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The Egoroff Theorem for Operator-Valued Measures in Locally Convex Spaces

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Abstract

The Egoroff theorem for measurable $\mathbf{X}$-valued functions and operator-valued measures $\mathbf{m} : \Sigma \to L(\mathbf{X}, \mathbf{Y})$ is proved, where Σ is a σ-algebra of subsets of $T \neq \emptyset$ and $\mathbf{X}$, $\mathbf{Y}$ are both locally convex spaces.

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1 Introduction

The classical Egoroff theorem states that almost everywhere convergent sequences of measurable functions on a finite measure space converge almost uniformly, i.e., for every $\varepsilon > 0$ the convergence is uniform on a set whose complement has measure less than ε, cf. [1]. When generalizing to functions taking values in more general spaces, some further problems appear arising from the fact that the classical relationship between the pointwise convergence and the convergence in measure is not saved. It is also well-known that the Egoroff theorem cannot hold for arbitrary nets of measurable functions without some restrictions on measure, net convergence of functions, or class of measurable functions. In [2,

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Definition 1.2] the first author introduced the so-called Condition (GB) under which everywhere convergence of net of measurable functions implies convergence of these functions in semivariation on a set of finite variation of measure in locally convex setting, cf. [3, Theorem 3.3]. This condition concerns families of submeasures and enables to work with nets of measurable functions instead of sequences. Recall that Condition (GB) is fulfilled in the case of atomic operator-valued measures, cf. [3]. Atomic measures are not of a great interest in the classical theory of measure and integral, because they lead only to considerations of absolutely convergent series. But when we consider measures with very general range space, e.g. a locally convex space, the situation changes. Thus, in this paper we prove the following Egoroff theorem for atomic operator-valued measures in locally convex topological vector spaces.

Theorem 1.1 (Egoroff)**.** *Let* $\mathbf{m} : \Sigma \to L(\mathbf{X}, \mathbf{Y})$ *be a purely atomic operator-valued measure, and let* $E \in \Sigma$ *be a set of finite variation of the measure* $\mathbf{m}$*. If* $\mathbf{f} : T \to \mathbf{X}$ *is a measurable function, and* $(\mathbf{f}_i : T \to \mathbf{X})_{i\in I}$ *is a net of measurable functions, such that*

$$\lim_{i\in I} p(\mathbf{f}_i(t) - \mathbf{f}(t)) = 0 \quad \textit{for every} \;\; t \in E \textit{ and } \; p \in P, \tag{1.1}$$

then the net $(\mathbf{f}_i)_{i\in I}$ *of functions* $\mathbf{m}$*-almost uniformly converges to* $\mathbf{f}$ *on* $E \in \Sigma$.

2 Preliminaries

By a *net* (with values in a set S) we mean a function from I to S, where I is a directed partially ordered set. Throughout this paper I is a directed index set representing direction of a net. Let Σ be a σ-algebra of subsets of $T \neq \emptyset$, and $\mathbf{X}$, $\mathbf{Y}$ be two Hausdorff locally convex topological vector spaces over the field $\mathbb{K}$ of all real $\mathbb{R}$ or complex numbers $\mathbb{C}$, with two families of seminorms P and Q defining the topologies on $\mathbf{X}$ and $\mathbf{Y}$, respectively. Then $L(\mathbf{X}, \mathbf{Y})$ denotes the space of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$. In this paper $\mathbf{m}$ always means an operator-valued measure $\mathbf{m} : \Sigma \to L(\mathbf{X}, \mathbf{Y})$ σ-additive in the strong operator topology of the space $L(\mathbf{X}, \mathbf{Y})$, i.e., $\mathbf{m}(\cdot)\mathbf{x} : \Sigma \to \mathbf{Y}$ is a $\mathbf{Y}$-valued vector measure for every $\mathbf{x} \in \mathbf{X}$. Let $(p, q) \in P \times Q$. For the measure $\mathbf{m}$ we introduce the following quantities:

(a) the (p, q)-*semivariation* $\hat{\mathbf{m}}_{p,q} : \Sigma \to [0, +\infty]$ defined as follows

$$\hat{\mathbf{m}}_{p,q}(E) := \sup q\left(\sum_{n=1}^{N} \mathbf{m}(E_n \cap E)\mathbf{x}_n\right),$$

where the supremum is taken over all finite disjoint partitions

$$\left\{E_n \in \Sigma;\, E = \bigcup_{n=1}^{N} E_n,\, E_n \cap E_m = \emptyset,\, n \neq m,\, m, n = 1, 2, \ldots, N\right\} \tag{2.1}$$

of $E \in \Sigma$ and all finite sets $\{\mathbf{x}_n \in \mathbf{X};\, p(\mathbf{x}_n) \leq 1,\, n = 1, 2, \ldots, N\}$, where $N \in \mathbb{N}$;

