\Lambda(\sigma)} .$$

We know that $r_{0\sigma}$ converges pointwise to a limit, r_0, and is bounded by the function b,

$$|r_{0\sigma}| \leqq b . \tag{3.3.15}$$

Lemma 3.3.1. *$r_{G\sigma}$ converges pointwise to a limit r_G and is bounded by* the majorant of $r_{0\sigma}$:

$$|r_{G\sigma}|, |r_G| \leqq b . \tag{3.3.16}$$

Now suppose the bound (3.3.13) has been proved. Each reduced graph G, with its associated integrand $r_{G\sigma}$, contributes a finite number of terms to (3.3.12), each of which is an integral of $\varphi r_{G\sigma} \psi$ over the variables of φ, $r_{G\sigma}$ and ψ. (Of course some of these variables will coincide; the ones which coincide change from term to term.) By Lemma 3.3.1 the integrands $\varphi r_{G\sigma} \psi$ converge pointwise. Consider the family of all integrands $\{\varphi r_{G\sigma} \psi\}$, where all possible reduced graphs G occur and all terms associated with a given graph G occur. The family $\{\varphi r_{G\sigma} \psi\}$ is a σ dependent function on a measure space (the direct sum of the measure spaces R^l associated with each term) and as $\sigma \to \infty$, the function converges pointwise. Now by (3.3.13) and (3.3.16), the σ dependent functions are bounded by a fixed (σ independent) function in L_1. By the bounded convergence theorem, the functions converge in L_1 and the limit (3.3.12) exists. We have proved

Lemma 3.3.2. *The limit* (3.3.12) *follows from* (3.3.13). *If we take $W = I$ then $P_\sigma = T_\sigma$ is just a special case and so Theorem 3.3.1 will follow from a bound on the right side of* (3.3.5).

Proof of Lemma 3.3.1. We have already proved that $|r_{G\sigma}| \leqq |r_{0\sigma}|$, so (3.3.16) follows from (3.3.15). Let

$$\exp_j(x) = \sum_{l=0}^{j} x^l/l!$$

and

$$\Lambda(\sigma, \Theta_j) = \Lambda_j(\sigma) = 4!\, \|\gamma v_{4\sigma}^{(j)}\|_2^2$$

[see (3.2.6), (3.2.8)]. We let $\xi = k_1, \ldots$ denote the variables of $r_{0\sigma}$ and we choose a fixed value of ξ. If j is large relative to ξ then $\Gamma V_4^{(j)}$ does not occur (has order zero) in a power series expansion of $r_{0\sigma}$. According to the truncations in the definition of T, $\Gamma V_{4\sigma}^{(j)}$ has order at most j in

$$\sum_{l=0}^{\infty} \int_{A} r_{l\sigma}(\xi) . \tag{3.3.17}$$

As in the proof of (3.3.3), we see that (3.3.17) is a product of $\exp_j(A_j(\sigma))$ times a function in which $\Gamma V_{4\sigma}^{(j)}$ has order zero, and for $\tau > \sigma$,

$$\sum_{l=0}^{\infty} \int_{A} r_{l\tau}(\xi) = \prod_{j} \exp_{j}(A_j(\tau) - A_j(\sigma)) \sum_{l=0}^{\infty} \int_{A} r_{l\sigma}(\xi) .$$

Thus

$$|r_{G\sigma}(\xi) - r_{G\tau}(\xi)| = |r_{G\sigma}(\xi)|$$
$$\cdot \left|1 - \exp\left(\sum_{j} (- A_j(\tau) + A_j(\sigma))\right) \prod_{j} \exp_{j}\left(A_j(\tau) - A_j(\sigma)\right)\right| .$$

Let

$$a_j = \exp(- A_j(\tau) + A_j(\sigma)) \exp_{j} (A_j(\tau) - A_j(\sigma)) .$$

Then $0 < a_j \leqq 1$ and $a_j = 1$ unless $\sigma \leqq 2^{j+1} \leqq 2\tau$ and

$$\begin{aligned}
|r_{G\sigma}(\xi) - r_{G\tau}(\xi)| &= |1 - \prod a_j|\, |r_{G\sigma}(\xi)| \\
&\leqq \sum (1 - a_j)\, |r_0(\xi)| \\
&\leqq \left(\sum (A_j(\tau) - A_j(\sigma))^{j+1}/(j+1)!\right) |r_0(\xi)| \\
&\leqq \left(\sum_{\sigma \leqq 2^{j+1}} \|\gamma v_4^{(j)}\|_2^{2j+2}/(j+1)!\right) |r_0(\xi)|\, 4!^{2j+2} \\
&= o(1)\, |r_0(\xi)|
\end{aligned}$$

since $\|\gamma v_4^{(j)}\|_2$ is bounded uniformly in j. This completes the proof.

Let b be the majorant of a reduced term R_σ in a product $T_\sigma^* T_\sigma$ or $P_\sigma^* P_\sigma$. Define $\prod_{e} \mu^{-2} \int_{I} b$ as in § 2.2; $I = I(G)$ is the set of internal variables of b or of r_σ. If R_σ has n vertices then

$$(|\psi|, B|\varphi|) \leqq L(n!)^L \left\| \prod_{e} \mu^{-3} \int_{I} b \right\|_2 , \tag{3.3.18}$$

where L is a constant which does not depend on R or σ, but which does depend on ψ and φ. We note that there are at most $L(n!)^L$ graphs with n vertices (with a new constant L).

Proof of Theorem 3.3.1. We assert that for some $\gamma > 0$ and some constant K,

$$\left\| \prod_{e} \mu^{-3} \int_{I} b \right\|_2 \leqq K(n!)^K\, 2^{-\gamma n^{3/2}} . \tag{3.3.19}$$

With a new constant L, we have from (3.3.18) and (3.3.19)

$$\begin{aligned}
\sum (|\psi|, B|\varphi|) &\leqq \sum_{n} L(n!)^{L_2 - \gamma n^{3/2}} \\
&\leqq L \sum_{n} e^{n L \operatorname{Log} n}\, 2^{-\gamma n^{3/2}} < \infty .
\end{aligned}$$

This gives a bound on the right side of (3.3.5) and the theorem follows by Lemma 3.3.2.

To prove (3.3.19) we introduce an order in the ΓV_4 and $(\Gamma V_4)^*$ vertices. Let m_j be the largest of the magnitudes of the momenta of the j^{th} ΓV_4 vertex. Then m_j is a function of the variables of r and the inequalities

$$m_1 \leqq m_2 \leqq \cdots \tag{3.3.20}$$

define a subset of the range of the variables of r. We estimate the contribution to the L_2 norm (3.3.14) coming from this subset (3.3.20) and a similar subset of the variables of the $(\Gamma V_4)^*$. Since there are $(n!)^2$ orderings and subsets, this is no loss of generality.

