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On the Connection Between Analyticity and Lorentz Covariance of Wightman Functions

J. Bros
Centre d'Etudes Nucléaires de Saclay

H. Epstein and V. Glaser
CERN — Geneva

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Abstract. We prove a conjecture of R. Streater [1] on the finite covariance of functions holomorphic in the extended tube which are Laplace transforms of two tempered distributions with supports in the future and past cones. A new, slightly more general proof is given for a theorem of analytic completion of [1].

A. Notations

1. Scalar product:

$$(z, z') = z^\mu z'_\mu = z^0 z'^0 - z^1 z'^1 - z^2 z'^2 - z^3 z'^3 = z^\mu g_{\mu\nu} z'^\nu$$

for z and z' real or complex four vectors.

2. Future cone:

$$V^+ = \{x : x \in \mathbb{R}^4, (x, x) > 0, x^0 > 0\} = - V^-$$

n-point future cone:

$$V_n^+ = \{x \in \mathbb{R}^{4n} : x = x_1, \ldots, x_n, x_j \in V^+ \ (j = 1, \ldots, n)\} = - V_n^- .$$

3. n-point forward tube:

$$\mathcal{T}_n^+ = \{z \in \mathbb{C}^{4n} : z = x + iy, y \in V_n^+\} = - \mathcal{T}_n^- .$$

4. $L_+^\uparrow$ = connected real Lorentz group. $L_+(\mathbb{C})$ = connected complex Lorentz group.

5. n-point extended tube:

$$\mathcal{T}'_n = \bigcup_{\Lambda \in L_+(\mathbb{C})} \Lambda \mathcal{T}_n^+$$

for $z = z_1, \ldots, z_n \in (\mathbb{C}^4)^n$, $\Lambda z = \Lambda z_1, \ldots, \Lambda z_n$.

6. For $z = z^0, z^1, z^2, z^3 = z^0, \mathrm{z}$, we denote

$$\|z\|^2 = \sum_{\mu=0}^{3} |z^\mu|^2 = |z^0|^2 + |\mathrm{z}|^2$$

for $z = z_1, \ldots, z_n \in (\mathbb{C}^4)^n$, $\|z\|^2 = \sum_{j=1}^{n} \|z_j\|^2$.

7. $\mathcal{J}_n$ = the set of Jost points.

B. Introduction

We recall the following theorem of Streater [1].

S Theorem. *Let $f(z)$ be a holomorphic function of z in $\mathcal{T}_n^+ \cup \mathcal{T}_n^- \cup \mathcal{N}$, where $\mathcal{N}$ is an open (complex) neighbourhood of the set of Jost points* [2]. *Then there exists a function, holomorphic in $\mathcal{T}'_n$, which coincides with f in $\mathcal{T}_n^+ \cup \mathcal{T}_n^-$ and at the Jost points.*

The second part of this paper is devoted to some comments on the proof of this theorem, particularly on the question of single valuedness.

In the first part of this paper, we shall prove the following theorem, which supplements the preceding one:

Theorem 1. *Let $f(z)$ be a holomorphic function in $\mathcal{T}'_n$, whose restrictions to $\mathcal{T}_n^+$ and $\mathcal{T}_n^-$ are the Laplace transforms of two temperate distributions on $\mathbb{R}^{4n}$, $\tilde{f}^+$ and $\tilde{f}^-$, respectively, the supports of $\tilde{f}^+$ and $\tilde{f}^-$ being contained in $\overline{V}_n^+$ and $\overline{V}_n^-$, respectively. Then the following formulae hold for all $z \in \mathcal{T}'_n$ and all $\Lambda \in L_+(\mathbb{C})$:*

$$f(\Lambda^{-1} z) = \sum_{r,s=0}^{L} \operatorname{tr} F^{(r,s)}(z)\, D^{(r,s)}(\Lambda)\,.$$

$$f(z) = \sum_{r,s=0}^{L} \operatorname{tr} F^{(r,s)}(z)$$

Here, $D^{(r,s)}(\Lambda)$ is the finite dimensional irreducible representation of $L_+(\mathbb{C})$ with indices r and s; $F^{(r,s)}(z)$ is a matrix operating in the same space as $D^{(r,s)}$ (i.e., an $(r+1)(s+1) \times (r+1)(s+1)$ complex matrix), whose matrix elements are holomorphic in z in $\mathcal{T}'_n$ and have the properties postulated for $f(z)$.

This theorem has been conjectured by Streater [1]. A special case ($n = 1$) has been proved by Bogoliubov and Vladimirov [3]. In this case it also follows easily from the Jost-Lehmann-Dyson representation. Another special case has been proved by Borchers [4].

Part I

1. Preliminary Remarks

1. We denote G the covering group $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. For $g = (A, B)$ in G, $\Lambda(g)$ denotes the corresponding Lorentz transformation, as described in Ref. [2], p. 14. We also denote

$$G_0 = SU(2, \mathbb{C}) \times SU(2, \mathbb{C})$$

$$G_r = \{(A, B) : B = \bar{A} \in SL(2, \mathbb{C})\}\,.$$

These are two subgroups of G. Every point of G_0 (resp. G_r) has a complex neighborhood V in G, where local analytic co-ordinates $\zeta_k = \xi_k + i\eta_k$ can be chosen so that

$$V \cap G_0 = \{\zeta : \eta_k = 0, \forall k\}$$

$$(\text{resp. } V \cap G_r = \{\zeta : \eta_k = 0, \forall k\})\,.$$

The image of G_r in $L_+(\mathbb{C})$ is $L_+^{\uparrow}$.

2. The irreducible finite dimensional representations $D^{(r,s)}$ are defined as follows: r and s are two integers ≥ 0; for $g = (A, B)$, $D^{(r,s)}(g)$ is the restriction of $(A \otimes \cdots \otimes A) \otimes (B \otimes \cdots \otimes B)$ to the space of all complex valued tensors $\xi_{i_1 \ldots i_r j_1 \ldots j_s}$ separately symmetric in $i_1, \ldots, i_r$ and in $j_1, \ldots, j_s$. Note that: $D^{(r,s)}(g)$ is holomorphic on G. For $u = (U, V) \in G_0$, $D^{(r,s)}(u)$ is unitary:

$$\overline{D^{(r,s)}_{\alpha\beta}(u)} = D^{(r,s)}_{\beta\alpha}(u^{-1}) .$$

The restrictions of $D^{(r,s)}$ to G_0 and G_r give all irreducible finite dimensional representations of G_0 and G_r.

3. Let $\varphi(g)$ be a holomorphic function in an open connected set Ω of G of the form $\Omega = \Omega G_0$. It can be shown [5] that $\varphi(g)$ has a unique expansion

$$\varphi(g) = \sum_{r,s=0}^{\infty} \operatorname{tr} \Phi^{(r,s)} D^{(r,s)}(g)$$

converging uniformly on any compact of Ω. This expansion is the analogue of the Laurent series in $\mathbb{C} - \{0\}$ (the complex Lorentz group for two dimensional spacetime). $\Phi^{(r,s)}$ is a $(r+1)(s+1) \times (r+1)(s+1)$ matrix given by

$$\Phi^{(r,s)}_{\alpha\beta} = (r+1)(s+1) \int_{G_0} D^{(r,s)}_{\alpha\beta}(u^{-1}g^{-1})\, \varphi(gu)\, du; \quad g \in \Omega$$

where du is the (left and right invariant) Haar measure on G_0 normalized by $\int_{G_0} du = 1$. The above expression is, in effect, independent of g; to see this, we replace g by gh and find a holomorphic function of h in a neighborhood V of the unit. This function is *constant* on $V \cap G_0$ because du is invariant. Using local co-ordinates mentioned in remark 2., we conclude that the function is independent of h in a neighborhood of the unit. Since Ω is connected, our assertion follows. We can write

$$\Phi^{(r,s)} = (r+1)(s+1) \left[\int_{G_0} D^{(r,s)}(u^{-1})\, \varphi(gu)\, du \right] D^{(r,s)}(g^{-1}) .$$

4. In the case when the $\Phi^{(r,s)}$ defined by the above formula happen to be all 0 when $r, s > L$, the identity

$$\varphi(g) = \sum_{r,s \leq L} \operatorname{tr} \Phi^{(r,s)} D^{(r,s)}(g), \quad (g \in \Omega),$$

is an immediate consequence of the Peter-Weyl theorem applied to $\varphi(gu)$ as a function on G_0. Then $\varphi(g)$ is evidently holomorphic on G.

5. If $\varphi(g)$ is a function of $A(g)$, i.e., if $\varphi(g) = \varphi(-g)$, only the $\Phi^{(r,s)}$ with $(-1)^{r+s} = 1$ appear in the expansion, since

$$D^{(r,s)}(g) = (-1)^{r+s} D^{(r,s)}(-g) .$$

6. We shall denote $[\lambda]$ the following element of $L_+(\mathbb{C})$

$$[\lambda] = \begin{bmatrix} \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) & 0 & 0 & \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) & 0 & 0 & \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) \end{bmatrix}.$$

If we use the variables: $u_j = z_j^0 + z_j^3$, $v_j = z_j^0 - z_j^3$, $[\lambda]$ is simply the transformation: $u_j \to \lambda u_j$, $v_j \to \frac{1}{\lambda} v_j$, all the other components remaining unchanged. It is easy to check that $[\lambda^2] = \Lambda(g)$ for

$$g = \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}\right).$$

2. Some Properties of Tempered Distributions with Support in $\overline{V}_n^+$

The theory of Laplace transforms of tempered distributions with support in a convex cone (such as $\overline{V}_n^+$) has been extensively treated in Refs. [2, 6, 7, 8, 9, 10, 11]. We shall give an elementary derivation of the results of the theory needed for our purposes.

We consider the functions of one four vector p defined by

$$\tilde{F}_k(p) = \frac{1}{4^k k!(k+1)!}(p, p)^k \tilde{F}_0(p) \quad (k \geqq 1, \text{ an integer})$$

$8\pi \tilde{F}_0(p) = \theta(p, p)\, \theta(p^0) =$ characteristic function of $\overline{V}^+$.

