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ROBUST FEEDBACK SYNTHESIS PROBLEM FOR SYSTEMS WITH A SINGLE PERTURBATION

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Abstract

The paper deals with the global robust feedback syntheses of a bounded control for a disturbed canonical system with an unknown bounded perturbation. Our approach is based on the controllability function method created by V. I. Korobov in 1979. We find a segment where the perturbation can vary and give a positional control which is independent of the perturbation and steers any initial point to the origin for any admissible perturbation from this segment. An estimate for the time of motion is given.

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1 Introduction

The paper deals with the problem of global robust control design for the disturbed canonical system with an unknown bounded perturbation. Specifically, we consider the following system:

$$\dot{x}_1 = (1 + p(t, x_1, x_2, \dots, x_n))x_2,\ \dot{x}_2 = x_3,\ \dots,\ \dot{x}_{n-1} = x_n,\ \dot{x}_n = u$$

or, in the matrix form,

$$\dot{x} = (A_0 + p(t,x)R)x + b_0 u, \tag{1.1}$$

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where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Here $t \geq 0$, $x \in \mathbb{R}^n$ is a state ($n \geq 2$), $u \in \mathbb{R}$ is a control satisfying the constraint $|u| \leq 1$, and $p(t,x)$ is an *unknown* bounded perturbation, which, however, satisfies the constraint $d_1 \leq p(t,x) \leq d_2$.

Below, for a pair of numbers $d_1 < d_2$, by $\mathcal{P}_{d_1,d_2}$ we denote the class of functions $p(t,x) : [0;+\infty) \times \mathbb{R}^n \to \mathbb{R}$, each of which satisfies the following conditions:
1) $p(t,x)$ is continuous on t;
2) in any domain $K_1(t_1,\rho_2) = \{(t,x) :\ 0 \leq t \leq t_1,\ \|x\| \leq \rho_2\}$, $\rho_2 > 0$, $t_1 > 0$, the function $p(t,x)$ satisfies the Lipschitz condition

$$|p(t,x'') - p(t,x')| \leq \ell_1(t_1,\rho_2)\|x'' - x'\|,$$

(where $\ell_1(t_1,\rho_2)$ depends on the function p);
3) the function $p(t,x)$ satisfies the constraint $d_1 \leq p(t,x) \leq d_2$ for all $(t,x) \in [0;+\infty) \times \mathbb{R}^n$.

Definition 1.1. The (d_1,d_2)-*global robust feedback synthesis problem (or robust finite-time stabilization problem)* for system (1.1) is to construct a control of the form $u = u(x)$, $x \in \mathbb{R}^n$ such that:
1) in any domain $K_2(\rho_1,\rho_2) = \{x :\ 0 < \rho_1 \leq \|x\| \leq \rho_2\}$, $0 < \rho_1 < \rho_2$, the function $u(x)$ satisfies the Lipschitz condition

$$|u(x'') - u(x')| \leq \ell_2(\rho_1,\rho_2)\|x'' - x'\|;$$

2) $|u(x)| \leq 1$ for any $x \in \mathbb{R}^n$;
3) for any $p(t,x) \in \mathcal{P}_{d_1,d_2}$ the trajectory $x(t)$ of the closed-loop system

$$\dot{x} = (A_0 + p(t,x)R)x + b_0 u(x), \tag{1.2}$$

starting at an arbitrary initial point $x(0) = x_0 \in \mathbb{R}^n$, ends at the origin at a finite time of motion $T(x_0,p) < \infty$, that is $\lim\limits_{t \to T(x_0,p)} x(t) = 0$.

Our approach is based on the *controllability function method*, suggested by V. I. Korobov in 1979 [10, 11] in connection with the feedback synthesis problem, and developed further in the works of V. I. Korobov, G. M. Sklyar and other authors [1, 4, 15, 17]. Later, Korobov's ideas were developed in many papers (see, for example, [19, 21]); an application to chaotic systems can be found in [3].

