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A Note on Embedded Trapped Modes for a Two-layer Fluid over a Rectangular Barrier

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Abstract

We prove that the system of equations describing waves in a two-layer fluid over a rectangular barrier of small height possesses embedded trapped modes (eigenvalues submerged in the continuous spectrum) for certain values of the width of the barrier and that these eigenvalues are analytic in the small parameter characterizing the height of the barrier. We do this by means of purely elementary considerations constructing explicit solutions and thus confirm the results of [7] obtained for general perturbations of the depth of the fluid in the particular case of a rectangular barrier.

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1 Introduction

The appearance of trapped modes (eigenfunctions or bound states) under perturbations has attracted a lot of attention in recent years (see, e.g., [2, 5]). As a rule, their existence is due to the nature of the continuous spectrum of the unperturbed problem: a cutoff of the continuous spectrum generates an eigenvalue under certain perturbations. This is the case of the one-dimensional Schrödinger equation [3, 8]; in those references it is shown that a fairly general shallow potential well possesses an eigenvalue (energy level) to the left of the

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continuous spectrum, which occupies the positive semi-axis. This result can be obtained by means of reducing the initial differential equation to an integral one and then applying the Neumann series in order to solve the latter. This procedure shows that the eigenvalue is an analytic function of the parameter ϵ characterizing the perturbation. On the other hand, in the "trivial" case of a rectangular potential well, exact solution can be obtained in terms of elementary functions (trigonometric and exponential) because the equation is a second-order ordinary differential equation with piecewise constant coefficients. The matching of the solutions at the points of discontinuity of the potential provides an equation for the energy level and this equation turns out to be solvable by means of the implicit function theorem, thus providing the analyticity of the eigenvalue in ϵ in this particular case, as was to be expected.

Apart from trapped modes corresponding to eigenvalues that lie outside the continuous spectrum (discrete eigenvalues), there are situations when the eigenvalues are embedded in it — the so-called embedded trapped modes [4, 6]. These trapped modes are as a rule associated with the embedded cutoffs of the continuous spectrum, which divide it into segments where the multiplicity is constant but varies when the spectral parameter passes through a cutoff. In our recent paper [7] we studied the problem of oblique waves in a two-layer fluid over a cylindrical protrusion of small height (of the order of ϵ) and general shape on the bottom. For this problem, the continuous spectrum occupies the ray $\lambda \geq \lambda_1 > 0$, where $\lambda = \omega^2/g$ is the spectral parameter, ω is the frequency and g is the acceleration due to gravity. The value of the cutoff λ_1 depends on the various parameters of the problem. Apart from the cutoff λ_1, there is another cutoff $\lambda_2 > \lambda_1$ such that the multiplicity of the spectrum is 2 for $\lambda_1 < \lambda < \lambda_2$ and 4 for $\lambda > \lambda_2$. Under the perturbation, the cutoff λ_1 generates a discrete eigenvalue to the left of it, which is analytic in ϵ. The second cutoff λ_2 also generates an eigenvalue to the left of it, but only if the perturbation satisfies a certain geometric condition which in the case of a rectangular barrier means that the width of the barrier should be almost equal to an integer multiple of the wavelength of the propagating mode. This eigenvalue is embedded in the continuous spectrum and is analytic in ϵ. These results were obtained in [7] by means of reducing the initial system to integral equations for fairly general perturbations.

In the case of a rectangular barrier, the equations of our system become second order ordinary differential equations with piecewise constant coefficients, and, as explained above for the Schrödinger equation, admit explicit solutions in terms of elementary functions. The matching at the points of discontinuity of the coefficients provides a system of equations for the eigenvalue and the width of the barrier. One expects that this system, just as in the case of the Schrödinger equation with a shallow rectangular potential well, can be solved by means of the implicit function theorem. Interestingly, this is not straightforward since the corresponding Jacobian vanishes. Nevertheless, due to a special structure of the system, the first equation can be shown to possess an analytic in ϵ solution. Substituting the result in the second equation, one sees that an expression vanishing at $\epsilon = 0$ can be factored out and then one can apply the implicit function theorem in order to prove the analyticity. The goal of this note is to explain the corresponding calculations. All our considerations are quite elementary but seem to us interesting, especially in view of possible numerical simulations (cf. [1]).

