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Sequential Convergence in the Dual of a W^*-Algebra

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Abstract. The present paper is the result of the author's attempt to extend Theorem 9 of [5] to the case of a non-abelian W^*-algebra. In [5] GROTHENDIECK proves that weak and weak* convergence are equivalent for sequences in the dual space of an abelian W^*-algebra. Theorem 4 of the present paper is only a partial result in that direction, but it is presented here because of its possible worth as a technical tool.

I. Preliminaries and Notation

Let A be a W^*-algebra. By [8, p. 1.74] the second dual A^{**} of A is also a W^*-algebra, and we shall consider the canonical imbedding of A into A^{**} as an identification. By [2, p. 126] there exists a central projection $z \in A^{**}$ which is the supremum of the set of minimal projections in A^{**}. Set $z' = 1 - z$; let $A_d^* = \{f \in A^* : f \mid z'A^{**} = 0\}$, and $A_c^* = \{f \in A^* : f | zA^{**} = 0\}$. Since z is central, $A^* = A_c^* \oplus A_d^*$, and both A_c^* and A_d^* are closed invariant subspaces [7, p. 439] of A^*. Thus by [7, p. 439] any positive $f \in A^*$ has a unique decomposition $f = f^d + f^c$ into positive functionals with $f^d \in A_d^*$ and $f^c \in A_c^*$.

Following EFFROS [4] we define an order ideal I in A^* to be a set of positive functionals in A^* with the property that if $f \in I$ and $0 \leqq g \leqq \lambda f$ for some $\lambda \geqq 0$, then $g \in I$. If I is a norm-closed order ideal in A^*, we define the support of I to be the complement of the largest projection p in A^{**} such that $f(p) = 0$ for all $f \in I$ [cf. 4, p. 405].

For any $a \in A^{**}$, let $a' = 1 - a$. Recall that a pure state of A is a positive f in A^* such that if $0 \leqq g \leqq \lambda f$ for some $\lambda \geqq 0$, then $g = \alpha f$ for some $\alpha \geqq 0$.

II. The Main Results

The first result characterizes those projections in A^{**} which support a weak* closed order ideal in A^*. We need only a special case for Theorem 4.

Proposition 1. *A projection p in A^{**} supports a weak* closed order ideal in A^* iff $p = \lim a_\alpha$ where $\{a_\alpha\}$ is a decreasing net of positive elements of A.*

Proof. If I is a weak* closed order ideal with support p, it follows from [4, p. 408] that $p'A^{**}p'$ is the weak* closure of $A_{p'} = \{a \in A : p'ap' = a\}$ in A^{**}. By [3, p. 15] there is an approximate identity $\{b_\alpha\}$ in $A_{p'}$ with $b_\alpha \geqq 0$ and $b_\alpha \uparrow$ for each α. Clearly $b_\alpha \uparrow p'$ in A^{**}, since multiplication is weak* continuous (in a single variable) in A^{**} [8, p. 1.12]. Setting $a_\alpha = 1 - b_\alpha$, we get $a_\alpha \downarrow p$.

Conversely, suppose $\{a_\alpha\} \subset A$ and $a_\alpha \downarrow p$. Define $I_\alpha = \{f \in A^* : f \geqq 0, f(a'_\alpha) = 0\}$. Each I_α is a weak* closed order ideal, so $\cap I_\alpha = I$ is also. Let q be the support of I. Now if $f \geqq 0$ in A^*, $f(p') = 0$ iff $f(a'_\alpha) = 0$ for all α iff $f \in I_\alpha$ for all α iff $f \in I$ iff $f(q') = 0$. Thus $p' = q'$, so $p = q$. Q.E.D.

Corollary 2. *If p is a minimal projection in A^{**}, then there is a decreasing net $\{a_\alpha\}$ in A such that $a_\alpha \downarrow p$.*

Proof. We need only show that p supports a weak* closed order ideal in A^*. If $I = \{f \in A^* : f \geqq 0$ and $f(p') = 0\}$, then p supports I. Let $f, g \in I$, $g \neq 0$. Then $f(a) = f(pap)$ and $g(a) = g(pap)$ for each $a \in A^{**}$. But since p is minimal, the W^*-algebra $pA^{**}p$ is one dimensional. Thus there is a scalar $\alpha \geqq 0$ such that $f(pap) = \alpha g(pap)$ for all $a \in A^{**}$, and hence $f(a) = \alpha g(a)$ for all $a \in A^{**}$. This proves that $I = \{\alpha g : \alpha \geqq 0\}$, so I is weak* closed. Q.E.D.

The next proposition has some independent interest as a technical result. In the case of abelian A, it follows from [5, p. 168]. It is only in this proposition that we use the W^* property of A. Otherwise A could be any C^*-algebra.