$\tilde{F}_k(p)$ is continuous and has continuous derivatives of the $2k - 2$ first orders for $k \geqq 1$. For $k \geqq 1$:

$$\Box \tilde{F}_k(p) = \tilde{F}_{k-1}(p)$$

in the sense of distributions.

Moreover:

$$\Box \tilde{F}_0(p) = \Delta_R(p; 0) \quad \text{(retarded function for the wave equation)}$$

$$\Box \Delta_R(p; 0) = \delta(p).$$

Their Fourier transforms are

$$F_k(x) = \lim_{\substack{y \to 0 \\ y \in V^+}} (x + iy, x + iy)^{-(k+1)}, \quad k \geqq 0,$$

the limits being taken in the sense of distributions.

Let $\tilde{f}^+$ be a tempered distribution on $\mathbb{R}^{4n} = (\mathbb{R}^4)^n$ with support in $\overline{V}_n^+$. There exists an integer N' and a constant $C > 0$ such that, for all $\varphi \in \mathscr{S}^{4n}$,

$$|\langle \tilde{f}^+, \varphi \rangle| < C \sup_{\substack{|\alpha| \leq N' \\ p \in \mathbb{R}^{4n}}} (1 + \|p\|)^{N'} |D^\alpha \varphi(p)| = C \|\varphi\|_{N'}.$$

It follows that $\tilde{f}^+$ can be extended to a continuous linear functional on the space of all functions φ having continuous derivatives of the N' first orders, such that $\|\varphi\|_{N'} < \infty$, with the topology defined by the norm $\|\varphi\|_{N'}$. The inequality $|\langle \tilde{f}^+, \varphi\rangle| \leqq C\|\varphi\|_{N'}$ continues to hold for such functions [12].

We notice that

$$\mathbf{F}_{N'}(p - p') \equiv \tilde{F}_{N'+1}(p_1 - p_1') \ldots \tilde{F}_{N'+1}(p_n - p_n')$$

considered as a function of p', has its support in $p - \overline{V}_n^+$. The intersection of this support with $\overline{V}_n^+$ (the support of $\tilde{f}^+$) is a compact set if $p \in \overline{V}_n^+$, and is empty otherwise. We can therefore [12] define the convolution $\tilde{f}^+ * \mathbf{F}_{N'}$ and obtain a continuous function (for $N' \geqq 1$, it will even be differentiable, but this is not relevant to our purpose). To see this we can define $\tilde{f}^+ * \mathbf{F}_{N'}$ as $\langle \tilde{f}^+, G_p\rangle$ where $G_p(p') = \mathbf{F}_{N'}(p - p')\,\chi_p(p')$ and $\chi_p(p')$ is a $\mathscr{C}^\infty$ function of p' with compact support, equal to 1 in a neighborhood of $\{p' : p' \in \overline{V}_n^+ \cap (p - \overline{V}_n^+)\}$. The result is evidently independent of the choice of χ_p. In particular χ_p may be chosen independent of p when p stays in a fixed compact. The function of p', $F_{N'}(p - p')\,\chi_p(p')$ then depends continuously on p in the topology of the norm $\|\,.\,\|_{N'}$. This proves the continuity of $\tilde{f}^+ * \mathbf{F}_{N'}$. To obtain some estimate on the growth of this function, we first choose a $\mathscr{C}^\infty$ function $\alpha(t)$ of a real variable with the following properties:

$$\begin{cases} 0 \leqq \alpha(t) \leqq 1 \\ \alpha(t) = 1 \quad \text{for} \quad t \leqq 2 \\ \alpha(t) = 0 \quad \text{for} \quad t \geqq 3\,. \end{cases}$$

We note that, for $p' \in \overline{V}_n^+ \cap (p - \overline{V}_n^+)$,

$$0 \leqq p_j'^0 \leqq p_j^0; \quad |\mathbf{p}_j'| \leqq p_j'^0; \quad \|p'\| \leqq \sqrt{2}\|p\|\,.$$

We can therefore define:

$$\chi_p(p') = \alpha(\|p'\|) \quad \text{for} \quad \|p\| \leqq 1$$

$$\chi_p(p') = \alpha\left(\frac{\|p'\|}{\|p\|}\right) \text{for} \quad \|p\| \geqq 1\,.$$

For a given multi-index β, $|D^\beta \chi_p(p')|$ is bounded by a constant K_β *independent of p and p'.*

It follows that (using Leibnitz's formula), for $\|p\| \geqq 1$,

$$\|G_p\|_{N'} \leqq K \sup_{\substack{\|p'\| \leqq 3\|p\| \\ |\beta| \leqq N'}} (1 + \|p'\|)^{N'}\, |D^\beta\,\mathbf{F}_{N'}(p - p')|\,.$$

Since $\mathbf{F}_{N'}$ behaves like a polynomial of degree $2n(N' + 1)$, there exists a constant $K''_{N'}$ such that

$$\|G_p\|_{N'} \leqq K''_{N'}(1 + \|p\|)^{(2n+1)N' + 2n}\,.$$

This gives the estimate:

$$|\tilde{f}^+ * \mathbf{F}_{N'}(p)| < C K''_{N'} (1 + \|p\|)^{(2n+1)N' + 2n} .$$

On the other hand:

$$\Box_{p_1}^{N'+2} \ldots \Box_{p_n}^{N'+2} (\tilde{f}^+ * \mathbf{F}_{N'}) = \tilde{f}^+ .$$

We have proved:

Lemma 1. *Any tempered distribution $\tilde{f}^+$ with support in $\overline{V}_n^+$ can be written in the form:*

$$\tilde{f}^+ = \Box_{p_1}^{N'} \ldots \Box_{p_n}^{N'} G$$

where G has its support in $\overline{V}_n^+$ and is a continuous function of at most polynomial increase. If $\tilde{f}^+$ is Lorentz-invariant, G can be chosen Lorentz invariant.

[The method we have used can be easily generalized to deal with tempered distributions with support in an arbitrary convex cone Γ with non-empty interior. One would then choose a basis with elements $e_1, \ldots, e_r$ in the interior of Γ, and use the functions

$$x_1^N x_2^N \ldots x_r^N \theta(x_1)\, \theta(x_2) \ldots \theta(x_r)$$

in the same way as we have used $\mathbf{F}_N$.]

We can now study the Laplace transform of a tempered distribution $\tilde{f}^+$ with support in $\overline{V}_n^+$. This is a function $f(z)$ holomorphic in $\mathscr{T}_n^+$, defined, for $z = x + iy \in \mathscr{T}_n^+$, by the heuristic formula:

$$f(z) = \int e^{i \sum_{j=1}^{n} (p_j, z_j)} \Box_1^{N'} \ldots \Box_n^{N'} G(p_1, \ldots, p_n)\, d^4 p_1 \ldots d^4 p_n$$

which has the precise meaning:

$$f(z) = (-1)^{nN'} (z_1, z_1)^{N'} \ldots (z_n, z_n)^{N'} \int e^{i \sum_{j=1}^{n} (p_j, z_j)} G(p)\, dp$$

where $G(p)$ is the continuous function, the existence of which is asserted by lemma 1 and which satisfies an inequality:

$$|G(p)| < C_0 (1 + \|p\|)^M .$$

Therefore

$$|f(z)| < C_0 \prod_{k=1}^{n} |(z_k, z_k)|^{N'} \int (1 + \|p\|)^M \exp\left[- \sum_j (p_j, y_j)\right] dp .$$

We note, that, for any fixed $\Lambda \in L_+^\uparrow$, $f_\Lambda(z) = f(\Lambda^{-1} z)$ satisfies a similar inequality, *with the same M and N'* but a different C_0. Because, in the integrand, $y_j \in V^+$ and $p_j \in \overline{V}^+$, we have:

$$(p_j, y_j) \geqq \frac{1}{2} \|p_j\| (y^0_j - |\mathbf{y}_j|) .$$

Denote

$$m = \min_{1 \leqq j \leqq n} (y^0_j - |\mathbf{y}_j|) .$$

We find: $\sum_j' (p_j, y_j) \geqq \frac{1}{2} m \|p\|$, so that:

$$|f(z)| < C_0 S_{4n} \prod_{j=1}^{n} |(z_j, z_j)|^{N'} \int_0^\infty (1+\tau)^{M+4n-1} e^{-\frac{1}{2} m\tau} d\tau$$

we finally get

$$|f(z)| < C_0 B \prod_{j=1}^{n} |(z_j, z_j)|^{N'} \left(1 + \frac{1}{m}\right)^{M'}$$

where $M' = M + 4n$; B is a numerical constant depending only on n and M'. Therefore, for any fixed $\Lambda \in L_+^\uparrow$, $f_\Lambda(z) = f(\Lambda^{-1} z)$ satisfies a similar inequality with the same N', M', B, but a different C_0. We can write:

$$|f_\Lambda(z)| < C_\Lambda \prod_{j=1}^{n} |(z_j, z_j)|^{N'} \left[1 + \frac{1}{\min_j (|y^0{}_j| - |\mathbf{y}_j|)}\right]^{M'} \tag{1}$$

where M' and N' are independent of Λ.

3. *Proof of Theorem 1*

We shall use two lemmas.

Lemma 2. *Let f satisfy the hypotheses of theorem* 1. *Then:*

1. *$f(M[\lambda]z)$ is holomorphic in M, λ, z in $L_+(\mathbb{C}) \times \{\mathbb{C} - \{0\}\} \times \mathscr{T}_n'$.*

2. *$f(M[\lambda]z) = \sum_{q=-N}^{N} a_q(M; z)\lambda^q$, where N is an integer $\geqq 0$, and $a_q(M; z)$ is holomorphic in $L_+(\mathbb{C}) \times \mathscr{T}_n'$.*

Proof. 1. The first statement is evident.

2. For fixed M and z, $f(M[\lambda]z)$ has a Laurent expansion

$$f(M[\lambda]z) = \sum_{q=-\infty}^{\infty}{}' a_q(M; z)\lambda^q$$

where:

$$a_q(M; z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-iq\theta} f(M[e^{i\theta}]z)\, d\theta$$

is holomorphic in $L_+(\mathbb{C}) \times \mathscr{T}_n'$.