2 Formulation and Main Results

We consider the linearized shallow-water system for a two-layer fluid (see [7] and references therein) describing waves over a rectangular barrier parallel to the y-axis. Looking for the solution in the form $\exp[i(ky-\omega t)]\boldsymbol{u}(x)$, $\boldsymbol{u}=(u,v)$, with $k>0$ (oblique incidence), where t is time, and x is the coordinate orthogonal to the barrier, we come to the following system of two coupled ordinary differential equations for $\boldsymbol{u}$:

$$\hat{\mathcal{L}}\boldsymbol{u}=\lambda\boldsymbol{u}, \tag{2.1}$$

where

$$\hat{\mathcal{L}}\boldsymbol{u}=\begin{pmatrix}-b(u''-k^2u)-(h(\beta u+\alpha v)')'+k^2h(\beta u+\alpha v)\\ -(h(\beta u+\alpha v)')'+k^2h(\beta u+\alpha v)\end{pmatrix},$$

primes denote the derivatives with respect to x, $b>0$ is the (constant) depth of the upper layer of density ρ_1, h is the depth of the lower layer of density ρ_2, $h=h_0>0$ for $|x|>L$, $h=h_0-\epsilon$ for $|x|<L$, $L>0$ is the half-width of the barrier, $\epsilon>0$ is a small parameter, $\beta=\rho_1/\rho_2$ (we assume that $0<\beta<1$), $\alpha=1-\beta$, u and v are the displacements of the free surface and the interface, respectively. The operator $\hat{\mathcal{L}}$ is symmetric (and even self-adjoint) with respect to the scalar product

$$\langle\boldsymbol{u}_1,\boldsymbol{u}_2\rangle_{\alpha\beta}=\frac{1}{\alpha}\langle u_1,u_2\rangle+\frac{1}{\beta}\langle v_1,v_2\rangle,\quad \boldsymbol{u}_1=(u_1,v_1)^\top,\quad \boldsymbol{u}_2=(u_2,v_2)^\top,$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product in $L_2(\mathbb{R})$, with the domain consisting of the vectors $\boldsymbol{u}=(u,v)^\top$ such that $u,u',u'',v,w\equiv h(\beta u'+\alpha v')$ and w' belong to $L_2(\mathbb{R})$. The last condition implies that u,u',v and w are continuous functions (physically, this means that the displacements and fluxes are continuous at the points of discontinuity of h).

System (2.1) has the following properties. In the absence of the perturbation ($\epsilon=0$), (2.1) has purely continuous spectrum for $\lambda\geq\lambda_1(h_0)=k^2\Omega_1^2(h_0)$, where

$$\Omega_1^2(h)=\frac{1}{2}\left(b+h-\sqrt{(b+h)^2-4\alpha bh}\right).$$

This spectrum has multiplicity 2 for $\lambda_1(h_0)<\lambda<\lambda_2(h_0)$, where $\lambda_2(h_0)=k^2\Omega_2^2(h_0)$,

$$\Omega_2^2(h)=\frac{1}{2}\left(b+h+\sqrt{(b+h)^2-4\alpha bh}\right),$$

and multiplicity 4 for $\lambda>\lambda_2$. The quantities $\Omega_{1,2}(h)$ satisfy the equation

$$\Omega^4-(b+h)\Omega^2+\alpha bh=0$$

and by the Viète theorem

$$\Omega_1^2(h)\Omega_2^2(h)=\alpha bh. \tag{2.2}$$

We note that the quantity

$$\Delta^2(h)=\Omega_2^2(h)-\Omega_1^2(h)=\sqrt{(b+h)^2-4\alpha bh}>0 \tag{2.3}$$

which measures the distance between the cut-offs is strictly positive, and the quantities

$$b_1 = \Omega_2^2 - b, \qquad b_2 = b - \Omega_1^2 \tag{2.4}$$

are also positive.

Both Ω_1^2 and Ω_2^2 are increasing functions of h, as can be verified by an elementary calculation. From the point of view of explicit solutions, system (2.1) has only exponential solutions for $\lambda < \lambda_1$, exponential and oscillating solutions for $\lambda_1 < \lambda < \lambda_2$ and only oscillating solutions for $\lambda > \lambda_2$.