The synthesis problem for systems with perturbations was first solved in [12]. Namely, for systems of the form $\dot{x} = Ax + b(u + v)$, where v is a bounded perturbation, a bounded control $u = u(x)$ solving the synthesis problem and independent of a perturbation was built. The robust feedback synthesis problem, in the statement close to the present paper, first appeared in [14], and this investigation was continued in [20], where the robust feedback synthesis problem for concrete oscillation systems was considered. The problem of asymptotically stable syntheses of a bounded control which transfers points from a neighborhood of the origin to the origin in a finite time first proposed in [11], and this investigation was continued in many papers (see, for example, [1]). In recent years, the problem of finite-time stabilization appears in various formulations [2, 6, 9, 13, 18, 19].

The purpose of the present paper is to propose a constructive control algorithm for solving feedback syntheses problem for the system (1.1) and study the robustness of this algorithm with respect to perturbation $p(t,x)$. We find d_1 and d_2 for which the (d_1,d_2)-global robust feedback synthesis problem is solvable. Obviously, if $p(t,x) = -1$, then the first coordinate x_1 in (1.1) is uncontrollable; hence, the problem is not solvable for all values d_1, d_2. We emphasize that the control $u(x)$, which is constructed, necessarily satisfies the preassigned constraint, $|u(x)| \leq 1$.

The paper is organized as follows. In Section 2, we recall the basic concepts of the controllability function method. Section 3 contains the main results. In Section 4, the obtained results are illustrated by the examples in dimensions 2 and 3. In Section 5, an auxiliary lemma is proved.

2 Background: the controllability function method

2.1 Statement of the feedback synthesis problem

We consider a control system of the form

$$\dot{x} = f(x,u), \tag{2.1}$$

where $x \in Q \subset \mathbb{R}^n$, Q is a certain neighborhood of the origin; $u \in \Omega \subset \mathbb{R}^r$, Ω is such that $0 \in$ *int* Ω. The *feedback synthesis problem* is to construct a control in the form $u = u(x)$, $x \in Q$, such that:
1) $u(x) \in \Omega$;
2) the trajectory $x(t)$ of the closed-loop system

$$\dot{x} = f(x,u(x)), \tag{2.2}$$

starting at an arbitrary point $x(0) = x_0 \in Q$, ends at the origin $x(T) = 0$ at a finite time of motion $T = T(x_0) < \infty$, that is, $\lim\limits_{t\to T(x_0)} x(t) = 0$. If $Q = \mathbb{R}^n$, this problem is referred to as *the global feedback synthesis problem*.

Remark 2.1. Since there exist infinitely many trajectories passing through the origin (recall that the time of motion is finite), the right-hand side of equation (2.2) cannot satisfy the Lipschitz condition in a neighborhood of the origin, due to the theorem on the uniqueness of a solution.

2.2 The controllability function method

We formulate a general theorem concerning the controllability function method.

Theorem 2.2 (Korobov [10, 11, 13])**.** *Consider the control system (2.1). Put $G = \{x : \|x\| \le r\}$ $(0 < r \le \infty)$. Assume that the vector-function $f(x,u)$ is continuous in $G \times \Omega$ and satisfies the Lipschitz condition*

$$\|f(x',u') - f(x'',u'')\| \le L_1(\rho_1,\rho_2)(\|x''-x'\| + \|u''-u'\|)$$

in any domain $\{(x,u) : 0 < \rho_1 \le \|x\| \le \rho_2, u \in \Omega\},\ 0 < \rho_1 < \rho_2$.