We now use the inequality (1) and the symmetric inequality which can be obtained in $\mathscr{T}_n^-$. Replacing C_Λ, M' and N' by the maximum of their values in $\mathscr{T}_n^+$ and $\mathscr{T}_n^-$, we find:

$$|f_\Lambda(z)| < C_\Lambda \prod_j |(z_j, z_j)|^{N'} \left[1 + \frac{1}{\min_j (|y^0{}_j| - |\mathbf{y}_j|)}\right]^{M'}$$

for $z \in \mathscr{T}_n^+ \cup \mathscr{T}_n^-$, $\Lambda \in L_+^\uparrow$.

Let Σ be a closed ball in $(\mathbb{R}^4)^n$ such that, for $x \in \Sigma$,

$$\begin{cases} x_j^3 > |x_j^\mu| \quad \text{for} \quad \mu = 0, 1, 2;\ j = 1, \ldots, n\,. \\ 0 < u < u_j = x_j^3 + x_j^0\,, \\ 0 < v < |v_j| = -v_j = x_j^3 - x_j^0;\ j = 1, \ldots, n\,. \end{cases}$$

Let $z' = x' + iy' = [\lambda]x$; $\lambda = \xi + i\eta$. We have, for all $j = 1, \ldots, n$,

$$y_j'^{\,0} + y_j'^{\,3} = \eta u_j$$

$$y_j'^{\,0} - y_j'^{\,3} = -\frac{\eta}{|\lambda|^2} v_j$$

$$y_j'^{\,1} = y_j'^{\,2} = 0\,.$$

Hence $z' \in \mathcal{T}_n^+$ for $\eta > 0$, $z' \in \mathcal{T}_n^-$ for $\eta < 0$. Moreover

$$|y_j'^{\,0}| - |y_j'^{\,3}| = \frac{1}{2}|\eta| \left\{ \left|u_j - \frac{1}{|\lambda|^2} v_j\right| - \left|u_j + \frac{1}{|\lambda|^2} v_j\right| \right\}$$

$$= |\eta| \min\left(u_j, \frac{-1}{|\lambda|^2} v_j\right) > |\eta| \min\left(u, \frac{1}{|\lambda|^2} v\right)$$

$$|y_j'^{\,0}| - |y_j'^{\,3}| > |\eta| \left[\frac{1}{u} + \frac{|\lambda|^2}{v}\right]^{-1}.$$

Therefore

$$|f_\Lambda([\lambda]x)| < C_\Lambda \prod_j |(x_j, x_j)|^{N'} \left[1 + \frac{1}{|\eta|}\left(\frac{1}{u} + \frac{|\lambda|^2}{v}\right)\right]^{M'}.$$

Since Σ is compact, there exists a constant $Q > 0$ such that $\left|\prod_j (x_j, x_j)\right| < Q$ in Σ. We get: for all $x \in \Sigma$, $|\eta| \neq 0$, $\lambda = \xi + i\eta$,

$$|f_\Lambda([\lambda]x)| < C_\Lambda Q^{N'} \left[1 + \frac{1}{|\eta|}\left(\frac{1}{u} + \frac{|\lambda|^2}{v}\right)\right]^{M'}.$$

Since every point $x \in \Sigma$ is a Jost point, $f_\Lambda([\lambda]x)$ is holomorphic in λ for $\lambda \neq 0$. Denote $h(\lambda)$ this function for fixed $x \in \Sigma$ and $\Lambda \in L_+^\uparrow$.

The function $h(\lambda)$ satisfies

$$|h(\lambda)| < H(1 + |\lambda|)^{2M'} |\eta|^{-M'}$$

in $\mathbb{C} - \mathbb{R}$. This is a well-known [9, 10] sufficient condition for $h(\lambda)$ to have tempered distribution boundary values on the real axis:

$$\int_{-\infty}^{+\infty} \varphi(\xi)\, h(\xi \pm i\eta)\, d\xi \to \langle h^\pm, \varphi\rangle \quad \text{when} \quad \eta \to 0,\ \eta > 0,\ \varphi \in \mathcal{S}\,,$$

where h^+ and h^- are derivatives of order $M' + 1$ of continuous functions. But $h^+ - h^-$ has its support at the origin. Therefore $\xi^{(M'+2)}(h^+ - h^-) = 0$. The function $\lambda^{M'+2} h(\lambda)$ is holomorphic in the whole complex plane and satisfies, for $\eta \neq 0$,

$$|\lambda^{M'+2} h(\lambda)| < H(1 + |\lambda|)^{2M'} |\lambda|^{M'+2} |\eta|^{-M'}\,. \tag{2}$$

We can now use an argument due to Vladimirov [13]. The above inequality implies that $\xi^{M'+2} h^+$ (resp. $\xi^{M'+2} h^-$) has a Fourier transform with support on the positive (resp. on the negative) real axis. Since they are equal, their Fourier transform has support at 0. Therefore $\lambda^{M'+2}h(\lambda)$ is a polynomial [12], the degree of which cannot exceed $3M' + 2$ because of the inequality (2). This means that:

$$f_\Lambda([\lambda]x) = \sum_{q=-M'-2}^{2M'} a_q(\Lambda^{-1}; x)\lambda^q$$

for all $\Lambda \in L_+^\uparrow$, $x \in \Sigma$. In other words, $a_q(M; z)$ vanishes, for $|q| > N = \max(2M', M' + 2)$, for all $M \in L_+^\uparrow$ and $z \in \Sigma$. This is sufficient to prove that $a_q(M; z)$ vanishes in $L_+(\mathbb{C}) \times \mathscr{T}'_n$, for $|q| > N$. Lemma 2 is thus established.

Lemma 3. *Let $\varphi(g)$ be a holomorphic function on the complex Lie group $G = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. Denote $g(\lambda)$ the element of G given by*

$$g(\lambda) = \left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}, \begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\right)$$

Assume that there is a positive integer N and, for any pair of compacts K and K' in G, a positive constant $A(K, K')$ such that

$$|\varphi(kg(\lambda)k')| < A(K, K')\left(1 + \frac{1}{|\lambda|^2} + |\lambda|^2\right)^N$$

holds for all $\lambda \neq 0$, $k \in K$, $k' \in K'$. Then

$$\Phi^{(r,s)} = (r + 1)(s + 1)\int_{G_0} D^{(r,s)}(u^{-1}h)\,\varphi(h^{-1}u)\,du$$

vanishes for $r + s > 2N$.

Proof. We already know that $\Phi^{(r,s)}$ is independent of $h \in G$. Every vector of the space, in which $D^{(r,s)}$ and $\Phi^{(r,s)}$ operate, can be written as a finite sum of vectors of the form $(\xi \otimes \cdots \otimes \xi) \otimes (\eta \otimes \cdots \otimes \eta) = \xi^{\otimes r} \otimes \eta^{\otimes s}$, ξ and η in $\mathbb{C}^2$. It is therefore sufficient to prove that, for $r + s > 2N$, $\Phi^{(r,s)}$ annihilates every vector $\psi = \xi^{\otimes r} \otimes \eta^{\otimes s}$ with $\|\xi\| = \|\eta\| = 1$. Choose $h = g(\lambda)v$, $v = (U, V)$ and U and V, two unitary unimodular matrices such that $U\xi = V\eta = \begin{bmatrix}1\\0\end{bmatrix}$. We have

$$\Phi^{(r,s)} = (r + 1)(s + 1)\left[\int_{G_0} D^{(r,s)}(u^{-1})\,\varphi(h^{-1}u)\,du\right] D^{(r,s)}(h),$$

$$\Phi^{(r,s)}\psi = (r + 1)(s + 1)\left[\int_{G_0} D^{(r,s)}(u^{-1})\,\varphi(v^{-1}g(\lambda^{-1})u)\right]\lambda^{r+s}\begin{bmatrix}1\\0\end{bmatrix}^{\otimes(r+s)}.$$

Using the hypotheses of the lemma, we have

$$\|\Phi^{(r,s)}\psi\| < (r + 1)(s + 1)A(G_0, G_0)\left(1 + \frac{1}{|\lambda|^2} + |\lambda|^2\right)^N |\lambda|^{r+s}$$

for all λ. Let λ tend to zero: we find $\Phi^{(r,s)}\psi = 0$ if $r + s > 2N$, and lemma 3 is proved.

Application to the Proof of Theorem 1

We apply lemma 3 to the case when $\varphi(h) = f(\Lambda(h)^{-1}z)$, for some fixed $z \in \mathcal{T}'_n$. Let $h = kg(\lambda)k'$. We have $\Lambda(h)^{-1} = \Lambda(k')^{-1}[\lambda^{-2}]\Lambda(k)^{-1}z$. When k and k' stay in fixed compacts K and K', $\Lambda(k)^{-1}z$ stays in a fixed compact of $\mathcal{T}'_n$. By virtue of lemma 2 we have:

$$|f(\Lambda(k')^{-1}[\lambda^{-2}]\Lambda(k)^{-1}z)| < A(K, K', z)\left(1 + \frac{1}{|\lambda|^2} + |\lambda|^2\right)^N .$$

Define

$$F^{(r,s)}(z) = (r+1)(s+1)\int_{G_0} D^{(r,s)}(u^{-1}h)\, f(\Lambda(u^{-1}h)z)\, du . \tag{3}$$

This matrix is independent of h, and holomorphic in z in $\mathcal{T}'_n$, since the integration is over a compact set. Lemma 3 shows that $F^{(r,s)}(z) \equiv 0$ for $r + s > 2N$. From remarks 3 and 4 of Section 1 we conclude:

$$f(\Lambda(g)^{-1}z) = \sum_{r+s \leqq 2N} \operatorname{tr} F^{(r,s)}(z)\, D^{(r,s)}(g) .$$

From remark 5, we see that only terms with even $r + s$ appear in the expansion and we write, with a slight abuse of notation:

$$f(\Lambda^{-1}z) = \sum_{\substack{r+s \leqq 2N \\ r+s \text{ even}}} \operatorname{tr} F^{(r,s)}(z)\, D^{(r,s)}(\Lambda) \tag{4}$$

for $z \in \mathcal{T}'_n$, $\Lambda \in L_+(\mathbb{C})$.