Consider the behavior of system (2.1) under the perturbation. If $\epsilon > 0$, the cutoffs λ_1 and λ_2 move to the left in the domain $|x| < L$. Thus there is a range of λ, $\lambda_1(h_0 - \epsilon) < \lambda < \lambda_1(h_0)$, for which there exists an oscillatory solution for $|x| < L$. This gives rise to a discrete eigenvalue $\lambda = \lambda_1(h_0) - \mu^2$ with $\mu = O(\epsilon)$ to the left of $\lambda_1(h_0)$. Indeed, using the reflection symmetry of the problem, we look for an even solution of (2.1). It has the form of a linear combination of two exponentially decreasing solutions for $|x| > L$ and of a linear combination of cos- and cosh-like solutions for $|x| < L$. Matching these at $x = L$ (that is, guaranteeing the continuity of u, v, u' and $h(\beta u' + \alpha v')$), we come to a 4×4 homogeneous system for the coefficients of these linear combinations. The determinant of this system depends analytically on ϵ, μ and its derivative with respect to μ is nonzero for $\epsilon = \mu = 0$ (we do not give here rather cumbersome calculations leading to this result since it is easier to obtain μ from the integral equations (see Example 5.1 in [7]), and since there is nothing unusual in this procedure: the implicit function theorem is directly applicable to the equation for μ).

Consider the embedded eigenvalue. For λ such that $\lambda_2(h_0 - \epsilon) < \lambda < \lambda_2(h_0)$, system (2.1) has one exponentially decreasing solution and two oscillating solutions for $|x| > L$, and four oscillating solutions for $|x| < L$. Using again the reflection symmetry, we look for the solution in the region $|x| < L$ in the form of a linear combination of cos-like solutions and in the form of a multiple of the single decreasing exponential in the region $|x| > L$. Matching these at $x = L$, we obtain four equations for three coefficients. This system has a nontrivial solution if the rank of its matrix is less or equal to 2. Let us write out the corresponding formulas. We look for the eigenvalue in the form $\lambda = \lambda_2(h_0) - \mu^2$ with $\mu \to 0$ as $\epsilon \to 0$. Inside the barrier, the solution has the form

$$D_1(1, B_1)^\top \cos m_1 x + D_2(1, B_2)^\top \cos m_2 x,$$

where $D_{1,2}$ are arbitrary constants,

$$B_1 = -\frac{\beta}{\alpha} \frac{\Omega_2^2(h_0 - \epsilon)}{\Omega_2^2(h_0 - \epsilon) - b}, \qquad B_2 = \frac{\beta}{\alpha} \frac{\Omega_1^2(h_0 - \epsilon)}{b - \Omega_1^2(h_0 - \epsilon)},$$

$$m_1^2(\epsilon, \mu) = \frac{k^2(\Omega_2^2(h_0) - \Omega_1^2(h_0 - \epsilon)) - \mu^2}{\Omega_1^2(h_0 - \epsilon)}, \qquad m_2^2(\epsilon, \mu) = \frac{k^2(\Omega_2^2(h_0) - \Omega_2^2(h_0 - \epsilon)) - \mu^2}{\Omega_2^2(h_0 - \epsilon)}.$$

For $|x| > L$, the solution is

$$D_3(1, C)^\top e^{-m|x|},$$

where D_3 is an arbitrary constant,

$$C = \frac{\beta}{\alpha}\frac{\Omega_1^2(h_0)}{b-\Omega_1^2(h_0)}, \qquad m = \frac{\mu}{\Omega_2(h_0)}.$$

The continuity conditions for u, v, u' and $h(\beta u' + \alpha v')$ give a 4×3 homogeneous system of linear algebraic equations for the coefficients D_js. The matrix of this system has the form (after factoring out the exponential $\exp(-mL)$ in the third column)

$$M = \begin{pmatrix} \cos m_1 L & \cos m_2 L & -1 \\ B_1 \cos m_1 L & B_2 \cos m_2 L & -C \\ -m_1 \sin m_1 L & -m_2 \sin m_2 L & m \\ -(h_0-\epsilon)(\beta+\alpha B_1)m_1 \sin m_1 L & -(h_0-\epsilon)(\beta+\alpha B_2)m_2 \sin m_2 L & mh_0(\beta+\alpha C) \end{pmatrix}.$$