Assume that there exists a function $\Theta(x)$, $x \in G$ such that:

1) *$\Theta(x) > 0$ at $x \neq 0$, and $\Theta(0) = 0$;*

2) *$\Theta(x)$ is continuous in G and continuously differentiable in $G\backslash\{0\}$;*

3) *there exists a number $c > 0$ such that the set $Q = \{x :\ \Theta(x) \le c\}$ is bounded and $Q \subset \{x :\ \|x\| < r\}$;*

4) *there exists a function $u(x) \in \Omega$, $x \in Q$, such that for some positive numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ the following inequalities hold:*

$$-\beta_1 \Theta^{1-\frac{1}{\alpha_1}}(x) \le \sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x,u(x)) \le -\beta_2 \Theta^{1-\frac{1}{\alpha_2}}(x) \tag{2.3}$$

5) *the function $u(x)$ satisfies the Lipschitz condition*

$$\|u(x'') - u(x')\| \le L_2(\rho_1,\rho_2)\|x''-x'\|$$

in any domain $K(\rho_1,\rho_2) = \{(x) :\ 0 < \rho_1 \le \|x\| \le \rho_2\},\ 0 < \rho_1 < \rho_2$.

Then the trajectory of the closed-loop system (2.2), starting at any initial point $x(0) = x_0 \in Q$, ends at the point $x_1(T) = 0$ and time of motion $T(x_0)$ is bounded as follows

$$\frac{\alpha_1}{\beta_1}\Theta(x_0)^{\frac{1}{\alpha_1}} \le T(x_0) \le \frac{\alpha_2}{\beta_2}\Theta^{\frac{1}{\alpha_2}}(x_0). \tag{2.4}$$

Remark 2.3. When ρ_1 tends to zero, Lipschitz constants $L_i(\rho_1,\rho_2),\ i = 1,2$ increase indefinitely (see Remark 2.1).

Remark 2.4. [1, 13] Theorem 2.2 can be extended to the case $f(t,x,u)$, where f is continuous in $[0,+\infty)\times G\times\Omega$ and satisfies the Lipschitz condition

$$\|f(t,x',u')-f(t,x'',u'')\|\le L_1(t_1,\rho_1,\rho_2)(\|x''-x'\|+\|u''-u'\|)$$

in any closed domain $\{(t,x,u):0\le t\le t_1, 0<\rho_1\le\|x\|\le\rho_2, u\in\Omega\},\ 0<\rho_1<\rho_2,\ t_1>0$.

Remark 2.5. Inequalities (2.3) mean that the system moves in the direction of decrease of the function $\Theta(x)$. If $\alpha_1=\alpha_2=\beta_1=\beta_2=1$, then (2.3) takes the form

$$\sum_{i=1}^{n}\frac{\partial\Theta(x)}{\partial x_i}f_i(x,u(x))=-1. \tag{2.5}$$

This means that $\frac{d}{dt}\Theta(x(t))=-1$, where $x(t)$ is a trajectory of the system (2.2). Therefore, $\Theta(x)=T(x)$, i.e., the controllability function equals the time of motion from the point x to the origin [13].

Remark 2.6. Suppose that, in addition to the conditions of Theorem 2.2, the control $u(x)$ is such that

$$\min_{u\in\Omega}\sum_{i=1}^{n}\frac{\partial\Theta(x)}{\partial x_i}f_i(x,u)=\sum_{i=1}^{n}\frac{\partial\Theta(x)}{\partial x_i}f_i(x,u(x))=-1.$$

Then, for the function $\omega(x)=-\Theta(x)=-T(x)$, the Bellman equation holds

$$\max_{u\in\Omega}\sum_{i=1}^{n}\frac{\partial\omega(x)}{\partial x_i}f_i(x,u)=1.$$

The Bellman equation can be interpreted as follows: we choose a control minimizing the angle between the direction of motion and the direction of decrease of the function $\Theta(x)$. In the controllability function method this angle is not necessarily minimal.

If we put $\alpha_2=\infty$ in inequalities (2.3), we get

$$\sum_{i=1}^{n}\frac{\partial\Theta(x)}{\partial x_i}f_i(x,u(x))\le-\beta\Theta(x). \tag{2.6}$$

This means that $\Theta(x)$ is a Lyapunov function.