From equation (3), it is easy to derive:

$$F^{(r,s)}(\Lambda^{-1}z) = F^{(r,s)}(z)\, D^{(r,s)}(\Lambda) . \tag{5}$$

Let $M_{\mu\nu} = -M_{\nu\mu}$ be an element of the usual basis of the Lie algebra of $L_+^{\uparrow}$ and $\Delta^{(r,s)}(M_{\mu\nu})$ the corresponding matrix in the representation $D^{(r,s)}$. We have

$$F^{(r,s)}(e^{-tM_{\mu\nu}}z) = F^{(r,s)}(z)\, D^{(r,s)}(e^{tM_{\mu\nu}}) .$$

Taking derivatives at $t = 0$ on both sides, we find:

$$\sum_{j=1}^{n}\left(z_{j\mu}\frac{\partial}{\partial z_j^\nu} - z_{j\nu}\frac{\partial}{\partial z_j^\mu}\right) F^{(r,s)}(z) = F^{(r,s)}(z)\, \Delta^{(r,s)}(M_{\mu\nu}) .$$

We denote $\Delta(M_{\mu\nu})$ the differential operator

$$\sum_j \left(z_{j\mu}\frac{\partial}{\partial z_j^\nu} - z_{j\nu}\frac{\partial}{\partial z_j^\mu}\right) .$$

The above equation reads:

$$\Delta(M_{\mu\nu})\, F^{(r,s)}(z) = F^{(r,s)}(z)\, \Delta^{(r,s)}(M_{\mu\nu}) .$$

Clearly we can re-apply the differential operator $\Delta(M_{\varrho\sigma})$ on both sides. We find:

$$\begin{aligned}\Delta(M_{\varrho\sigma})\,\Delta(M_{\mu\nu})\, F^{(r,s)}(z) &= [\Delta(M_{\varrho\sigma})\, F^{(r,s)}(z)]\, \Delta^{(r,s)}(M_{\mu\nu}) \\ &= F^{(r,s)}(z)\, \Delta^{(r,s)}(M_{\varrho\sigma})\, \Delta^{(r,s)}(M_{\mu\nu}) .\end{aligned}$$

This process can be continued. It can be shown [14] that the mappings $M_{\mu\nu} \to \Delta(M_{\mu\nu})$ and $M_{\mu\nu} \to \Delta^{(r,s)}(M_{\mu\nu})$ can be extended to homomorphisms of the enveloping algebra of $L_+^{\uparrow}$. This means that to every element P of this algebra we can associate a differential operator $\Delta(P)$ and a matrix $\Delta^{(r,s)}(P)$ such that:

$$\Delta(P)\,\Delta(Q) = \Delta(PQ)$$
$$\Delta(\alpha P + \beta Q) = \alpha\Delta(P) + \beta\Delta(Q)$$

and similar conditions for $\Delta^{(r,s)}$; for $P = M_{\mu\nu}$, $\Delta(P)$ and $\Delta^{(r,s)}(P)$ are those already defined. $\Delta(P)$ is a differential operator whose coefficients are polynomials in z. The computations we have performed above show that:

$$\Delta(P)\,F^{(r,s)}(z) = F^{(r,s)}(z)\,\Delta^{(r,s)}(P)\,.$$

We denote [15]

$$\mathbf{F}^2 = \frac{1}{4}\sum{}' (M_{0\mu} + iM_{\nu\varrho})^2$$
$$\mathbf{G}^2 = \frac{1}{4}\sum{}' (M_{0\mu} - iM_{\nu\varrho})^2$$

where the summations are over all cyclic permutations μ, ν, ϱ of 1, 2, 3. $\mathbf{F}^2$ and $\mathbf{G}^2$ are in the centre of the enveloping algebra, and

$$\Delta^{(r,s)}(\mathbf{F}^2) = \frac{r}{2}\left(\frac{r}{2}+1\right) I$$
$$\Delta^{(r,s)}(\mathbf{G}^2) = \frac{s}{2}\left(\frac{s}{2}+1\right) I\,.$$

Denote, for r, s two integers $\geqq 0$:

$$A(r,s) = \prod_{\substack{r'<r\\ s'<s}} \frac{\left[\mathbf{F}^2 - \frac{r'}{2}\left(\frac{r'}{2}+1\right)\right]}{\left[\frac{r}{2}\left(\frac{r}{2}+1\right) - \frac{r'}{2}\left(\frac{r'}{2}+1\right)\right]}\,\frac{\left[\mathbf{G}^2 - \frac{s'}{2}\left(\frac{s'}{2}+1\right)\right]}{\left[\frac{s}{2}\left(\frac{s}{2}+1\right) - \frac{s'}{2}\left(\frac{s'}{2}+1\right)\right]}$$

Let $f(z)$ be a function satisfying the hypotheses of theorem 1. It has a finite expansion of the form (4). Let R and S be two integers such that $F^{(r,s)}(z) \equiv 0$ for $r > R$ or $s > S$. Then

$$\Delta(A(R,S))\,f(z) = \operatorname{tr} F^{(R,S)}(z)$$

and

$$\Delta(A(R,S))\,f(\Lambda^{-1}z) = \operatorname{tr} F^{(R,S)}\,D^{(R,S)}(\Lambda)\,.$$

[Note that $\Delta(\mathbf{F}^2)$, $\Delta(\mathbf{G}^2)$, $\Delta(A(r,s))$ are *Lorentz invariant* differential operators.] We may calculate in the same way

$$\operatorname{tr} F^{(R-1,S)}(z) = \Delta(A(R-1,S))\,[1 - \Delta(A(R,S))]\,f(z)\,.$$

By induction, we can thus construct, for each $r \leqq R$, $s \leqq S$, a Lorentz invariant differential operator $\boldsymbol{\Delta}^{(r,s)}_{(R,S)}$, such that, for *every* $f(z)$ verifying:

$$\begin{cases} f(z) = \sum\limits_{\substack{r\leqq R\\ s\leqq S}} \operatorname{tr} F^{(r,s)}(z) \\ F^{(r,s)}(\Lambda^{-1}z) = F^{(r,s)}(z)\,D^{(r,s)}(\Lambda)\,, \end{cases}$$

the following identity holds:

$$\Delta_{R,S}^{(r,s)} f(z) = \mathrm{tr} F^{(r,s)}(z) \; .$$

By Burnside's theorem [16], for fixed r and s, α and β, one can find a finite sequence $\Lambda_1, \ldots, \Lambda_p$ in $L_+^{\uparrow}$, and numbers $c_l (l = 1, \ldots, p)$ such that, for every $(r+1)(s+1) \times (r+1)(s+1)$ matrix Ξ, the matrix element $\Xi_{\alpha\beta}$ can be computed from the formula:

$$\Xi_{\alpha\beta} = \sum_{l=1}^{p} c_l \, \mathrm{tr} \Xi D^{(r,s)}(\Lambda_l) \; .$$

In particular

$$F_{\alpha\beta}^{(r,s)}(z) = \sum_{l=1}^{p} c_l \Delta_{R,S}^{(r,s)} f(\Lambda_l^{-1} z) \; . \tag{6}$$

Since $\Delta_{R,S}^{(r,s)}$ is an operator with *polynomial coefficients*, and since $\Lambda_l \in L_+^{\uparrow}$, the restriction of $F^{(r,s)}(z)$ to $\mathscr{T}_n^{\pm}$ is the Laplace transform of a tempered distribution $\tilde{F}^{(r,s)\pm}$ with support in $\overline{V}_n^{\pm}$. Let $F^{(r,s)\pm}$ be the Fourier transform of $\tilde{F}^{(r,s)\pm}$. For $\varphi \in \mathscr{S}^{4n}$, we have:

$$\begin{aligned}
&\lim_{\substack{y \to 0 \\ y \in V_n^+}} \int F_{\alpha\beta}^{(r,s)}(x+iy)\, \varphi(x)\, dx = \langle F_{\alpha\beta}^{(r,s)\pm}, \varphi \rangle \,, \\
&\lim_{\substack{y \to 0 \\ y \in V_n^+}} \int f(x+iy)\, \varphi(x)\, dx = \langle f^{\pm}, \varphi \rangle \,, \\
&f^{\pm} = \sum_{\substack{r \leqq R \\ s \leqq S}} \mathrm{tr}\, F^{(r,s)\pm} \; .
\end{aligned} \tag{7}$$

If we define, for any distribution T and $\Lambda \in L_+^{\uparrow}$,

$$\langle T_\Lambda, \varphi \rangle = \langle T, \varphi_{\Lambda^{-1}} \rangle; \;\; \varphi_{\Lambda^{-1}}(x) = \varphi(\Lambda x)$$

we also have:

$$F_{\alpha\beta}^{(r,s)\pm} = \sum_{l=1}^{p} c_l \, \Delta_{R,S}^{(r,s)} f_{\Lambda_l}^{\pm} \, .$$

This completes the proof of theorem 1.

We note that, by a theorem of Hepp [17], $F^{(r,s)}(z)$ can always be written in the form

$$F_{\alpha\beta}^{(r,s)} = \sum_{\varkappa=1}^{K} Q_{\alpha\beta}^{(\varkappa)}(z)\, f_{\varkappa}^{(r,s)}(z)$$

where the $Q_{\alpha\beta}^{(\varkappa)}(z)$ are covariant polynomials of type (r, s), chosen in advance, and the $f_{\varkappa}^{(r,s)}$ are holomorphic in $\mathscr{T}_n'$ and Lorentz invariant. Hence:

Theorem 2. *Every $f(z)$ satisfying the hypotheses of theorem 1 can be written as a finite sum*

$$\sum_{j=1}^{J} Q_j(z)\, f_j(z)$$

where the Q_j are polynomials and the f_j are Lorentz invariant holomorphic functions in $\mathscr{T}_n'$.