(b) the (p, q)-*variation* $\mathbf{var}_{p,q}(\mathbf{m}, \cdot) : \Sigma \to [0, +\infty]$ defined by the equality

$$\mathbf{var}_{p,q}(\mathbf{m}, E) := \sup \sum_{n=1}^{N} q_p\left(\mathbf{m}(E_n \cap E)\right),$$

where the supremum is taken over all finite disjoint partitions (2.1) of $E \in \Sigma$ with

$$q_p(\mathbf{m}(F)) := \sup_{p(\mathbf{x}) \leq 1} q(\mathbf{m}(F)\mathbf{x}), \quad F \in \Sigma.$$

Clearly, the (p,q)-variation of $\mathbf{m}$ is a monotone and σ-additive set function, whereas the the (p,q)-semivariation of $\mathbf{m}$ is a monotone and σ-subadditive set function with $\hat{\mathbf{m}}_{p,q}(\emptyset) = \mathbf{var}_{p,q}(\mathbf{m},\emptyset) = 0$ for every $(p,q) \in P \times Q$. Also, $\hat{\mathbf{m}}_{p,q}(E) \leq \mathbf{var}_{p,q}(\mathbf{m},E)$ for every $(p,q) \in P \times Q$ and $E \in \Sigma$.

In this paper we consider the sets $E \in \Sigma$ of positive variation of the measure $\mathbf{m}$, i.e., there exist $(p,q) \in P \times Q$, such that $\mathbf{var}_{p,q}(\mathbf{m},E) > 0$, and of finite variation of the measure $\mathbf{m}$, i.e., if for every $q \in Q$ there exists $p \in P$, such that $\mathbf{var}_{p,q}(\mathbf{m},E) < +\infty$. We will denote this relation shortly $Q \to_E P$, or, $q \mapsto_E p$ for $(p,q) \in P \times Q$.

Definition 2.1. We say that a set $E \in \Sigma$ of positive semivariation of the measure $\mathbf{m}$ is an $\hat{\mathbf{m}}$-*atom* if every proper subset A of E is either $\emptyset$ or $A \notin \Sigma$. We say that the measure $\mathbf{m}$ is *purely atomic* if each $E \in \Sigma$ can be expressed in the form $E = \bigcup_{k=1}^{\infty} A_k$, where $A_k, k \in \mathbb{N}$, are $\hat{\mathbf{m}}$-atoms.

In what follows we consider only measurable functions in the following sense: a function $\mathbf{f} : T \to \mathbf{X}$ is *measurable* if the set $\{t \in T; p(\mathbf{f}(t)) \geq \eta\}$ belongs to Σ for every $\eta > 0$ and $p \in P$.

Definition 2.2. A net $(\mathbf{f}_i)_{i\in I}$ of measurable functions is said to be $\mathbf{m}$-*almost uniformly convergent* to a measurable function $\mathbf{f}$ on $E \in \Sigma$ if for every $\varepsilon > 0$ and every $(p,q) \in P \times Q$ with $q \mapsto_E p$ there exist measurable sets $F = E(\varepsilon, p, q)$, such that $\lim_{i\in I} ||\mathbf{f}_i - \mathbf{f}||_{E\setminus F,p} = 0$ and $\hat{\mathbf{m}}_{p,q}(F) < \varepsilon$, where $||\mathbf{g}||_{G,p} := \sup_{t\in G} p(\mathbf{g}(t))$.

The concept of generalized strong continuity of semivariation of a measure is introduced in [4]. This notion enables development of the concept of an integral with respect to the $L(\mathbf{X},\mathbf{Y})$-valued measure based on the net convergence of simple functions. For this purpose the notion of inner semivariation is used for this generalization. Recall that for $(p,q) \in P \times Q$ the set function $\hat{\mathbf{m}}^*_{p,q} : 2^T \to [0,+\infty]$ given by

$$\hat{\mathbf{m}}^*_{p,q}(E) = \sup_{F\subset E, F\in\Sigma} \hat{\mathbf{m}}_{p,q}(F), \quad E \in 2^T,$$

is said to be the *inner* (p,q)-*semivariation of the measure* $\mathbf{m}$.

Definition 2.3. We say that the semivariation of the measure $\mathbf{m}$ is *generalized strongly continuous* (GS-*continuous, for short)* if for every set $E \in \Sigma$ of finite variation of the measure $\mathbf{m}$ and every monotone net of sets $(E_i)_{i\in I} \subset T$, $E_i \subset E$, $i \in I$, the following equality

$$\lim_{i\in I} \hat{\mathbf{m}}^*_{p,q}(E_i) = \hat{\mathbf{m}}^*_{p,q}\left(\lim_{i\in I} E_i\right)$$

holds for every couple $(p,q) \in P \times Q$, such that $q \mapsto_E p$.