2.3 The controllability function method for linear systems

Let us suppose that system (2.1) is linear

$$\dot{x}=Ax+Bu \tag{2.7}$$

and completely controllable. Suppose that $\{u:\ \|u\|\le d\}\subset\Omega$. We describe only one possible way for constructing the controllability function, which is used in the present paper.

Theorem 2.7 (Korobov, Sklyar [17]). *Suppose that all eigenvalues of the matrix A have nonpositive real part. For any $x\neq0$, define the controllability function $\Theta=\Theta(x)$ as the unique positive solution of the equation*

$$2a_0\Theta=(N^{-1}(\Theta)x,x),\ x\neq0,\quad \Theta(0)=0, \tag{2.8}$$

where

$$N(\Theta)=\int_0^{\Theta}\left(1-\frac{t}{\Theta}\right)e^{-At}BB^*e^{-A^*t}\,dt.$$

Then for a sufficiently small a_0, $0 < a_0 \le a$, *the control*

$$u(x) = -\frac{1}{2}\,B^* N^{-1}(\Theta(x))x \tag{2.9}$$

solves the global feedback synthesis problem for system (2.7) and satisfies the constraint $\|u(x)\| \le d$.

Moreover, in this case (2.5) holds, i.e., $\Theta(x)$ *equals the time of motion from the point x to the origin.*

Remark 2.8. Usually, a Lyapunov function is defined explicitly, while usually $\Theta(x)$ is defined implicitly by the equation (2.8). Recall in this connection, that the optimal time of motion for a linear time-optimal control problem is also defined implicitly [16].

Let us apply Theorem 2.7 to the canonical system

$$\dot{x}_1 = x_2,\ \ldots,\ \dot{x}_{n-1} = x_n,\ \dot{x}_n = u,$$

or, in the matrix form,

$$\dot{x} = A_0 x + b_0 u, \tag{2.10}$$

where $u \in \mathbb{R}$ satisfies the constraint $|u| \le 1$. It should be noted that for $p(t,x) = 0$ system (1.1) coincides with (2.10). It can be easily shown that in this case $N(\Theta) = (D(\Theta)FD(\Theta))^{-1}$, where

$$D(\Theta) = diag\left(\Theta^{-\frac{2n-2i+1}{2}}\right)_{i=1}^{n}, \tag{2.11}$$

$$F^{-1} = \int_0^1 (1-t)e^{-A_0 t} b_0 b_0^* e^{-A_0^* t}\,dt = \left(\frac{(-1)^{2n-i-j}}{(n-i)!(n-j)!(2n-i-j+1)(2n-i-j+2)}\right)_{i,j=1}^{n}. \tag{2.12}$$

The elements f_{ij} of the matrix F can be found explicitly [23], however, we do not use their precise form in this paper. It can be shown that is this case $a = \frac{2}{f_{nn}}$. So, let us choose any a_0 such that

$$0 < a_0 \le \frac{2}{f_{nn}} \tag{2.13}$$

and define the controllability function $\Theta = \Theta(x)$ as the unique positive solution of the equation

$$2a_0\Theta = (D(\Theta)FD(\Theta)x,x),\ x \ne 0, \quad \Theta(0) = 0. \tag{2.14}$$

Then the control

$$u(x) = -\frac{1}{2}\,b_0^* D(\Theta(x))FD(\Theta(x))x \tag{2.15}$$

solves the global feedback synthesis problem for system (2.10) and satisfies the constraint $|u(x)| \le 1$.