Now we use the truncations in the definition of T. For any value of the variables of r_σ for which $b \neq 0$, there are at most

$$\sum_{l=1}^{j-1} l = j(j-1)/2$$

ΓV_4 vertices from T with the corresponding m_l less than 2^j. Thus

$$2^j \leqq m_{1+j(j-1)/2} \quad \text{or} \quad b = 0$$

and so

$$2^{j^{1/2}} \leqq m_j \quad \text{or} \quad b = 0\,, \tag{3.3.21}$$

for large j. To get an upper bound on (3.3.19) we increase the region of integration by replacing the truncation in the ΓV_4 variables by (3.3.21) (for large j). We apply the same reasoning to the other variables in r_σ. For the p^{th} V_3, V_2 or $\Delta_{2\,\text{ren}}$ vertex, we have

$$2^{p^2} \leqq |k|$$

if $|k|$ is the largest of the magnitudes of the momenta of the created particles [cf. (3.2.10)]. Thus in the region (3.3.21). $1 \leqq \mu(m_j)^{\varepsilon_2 - \varepsilon j^{1/2}}$ and so the contribution of the region (3.3.20) to (3.3.19) is bounded by

$$\left\| \prod_e \mu^{-2} \int_I \prod \mu^{\varepsilon} b \right\|_2 \prod_{j=1}^{n/4} 2^{-\varepsilon j^{1/2}} K_1^n \leqq \left\| \prod_e \mu^{-2} \int_I \prod \mu^{\varepsilon} b \right\|_2 2^{-\gamma n^{3/2}} K_1^n$$

where $\prod \mu^{\varepsilon}$ is a product over the internal variables of b. There are no divergent variables in the graph of b, and so the quantity above is

$$\left\| \prod_e \mu^{-2} \int_I \mu^{\theta} b \right\| 2^{-\gamma n^{3/2}} K_1^n \leqq K^n 2^{-\gamma n^{3/2}}$$

by Theorem 2.2.1, and the proof is complete.

Our proof of Theorem 3.3.1 also gives us

Theorem 3.3.3. (3.3.13) *follows from the bound*

$$\sup \left\{ K^{-n} \left\| \prod_e \mu^{-2} \int_I \prod \mu^{\varepsilon} b \right\|_2 \right\} \leqq c < \infty \tag{3.3.22}$$

where the sup is taken over the majorants b of all reduced terms of the product $P^* P$. Also

$$\|P\psi\|_{\mathrm{ren}}^2 \leqq \text{const. } c \tag{3.3.23}$$

where the constant depends only on ψ, ε, K and the number of variables in w.

Proof of Theorem 3.3.2. Let

$$\varphi = 0, \ldots, 0, \varphi_n, \varphi_{n+1}, \ldots$$

with $\varphi_n \neq 0$, $\varphi_j \in \mathscr{F}_j$, $\varphi \in \mathscr{D}$. Let

$$a = \inf\left\{\sum_{i=1}^{n} \mu(k_i) : k \in \text{suppt. } \varphi_n\right\}$$

and let φ_n' be φ_n times the characteristic function of the set

$$\sum_{i=1}^{n} \mu(k_i) \in [a, a + \mu_0] .$$

Choose ϱ large and write

$$\theta = T_\sigma \varphi = \theta(\varrho) + (\theta - \theta(\varrho)) \tag{3.3.24}$$

where the j particle component, $\theta(\varrho)_j$, of $\theta(\varrho)$ has exactly n particles whose total free energy $\sum_{i=1}^{n} \mu(k_{l_i})$ is in $[a, a + \mu_0]$ and the remaining $j - n$ particles have energy $\mu(k) \geqq \varrho > a + \mu_0$. The rest of θ, $\theta - \theta(\varrho)$, violates this condition and so (3.3.24) is an orthogonal decomposition and

$$\|T_\sigma \varphi\|^2 \geqq \|\theta(\varrho)\|^2 .$$

Terms contribute to $\theta(\varrho)$ as follows. Terms from Q act on φ annihilating and creating low energy ($\mu < \varrho$) or high energy ($\mu \geqq \varrho$) particles, but eventually all low energy particles (except $m \leqq n$ of them) must be annihilated. Then ΓV_4^j from the T_1 part of T acts, and creates $n - m$ low energy particles ($\sum \mu \leqq a + \mu_0$) and the rest of the particles it creates must have high energy ($\mu \geqq \varrho$). The quantity $\|\theta(\varrho)\|^2$ can be written as a sum of contributions from the graphs of $T^* T$. Let

$$\Theta_\varrho = \{k_1, \ldots, k_4 : \mu(k_i) < \varrho \text{ for some } i\} .$$

By a reduced graph we mean a graph with no $\Lambda(\sim \Theta_\varrho)$ component. First we estimate contributions to $\|\theta(\varrho)\|^2$ whose reduced graph contains at least on T vertex with a particle of high energy ($\mu \geqq \varrho$). This contribution is bounded by

$$(|\varphi|, \sum B|\varphi|) \exp(\Lambda(\sigma, \sim \Theta_\varrho))$$

where we sum over the relevant reduced graphs, and it is bounded by

$$\text{const.} \exp(\Lambda(\sigma, \sim \Theta_\varrho)) \sup\left\{K^{-n} \left\|\prod_e \mu^{-2\frac{1}{2}} \int_I \prod \mu^\varepsilon b\right\|_2\right\}$$

$$\leqq \varrho^{-\varepsilon} \text{ const.} \exp(\Lambda(\sigma, \sim \Theta_\varrho)) \sup\left\{K_1^{-n} \left\|\prod_e \mu^{-2} \int_I \prod \mu^{2\varepsilon} b\right\|_2\right\}$$

with the sup taken over the relevant reduced graphs. The $\varrho^{-\varepsilon}$ occurs because there is at least one variable k with $|k| \geqq \varrho$ whenever $b \neq 0$. The reduced graph may contain $\Lambda(\Theta_\varrho)$ components, but because of our definition of $\theta(\varrho)$, such a component is integrated only over $\Theta_{a+\mu_0}$ and there can be at most n of these components. The integral over each of these $\Lambda(\Theta_\varrho)$ components is bounded by

$$4! \int_{\Theta_{a+\mu_0}} |\Pi \mu^\varepsilon \gamma v|^2$$

which is finite and bounded independently of ϱ, so that the integral over all of the $\Lambda(\Theta_\varrho)$ components is bounded by K^n. By Theorem 2.2.1, the integral over the remaining variables of the graph is also bounded by K^n and so the sup above is finite for small ε and large K_1. Thus with a new constant, independent of ϱ,

$$\text{const.}\ \varrho^{-\varepsilon} \exp(\Lambda(\sigma, \sim \Theta_\varrho)) \tag{3.3.25}$$

is a bound for the contribution to $\|\theta(\varrho)\|^2$ which we are estimating.