As mentioned above, there exists a nontrivial solution for the D_js if the rank of M is less than 3. For $\epsilon = 0$ this will be the case if $m_1 L = \pi n + \theta$, $\theta \to 0$ as $\epsilon \to 0$, $n = 1, 2, \ldots$. Indeed, in this case we have $m_2 \to 0$, $B_2 \to C$, $m \to 0$ as $\epsilon \to 0$ and for $\epsilon = 0$ the matrix M reduces to

$$M|_{\epsilon=0} = \begin{pmatrix} (-1)^n & 1 & -1 \\ (-1)^n B_1^0 & C & -C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_1^0 = \lim_{\epsilon\to 0} B_1,$$

and the rank of the last matrix is obviously equal to 2 (since $B_1^0 \neq C$). We see that the first two rows of M are linearly independent for $\epsilon = 0$. Hence, its rank will be equal to 2 when the determinants formed by first two rows and the third or the fourth row are zero. This gives us the following two equations for θ, μ:

$$M_1(\epsilon,\mu,\theta) = \begin{vmatrix} \cos m_1 L & \cos m_2 L & -1 \\ B_1 \cos m_1 L & B_2 \cos m_2 L & -C \\ -m_1 \sin m_1 L & -m_2 \sin m_2 L & m \end{vmatrix} = 0, \tag{2.5}$$

$$M_2(\epsilon,\mu,\theta) = \begin{vmatrix} \cos m_1 L & \cos m_2 L & -1 \\ B_1 \cos m_1 L & B_2 \cos m_2 L & -C \\ -(h_0-\epsilon)(\beta+\alpha B_1)m_1 \sin m_1 L & -(h_0-\epsilon)(\beta+\alpha B_2)m_2 \sin m_2 L & mh_0(\beta+\alpha C) \end{vmatrix} = 0 \tag{2.6}$$

(here $L = (\pi n + \theta)/m_1$; note that $m_1(0,0) \neq 0$ and m_1 is analytic in ϵ, μ). As noted above, these equations are satisfied for $\epsilon = \mu = \theta = 0$, but it turns out that the Jacobian vanishes,

$$\left.\frac{\partial(M_1, M_2)}{\partial(\mu,\theta)}\right|_{\epsilon=\mu=\theta=0} = 0$$

(this is shown by a rather lengthy elementary calculation). Thus the implicit function theorem is not directly applicable to system (2.5), (2.6). Nevertheless, the following statement is valid.

Theorem 2.1. *System (2.5), (2.6) possesses an analytic in ϵ solution $\mu(\epsilon), \theta(\epsilon)$ such that $\theta \to 0$ and $\mu \to 0$ as $\epsilon \to 0$.*

This theorem is the main result of this note. Its proof is given in the next section.

3 Proof of Theorem 2.1

The proof is based on the following simple observation. Consider equation (2.5) as an equation for μ with θ small and independent of ϵ.

Lemma 3.1. *The solution $\mu(\epsilon,\theta)$ of equation (2.5) such that $\mu \to 0$ as $\epsilon,\theta \to 0$ exists, is unique and analytic in ϵ,θ and has the form $\mu(\epsilon,\theta) = \epsilon f(\epsilon,\theta)$ with f analytic in ϵ,θ.*

Proof. By direct substitution, $M_1(\epsilon,\epsilon f,\theta)$ admits division by ϵ, that is, $N_1(\epsilon, f,\theta) = \epsilon^{-1}M_1(\epsilon,\epsilon f,\theta)$ is analytic in ϵ, f,θ. Moreover, $N_1(0, f_0,0) = 0$, where

$$f_0 = \frac{\pi n k^2}{m_1^0 \Omega_2(h_0)} \lim_{\epsilon\to 0} \frac{\Omega_2^2(h_0) - \Omega_2^2(h_0-\epsilon)}{\epsilon}, \qquad m_1^0 = m_1(0,0) \neq 0,$$

and $\partial N_1/\partial f|_{0,f_0,0} \neq 0$. The lemma is proven. □

We are now able to prove Theorem 2.1. Indeed, substituting $\mu = \epsilon f$ in (2.6), we see by inspection that $M_2(\epsilon,\epsilon f(\epsilon,\theta),\theta)$ admits division by ϵ, that is, $N_2(\epsilon,\theta) = \epsilon^{-1}M_2(\epsilon,\epsilon f(\epsilon,\theta),\theta)$ is analytic in ϵ,θ, $N_2(0,0) = 0$ and $\partial N_2/\partial\theta|_{\epsilon=\theta=0} \neq 0$. The theorem is proven.

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