3 Main results

Let us return to the system (1.1). Our goal is to find d_1, d_2 such that the control (2.15) solves the (d_1,d_2)-global robust feedback synthesis problem for the system (1.1). Namely, let us choose a_0 satisfying (2.13) and consider the closed-loop system (1.2) with the control (2.15), where $\Theta(x)$ is defined as the unique positive solution of the equation (2.14). Put $y(\Theta,x) = D(\Theta)x$, where $D(\Theta)$ is given by (2.11). Then equation (2.14) takes the following form

$$2a_0\Theta = (Fy(\Theta,x), y(\Theta,x)). \tag{3.1}$$

Let us denote by $x(t)$ the trajectory of the system (1.2) and find the total derivative $\dot{\Theta} = \frac{d}{dt}\Theta(x(t))$. Equation (3.1) gives

$$2a_0\dot{\Theta} = (F\dot{y}(\Theta,x), y(\Theta,x)) + (Fy(\Theta,x), \dot{y}(\Theta,x)).$$

Let us find $\dot{y}(\Theta,x)$. Put $H = diag\left(-\frac{2n-2i+1}{2}\right)_{i=1}^{n}$, then $\dfrac{d}{d\Theta}D(\Theta) = \dfrac{1}{\Theta}HD(\Theta)$. Therefore,

$$\dot{y}(\Theta,x) = \dot{D}(\Theta)x + D(\Theta)\dot{x} = \frac{\dot{\Theta}}{\Theta}HD(\Theta)x + D(\Theta)A_0x + p(t,x)D(\Theta)Rx + D(\Theta)b_0u(x) =$$

$$= \frac{\dot{\Theta}}{\Theta}Hy(\Theta,x) + D(\Theta)A_0D^{-1}(\Theta)y(\Theta,x) + p(t,x)D(\Theta)RD^{-1}(\Theta)y(\Theta,x) - \frac{1}{2}D(\Theta)b_0b_0^*D(\Theta)Fy(\Theta,x).$$

Let us introduce the notation $S(\Theta) = \Theta(FD(\Theta)RD^{-1}(\Theta) + D^{-1}(\Theta)R^*D(\Theta)F)$. One can show that [13] $D(\Theta)RD^{-1}(\Theta) = \Theta^{-1}R$, so we have

$$S(\Theta) = S = FR + R^*F.$$

We emphasize that in the considered case the matrix $S(\Theta)$ does not depend on Θ. This observation is crucial for our method of solving the robust feedback synthesis problem. In fact,

$$S = \begin{pmatrix} 0 & f_{11} & 0 & \dots & 0 \\ f_{11} & 2f_{12} & f_{13} & \dots & f_{1n} \\ 0 & f_{13} & 0 & \dots & 0 \\ & & \dots & & \\ 0 & f_{1n} & 0 & \dots & 0 \end{pmatrix}.$$

We denote

$$F^1 = F - FH - HF = ((2n-i-j+2)f_{ij})_{i,j=1}^{n} = \begin{pmatrix} 2nf_{11} & (2n-1)f_{12} & \dots & (n+1)f_{1n} \\ (2n-1)f_{21} & (2n-2)f_{22} & \dots & nf_{2n} \\ & & \dots & \\ (n+1)f_{1n} & nf_{2n} & \dots & 2f_{nn} \end{pmatrix}. \tag{3.2}$$

One can show that [13]

$$D(\Theta)A_0D^{-1}(\Theta) = \frac{1}{\Theta}A_0, \quad D(\Theta)b_0 = \Theta^{-1/2}b_0, \quad FA_0 + A_0^*F - Fb_0b_0^*F = -F^1,$$

hence,

$$\dot{\Theta}(2a_0 - \frac{1}{\Theta}((FH+HF)y(\Theta,x), y(\Theta,x))) = \frac{1}{\Theta}((-F^1 + p(t,x)S)y(\Theta,x), y(\Theta,x)).$$

Taking into account equation (3.1), we get

$$\dot{\Theta} = \frac{(-F^1 + p(t,x)S)y(\Theta,x), y(\Theta,x))}{(F^1y(\Theta,x), y(\Theta,x))}. \tag{3.3}$$

Theorem 3.1. *Let us consider the equation* $\det(F^1 - \tilde{p}S) = 0$ *with respect to* $\tilde{p}$. *Let* $\tilde{d}_1^0$ *and* $\tilde{d}_2^0$ *be the smallest and largest roots of this equation respectively. Let us choose* $0 < \gamma_1 < 1$, $\gamma_2 > 1$. *Put*

$$d_1^0 = \max\{(1-\gamma_1)\tilde{d}_1^0;\ (1-\gamma_2)\tilde{d}_2^0\}, \quad d_2^0 = \min\{(1-\gamma_1)\tilde{d}_2^0;\ (1-\gamma_2)\tilde{d}_1^0\}; \tag{3.4}$$