The remaining contribution to $\|\theta(\varrho)\|^2$ is the leading contribution and must be bounded from below. We assert that it is the sum of all terms with reduced graph $\emptyset$, the empty set. These terms are independent of $\varphi - \varphi_n'$ because $\varphi - \varphi_n'$ contains too many particles or particles with the wrong momenta. To occur in $\theta(\varrho)$ these extraneous particles must be annihilated by operators from Q. However Q increases the number of particles or their energy and so Q makes matters worse unless the new particles created by Q have high energy ($\mu \geqq \varrho$); in the latter case there is a vertex in the reduced graph with a high energy particle. Thus the remaining contribution to $\|\theta(\varrho)\|^2$ comes only from φ_n'. If Q annihilates particles from φ_n' it must either create new particles of high energy or else new particles incompatible with the definition of $\theta(\varrho)$. For a term to occur in $\|\theta(\varrho)\|^2$, these new particles (in the second case) must be annihilated and replaced by high energy particles. In either case the reduced graph will have at least one high energy particle. Thus our contribution to $\|\theta(\varrho)\|^2$ contains only ΓV_4 and $(\Gamma V_4)^*$ vertices and these vertices are integrated only over the region $\sim \Theta_\varrho$. If the reduced graph is not $\emptyset$ there will again be high energy particles, so our assertion is proved and the reduced graph is $\emptyset$. Hence our contribution to $\|\theta(\varrho)\|^2$ can be written in closed form as

$$\prod_j \exp_j(\Lambda(\sigma, \Theta_j \sim \Theta_\varrho)) \|\varphi_n'\|^2 . \tag{3.3.26}$$

For small j, $\Theta_j \sim \Theta_\varrho = \emptyset$ and as in the proof of Lemma 3.3.1, we see that (3.3.26) is

$$\exp(\Lambda(\sigma, \sim \Theta_\varrho)) (\|\varphi_n'\|^2 + o(1)) .$$

Combining this with our previous estimate (3.3.25), we have

$$\begin{aligned}\|T_\sigma \varphi\|^2 e^{-A(\sigma)} &\geqq \|\theta(\varrho)\|^2 e^{-A(\sigma)} \\ &\geqq \exp(-A(\sigma, \Theta_\varrho))\,(\|\varphi_n'\|^2 - o(1) - \text{const.}\ \varrho^{-\varepsilon})\,.\end{aligned}$$

We choose ϱ large enough so that

$$0 < \|\varphi_n'\|^2 - o(1) - \text{const.}\ \varrho^{-\varepsilon}$$

and then let $\sigma \to \infty$. Since $A(\Theta_\varrho)$ is finite, this completes the proof.

§ 4. The Definition of H_{ren}

4.1 Introduction

The cutoff Hamiltonian is given by the formula

$$H_{\text{ren}}(\sigma) = H_0 + V_\sigma + \Delta(\sigma) + c_2(\sigma) I + c_3(\sigma) I \tag{4.1.1}$$

where

$$c_2(\sigma) = 4!\int |v_{0\sigma}|^2 \left(\sum \mu_i\right)^{-1} dk \tag{4.1.2}$$

$$c_3(\sigma) I = -\ V_{0\sigma} \underset{4}{-\!\circ\!-}\ \Gamma(V_{2\sigma} \underset{2}{-\!\circ\!-}\ \Gamma V_{4\sigma}) \tag{4.1.3}$$

$$\Delta(\sigma) = \delta_m^2(\sigma) \int :\Phi_\sigma(x)^2: h^2(x)\, dx\,, \tag{4.1.4}$$

see Fig. 4. Here $\Phi_\sigma(x)$ is the cutoff field

$$\Phi_\sigma(x) = \int_{|k| \leqq \sigma} e^{ikx} \mu^{-1/2}(a^*(-k) + a(k))\, dk\,.$$

To define $\delta_m^2(\sigma)$ we first let $\zeta = (k_1 + k_2 + k_3)/3$. Then $3^{1/2}\zeta$ is the distance from k_1, k_2, k_3 to the linear space $\zeta = 0$ and $k_1 - \zeta$, $k_2 - \zeta$, $k_3 - \zeta$ is the perpendicular projection of k_1, k_2, k_3 onto that linear space. Let

$$\begin{aligned}\delta m^2(\sigma) = \delta_f m^2 + 4(4!) \int_{Z(\sigma)} \prod_{i=1}^{3} \mu(k_i - \zeta)^{-1} \\ \cdot \left(\sum_{i=1}^{3} \mu(k_i - \zeta)\right)^{-1} d_\zeta k\end{aligned} \tag{4.1.5}$$

where $d_\zeta k$ is proportional to Euclidean measure on the space $\zeta = 0$,

$$3\, d\zeta\, d_\zeta k = dk_1\, dk_2\, dk_3\,,$$

and $Z(\sigma)$ is the subset of the space $\zeta = 0$ defined by

$$|k_i - \zeta| \leqq \sigma\,, \quad 1 \leqq i \leqq 3\,. \tag{4.1.6}$$

$\delta_f m^2$ is any finite number. It represents a finite renormalization to be determined at some later point in the development of the theory. We write

$$\Delta(\sigma) = \Delta_0(\sigma) + \Delta_1(\sigma) + \Delta_2(\sigma)$$

where $\Delta_j(\sigma)$ is the part of $\Delta(\sigma)$ which creates j particles. As $\sigma \to \infty$, $\delta m^2(\sigma)$ becomes logarithmically infinite, and so

$$\Delta = \lim_{\sigma \to \infty} \Delta(\sigma)$$

is infinite also. Let

$$\left.\begin{aligned} D_0(\sigma) &= V_{0\sigma} \underset{3}{-\!\circ\!-} \Gamma V_{3\sigma} \\ D_1'(\sigma) &= V_{0\sigma} \underset{3}{-\!\circ\!-} \Gamma V_{4\sigma} \\ D_1''(\sigma) &= V_{1\sigma} \underset{3}{-\!\circ\!-} \Gamma V_{3\sigma} \\ D_2(\sigma) &= V_{1\sigma} \underset{3}{-\!\circ\!-} \Gamma V_{4\sigma} \\ D(\sigma) &= D_0(\sigma) + D_1'(\sigma) + D_1''(\sigma) + D_2(\sigma)\,, \end{aligned}\right\} \tag{4.1.7}$$

see Fig. 6, for example. D is also infinite and its infinite

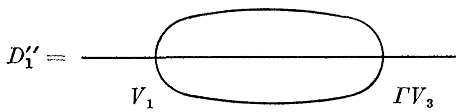


Fig. 6

part coincides with Δ. This means that the operators

$$\left.\begin{aligned} \Delta_{\mathrm{ren}}(\sigma) &= \Delta(\sigma) - D(\sigma) \\ \Delta_{0\,\mathrm{ren}}(\sigma) &= \Delta_0(\sigma) - D_0(\sigma) \\ \Delta'_{1\,\mathrm{ren}}(\sigma) &= 2^{-1}\Delta_1(\sigma) - D_1'(\sigma) \\ \Delta''_{1\,\mathrm{ren}}(\sigma) &= 2^{-1}\Delta_1(\sigma) - D_1''(\sigma) \\ \Delta_{2\,\mathrm{ren}}(\sigma) &= \Delta_2(\sigma) - D_2(\sigma) \end{aligned}\right\} \tag{4.1.8}$$

have finite limits as $\sigma \to \infty$, as we will prove in § 4.5.

