

On the Continuity of Causal Automorphisms of Space-Time

J. L. ALONSO and F. J. YNDURÁIN*

University of Zaragoza, Spain

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Abstract. We prove that any causal automorphism of the (curved or not) space-time in any dimension is continuous.

In a recent paper, ZEEMAN [1] has shown that relativistic invariance is implied by causality alone. More precisely, he has proved that the group of automorphisms of the Minkowski space M_4 preserving the partial ordering $x < y$ (we say that $x < y$ iff $(x - y) \cdot (x - y) > 0$ and $x_0 < y_0$) coincides with the group G consisting of the connected Poincaré group plus space reversal and dilatations. Here $x = (x_0, x_1, x_2, x_3)$ is a vector in M_4 and $x \cdot y$ is the Lorentz scalar product of x and y . Linearity or continuity are not assumed a priori.

ZEEMAN's proof leans essentially on the four-dimensional character of M_4 , and, in fact (as pointed out by himself) the result fails for the M_2 space. On the other hand, the (pseudo) euclidean character of the whole of M_4 is also essential for his proof, and thus, as was to be expected, the theorem does not apply to the space-time of general relativity (V_4) either. A less strong result, however, may still be shown to hold in the more general cases. In fact, we are going to prove:

Theorem. *Let the $(n + 1)$ -dimensional Minkowski space with the metric $x \cdot y = \sum_0^n x_\mu y_\nu g_{\mu\nu}$, $g_{\mu\nu} = \text{diag}(+1, -1, \dots, -1)$ be denoted by M_{n+1} , and let the space V_{n+1} be locally homeomorphic to M_{n+1} . If the (bijective) automorphism f of V_{n+1} is causal, then it is continuous.*

We assume the euclidean topology for M_{n+1} . By locally homeomorphic we mean that for every p in V_{n+1} there exist a x in M_{n+1} , two open neighborhoods of p , x , in V_{n+1} , M_{n+1} denoted by Γ , Γ' , and a mapping $\varphi: \Gamma \rightarrow \Gamma'$ continuous and one-to-one such that φ^{-1} is also continuous. Finally, to define causal mappings we proceed as follows: it is clear that, due to the topological manifold structure of V_{n+1} , we must define causality in terms of local causality. Thus we begin by defining locally causal mappings: f is locally causal if a) for every point p in V_{n+1} and x in M_{n+1} there exists a Γ around p as above and the open

* Present address: Physics Department, New York University.

set \mathcal{A} around p such that both \mathcal{A} and $f\mathcal{A}$ are contained in Γ (locality); and b) given these Γ , \mathcal{A} , and Γ' , φ being as above, the mapping $\varphi f \varphi^{-1}$ in $\varphi\mathcal{A}$ is causal in the ordinary sense, i.e., it preserves the ordering $x < y$ ($x < y$ whenever $(x - y) \cdot (x - y) > 0$ and $x_0 < y_0$). Now, given a mapping f of V_{n+1} , we say that it is causal if it may be decomposed as a product of locally causal mappings.

Remark 1. We may use the partial ordering just described or the ordering $x < y$ whenever $(x - y) \cdot (x - y) = 0$ and $x_0 < y_0$ (note that the last is not properly an ordering since it is not transitive), i.e., causality for massive or massless particles. The reason is that f preserves one ordering iff it preserves the other (ZEEMAN's proof of this fact remains unaltered in the more general cases).

Remark 2. The requirement of f being one-to-one is essential. It is not a difficult task to show this by explicit counterexamples.

We divide the proof of the Theorem into two steps:

1. Let X be a topological space and let \mathcal{B} be a basis of its topology (in the case of M_{n+1} , \mathcal{B} may consist of the open balls). Let " \subset " be a partial ordering in X ; we write $x < y$ whenever $x \subset y$ but $y \not\subset x$. If $x < y$, we define the range R_{xy} as the set of all z with $x < z < y$. We say that the ordering is continuous in the case that the following be true: whenever $x < y$ there exists B_x in \mathcal{B} such that x is in B_x and $B_x < y$, and a B_y also in \mathcal{B} with $x < B_y$ and y contained in B_y . One has:

Lemma. *If (i) the ordering " $<$ " is continuous and (ii) for every z in B (B arbitrary in \mathcal{B}) there exist x, y such that R_{xy} is contained in B and z is in R_{xy} ; if $f: X \rightarrow X$ is one-to-one and f, f^{-1} preserve the ordering, then f (and a fortiori, f^{-1}) is continuous.*

Note 1. An alternative formulation of the Lemma would be the following: let X be a topological space with an ordering " $<$ "; assume that a) for all x, y , the set R_{xy} is open; and b) the sets R_{xy} form a basis of the topology of X . Then, if f is a one-to-one mapping of X , and if f, f^{-1} preserve the ordering, they are continuous.

This form has the virtue of making apparent the simplicity of the Lemma; it is not difficult, on the other hand, to prove that both formulations are equivalent. For purposes of applications, however, we believe the former to be more appropriate.

Note 2. The Lemma may be trivially generalised to mappings from a space X_1 to another X_2 , both with topology and ordering.

The alternative formulation and generalisation of the Lemma are due to D. RUELLÉ.

Proof of the Lemma. Since for any z in R_{xy} there exist $B_z < y$ and $B'_z > x$, z being in both, the set $B = B_z \cap B'_z$ is open, is contained in R_{xy} and contains z . Consequently, R_{xy} may be obtained as the union of such B and thus it is open. It is also clear that, in virtue of the hypo-

theses, for any open set U and every point z in U there exists a R_{xy} around z and contained in U , i.e., any open set is the union of R_{xy} 's. The continuity of f, f^{-1} then follows from the fact that the transformed of a R_{xy} by any of them is again a $R_{x'y'}$, and that a mapping is continuous whenever the inverse image of an open set is open.

2. To prove the Theorem it is now sufficient to show that the causality ordering verifies conditions (i), (ii) in the Lemma. As to the first, it is obvious in virtue of the continuity of the functions $x_0 - y_0$ and $(x - y) \times (x - y)$. Let us prove (ii); we will do it for M_{n+1} as X . Let z be in B and take $x < z < y$ with x, y lying on the parallel to the 0-axis through z ; let $d_1 = y_0 - z_0$ and $d_2 = z_0 - x_0$. Let finally z' be any point in R_{xy} . Now:

$$\begin{aligned} (d_1 + d_2)^2 &= (y_0 - x_0)^2 = ((y_0 - z'_0) + (z'_0 - x_0))^2 > \\ &> (y_0 - z'_0)^2 + (z'_0 - x_0)^2 > \sum_1^n (z'_i - y'_i)^2 + \sum_1^n (x_i - z'_i)^2 > \\ &> \sum_1^n (z'_i - z_i)^2 > \max |z'_i - z_i|^2, \end{aligned}$$

the last because of z being on the xy line. On the other hand, $y_0 - z'_0 > 0$ and $z'_0 - x_0 > 0$, so that also $|z'_0 - z_0| < d_1 + d_2$, i.e., $|z'_\mu - z_\mu| < d_1 + d_2$ for any μ . It then suffices to take d_1, d_2 such that

$$(d_1 + d_2)^2 \leq \inf \left\{ \sum_0^n |z_\mu - u_\mu|, u \notin B \right\} / (n + 1)$$

to obtain that

$$\sum_0^n |z'_\mu - z_\mu| < \inf \left\{ \sum_0^n |z_\mu - u_\mu|, u \notin B \right\}$$

i.e., that R_{xy} is contained in B . This proves the Theorem for automorphisms of M_{n+1} . To prove it when f is a locally causal automorphism of V_{n+1} , let z be any point in V_{n+1} and let A, Γ, Γ' and φ be as before. Take B contained into φA ; then, the previous analysis shows that $\varphi f \varphi^{-1}$ is continuous at φz . The continuity of f at z is now clear since $f = \varphi^{-1}(\varphi f \varphi^{-1})\varphi$, and all φ, φ^{-1} and $\varphi f \varphi^{-1}$ are continuous, i.e., f is the product of continuous mappings. Finally, in the most general case of any causal automorphism the theorem is proved by simply remarking that f is a product of locally causal (and thus continuous) transformations.

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References

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