4. Some Remarks on Finitely Covariant Distributions

We call *finitely covariant* a tempered distribution T on $\mathbb{R}^{4n}$ having a finite expansion

$$T = \sum_{\substack{r \leqq R \\ s \leqq S}} \operatorname{tr} T^{(r,s)} \tag{8}$$

where the $T^{(r,s)}$ are matrix valued tempered distributions with the property

$$T^{(r,s)}_{\Lambda} = T^{(r,s)} D^{(r,s)}(\Lambda), \quad \Lambda \in L_+^{\uparrow} . \tag{9}$$

The finitely covariant distributions form a linear subspace of $\mathscr{S}'$. By the same arguments as in the preceding Section, we have:

$$\operatorname{tr} T^{(r,s)} = \Delta^{(r,s)}_{R,S}\, T \tag{10}$$

and

$$T^{(r,s)}_{\alpha\beta} = \sum_l c_l \operatorname{tr} T^{(r,s)}_{\Lambda_l} \tag{11}$$

where the Λ_l and c_l depend only on (r, s, α, β). We conclude that if T has two expansions of the form (8), they must coincide, since we may always consider them to have the same R and S. In particular suppose that $T_1, \ldots, T_p$ are finitely covariant and satisfy a system of linear equations

$$\sum_k \alpha_{jk} T_k = G_j$$

where the G_j are Lorentz invariant (the α_{jk} are complex numbers). Then:

$$\sum_k \alpha_{jk} T^{(r,s)}_k = 0 \quad \text{for} \quad (r, s) \neq (0, 0)$$

$$\sum_k \alpha_{jk} T^{(0,0)}_k = G_j .$$

As an application, consider the problem of defining generalized retarded functions (Steinmann functions) [18—21] from Lorentz invariant Wightman functions. The problem is to find a set of tempered distributions R_k satisfying the following conditions:

a) $\sum_k \alpha_{jk} R_k = C_j$,

b) (support of R_k) $\subset S_k$ (x-space support conditions),

c) $\tilde{R}_k = \tilde{R}_l$ in Ω_{kl} (p-space coincidence conditions).

d) Lorentz invariance.

Here the Lorentz-invariant tempered distributions C_j are given linear combinations of Wightman functions ["multiple commutators" — a) also contains the Steinmann identities]. The sets S_k are given closed cones in $\mathbb{R}^{4n}$. The Ω_{kl} are given open sets in $\mathbb{R}^{4n}$. The α_{jk} are given numbers. $\tilde{R}_k$ is the Fourier transform of R_k. It is not very difficult to solve a) and b) simultaneously. STEINMANN [22] has even succeeded in solving a), b) and d) in two-dimensional space time. Stora has given a complete solution for $n = 2$. If a given set of R_k satisfies a), b), c), the $\tilde{R}_k$ are boundary values of a unique function, holomorphic in a domain containing several pairs of opposite tubes of the type $\mathcal{T}_n^{\pm}$, and the corresponding Jost points. Streater's theorem and theorem 1 prove that the distributions $\tilde{R}_k$ (and R_k) are finitely covariant. Therefore the corresponding $R_k^{(0,0)}$ are a solution of a), b), c), d).

We shall now find some precisions on the maximum number of non-zero terms in the expansion (8) of a finitely covariant distribution T. Suppose that there is an integer p such that, for every $\varphi \in \mathcal{S}^{4n}$,

$$|\langle T_\Lambda, \varphi\rangle| < C_\varphi (1 + |\Lambda|)^p \quad \text{for all} \quad \Lambda \in L_+^{\uparrow}$$

where C_φ is a constant which may depend on φ. If $\Lambda = \Lambda(g)$, $g = (A, \bar{A})$, we define $|\Lambda| = \|A\|^2$. Because $T^{(r,s)}$ can be computed from formulae (10) and (11), where $\Delta^{(r,s)}_{(R,S)}$ is Lorentz invariant, there exists a constant C'_φ (depending on φ, r, s, R, S), such that

$$\|\langle T_\Lambda^{(r,s)}, \varphi\rangle\| < C'_\varphi (1 + |\Lambda|)^p$$

or

$$\|\langle T^{(r,s)}, \varphi\rangle D^{(r,s)}(\Lambda)\| < C'_\varphi (1 + |\Lambda|)^p .$$

Let ψ be a vector in $\mathbb{C}^{(r+1)(s+1)}$. Since $D^{(r,s)}$ is an irreducible representation of $L_+^{\uparrow}$, we can find a finite sequence $\Lambda_1, \ldots, \Lambda_q$ in $L_+^{\uparrow}$, and numbers $a_1, \ldots, a_q$ such that

$$\psi = \sum_{j=1}^{q} \psi_j, \quad \psi_j = a_j D^{(r,s)}(\Lambda_j) \begin{bmatrix}1\\0\end{bmatrix}^{\otimes r+s} .$$

We have:

$$\langle T^{(r,s)}, \varphi\rangle \psi_j = a_j \langle T^{(r,s)}, \varphi\rangle D^{(r,s)}(\Lambda_j) \begin{bmatrix}1\\0\end{bmatrix}^{\otimes r+s}$$

$$= a_j \langle T^{(r,s)}, \varphi\rangle D^{(r,s)}(\Lambda) D^{(r,s)}(\Lambda^{-1}\Lambda_j) \begin{bmatrix}1\\0\end{bmatrix}^{\otimes r+s} .$$

Take:

$$\Lambda^{-1}\Lambda_j = [\lambda^2], \quad \Lambda = \Lambda_j[\lambda^{-2}]; \quad |\Lambda| \leq \left(|\lambda|^2 + \frac{1}{|\lambda|^2}\right) |\Lambda_j|;$$

$$D^{(r,s)}([\lambda^2]) \begin{bmatrix}1\\0\end{bmatrix}^{\otimes r+s} = \lambda^{(r+s)} \begin{bmatrix}1\\0\end{bmatrix}^{\otimes r+s} .$$

It follows that:

$$\|\langle T^{(r,s)}, \varphi\rangle \psi_j\| < \text{constant} \times (|\lambda|^2 + |\lambda|^{-2})^p \, |\lambda|^{r+s} .$$

Letting λ tend to zero, we conclude that $\langle T^{(r,s)}, \varphi\rangle \psi_j = 0$ for $r + s > 2p$. It follows that: $T^{(r,s)} \equiv 0$ for $(r + s) > 2p$.

In particular, if $p = 0$, T has to be Lorentz invariant. This coincides with the result of Borchers [4].

We also note that a finitely covariant distribution T has a support which is invariant under $L_+^{\uparrow}$. For, if a point x is not in the support of T, there is an open real ball B centered at x such that $\langle T, \varphi\rangle = 0$ whenever the support of φ is a compact K contained in B. If such is the case, there is a neighbourhood W of the unit in $L_+^{\uparrow}$ such that the support of $\varphi_{\Lambda^{-1}}$ is still in B for all $\Lambda \in W$, so that $\langle T_\Lambda, \varphi\rangle = 0$ for all Λ in W. But

$$\langle T_\Lambda, \varphi\rangle = \sum_{\substack{r \leq R \\ s \leq S}} \operatorname{tr}\langle T^{(r,s)}, \varphi\rangle\, D^{(r,s)}(\Lambda)$$

is a holomorphic function of Λ in $L_+(\mathbb{C})$ and since it vanishes in the "real environment" W, it vanishes everywhere. It follows that $\langle T_\Lambda, \varphi\rangle = 0$ for all $\Lambda \in L_+^{\uparrow}$ so that every point of the form Λy, $\Lambda \in L_+^{\uparrow}$, $y \in B$ is outside the support of T.

Part II

In the second part of this paper we shall review the proof of Streater's theorem. A proof of this theorem can be found in [1]. Different proofs are due to Jost [23] and to Ruelle [24]. See also [25]). We shall follow a slightly different approach with the purpose of stressing the question of single valuedness and of showing that this method applies to more general situations.

In the following, a function will always be associated with the set in which it is defined: we shall not identify a function with its restriction to a smaller subset or with an extension to a larger subset. We shall say that a function defined and holomorphic in an open set U has a single valued analytic continuation in an open set V if $V \cap U \neq \emptyset$ and if there exists a function g, holomorphic in V, such that f and g coincide in $V \cap U$.

We consider, as in the first part, $\mathbb{C}^{4n}$ as the topological product of n four dimensional Minkowski spaces, where $L_+(\mathbb{C})$ acts as follows: if $z = (z_1, \ldots, z_n) \in \mathbb{C}^{4n}$ and $\Lambda \in L_+(\mathbb{C})$, $\Lambda z = (\Lambda z_1, \ldots, \Lambda z_n)$. We denote $\overline{L_+(\mathbb{C})}$ and $\bar{L}_+^{\uparrow}$ the covering groups of $L_+(\mathbb{C})$ and $L_+^{\uparrow}$, respectively and $A \to \Lambda(A)$ the canonical homomorphisms of $\overline{L_+(\mathbb{C})}$ onto $L_+(\mathbb{C})$. Note that $\bar{L}_+^{\uparrow}$ is a subgroup of $\overline{L_+(\mathbb{C})}$ (see Part I).

Let e_0 and e_1 be two real four vectors with $(e_0, e_0) = -(e_1, e_1) = 1$; $(e_0, e_1) = 0$. Let $T = e_0 \wedge e_1$ be the linear operator given by $T^\mu{}_\nu = e_0^\mu e_{1\nu} - e_{0\nu} e_1^\mu$. For every complex ζ, $\exp[\zeta T]$ defines a complex Lorentz transformation $\chi(\zeta) = \hat{\chi}(e^\zeta)$. If we take a Lorentz frame in which $e_0 = (1, 0, 0, 0)$ and $e_1 = (0, 1, 0, 0)$, we find:

$$\hat{\chi}(\lambda) = \begin{pmatrix} \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) & \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) & 0 & 0 \\ \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) & \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

Such a complex one-parameter subgroup of $L_+(\mathbb{C})$ will be called a *time-like subgroup.*

We shall prove:

Theorem 3. I. *Let D be a domain in $\mathbb{C}^{4n}$ having the following properties:*

1. *For each time-like subgroup χ of $L_+(\mathbb{C})$, there exists an open non-empty sub-set E_χ of D invariant under χ, i.e., for every $\lambda \neq 0$, $\hat{\chi}(\lambda)\, E_\chi \subset E_\chi$.*
2. *D is invariant under $L_+^\uparrow$, the real connected Lorentz group.*

Then, for any function $f(z)$ holomorphic in D, there exists a function $F(\Lambda, z)$ holomorphic in $L_+(\mathbb{C}) \times D$ which, for $\Lambda \in L_+^\uparrow$, coincides with $f(\Lambda z)$.