The main result of this paper is

Theorem 4.1.1. *H_{ren} is a densely defined symmetric operator on $\mathscr{F}_{\mathrm{ren}}$ approximated by the cutoff Hamiltonians in the sense that*

$$\lim_{\sigma} (T_\sigma \psi, H_{\mathrm{ren}}(\sigma)\, T_\sigma \varphi)\, e^{-\Delta(\sigma)} = (T\psi, H_{\mathrm{ren}}\, T\varphi)_{\mathrm{ren}} \tag{4.1.9}$$

$$\|H_{\mathrm{ren}}(\sigma)\, T_\sigma \psi\|^2\, e^{-\Delta(\sigma)} \leq \text{const.} \tag{4.1.10}$$

with the constant independent of σ. The domain of H_{ren} is $T\mathscr{D}$.

To prove the theorem we consider separately different terms contributing to H_{ren}. We show that

$$\Delta_{0\,\mathrm{ren}},\ \Delta'_{1\,\mathrm{ren}},\ \Delta''_{1\,\mathrm{ren}} \tag{4.1.11}$$

$$V_2 + V_3 + V_4 + \Delta_{2\,\mathrm{ren}} + H_0 \tag{4.1.12}$$

$$V_1 + 2^{-1}\Delta_1 + \Delta_2 - \Delta_{2\,\mathrm{ren}} \tag{4.1.13}$$

$$V_0 + \Delta_0 + 2^{-1}\Delta_1 + c_2 I + c_3 I \tag{4.1.14}$$

are each operators on $\mathscr{F}_{\mathrm{ren}}$ and are approximated by cutoff operators.

4.2 *Finite Contributions to* H_{ren}

We consider truncations P and P_σ of products WT and $W_\sigma T_\sigma$ under the assumptions of § 3. In particular let:

A. $W = V_0$ and the truncation omits all graphs of the following types

a. $l1)$ is a subgraph

b. $ln3)$ is a subgraph and the V_2 vertex of this subgraph is the last T_2 vertex of the graph

c. V_0 has 3 legs contracted to a ΓV_4 in the T_1 part of T (a subcase of $ln2)$)

d. V_0 has 3 legs contracted to a ΓV_3 and this ΓV_3 is the last Q vertex of the graph (a subcase of $ln2)$).

B. $W = V_1$ and the truncation omits all graphs in which

a. V_1 has 3 legs contracted to a ΓV_4 in the T_1 part of T (a subcase of $ln2)$)

b. V_1 has 3 legs contracted to a ΓV_3 vertex and this vertex is the last Q vertex of the graph (a subcase of $ln2)$).

C. $W = V_2$, V_3 or Δ_{ren} and the truncation restricts the integration to the set

$$\left.\begin{aligned} &|k| \in [2^{j_p}, 2^{j_p+1}), \quad |l_i| \in [2^{j_i}, 2^{j_i+1}) \\ &j_p \leqq \left(\sum_{i=1}^{p-1} j_i\right)^{3/4} \end{aligned}\right\} \tag{4.2.1}$$

where k is the momentum of largest magnitude created by W, l_i is the momentum of largest magnitude created by the i^{th} of the V_2, V_3 or $\Delta_{2\,\text{ren}}$ vertices [cf. (3.2.9)] and $p-1$ is the number of these vertices.

D. $W = \Delta_{0\,\text{ren}}$, $\Delta'_{1\,\text{ren}}$ or $\Delta''_{1\,\text{ren}}$ with no truncation ($P = WT$).

Theorem 4.2.1. *In cases $A, \ldots, D$, PT^{-1} is a densely defined operator on $\mathscr{F}_{\text{ren}}$ approximated by $P_\sigma T_\sigma^{-1}$.*

Proof. We need only establish (3.3.22); for graphs with no divergent variables this follows from Theorem 2.2.1. Thus D is proved because the only divergent graph, $ln4)$, cannot occur in $P^* P$ in view of the fact that the Δ's in D create at most one particle. $ln1)$ never occurs because we consider only reduced graphs. In C we have

$$\begin{aligned} \mu(k)^\varepsilon &\leqq \mu(k)^{-1-\varepsilon}\, 2^{2((\Sigma j_i)^{3/4}+1)} \\ &\leqq \mu(k)^{-1-\varepsilon}\, 2^{2^{-1}\varepsilon \Sigma j_i} \leqq \mu(k)^{-1-\varepsilon} \prod_i \mu(l_i)^{\varepsilon/2} \end{aligned}$$

for large $|k|$ and again (3.3.22) follows from Theorem 2.2.1. In B we have to consider the remaining cases in $ln2)$, and we have to take into account the fact that the γ factors in the definition of T_2 have the form

$$\left(\sum_{i=1}^{l} \mu_i\right)^{-1}$$

where the summation ranges not only over the variables of a single vertex but also includes uncontracted variables of preceding vertices. As a typical case, consider graphs with V_1 contracted with 3 legs of a ΓV_4 vertex in the T_2 part of T and let the other leg of this ΓV_4 be contracted to an annihilating leg of a ΓV_3 vertex, for example. Let

$$y_1 = \left(\sum_{i=1}^{3} \mu_i\right)^{-1} v_3(k_1, k_2, k_3, k_7)\, v_4(k_4, \ldots, k_7)$$

$$b_1 = \left(\sum_{i=1}^{6} \mu_i\right)^{-1} |v_3(k_1, k_2, k_3, k_7)\, v_4(k_4, \ldots, k_7)|$$

and recall that y_1 is a factor of y in Theorem 2.2.1 while b_1 is a factor of b in Theorem 3.3.4. Now

$$\prod_{i=1}^{7} \mu_i^{\varepsilon} b_1 \leqq \text{const.} \left(\prod_{i=1}^{3} \mu_i^{7\varepsilon}\right) \left(\prod_{i=4}^{6} \mu_i^{-\varepsilon}\right) \mu_7^{\varepsilon}\, |y_1|$$

and so (3.3.22) follows from Theorem 2.2.1. The remaining cases are similar and the theorem is proved.