Theorem 2.4. *If* $\mathbf{m}$ *is a (countable) purely atomic measure, then its semivariation is* GS-*continuous.*

Proof. Let $E \in \Sigma$ be a set of finite and positive variation of the measure $\mathbf{m}$, and $(E_i)_{i\in I}$ be an arbitrary decreasing net of sets from Σ. Recall that $E_i \searrow G(\in 2^G)$ if and only if $i \le j \Rightarrow E_i \supset E_j$, and $\bigcap_{i\in I} E_i = G$. It is clear that it is enough to consider the case $G = \emptyset$, because $E_i \searrow G \Leftrightarrow G \subset E_i$, $E_i \setminus G \searrow \emptyset$. First, in the case $E_i \in \Sigma$, $i \in I$, we have

$$\lim_{i\in I} \hat{\mathbf{m}}^*_{p,q}(E_i) = \lim_{i\in I} \hat{\mathbf{m}}_{p,q}(E_i), \tag{2.2}$$

and since the family of atoms is at most a countable set, there is $\lim_{i\in I} E_i = \bigcap_{i\in I} E_i \in \Sigma$, and therefore

$$\hat{\mathbf{m}}^*_{p,q}\left(\lim_{i\in I} E_i\right) = \hat{\mathbf{m}}_{p,q}\left(\lim_{i\in I} E_i\right) \tag{2.3}$$

for every $(p,q) \in P\times Q$, such that $q \mapsto_E p$.

Denote by $\mathcal{A}$ the set of all $\hat{\mathbf{m}}$-atoms, and put $\ell(i,E) = (\mathcal{A}\cap E)\setminus E_i$, $i \in I$. Clearly for $i, j \in I$ we get $i \le j \Rightarrow \ell(i,E) \subset \ell(j,E)$, and there exist atoms $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, such that $\ell(i,E) = \{A_1, A_2, \ldots, A_n, \ldots\}$. Since for every $(p,q) \in P\times Q$ the (p,q)-variation of the measure $\mathbf{m}$ is σ-additive, then

$$\mathbf{var}_{p,q}(\mathbf{m}, E) = \mathbf{var}_{p,q}(\mathbf{m}, E_i) + \sum_{A_n\in\ell(i,E)} \mathbf{var}_{p,q}(\mathbf{m}, A_n),$$

for $i \in I$, and $(p,q) \in P\times Q$. The inequality $\hat{\mathbf{m}}_{p,q}(E_i) \le \mathbf{var}_{p,q}(\mathbf{m}, E_i)$ for $i \in I$ and $(p,q) \in P\times Q$ implies

$$\hat{\mathbf{m}}_{p,q}(E_i) \le \mathbf{var}_{p,q}(\mathbf{m}, E) - \sum_{A_n\in\ell(i,E)} \mathbf{var}_{p,q}(\mathbf{m}, A_n).$$

Since $\mathbf{var}_{p,q}(\mathbf{m}, E\cap\cdot) : \Sigma \to [0,+\infty)$ is a finite real measure for every $(p,q) \in P\times Q$ with $q \mapsto_E p$, then for every $\varepsilon > 0$ and $(p,q) \in P\times Q$ with $q \mapsto_E p$ there exists an index $i_0 := i_0(\varepsilon, p, q, E) \in I$, such that

$$\hat{\mathbf{m}}_{p,q}(E_i) < \varepsilon \tag{2.4}$$

holds for every $i \ge i_0$, $i \in I$. Combining (2.2), (2.3), (2.4) and Definition 2.3 we see that the assertion is proved for the case when $(E_i)_{i\in I}$ is a decreasing net of sets from Σ. The other cases of monotone nets of sets may be proved analogously.

Let now $G \subset T$ be an arbitrary set. Then there is exactly one (countable) set $F^* = \mathcal{A}\cap G$ with the property

$$\hat{\mathbf{m}}^*_{p,q}(G) = \sup_{F\subset G, F\in\Sigma} \hat{\mathbf{m}}_{p,q}(F) = \hat{\mathbf{m}}_{p,q}(F^*), \quad (p,q) \in P\times Q.$$

The proof for the inner measure and the arbitrary net of subsets $(E_i)_{i\in I}$ goes by the same procedure as in the previous part of proof concerning the set system Σ. □

3 Proof of Theorem 1.1

We have to prove that for a given $\varepsilon > 0$ and every $q \in Q$, $p \in P$, such that $q \mapsto_E p$ there exist measurable sets $F = E(\varepsilon, p, q) \in \Sigma$, such that $\lim_{i\in I} \|\mathbf{f}_i - \mathbf{f}\|_{E\setminus F,p} = 0$, and $\hat{\mathbf{m}}_{p,q}(F) < \varepsilon$.