II. *Let D_0 be a non-empty sub-domain of D having the following property: for every $z \in D_0$, the set of all $\Lambda \in L_+(\mathbb{C})$ such that $\Lambda^{-1} z \in D_0$ is connected. Then there exists a function $g(z)$ holomorphic in*

$$D_0' = \bigcup_{\Lambda \in L_+(\mathbb{C})} \Lambda D_0$$

which coincides with f in D_0.

The proof of this theorem necessitates several steps. The class of all functions holomorphic in D will be denoted $\mathscr{H}$.

(i) *Continuation Using One Timelike Subgroup*

In this section χ is a timelike subgroup chosen once and for all. Let Ω_χ be the following *domain* in $\mathbb{C} \times \mathbb{C}^{4n}$

$$\Omega_\chi = \{\zeta, z : z \in D, \zeta \in \Delta(z)\}$$

where $\Delta(z)$ is the connected component of 0 in the open set of $\mathbb{C}$:

$$\Delta'(z) = \{\zeta : \chi(\zeta)\, z \in D\}$$

$\Delta'(z)$ is invariant under real translations for, if t is real and $\chi(\zeta)\, z \in D$, $\chi(\zeta + t) z = \chi(t)\, \chi(\zeta)\, z \in D$ due to the invariance of D under $L_+^\uparrow$. When $z \in D$, $\Delta'(z)$ contains 0 and $\Delta(z)$ is a non-empty open strip parallel to the real axis. It is easy to verify that Ω_χ is open and connected; it is a semitube. When $z \in E_\chi$, $\Delta(z) = \mathbb{C}$. Using Bremermann's semitube theorem [26, 27], it is then clear that every function φ holomorphic in Ω_χ has a single valued holomorphic continuation in $\mathbb{C} \times D$. In order to exhibit the single valuedness of this continuation we go through an elementary proof of this fact.

Let S be the set of all points $z \in D$ with the following property: there exists a polycylinder $P(z; \varrho(z))$ contained in D, of the form $\{z' : |z_j'^\mu - z_j^\mu| < \varrho(z)\}$ such that for every function φ holomorphic in Ω_χ, there exists a function φ_z holomorphic in $\mathbb{C} \times P(z; \varrho(z))$ which coincides with φ in the domain $\Omega_\chi \cap \{\mathbb{C} \times P(z; \varrho(z))\}$.

The set S is obviously open and non-empty since it contains E_χ. Let z and $z' \in S$. If $\mathbb{C} \times P(z; \varrho(z))$ and $\mathbb{C} \times P(z'; \varrho(z'))$ intersect, their intersection is connected and has a non-empty intersection with Ω_χ where

φ_z and $\varphi_{z'}$ coincide with φ. Hence φ_z and $\varphi_{z'}$ coincide wherever they are both defined. It follows that there is a function $\hat{\varphi}$ holomorphic in $(\mathbb{C} \times S) \cup \Omega_\chi$ which coincides with φ in Ω_χ.

Let $z \in \bar{S} \cap D$. There is a polycylinder $P(z; 2\varrho)$ centered at z, with radius 2ϱ and a number $\varepsilon > 0$ such that

$$\{\zeta, z : |\mathrm{Im}\zeta| < \varepsilon, z \in P(z; 2\varrho)\} \subset \Omega_\chi .$$

$S \cap P(z; \varrho)$ contains a polycylinder $P(\hat{z}; \sigma)$ $(0 < \sigma < \varrho)$ so that $\hat{\varphi}$ is holomorphic in $\{\mathbb{C} \times P(\hat{z}; \sigma)\} \cup \{\zeta, z' : |\mathrm{Im}\zeta| < \varepsilon, z' \in P(\hat{z}; \varrho)\}$. Standard methods of analytic completion show that there exists a function $\hat{\varphi}_1$, holomorphic in $\mathbb{C} \times P(\hat{z}; \varrho)$, which coincides with $\hat{\varphi}$ (hence with φ) in $\{\zeta, z' : |\mathrm{Im}\zeta| < \varepsilon, z' \in P(\hat{z}; \varrho)\}$. It therefore coincides with φ in $\mathbb{C} \times P(z; \sigma) \cap \Omega_\chi$ so that $z \in S$. [To perform the analytic continuation we may introduce the variables $w_j^\mu = i \log(z_j'^\mu - \hat{z}_j^\mu)$. The function $\hat{\varphi}(\zeta, (e^{-iw_j^\mu} + \hat{z}_j^\mu))$ is holomorphic in the tube

$$\{\zeta, w : \mathrm{Im} w_j^\mu < \log\sigma\} \cup \{\zeta, w : |\mathrm{Im}\zeta| < \varepsilon, \mathrm{Im} w_j^\mu < \log\varrho\}$$

and is invariant under the change: $w_j^\mu \to w_j^\mu + 2 n_j^\mu \pi$ (n_j^μ integer). This function has a single valued holomorphic continuation ψ in $\{\zeta, w : \mathrm{Im} w_j^\mu < \log\varrho\}$ obviously also periodic. Thus,

$$\psi(\zeta, w) = \varphi_1(\zeta, (e^{-iw_j^\mu} + \hat{z}_j^\mu)) ,$$

where $\varphi_1(\zeta, z')$ is holomorphic in $\mathbb{C} \times \{z' : 0 < |z_j'^\mu - \hat{z}_j^\mu| < \varrho\}$ and coincides with $\hat{\varphi}$ wherever they are both defined. Using the continuity theorem one obtains the announced result.]

Since D is connected we have proved: $S = D$. Thus:

Every function holomorphic in Ω_χ has a single valued continuation in $\mathbb{C} \times D$.

Remark. If a function φ holomorphic in Ω_χ is such that, for $z \in E_\chi$ and for any integer m, $\varphi(\zeta, z) = \varphi(\zeta + 2im\pi, z)$, then its continuation $\hat{\varphi}(\zeta, z)$ in $\mathbb{C} \times D$ is also periodic and can be cast in the form $\varphi_1(e^\zeta, z)$, $\varphi_1(\lambda, z)$ being holomorphic in $(\mathbb{C} - \{0\}) \times D$.

(ii) *Products of Several Timelike Subgroups*

Let $\Phi = (\chi_1, \ldots, \chi_N)$ be a finite sequence of timelike subgroups. Let f be a function holomorphic in D. $\zeta_1, z \to f(\chi_1(\zeta_1)z)$ defines a function holomorphic in Ω_{χ_1}. Hence there exists a function $F_1(\zeta_1, z)$ holomorphic in $\{\zeta_1, z : z \in D\}$ which coincides with $f(\chi_1(\zeta_1)z)$ in Ω_{χ_1}, in particular in a neighborhood of $\{\zeta_1, z : z \in D, \zeta_1 \in \mathbb{R}\}$. $F_1(\zeta_1, z)$ can be considered as a holomorphic function of $z \in D$ with values in the Fréchet space of functions holomorphic in $\mathbb{C}$. By the same argument as above, there exists a function with values in the same Fréchet, holomorphic in $\mathbb{C} \times D$ which coincides with $F_1(\zeta_1, \chi_2(\zeta_2) z)$ near real values of ζ_2, i.e., a function $F_2(\zeta_1, \zeta_2, z)$ holomorphic in $\mathbb{C} \times \mathbb{C} \times D$ and coinciding with $f(\chi_1(\zeta_1)\, \chi_2(\zeta_2)z)$ in a neighborhood of $\mathbb{R}^2 \times D$. By induction we can prove:

For every finite sequence $\Phi = \{\chi_1, \ldots, \chi_N\}$ *of timelike subgroups and for every* $f \in \mathscr{H}$, *there exists a function* $F_\Phi(\zeta_1, \ldots, \zeta_N, z)$ *holomorphic in* $\{\zeta, z : z \in D\} = \mathbb{C}^N \times D$ *which coincides with* $f(\chi_1(\zeta_1) \ldots \chi_N(\zeta_N) z)$ *in a neighborhood of* $\mathbb{R}^N \times D$.

(iii) *Existence of* $F(\Lambda, z)$

Denote $G_\Phi(\zeta, z)$ the function defined by

$$G_\Phi(\zeta, z) = F_\Phi(\zeta, \chi_N(-\zeta_N) \ldots \chi_1(-\zeta_1) z)\,,$$

or

$$G_\Phi(\zeta, \chi_1(\zeta_1) \ldots \chi_N(\zeta_N) z) = F_\Phi(\zeta, z)\,.$$

$G_\Phi(\zeta, z)$ is holomorphic in: $\{\zeta, z : \chi_N(-\zeta_N) \ldots \chi_1(-\zeta_1)\, z \in D\}$ and coincides with $f(z)$ for real ζ. Therefore it is *locally* a function of z only, i.e., its derivatives $\frac{\partial}{\partial \zeta_j} G_\Phi(\zeta, z)$ vanish everywhere. Define:

$$\tilde{G}_\Phi(\zeta, \Lambda, z) = G_\Phi(\zeta, \Lambda z)\,.$$

This is a function of ζ, z, and $\Lambda \in L_+(\mathbb{C})$ holomorphic in:

$$\Delta = \{\zeta, \Lambda, z : \chi_N(-\zeta_N) \ldots \chi_1(-\zeta_1)\, \Lambda z \in D\}$$

and its derivatives $(\partial/\partial \zeta_j)\tilde{G}_\Phi$ vanish there. Hence, in any open subset of Δ of the form $\{\zeta, \Lambda, z : \zeta \in V_1, \Lambda \in V_2, z \in V_3\}$ where V_1, V_2, V_3 are open, and V_1 is connected, the function $\tilde{G}_\Phi(\zeta, \Lambda, z)$ coincides with a function of Λ and z holomorphic in $V_2 \times V_3$.