4.3 Renormalizing the Creation Part of V

We show that (4.1.12) is an operator on $\mathscr{F}_{\text{ren}}$. There are no infinite constants in (4.1.12) and this operator is renormalized merely by the choice of the new domain $T\mathscr{D}$ disjoint from the domain if H_0. We break the product $H_0 T$ into five parts (truncated products). The first part, $P_1 = {:}TH_0{:} = {:}H_0 T{:}$ is an operator from $\mathscr{D}$ to $\mathscr{F}_{\text{ren}}$ for essentially the same reason that T is. If we write the full product $H_0 T\varphi$ as

$$\sum_i \mu_i T\varphi \tag{4.3.1}$$

with the sum extending over all variables of $T\varphi$, then the Wick product ${:}H_0 T{:}$ is obtained by restricting the range of the summation to variables of φ which have not been contracted in forming the product $T\varphi$. We note that the variables in $T\varphi$ have been symmetrized and so it has no meaning to say that a variable in $T\varphi$ comes from φ. However the sum $H_0 = \sum_i \mu_i$ is symmetric and so it commutes with symmetrization; we apply H_0 to the unsymmetrized product and symmetrize later. The second part, P_2, of the product $H_0 T$ comes from restricting the summation in (4.3.1) to variables in the T_1 part of T. Again we must apply H_0 before symmetrization in order that this make sense. In each of the remaining parts P_3, P_4 and P_5 we sum over all remaining variables, those from the T_2 part of T, but we admit only the terms in which the last T_2 vertex is V_3 (in case of P_3), or V_2 (in case of P_4) or $\Delta_{2\,\text{ren}}$ (in case of P_5). We show that

$$V_4 + P_2 T^{-1}, \quad V_3 + P_3 T^{-1}, \quad V_2 + P_4 T^{-1}, \quad \Delta_{2\,\mathrm{ren}} + P_5 T^{-1} \tag{4.3.2}$$

are each operators on $\mathscr{F}_{\mathrm{ren}}$, as required. In the absense of the truncation in T, these operators would be zero (with the exception of $V_2 + P_4 T^{-1}$), because of (3.2.4) and (3.2.5). The ΓV_4 truncation in T does not affect the last three operators in (4.3.2) because $V_4^{(j)}$ occurs to the same power in $V_3 T$ and P_3 or in $V_2 T$ and P_4 or in $\Delta_{2\,\mathrm{ren}}\, T$ and P_5. Thus $V_3 + P_3 T^{-1}$ and $\Delta_{2\,\mathrm{ren}} + P_5 T^{-1}$ contain only terms introduced by the truncation (3.2.9) and they are exactly the operators treated in C of § 4.2. $V_2 + P_4 T^{-1}$ is handled in two parts. The first is the operator of C, § 4.2. The second part is $P T^{-1}$ where P is a truncation of the product $(V_2 - V_2') T$ and the truncation is defined by omitting all terms in which $V_2 - V_2'$ is contracted with a T_1 vertex. This part of $V_2 + P_4 T^{-1}$ arises from the presence of V_2' (or the absence of $V_2 - V_2'$) in (3.2.1). To show that $P T^{-1}$ is an operator we must verify (3.3.22) for reduced graphs of $P^* P$. The divergent graphs $ln 3$) and $ln 6$) of § 2.1 are excluded by the truncation and $ln 5$) is the only remaining divergent graph involving a V_2 vertex. In $ln 5$), two legs of $V_2^* = V_2$ are contracted with the other V_2 vertex. Let k_1 and k_2 be these divergent variables of V_2 and let k_3 and k_4 be the other variables. Then

$$\prod_{i=1}^{4} \mu_i^{\varepsilon} \leqq \text{const. } \mu_1^{-\varepsilon} \mu_2^{-\varepsilon} \mu_3^{5\varepsilon} \mu_4^{5\varepsilon}$$

for values of momenta contributing to $V_2 - V_2'$, by use of the definition (3.2.3) of V_2' and so (3.3.22) follows from Theorem 2.2.1.

The sum $V_4 T + P_2$ is nonzero because of the ΓV_4 truncation in T. In fact terms in $V_4 T + P_2$ of order $j + 1$ in $V_4^{(j)}$ cannot cancel since P_2 contains terms of order at most j in $V_4^{(j)}$. However the terms in $V_4 T$ of order j or less in $V_4^{(j)}$ cancel exactly with the corresponding terms in P_2. Thus $V_4 T + P_2 = P$ is a truncation of the product $V_4 T$. To describe this truncation we let $P_2^{(j)}$ be the result of restricting the summation in (4.3.1) to variables of $\Gamma V_4^{(j)}$ vertices from the T_1 part of T and we let

$$P^{(j)} = V_4^{(j)} T + P_2^{(j)} .$$

Then

$$P = \sum P^{(j)} \tag{4.3.3}$$

and $P^{(j)}$ is the truncation of $V_4^{(j)} T$ defined by retaining only terms of order j in $\Gamma V_4^{(j)}$. We assert that for $\varphi \in \mathscr{D}$,

$$\| P^{(j)} \varphi \|_{\mathrm{ren}} \leqq \text{const. } K^j [(j/2)!^{-1} + 2^{-\varepsilon j^2/2}] \tag{4.3.4}$$

with a constant independent of j. Each $P^{(j)} T^{-1}$ will be shown to be an operator on $\mathscr{F}_{\mathrm{ren}}$ by Theorems 2.2.1, 3.3.3 and 3.3.4 as before and by (4.3.3), (4.3.4), $P T^{-1} = V_4 + P T^{-1}$ is an operator also. Our proof of

(4.3.4) will also give us

$$\|P_\sigma^{(j)}\varphi\| \, e^{-\Lambda(\sigma)/2} \leqq \text{const. } K^j [(j/2)!^{-1} + 2^{-\varepsilon j^2/2}] \tag{4.3.5}$$

with a constant independent of j and σ. Then for φ and ψ in $\mathscr{D}$ we have

$$\begin{aligned} &|(T_\sigma\psi, P_\sigma\varphi)\, e^{-\Lambda(\sigma)} - (T\psi, P\varphi)_{\text{ren}}| \\ &\quad \leqq \sum_j |(T_\sigma\psi, P_\sigma^{(j)}\varphi)\, e^{-\Lambda(\sigma)} - (T\psi, P^{(j)}\varphi)_{\text{ren}}| \\ &\quad \leqq \sum_{j=1}^{J} |(T_\sigma\psi, P_\sigma^{(j)}\varphi)\, e^{-\Lambda(\sigma)} - (T\psi, P^{(j)}\varphi)_{\text{ren}}| \\ &\quad + \sum_{j=J+1}^{\infty} (|(T_\sigma\psi, P_\sigma^{(j)}\varphi)|\, e^{-\Lambda(\sigma)} + |(T\psi, P^{(j)}\varphi)_{\text{ren}}|) \,. \end{aligned}$$

We choose J so that the second term is small, uniformly in σ, and then the first term is small for large σ by Lemma 3.3.2. Thus $V_4 + P_2 T^{-1}$ is approximated by cutoff operators, as required for Theorem 4.1.1.