Proposition 3. *Suppose $\{a_\alpha\}_{\alpha \in I} \subset A$ is an increasing net with $a_\alpha \uparrow a$ in A^{**}. Suppose $\{f_N\}$ is a sequence in A^* with $f_N \to f$ weak* for some $f \in A^*$. Then $f_N(a_\alpha) \xrightarrow[\alpha]{} f_N(a)$ uniformly in N (and hence $f_N(a) \xrightarrow[N\to\infty]{} f(a)$).*

Proof. Suppose the proposition is false. Then there exists $\varepsilon > 0$ such that for all $\alpha_0 \in I$ there is $\alpha_1 \geqq \alpha_0$ in I and N such that $|f_N(a_{\alpha_1}) - f_N(a_{\alpha_0})| \geqq \varepsilon$. By induction we get $\{a_K\}$, an increasing sequence taken from $\{a_\alpha\}_{\alpha \in I}$, and a subsequence $\{f_{N_K}\}$ of $\{f_N\}$ such that for each K, $|f_{N_K}(a_{K+1} - a_K)| \geqq \varepsilon$. Write $q_K = a_{K+1} - a_K$. By [1, p. 297], $\sum_{i=1}^{\infty} |f_{N_K}(q_i)|$ converges uniformly in K since $\sum_{i=1}^{\infty} q_i$ exists in A. But $|f_{N_K}(q_K)| = f_{N_K}(a_{K+1} - a_K)| \geqq \varepsilon$, a contradiction. Q.E.D.

We can now prove the main result fairly easily. It extends the author's result [1, p. 298].

Theorem 4. *If $\{f_N\}$ is a sequence of positive functionals in A^* with $f_N \xrightarrow[N\to\infty]{} f$ weak* for some $f \in A^*_d$, then $f_N \xrightarrow[N\to\infty]{} f$ uniformly.*

Proof. Let q be the support of f. Fix $\varepsilon > 0$. By the definition of A^*_d, there exists a projection $p \leqq q$ such that p is a finite sum of minimal projections and $|f(q) - f(p)| < \varepsilon/8$. If p_0 is any minimal projection, Corollary 2 and Proposition 3 imply that $f_N(p_0) \xrightarrow[N\to\infty]{} f(p_0)$ and

$f_N(p_0') \xrightarrow[N\to\infty]{} f(p_0')$. Thus $f_N(p') \xrightarrow[N\to\infty]{} f(p')$ as well. Since q supports f, $f(p') < \varepsilon/8$, so $f_N(p') < \varepsilon/8$ for $N \geqq N_0$ for some N_0. Therefore, $|(f_N - f)(p')| < \varepsilon/8$ for $N \geqq N_0$. If $b \in A^{**}$ with $\|b\| \leqq 1$, the Schwarz inequality gives

$$|(f_N - f)(b)| \leqq |(f_N - f)[p'bp + pbp' + p'bp']| + |(f_N - f)(pbp)| \leqq$$
$$\leqq 3|(f_N + f)(p')|^{\frac{1}{2}} \cdot \|f_N + f\| + |(f_N - f)(pbp)|\,.$$

Since $f_N \xrightarrow[N\to\infty]{} f$ weak*, $\{\|f_N + f\|\}_{N=1}^{\infty}$ is a bounded sequence, say with bound M. Thus we have

$$|(f_N - f)(b)| \leqq 6 \cdot (\varepsilon/8)^{\frac{1}{2}} \cdot M + |(f_N - f)(pbp)|\,.$$

Since $\varepsilon > 0$ was arbitrary, we need only show that $(f_N - f)(pbp) \to 0$ uniformly for $b \in A^{**}$ with $\|b\| \leqq 1$ in order to complete the proof of the theorem. But $pA^{**}p$ is finite-dimensional and $f_N(p_0) \xrightarrow[N\to\infty]{} f(p_0)$ for each minimal projection in $pA^{**}p$, so the spectral theorem gives that $f_N \xrightarrow[N\to\infty]{} f$ uniformly on $pA^{**}p$. This means $f_N(pbp) \xrightarrow[N\to\infty]{} f(pbp)$ uniformly for $\|b\| \leqq 1$. Q.E.D.

Corollary 5. *If f is a pure state of A and $\{f_N\}$ is a sequence of positive functionals in A^* such that $f_N \xrightarrow[N\to\infty]{} f$ weak*, then $f_N \xrightarrow[N\to\infty]{} f$ uniformly.*

Proof. We need only prove that the support of f is a minimal projection. Suppose $p =$ support of f and $q < p$ is a non-zero projection. Then the functional g defined by $g(a) = f(qaq)$ is positive and $g \leqq f$, but $g \neq \alpha f$ for any α since $f(p - q) > 0$ and $g(p - q) = 0$. Q.E.D.

Bibliography

1. Akemann, C.: The dual space of an operator algebra. Trans. Am. Math. Soc. **126** (2), 286—302 (1967).
2. Dixmier, J.: Les algebres d'operateurs dans l'espace Hilbertien. Paris: Gauthier-Villars 1957.
3. — Les C^*-algebres et leurs representations. Paris: Gauthier-Villars 1964.
4. Effros, E.: Order ideals in a C^*-algebra and its dual. Duke Math. J. **30**, 391—412 (1963).
5. Grothendieck, A.: Sur les applications lineaires faiblement compactes d'espaces du type $C(K)$. Canad. J. Math. **5**, 129—173 (1953).
6. Phillips, R. S.: On linear transformations. Trans. Am. Math. Soc. **48**, 516—541 (1940).
7. Sakai, S.: On topological properties of W^*-algebras. Proc. Japan Acad. **33**, 439—444 (1958).
8. — The theory of W^*-algebras. Yale Notes. New Haven, Conn.: Yale University 1962.

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