For a fixed $z \in D$, the set $\mathscr{G}_z$ of all germs of analytic functions of Λ obtained in this fashion (starting from a given $f \in \mathscr{H}$), by using all possible finite sequences Φ of timelike subgroups, defines a "Riemann domain" [28] on which $f(\Lambda z)$ (considered as a function of Λ in a neighborhood of $\Lambda = 1$) can be continued. Technically, this Riemann domain is an open connected subset of the sheaf of germs of analytic functions over the complex Lorentz group $L_+(\mathbb{C})$. We shall now see that any continuous path γ in $L_+(\mathbb{C})$ connecting the identity 1 to some element of the group can be "lifted" into the Riemann domain $\mathscr{G}_z$. This means that we can patch together germs of analytic functions taken from $\mathscr{G}_z$ all along γ and thus obtain an analytic continuation of $f(\Lambda z)$ along γ. This will be done in two steps. (In the following, z is held fixed in D.)

First step. Suppose that γ is defined by a continuous mapping $t \to \Lambda(t)$ of $[0, 1]$ into $L_+(\mathbb{C})$ of the form $\Lambda(t) = \chi_1(\zeta_1(t)) \ldots \chi_N(\zeta_N(t))$, where $\Phi = \{\chi_1, \ldots, \chi_N\}$ is a finite sequence of timelike subgroups and $t \to \zeta_j(t)$ are continuous complex functions of $t \in [0, 1]$, with $\zeta_j(0) = 0$, $(j = 1, \ldots, N)$. The path defined in $\mathbb{C}^N \times L_+(\mathbb{C})$ by $t \to (\zeta(t), \Lambda(t))$ lies in the domain of definition of $\tilde{G}_\Phi(\zeta, \Lambda, z)$, as a function of ζ and Λ. This is an open set, so that, for each t, $(\zeta(t), \Lambda(t))$ has a neighborhood $V_1(t) \times V_2(t)$ contained in this domain: $V_2(t)$ is an open neighborhood of $\Lambda(t)$ and $V_1(t)$ is an open ball centered at $\zeta(t)$. Because $t \to (\zeta(t), \Lambda(t))$ is continuous, for each $t \in [0, 1]$, a number $r(t) > 0$ can be found such that $|s - t| < r(t)$ and $0 \leqq s \leqq 1$ imply $\zeta(s) \in V_1(t)$ and $\Lambda(s) \in V_2(t)$. We can find a finite sequence $0 = t_0 < t_1 < \cdots < t_{p-1} < t_p = 1$ having the following properties: the sets $J_k = \{s : |t_k - s| < r(t_k), 0 \leqq s \leqq 1\}$ form a covering of $[0, 1]$ and $J_{k-1} \cap J_k \neq \emptyset$ for $k \geqq 1$. For $p \geqq k \geqq 1$ let θ_k be a number such that $t_{k-1} < \theta_k < t_k$, $\theta_k \in J_{k-1} \cap J_k$; $\theta_0 = 0$; $\theta_{p+1} = 1$. Because $V_1(t_k)$ is connected, $\tilde{G}_\Phi(\zeta, \Lambda, z)$ coincides in $V_1(t_k) \times V_2(t_k)$ with a function of Λ, denoted $g_k(\Lambda, z)$, holomorphic for $\Lambda \in V_2(t_k)$. For any $\Lambda \in V_2(t_{k-1}) \cap V_2(t_k)$, we have $g_{k-1}(\Lambda, z) = \tilde{G}_\Phi(\zeta(\theta_k), \Lambda, z) = g_k(\Lambda, z)$. We now associate with every $t \in [0, 1]$ a germ $\hat{g}(t)$ of analytic function of Λ as follows: for $\theta_k \leqq t \leqq \theta_{k+1}$, $0 \leqq k \leqq p$, $\hat{g}(t)$ is the germ of $g_k(\Lambda, z)$ at $\Lambda(t)$. This defines a continuous mapping of $[0, 1]$ into $\mathscr{G}_z$ (the continuity of the mapping is an immediate consequence of the definition of the topology of the sheaf of germs of analytic functions). We have thus achieved the "lifting" of the path γ into $\mathscr{G}_z$. Note that $\hat{g}(0)$ is the germ of $f(\Lambda z)$ at $\Lambda = 1$. Intuitively what has been done is to define an analytic continuation of $f(\Lambda z)$ by taking $g_k(\Lambda, z)$ along the subset $\theta_k \leqq t \leqq \theta_{k+1}$ of the path γ.

Second Step. We first prove the following property: there exists a neighborhood W of 1 in $L_+(\mathbb{C})$ and six timelike subgroups $\chi_1^0, \ldots, \chi_6^0$ such that the mapping $\zeta \to \chi_1^0(\zeta_1) \ldots \chi_6^0(\zeta_6)$ is a biholomorphic map of an open ball (centered at 0) of $\mathbb{C}^6$ onto W.

Indeed, let $\chi_j^0(\zeta_j) = \exp[M_j \zeta_j]$. Then, at $\zeta = 0$,

$$d[\chi_1^0(\zeta_1) \ldots \chi_6^0(\zeta_6)] = \sum_{j=1}^{6} M_j \, d\zeta_j .$$

In view of the implicit function theorem, the mapping $\zeta \to \Lambda(\zeta) = \chi_1^0(\zeta_1) \ldots \chi_6^0(\zeta_6)$ is biholomorphic in a sufficiently small neighborhood of $\zeta = 0$ if $M_1, \ldots, M_6$ are linearly independent. Our statement will be proved if we can find $M_1, \ldots, M_6$ linearly independent and having the required form, i.e., $M_j = e_0{}^{(j)} \wedge e_1{}^{(j)}$, where $e_0{}^{(j)}$ and $e_1{}^{(j)}$ are real and $e_0{}^{(j)}$ is timelike. To do this, let e_0, e_1, e_2, e_3, be real four vectors with $(e_\mu, e_\nu) = g_{\mu\nu}$. Define:

$$M_j = e_0 \wedge e_j \quad \text{for} \quad 1 \leqq j \leqq 3 ;$$

$$M_4 = e_1 \wedge e_2 + \sqrt{2}\, e_0 \wedge e_2; \quad M_5 = e_1 \wedge e_3 + \sqrt{2}\, e_0 \wedge e_3 ;$$

$$M_6 = e_2 \wedge e_3 + \sqrt{2}\, e_0 \wedge e_3 .$$

This defines a basis of the (complex) Lie algebra of $L_+(\mathbb{C})$ and our statement is proved. [In the case of q-dimensional space time we define a basis of the Lie algebra having the required properties, as follows: $M'_{\mu\nu}$ is given, for $0 \leqq \mu < \nu < q$, by:

$$M'_{0\nu} = e_0 \wedge e_\nu ;$$

for $\mu \geqq 1$, $M'_{\mu\nu} = e_\mu \wedge e_\nu + \sqrt{2}\, e_0 \wedge e_\nu$.]

We now show that every continuous path γ in $L_+(\mathbb{C})$, originating at 1, can be cast into the special form used in the first step. Let γ be given by a continuous mapping $t \to A(t)$ of $[0, 1]$ into $L_+(\mathbb{C})$, with $A(0) = 1$. We shall prove the existence of a finite sequence $\chi_1, \ldots, \chi_N$ of timelike subgroups and of N continuous complex functions $\zeta_1(t), \ldots, \zeta_N(t)$ such that $A(t) = \chi_1(\zeta_1(t)) \ldots \chi_N(\zeta_N(t))$.

Let V be a neighborhood of 1 in $L_+(\mathbb{C})$ such that $VV \subset W$ and $V^{-1} = V$. By the same technique as in the first step, we define two finite sequences $0 = t_0 = \theta_0 < \theta_1 < t_1 < \theta_2 < t_2 < \cdots < \theta_p < t_p = \theta_{p+1} = 1$, such that $A(s) \in A(t_k)V$ for $\theta_k \leqq s \leqq \theta_{k+1}$, $k = 0, \ldots, p$. For $s \in [\theta_k, \theta_{k+1}]$, $0 \leqq k \leqq p$, we have $A(t_k)^{-1}A(s) \in V$, hence:

$$\begin{aligned} A(\theta_k)^{-1}A(s) &= A(\theta_k)^{-1}A(t_k)A(t_k)^{-1}A(s) \\ &= [A(t_k)^{-1}A(\theta_k)]^{-1}A(t_k)^{-1}A(s) \in V^{-1}V = VV \subset W . \end{aligned}$$

Therefore we can define six continuous complex functions:

$$\zeta_{6k+1}(s), \ldots, \zeta_{6k+6}(s) \quad \text{of} \quad s \in [\theta_k, \theta_{k+1}]$$

satisfying:

$$\chi_1^0(\zeta_{6k+1}(s)) \ldots \chi_6^0(\zeta_{6k+6}(s)) = A(\theta_k)^{-1}A(s) .$$

We extend these functions to the whole of $[0, 1]$ by setting:

$$\begin{aligned} \zeta_{6k+r}(s) &= \zeta_{6k+r}(\theta_k) = 0 \quad \text{for} \quad s \leqq \theta_k, \quad (k \geqq 1) , \\ \zeta_{6k+r}(s) &= \zeta_{6k+r}(\theta_{k+1}) \quad \text{for} \quad s \geqq \theta_{k+1}, \quad (k \leqq p-1) . \end{aligned}$$

Defining $\chi_{6k+r} = \chi_r^0$ for $k = 0, \ldots, p$ we have:

$$\chi_1(\zeta_1(s)) \ldots \chi_{6p+6}(\zeta_{6p+6}(s)) = A(s) .$$

The second step is now completed. Every germ in $\mathscr{G}_z$ can be obtained by analytic continuation along a certain path, starting from the germ of $f(Az)$ at $A = 1$. It therefore follows from the result obtained above that:

If $t \to A(t)$ is any continuous mapping of $[0, 1]$ into $L_+(\mathbb{C})$ and if $\hat{g}_0 \in \mathscr{G}_z$ is a germ projecting onto $A(0)$, there exists a continuous mapping $t \to \hat{g}(t)$ of $[0, 1]$ into $\mathscr{G}_z$ such that $g(t)$ projects onto $A(t)$ and $\hat{g}(0) = \hat{g}_0$.

[In other words, given any path γ in $L_+(\mathbb{C})$ originating at $A(0)$ and any germ of analytic function at $A(0)$, belonging to $\mathscr{G}_z$, it is possible to continue analytically this function along the path γ.]