We now prove (4.3.4). Let $T^{(l,m)}$ be the sum of all terms in T which have $l\ \Gamma V_4^{(j)}$ vertices in the T_1 part of their graph and $m\ \Gamma V_4^{(j)}$ vertices in the T_2 part of their graph and let

$$P^{(j,l)} = V_4^{(j)} (-\Gamma V_4^{(j)})^l \, l!^{-1} T^{(0,j-l)} \,.$$

Then

$$P^{(j)} = \sum_{l=0}^{j} P^{(j,l)} \,. \tag{4.3.6}$$

We want to estimate (3.3.13) for the operator $P^{(j)}$. Recall that the reduced graphs may contain $(\Gamma V_4^{(j)})^* \mathbin{-\!\circ\!-} \Gamma V_4^{(j)} = \Lambda_j$ components, but cannot contain $\Lambda - \Lambda_j$ components. We call a graph completely reduced if it has neither $\Lambda - \Lambda_j$ nor Λ_j components. Then (3.3.13) for $P^{(j)}$ is bounded by

$$(|\varphi|, \sum_n \Lambda_j^n n!^{-1} \sum B(j, l, l', n)\, |\varphi|) \,, \tag{4.3.7}$$

where $0 \leqq n \leqq \min\{l, l'\} \leqq j$ in the first sum and the second sum extends over all majorants $B(j, l, l', n)$ of completely reduced terms of the product

$$\begin{aligned} &(l-n)!^{-1} [V_4^{(j)} (-\Gamma V_4^{(j)})^{l-n} T^{(0,j-l)}]^* \\ &\cdot (l'-n)!^{-1} [V_4^{(j)} (-\Gamma V_4^{(j)})^{l'-n} T^{(0,j-l')}] \,. \end{aligned} \tag{4.3.8}$$

The convergent graphs contribute at most

$$\text{const.} \sum_{n=0}^{j} \Lambda_j^n \, n!^{-1} \, 2^{-2\varepsilon j (j-n)} \tag{4.3.9}$$

to (4.3.7) by Theorems 2.2.1 and 3.3.4. The factor $2^{-2\varepsilon j(j-n)}$ arises as follows. There are $2(j-n)$ vertices, the $\Gamma V_4^{(j)}$ vertices, with a lower cutoff at $|k| = 2^j$. In each of these $2(j-n)$ variables use the estimate

$$\mu(k)^\varepsilon \leqq \mu(k)^{2\varepsilon} 2^{-\varepsilon j} .$$

For large j and for $n \leqq j/2$, we have

$$\mu(k)^\varepsilon \leqq \mu(k)^{-2-\varepsilon} 2^{2(l+\varepsilon)(j+1)} \leqq \mu(k)^{-2-\varepsilon} 2^{\varepsilon j(j-n)} .$$

We substitute this in (3.3.22) and estimate graphs with divergent subgraphs $ln2$), $ln3$), $ln5$), $ln6$), $l1$), $l2$) and $q1$) by Theorem 2.2.1. As in (4.3.9), there is a factor $2^{-2\varepsilon j(j-n)}$ from the lower cutoffs in the $\Gamma V_4^{(j)}$ vertices. This factor dominates the factor $2^{\varepsilon j(j-n)}$ above and these graphs contribute

$$\text{const.} \sum_{n=0}^{j/2} A_j^n n!^{-1} 2^{-\varepsilon j^2/2} \tag{4.3.10}$$

to (4.3.7). For $n \geqq j/2$ and for large j,

$$\mu(k)^\varepsilon \leqq \mu(k)^{-2-\varepsilon} 2^{2(1+\varepsilon)(j+1)} \leqq \mu(k)^{-2-\varepsilon} 8^j$$

and so the graphs above contribute

$$\text{const.}\ (j/2)\ 8^j (A_j + 1)^j (j/2)!^{-1} \tag{4.3.11}$$

to (4.3.7). We add (4.3.9)–(4.3.11) to get the bound

$$\text{const.}\ j\ 8^j (A_j + 1)^j [(j/2)!^{-1} + 2^{-\varepsilon j^2/2}]$$

for (3.3.13) and $\|P^{(j)}\varphi\|_{\text{ren}}$. Since A_j is bounded independently of j, we have shown that $P^{(j)} T^{-1}$ is an operator on $\mathscr{F}_{\text{ren}}$ and we have proved (4.3.4).

4.4 Renormalizing the Annihilation Part of V

We prove that (4.1.13) and (4.1.14) are operators with the required properties. By Theorem 4.2.1 we may consider instead

$$V_1 + D_1'' + D_2 , \tag{4.4.1}$$

$$V_0 + D_0 + D_1' + c_2 I + c_3 I . \tag{4.4.2}$$

Let P_{Aa} be the terms omitted from WT in § 4.2, Aa, and define P_{Ab}, etc. similarly. Let P_{Aa} be the corresponding term from $V_0^{(j)} T$ and let

$$c_2^{(j)} = 4! \int |v_0^{(j)}|\ |\gamma v_4^{(j)}|\ dk .$$

Because of the cancellation we find

$$\begin{aligned} P_{Aa}^{(j)} + c_2^{(j)} T &= \sum_{l=0}^{j} c_2^{(j)} T^{(l,j-l)} \\ &= \sum_{l=0}^{j} c_2^{(j)} (-\Gamma V_4^{(j)})^l\, l!^{-1} T^{(0,j-l)} \end{aligned}$$

in the notation of § 4.3. The quantity (3.3.22) corresponding to the product

$$(P^{(j)}_{Aa} + c^{(j)}_2 T)^* (P^{(j)}_{Aa} + c^{(j)}_2 T)$$

is bounded as in § 4.3, and $P_{Aa} + c_2 I$ is an operator approximated by cutoff operators, as required. The cases $P_{Ac} + D'_1$, $P_{Ba} + D_2$ are similar.