Suppose that (1.1) holds. For every $m \in \mathbb{N}$, $p \in P$, and $j \in I$, put

$$\begin{aligned} B^p_{m,j} &= E \cap \left\{ t \in T;\, p(\mathbf{f}_i(t) - \mathbf{f}(t)) < \frac{1}{m}, \quad i \geq j \right\} \\ &= E \cap \bigcap_{i \geq j} \left\{ t \in T;\, p(\mathbf{f}_i(t) - \mathbf{f}(t)) < \frac{1}{m}, \quad i \in I \right\}. \end{aligned}$$

Since there are countable many of atoms, $B^p_{m,j} \subset \Sigma$ and $\# B^p_{m,j} = \aleph_0$. Clearly, if $i, j \in I$ such that $i \leq j$, then $B^p_{m,i} \subset B^p_{m,j}$ for every $m \in \mathbb{N}$ and $p \in P$. Put

$$E^p_m := \bigcup_{j \in I} B^p_{m,j}.$$

The net $(E^p_m \setminus B^p_{m,i})_{i \in I}$ clearly tends to void set for every $m \in \mathbb{N}$ and $p \in P$. Since $\mathbf{m}$ is a purely atomic operator-valued measure, then according to Theorem 2.4 its semivariation is GS-continuous, and therefore

$$\lim_{i \in I} \hat{\mathbf{m}}^*_{p,q}(E^p_m \setminus B^p_{m,i}) = 0, \quad (p,q) \in P \times Q, \text{ such that } q \mapsto_E p.$$

Let $\varepsilon > 0$ be given. For each $p \in P$ and $m \in \mathbb{N}$ there exists an index $j(m,p) \in I$, such that for $q \mapsto_E p$ the inequality $\hat{\mathbf{m}}_{p,q}(E^p_m \setminus B^p_{m,i}) < \varepsilon \cdot \alpha_p \cdot \beta_m$ holds for every $i \geq j(m,p)$, where $\{\alpha_p;\, p \in P\}$ is a summable system of positive numbers in the sense of Moore–Smith and $\{\beta_m;\, m \in \mathbb{N}\}$ is an absolutely convergent series of positive numbers. Putting

$$F := \bigcup_{m \in \mathbb{N}} \bigcup_{p \in P} \left(E^p_m \setminus B^p_{m,j(m,p)} \right)$$

we have

$$\begin{aligned} \hat{\mathbf{m}}^*_{p,q}(F) &= \hat{\mathbf{m}}^*_{p,q}\left(\bigcup_{m \in \mathbb{N}} \bigcup_{p \in P} \left(E^p_m \setminus B^p_{m,j(m,p)} \right) \right) \\ &= \lim_{K = \{p_1, \ldots, p_m\}} \hat{\mathbf{m}}_{p,q}\left(\sum_{m=1}^{\infty} \sum_{p \in K} \left(E^p_m \setminus B^p_{m,j(m,p)} \right) \right) \\ &\leq \lim_{K = \{p_1, \ldots, p_m\}} \sum_{m=1}^{\infty} \sum_{p \in K} \hat{\mathbf{m}}_{p,q}\left(E^p_m \setminus B^p_{m,j(m,p)} \right) < \varepsilon. \end{aligned}$$

Let us show that the convergence of net of functions $(\mathbf{f}_i)_{i \in I}$ is uniform on $E \setminus F$. Note that $\bigcup_{m=1}^{\infty} E^p_m = E$ for every $p \in P$. For a given $\eta > 0$ choose an $m_0 \in \mathbb{N}$, such that $\frac{1}{m_0} < \eta$. Then

$$E \setminus F = E \setminus \bigcup_{m \in \mathbb{N}} \bigcup_{p \in P} \left(E^p_m \setminus B^p_{m,j(m,p)} \right) = \bigcap_{m \in \mathbb{N}} \bigcap_{p \in P} B^p_{m,j(m,p)} \subset B^p_{m_0, j(m_0,p)}$$

for every $p \in P$. By definition of the set $B^p_{m_0,j(m_0,p)}$ we have that if $t \in B^p_{m_0,i}$, then $p(\mathbf{f}_i(t) - \mathbf{f}(t)) < \eta$ for every $i \geq j(m_0,p)$. So, (1.1) implies that for every $\eta > 0$ and $p \in P$ there exists an index $j(\eta,p)$, such that for every $i \geq j(\eta,p)$, $i \in I$, there is $p(\mathbf{f}_i(t) - \mathbf{f}(t)) < \eta$ for $t \in E \setminus B^p_{m_0,i} \supset E \setminus F$, i.e., the net $(\mathbf{f}_i)_{i \in I}$ converges uniformly on $E \setminus F$. □

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