Then for all d_1 *and* d_2 *such that* $d_1^0 < d_1 < d_2 < d_2^0$, *the control (2.15), where* $\Theta(x)$ *is defined by (2.14), solves the* (d_1,d_2)*-global robust feedback synthesis problem for the system (1.1). Moreover, the trajectory of the closed-loop system (1.2), starting at any initial point* $x(0) = x_0 \in \mathbb{R}^n$, *ends at the point* $x(T) = 0$, *where the time of motion* $T = T(x_0,d_1,d_2)$ *satisfies the estimate*

$$\frac{\Theta(x_0)}{\gamma_2} \le T(x_0,d_1,d_2) \le \frac{\Theta(x_0)}{\gamma_1}. \tag{3.5}$$

Remark 3.2. Notice that the equation $\det(F^1 - \tilde{p}S) = 0$ is quadratic with respect to $\tilde{p}$.

Remark 3.3. Formula (3.4) gives the exact estimates for d_1^0 and d_2^0.

Proof. We apply Theorem 2.2 and Remark 2.4. Below, for any symmetric matrix M, the notation $M > 0$ (or $M < 0$) means that M is positive definite (or negative definite). Suppose $p(t,x) \in \mathcal{P}_{d_1,d_2}$, where $d_1^0 < d_1 < d_2 < d_2^0$. Let us use (3.3) and prove that

$$-\gamma_2 < \dot{\Theta} < -\gamma_1. \tag{3.6}$$

It can be shown that $F^1 > 0$ [10, 5, 13]. Hence, taking into account (3.3) and denoting $y = y(\Theta, x)$, we can rewrite the required this inequality as

$$(-\gamma_2 F^1 y, y) < ((-F^1 + p(t,x)S)y, y) < (-\gamma_1 F^1 y, y),$$

or, what is the same,

$$(((1-\gamma_1)F^1 - p(t,x)S)y, y) > 0, \quad (((1-\gamma_2)F^1 - p(t,x)S)y, y) < 0.$$

Therefore, it suffices to prove that

$$((1-\gamma_1)F^1 - pS) > 0, \quad ((1-\gamma_2)F^1 - pS) < 0 \quad \text{ for all } d_1^0 < p < d_2^0,$$

or, what is the same,

$$(F^1 - \tilde{p}_1 S) > 0 \quad \text{ for all } \frac{d_1^0}{(1-\gamma_1)} < \tilde{p}_1 < \frac{d_2^0}{(1-\gamma_1)}$$

and

$$(F^1 - \tilde{p}_2 S) > 0 \quad \text{ for all } \frac{d_2^0}{(1-\gamma_2)} < \tilde{p}_2 < \frac{d_1^0}{(1-\gamma_2)}.$$

Instead, taking into account (3.4), we prove that

$$(F^1 - \tilde{p}S) > 0 \quad \text{ for all } \tilde{d}_1^0 < \tilde{p} < \tilde{d}_2^0, \tag{3.7}$$

or what is the same,

$$(F^1 - \tilde{p}S) > 0 \quad \text{ for all } \tilde{p} \text{ such as } \min\left\{\frac{d_1^0}{(1-\gamma_1)}, \frac{d_2^0}{(1-\gamma_2)}\right\} < \tilde{p} < \max\left\{\frac{d_2^0}{(1-\gamma_1)}, \frac{d_1^0}{(1-\gamma_2)}\right\}.$$