Next consider $P_{Ab} + c_3 T$. We first explain why there is cancellation in this expression. The untruncated series $T^\sim_2$ is a solution of the equation

$$T_2 = I - \Gamma(Q T^\sim_2) . \tag{4.4.3}$$

Consider the terms in Ab of § 4.2 for the product $V_0 T^\sim = V_0 T^\sim_1 T^\sim_2$. Since V_0 is to be contracted four times with $T^\sim_2$ vertices, these are the same terms as the terms of type Ab in the product $T^\sim_1 V_0 T^\sim_2$, and using (4.4.3) we see that they are the same as the terms of type Ab in the product $-T^\sim_1 V_0 \Gamma\big(V_2 \underset{2}{-\circ-} (\Gamma V_4)\, T^\sim_2\big)$; the latter terms are nearly equal to

$$-c_3 T^\sim = -\, T^\sim_1 \big(V_0 \underset{4}{-\circ-} (\Gamma(V_2 \underset{2}{-\circ-} \Gamma V_4))\big)\, T^\sim_2 ,$$

the difference being due to the fact that different variables are affected by one of the Γ operations in these two expressions. Thus there is a contribution to $P_{Ab} + c_3 I$ coming from this difference in the Γ factors. To estimate this contribution let $k_1, \ldots, k_4$ be the variables of V_0. Then c_3 contains the γ factor

$$a = \sum_{i=1}^{4} \mu_i$$

while P_{Ab} contains the γ factor

$$a + b = \sum_{i=1}^{l} \mu_i ,$$

where $k_5, \ldots, k_l$ are uncontracted variables from the T_2 part of T. We have

$$|a^{-1} - (a+b)^{-1}| \leqq a^{-1-\varepsilon}\, b^{\varepsilon} .$$

Now

$$b = \sum_{i=5}^{l} \mu_i \leqq K^l \prod_{i=5}^{l} \mu_i$$

for some constant K and so

$$|a^{-1} - (a+b)^{-1}| \leqq a^{-1-\varepsilon} K^l \prod_{i=5}^{l} \mu_i^{\varepsilon} . \tag{4.4.4}$$

With this bound, (3.3.22) follows from Theorem 2.2.1 for the present contributions to $P_{Ab} + c_3 T$. There are also contributions to $P_{Ab} + c_3 T$ due to the truncations. In fact the integration in c_3 is unrestricted while the integration in the corresponding variables of P_{Ab} is restricted by the ΓV_4 truncation and by the Q truncation (3.2.9). These contributions can be estimated as in § 4.2, § 4.3 and so $P_{Ab} + c_3 T$ is an operator.

The remaining cases $P_{Ad} + D_0 T$ and $P_{Bb} + D_1'' T$ are similar. After cancellation there are terms due to the truncation (3.2.9) and there are terms due to the difference in the variables affected by one of the Γ operations; both types of terms can be estimated as above. We show that cancellation does occur. Consider terms of type Ad in the product $V_0 T^\sim = V_0 T_1^\sim (I - \Gamma(Q T_2^\sim))$. Since the last Q vertex must be ΓV_3, it is equivalent to consider terms of type Ad in the product

$$- V_0 T_1^\sim \Gamma\big((V_3 - V_3 \multimap\!\circ\!- \Gamma V_4)\, T_2^\sim\big)\,. \tag{4.4.5}$$

However $D_0 T^\sim = \big(V_0 \underset{3}{-\!\circ\!-} \Gamma V_3\big)\, T_1^\sim T_2^\sim$ is exactly the sum of all terms of type Ad in the product $V_0 (\Gamma V_3)\, T_1^\sim T_2^\sim$. Now

$$(\Gamma V_3)\, T_1^\sim = T_1^\sim (\Gamma V_3 - \Gamma V_3 -\!\circ\!- \Gamma V_4)$$

and so $D_0 T^\sim$ is the sum of all terms of type Ad in the product

$$V_0 T_1^\sim (\Gamma V_3 - \Gamma V_3 -\!\circ\!- \Gamma V_4)\, T_2^\sim\,. \tag{4.4.6}$$

Since (4.4.5) and (4.4.6) have an opposite sign and otherwise differ only in the variables affected by a single Γ operation, cancellation occurs in $P_{Ad} + D_0 T$ as asserted. The proof that cancellation occurs in $P_{Bb} + D_1'' T$ is similar.

4.5 Renormalizing the Self Energy

Theorem 4.5.1. *Let $\delta_{\mathrm{ren}}(\sigma)$ be the kernel of one of the four operators $\Delta_{0\,\mathrm{ren}}(\sigma), \ldots, \Delta_{2\,\mathrm{ren}}(\sigma)$ of (4.1.8). Then $\delta_{\mathrm{ren}}(\sigma)$ converges pointwise to a limit δ_{ren} as $\sigma \to \infty$ and for any $\beta > 0$ and any N,*

$$|\delta_{\mathrm{ren}}(\sigma, k_1, k_2)| \leqq \text{const. } \mu_1^{\beta - 1/2}\, \mu_2^{-1/2}\, \mu(k_1 + k_2)^{-N} \tag{4.5.1}$$

with a constant independent of σ.

Proof. The finite renormalization $\delta_f m^2$ in (4.1.4) contributes to $\delta_{\mathrm{ren}}(\sigma)$ a function dominated by

$$|\hat{h}^* \hat{h}(\pm k_1 \pm k_2)|\, (\mu_1 \mu_2)^{-1/2}$$

which is bounded by the right side of (4.5.1). Thus we can take $\delta_f m^2 = 0$. We consider the operator $\Delta_{2\,\mathrm{ren}}(\sigma)$. For $|k_4|, |l| \leqq \sigma$, the kernel $\delta(\sigma, \ldotp, \ldotp)$ of $\Delta_2(\sigma)$ is given by

$$\begin{aligned}
\delta(\sigma, k_4, l) &= 4(4!) \int_{Z(\sigma)} \hat{h}^* \hat{h}(k_4 + l) \prod_{i=1}^{3} \mu(k_i - \zeta)^{-1} \Big(\sum_{i=1}^{3} \mu(k_i - \zeta)\Big)^{-1} \\
&\quad \cdot \mu_4^{-1/2}\, \mu(l)^{-1/2}\, d_\zeta k \\
&= 4(4!) \int_{Z(\sigma)} \hat{h}(k_4 + 3\zeta)\, \hat{h}(-3\zeta + l) \prod_{i=1}^{3} \mu(k_i - \zeta)^{-1} \qquad (4.5.2) \\
&\quad \cdot \Big(\sum_{i=1}^{3} \mu(k_i - \zeta)\Big)^{-1} \mu_4^{-1/2}\, \mu(l)^{-1/2}\, dk_1\, dk_2\, dk_3\,.
\end{aligned}$$

In (4.5.2), $Z(\sigma)$ is now the subset of R^6 defined by the same inequalities (4.1.6); the equality comes from taking $3\zeta = k_1 + k_2 + k_3$ as the variable of integration in the integral defining $\hat{h}^* \hat{h}$. The kernel d_σ of $D_2(\sigma)$ is given by the similar expression

$$d_\sigma(k_4, l) = 4(4!) \int_{Y(\sigma)} \hat{h}(k_4 + 3\zeta)\, \hat{h}(-3\zeta + l) \prod_{i=1}^{3} \mu_i^{-1} \tag{4.5.3}$$

$$\cdot \left(\sum_{i=1}^{4} \mu_i\right)^{-1} \mu_4^{-1/2} \mu(l)^{-1/2}\, dk_1\, dk_2\, dk_3$$

for $|k_4|$, $|l| \leqq \sigma$, where

$$Y(\sigma) = \{k_1, k_2, k_3 \in R^6 \colon |k_i| \leqq \sigma, \quad 1 \leqq i \leqq 3\}\,.$$

Integrals over the differences $Y(\sigma) \sim Z(\sigma)$, and $Z(\sigma) \sim Y(\sigma)$ are part of the bound on $\delta_{\mathrm{ren}}(\sigma)$. The rest of the bound comes from estimating the difference between the two integrands over the same region $Y(\sigma) \cap Z(\sigma)$.