This is what is needed to apply the *monodromy theorem* [29]. We conclude that the continuations of $f(Az)$ (as an analytic function of A near $A = 1$) along two *homotopic paths* in $L_+(\mathbb{C})$ lead to the same germ. This means that $\mathscr{G}_z$ is the set of germs of a holomorphic function $\hat{F}(A, z)$ on the covering group $\overline{L_+(\mathbb{C})}$ of $L_+(\mathbb{C})$ [in the case of four-dimensional space-time, this is $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$]. It is also holomorphic in z, since it always locally coincides with a function holomorphic in A and z. When A is in a sufficiently small neighborhood of 1, we have: $\hat{F}(A, z)$

$= f(\Lambda(A)z)$. By analytic continuation along any path lying in $\bar{L}_+^\uparrow$, we find: $\hat{F}(A, z) = f(\Lambda(A)z)$ for all $A \in \bar{L}_+^\uparrow$. The kernel N of the canonical mapping $A \to \Lambda(A)$ of $\overline{L_+(\mathbb{C})}$ onto $L_+(\mathbb{C})$ is contained in $\bar{L}_+^\uparrow$. Hence, if $A_0 \in N$, we have $\hat{F}(AA_0, z) = \hat{F}(A, z)$ for all $A \in \bar{L}_+^\uparrow$. It follows that $\hat{F}(AA_0, z) = \hat{F}(A, z)$ for all $A \in \overline{L_+(\mathbb{C})}$, so that $\hat{F}(A, z) = F(\Lambda(A), z)$, where $F(\Lambda, z)$ is a holomorphic function in $L_+(\mathbb{C}) \times D$. This proves the first part of the theorem.

The function $F(\Lambda, z)$ is locally a function of Λz. Define $G(\Lambda, z) = F(\Lambda, \Lambda^{-1}z)$. Then $G(\Lambda, z)$ is defined and holomorphic in $\{\Lambda, z : \Lambda^{-1}z \in D\}$, and is locally independent of Λ. Let D_0 be a subdomain of D such that, for every $z \in D_0$, $\{\Lambda : \Lambda^{-1}z \in D_0\}$ is connected. Let $D_0' = \bigcup_{\Lambda \in L_+(\mathbb{C})} \Lambda D_0$. Then $\{\Lambda : \Lambda^{-1}z \in D_0\}$ is connected for any $z \in D_0'$. The restriction of $G(\Lambda, z)$ to $\{\Lambda, z : \Lambda^{-1}z \in D_0\}$ is therefore independent of Λ. It defines a function $g(z)$ holomorphic in D_0', which coincides with $f(z)$ in D_0 [since $G(1, z) = f(z)$]. The theorem is thus proved.

Application to the Proof of Streater's Theorem

Let f satisfy the hypotheses of the S-theorem. For each Jost point $w \in \mathscr{J}_n$, there exists, by the edge-of-the-wedge theorem, an open set $B(w) = \{z : \|x - w\| < \varrho(w), \|y\| < \varrho(w)\}$ independent of f, and a function holomorphic in $B(w)$ which coincides with f in $B(w) \cap \{\mathscr{T}_n^+ \cup \mathscr{T}_n^- \cup \mathscr{J}_n\}$. Since this is also true for f_Λ, defined by $f_\Lambda(z) = f(\Lambda^{-1}z)$, $\Lambda \in L_+^\uparrow$, there exists, for every $w \in \mathscr{J}_n$ and every $\Lambda \in L_+^\uparrow$ a function holomorphic in $\Lambda B(w)$ which coincides with f in $\Lambda B(w) \cap \{\mathscr{T}_n^+ \cup \mathscr{T}_n^- \cup \mathscr{J}_n\}$. Let

$$D = \bigcup_{\substack{\Lambda \in L_+^\uparrow \\ w \in \mathscr{J}_n}} \Lambda B(w) \cup \mathscr{T}_n^+ \cup \mathscr{T}_n^- .$$

The intersection of $\Lambda B(w)$ and $\Lambda' B(w')$ (where Λ and $\Lambda' \in L_+^\uparrow$, w and $w' \in \mathscr{J}_n$) when non-empty, is convex, hence connected, and intersects $\mathscr{T}_n^+$ and $\mathscr{T}_n^-$ because it contains real points. It follows that there exists a function holomorphic in D which coincides with f in $D \cap \{\mathscr{T}_n^+ \cup \mathscr{T}_n^- \cup \mathscr{J}_n\}$. The theorem will be proved by applying theorem 3 to D. This domain is clearly invariant under $L_+^\uparrow$. Let χ be a timelike subgroup of $L_+(\mathbb{C})$. By a real Lorentz transformation, we choose co-ordinates in which $\hat{\chi}(\lambda)$ has the form (12). For any $z = (z_1, \ldots, z_n)$, $z_j = \{z_j^\mu\}$, $(\mu = 0, 1, 2, 3)$, we set $u_j = z_j^0 + z_j^1$, $v_j = z_j^0 - z_j^1$. Then $z' = \hat{\chi}(\lambda)z_j$ is given by: $u_j' = \lambda u_j$, $v'_j = \lambda^{-1}v_j$; the other co-ordinates are unchanged. Let A_χ be the set of all points z such that:

$$u_j = i|u_j| \neq 0$$

$$v_j = i|v_j| \neq 0$$

and having all other co-ordinates real. For $z \in A_\chi$ and $z' = \hat{\chi}(\lambda)z$, we have:

$$\operatorname{Im} u_j' = |u_j| \operatorname{Re}\lambda$$

$$\operatorname{Im} v_j' = |v_j| \frac{1}{|\lambda|^2} \operatorname{Re}\lambda .$$

For $\operatorname{Re}\lambda > 0$, $z' \in \mathscr{T}_n^+$; for $\operatorname{Re}\lambda < 0$, $z' \in \mathscr{T}_n^-$; for $\lambda = i\varrho$, ϱ real $\neq 0$, $z' \in \mathscr{J}_n$. Therefore: D contains all points $\hat{\chi}(\lambda)z$, where $z \in A_\chi$ and $\lambda \neq 0$. In particular, for any $z \in A_\chi$, D contains the compact set $\{z' : z' = \hat{\chi}(\lambda)z, |\lambda| = 1\}$. It therefore contains an open set of the form $\{z' : z' = \hat{\chi}(\lambda)z'', \|z'' - z\| < \varepsilon(z), |\lambda| = 1\}$ where $\varepsilon(z) > 0$. It follows that D contains an open set of the form $E_\chi = \bigcup_{\lambda \neq 0} \hat{\chi}(\lambda)\, V$ where V is an open neighbourhood of A_χ.

Now we take $D_0 = \mathscr{T}_n^+$. A theorem of Bargmann, Hall and Wightman [30] shows that for any $z \in \mathscr{T}_n^+$, $\{\Lambda : \Lambda \in L_+(\mathbb{C}), \Lambda z \in \mathscr{T}_n^+\}$ is connected. Therefore, for any function f holomorphic in D, there exists a function g, holomorphic in $\mathscr{T}_n'$, which coincides with f in $\mathscr{T}_n^+$. Because D is connected and $\mathscr{T}_n'$ contains $\mathscr{T}_n^\pm$ and $\mathscr{J}_n$, g also coincides with f in $\mathscr{T}_n^-$ and in $\mathscr{J}_n$. This is the S-theorem. We now make some remarks on the validity of these theorems for q-dimensional space-time.

The first part of theorem 3 has been proved for $q = 2$ in the remark at the end of the first step of the proof. For $q \geqq 3$ it rests essentially on the following property: if we denote ψ the canonical homomorphism of $\overline{L_+(\mathbb{C})}$ onto $L_+(\mathbb{C})$, and N the kernel of ψ, then $N \subset \psi^{-1}(L_+^\uparrow)$ and $\psi^{-1}(L_+^\uparrow)$ is connected. This fact is easily checked for $q = 3$ and 4. For $q > 4$ it is possible to prove it by induction [one proves that every closed path originating at 1, lying in $L_+^\uparrow$ and homotopic to 0 in $L_+(\mathbb{C})$ is homotopic to 0 in $L_+^\uparrow$. Thus for $q \geqq 4$, $\psi^{-1}(L_+^\uparrow) = \bar{L}_+^\uparrow$, the covering group of $L_+^\uparrow$, and $N \subset \psi^{-1}(L_+^\uparrow)$.] The second part of theorem 3 does not depend on q. The application to the S-theorem rests essentially on the theorem of Bargmann, Hall and Wightman which has been extended by R. Jost [23, 31] to q-dimensional space-time.

Finally we note that the proof of theorem 3 can be adapted by trivial modifications to the following case:

Let G be a connected complex Lie group, $\tilde{G}$ its covering group, ψ the canonical map: $\tilde{G} \to G$. Let G_r be a closed subgroup of G, G being a complexification of G_r; G_r is a real Lie group with Lie algebra $\mathscr{L}_r$, a real vector space with dimension p; the complex Lie algebra of G is the complexified of $\mathscr{L}_r$. Let $\tilde{G}_r$ be the connected component of the unit in $\psi^{-1}(G_r)$. Suppose G acts as a group of holomorphic automorphisms of some domain $\mathscr{O}$ of $\mathbb{C}^n$; more precisely we assume that there is a holomorphic mapping $(g, z) \sim\to gz$ of $(G \times \mathscr{O})$ into $\mathscr{O}$ such that $g_1(g_2 z) = (g_1 g_2)z$ for any two $g_1, g_2 \in G$ and $ez = z$ if e is the identity in G. Let $\mathscr{F}$ be a total family of elements of $\mathscr{L}_r$. One can then prove:

Theorem 4. *Let D be a domain contained in $\mathcal{O}$, invariant under G_r and such that, for every $X \in \mathcal{F}$, there is a non-empty open set $E_X \subset D$ such that $(\exp \zeta X) E_X \subset E_X$ for all $\zeta \in \mathbb{C}$. Then, for every function f holomorphic in D there exists a function F holomorphic in $\tilde{G} \times D$ such that $F(g, z)$ coincides with $f(\psi(g) z)$ in some neighborhood of $\tilde{G}_r \times D$.*

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