We break $Z(\sigma) \sim Y(\sigma)$ into two parts: $|\zeta| < \sigma^{3/4}$ and $|\zeta| \geqq \sigma^{3/4}$. If $k \in Z(\sigma) \sim Y(\sigma)$ and $|\zeta| < \sigma^{3/4}$ then

$$\sigma \leqq |k_i| \leqq \sigma + \sigma^{3/4}$$

$$\sigma - \sigma^{3/4} \leqq |k_i - \zeta| \leqq \sigma\,,$$

for some i, $1 \leqq i \leqq 3$, for example for $i = 1$. Also

$$\int_{\sigma - \sigma^{3/4} \leqq |k_1 - \zeta| \leqq \sigma} \mu(k_1 - \zeta)^{-1}\, dk_1 \leqq \text{const.}\ \sigma^{-1/4}$$

and so the contribution of $(Z(\sigma) \sim Y(\sigma)) \cap \{|\zeta| < \sigma^{3/4}\}$ to (4.5.2) is bounded by

$$\text{const.}\ \sigma^{-1/4} \ln \sigma\, \mu(k_4 + l)^{-N} \mu_4^{-1/2} \mu(l)^{-1/2}\,.$$

Next consider $Z(\sigma) \cap \{|\zeta| \geqq \sigma^{3/4}\}$. If $|k_4|$ is bounded by $\sigma^{1/2}$, then the factor $\hat{h}(k_4 + 3\zeta)$ is bounded by $|\zeta|^{-N} \leqq \sigma^{-3N/4}$ and our contribution to (4.5.2) is bounded by

$$\text{const.}\ \sigma^{-3N/4} \ln \sigma\, \mu(k_4 + l)^{-N} \mu_4^{-1/2} \mu(l)^{-1/2}\,.$$

If $|k_4|$ is greater than $\sigma^{1/2}$ then

$$\mu_4^{-1/2} \leqq \text{const.}\ \mu_4^{\beta - 1/2} \sigma^{-\beta/2}$$

and the contribution to (4.5.2) is bounded by

$$\text{const.}\ \sigma^{-\beta/2} \ln \sigma\, \mu(k_4 + l)^{-N} \mu_4^{\beta - 1/2} \mu(l)^{-1/2}\,.$$

In the same way we bound (4.5.3) in the regions $\sim Z(\sigma)$ and $|\zeta| \geqq \sigma^{3/4}$.

It remains to bound the difference between the integrands (4.5.2) and (4.5.3) and it is sufficient to do this in the region

$$Y(\sigma) \cap Z(\sigma) \cap \{|\zeta| \leqq \sigma^{3/4}\}\,.$$

Because of the rapid decrease of $\hat{h}(k_4 + 3\zeta)$, we may restrict the integration to the region $\mu(\zeta) \leqq \mu_4^2$. Then

$$|\mu_i - \mu(k_i - \zeta)| \leqq \text{const.}\ \mu(\zeta) \leqq \text{const.}\ \mu_4^2$$

for $1 \leqq i \leqq 3$ and

$$\left|\left(\sum_{i=1}^{4} \mu_i\right)^{-1} - \left(\sum_{i=1}^{3} (k_i - \zeta)\right)^{-1}\right|$$
$$\leqq \text{const.}\ \mu_4^{\beta} \left(\sum_{i=1}^{4} \mu_i\right)^{-\beta/2} \left(\sum_{i=1}^{3} \mu(k_i - \zeta)\right)^{-\beta/2}$$
$$\cdot \left[\left(\sum_{i=1}^{4} \mu_i\right)^{-1+\beta/2} + \left(\sum_{i=1}^{3} \mu(k_i - \zeta)\right)^{-1+\beta/2}\right],$$

and the desired bound on the difference between the integrands follows. The same estimates together with the bounded convergence theorem shows that the difference of the integrals (4.5.2) and (4.5.3) converges to the integral of the difference of the integrands, or in other words $\delta_{\mathrm{ren}}(\sigma)$ converges pointwise.

Bibliography

1. Friedrichs, K.: Perturbation of spectra in Hilbert space. Providence: Am. Math. Soc. 1965.
2. Glimm, J.: Yukawa coupling of quantum fields in two dimensions, I. Commun. Math. Phys. **5**, 343—386 (1967).
3. — Yukawa coupling of quantum fields in two dimensions, II. Commun. Math. Phys. **6**, 61—76 (1967).
4. — Boson fields with nonlinear selfinteraction in two dimensions. Commun. Math. Phys. 8, 12—25 (1968).
5. —, and A. Jaffe: A Yukawa interaction in infinite volume. Commun. Math. Phys. To appear.
6. Jaffe, A.: Wick polynomials at a fixed time. J. Math. Phys. **7**, 1250—1255 (1966).
7. —, and R. Powers: Infinite volume limit of a Φ^4 field theory. Commun. Math. Phys. **7**, 218—221 (1968).
8. Nelson, E.: A quartic interaction in two dimensions. In: Mathematical theory of elementary particles, ed. by R. Goodman, and I. Segal, pp. 69—73. Cambridge: M.I.T. Press 1966.
9. Kristensen, P., L. Mejlbo, and E. Poulsen: Tempered distributions in infinitely many dimensions II. Math. Scand. **14**, 129—150 (1964).
10. — — — Tempered distributions in infinitely many dimensions. III. Commun. Math. Phys. **6**, 29—48 (1967).
11. Symanzik, K.: Euclidean quantum field theory I. J. Math. Phys. **7**, 510—525 (1966).
12. Wightman, A.: Introduction to some aspects of the relativistic dynamics of quantized fields. Institute des Hautes Etudes Scientifiques, Bures-sur-Yvette (Revised notes for lectures at the French summer school of theoretical physics, Cargese, Corsica, July 1964).

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