The matrix $(F^1 - \tilde{p}S)$ has the form

$$F^1 - \tilde{p}S = \begin{pmatrix} 2nf_{11} & (2n-1)f_{12} - \tilde{p}f_{11} & (2n-2)f_{13} & \dots & (n+1)f_{1n} \\ (2n-1)f_{12} - \tilde{p}f_{11} & (2n-2)f_{22} - 2\tilde{p}f_{12} & (2n-3)f_{23} - \tilde{p}f_{13} & \dots & nf_{2n} - \tilde{p}f_{1n} \\ (2n-2)f_{13} & (2n-3)f_{23} - \tilde{p}f_{13} & (2n-4)f_{33} & \dots & (n-1)f_{3n} \\ & & \dots & & \\ (n+1)f_{1n} & nf_{2n} - \tilde{p}f_{1n} & (n-1)f_{3n} & \dots & 2f_{nn} \end{pmatrix}.$$

Put $g(\tilde{p}) = \det(F^1 - \tilde{p}S)$; obviously, $g(\tilde{p})$ is a quadratic function. Let us find its leading coefficient. To this end, we divide the 2^{nd} line and 2^{nd} column of the matrix $(F^1 - \tilde{p}S)$ by $\tilde{p}$ and then tend $\tilde{p}$ to ∞. We get that the coefficient of $\tilde{p}^2$ in the polynomial $g(\tilde{p})$ equals $\det\tilde{\Delta}_n$, where

$$\tilde{\Delta}_n = \begin{pmatrix} 2nf_{11} & -f_{11} & (2n-2)f_{13} & \dots & (n+1)f_{1n} \\ -f_{11} & 0 & -f_{13} & \dots & -f_{1n} \\ (2n-2)f_{13} & -f_{13} & (2n-4)f_{33} & \dots & (n-1)f_{3n} \\ & & \dots & & \\ (n+1)f_{1n} & -f_{1n} & (n-1)f_{3n} & \dots & 2f_{nn} \end{pmatrix}. \tag{3.8}$$

One can prove (see Lemma 5.1 in Appendix) that $\det\tilde{\Delta}_n < 0$. Hence, the function $g(\tilde{p})$ is quadratic with respect to $\tilde{p}$ with the negative leading coefficient. Recall that, by definition, $\tilde{d}_1^0$ and $\tilde{d}_2^0$ are the smallest and largest roots of the equation $\det(F^1 - \tilde{p}S) = g(\tilde{p}) = 0$. Moreover, $g(0) = \det F^1 > 0$ since $F^1 > 0$. Hence, $\tilde{d}_1^0 < 0$, $\tilde{d}_2^0 > 0$, and $\det(F^1 - \tilde{p}S) > 0$ for all $\tilde{d}_1^0 < \tilde{p} < \tilde{d}_2^0$.

Let us now prove (3.7). For $\tilde{p} = 0$ the matrix $F^1 - \tilde{p}S$ equals F^1 and is positive definite, hence, all its eigenvalues are positive. Since eigenvalues continuously depend on the parameter $\tilde{p}$ and for all $\tilde{p} \in (\tilde{d}_1^0, \tilde{d}_2^0)$ the matrix $F^1 - \tilde{p}S$ is nonsingular, its eigenvalues still positive. Hence, $F^1 - \tilde{p}S > 0$ for all $\tilde{p} \in (\tilde{d}_1^0, \tilde{d}_2^0)$.

Thus, for any numbers d_1, d_2 such that $d_1^0 < d_1 < d_2 < d_2^0$, and for any $p(t,x) \in \mathcal{P}_{d_1,d_2}$ inequality (3.6) holds. This means that inequality (2.3) holds for $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \gamma_2$, $\beta_2 = \gamma_1$.

The rest of the proof can be carried out similarly to Theorem 2.2 and Remark 2.4 [1, 13]. □

Corollary 3.4. *The values of $\tilde{d}_1^0$ and $\tilde{d}_2^0$ can be found as $\tilde{d}_1^0 = 1/\lambda_{min}((F^1)^{-1}S)$, $\tilde{d}_2^0 = 1/\lambda_{max}((F^1)^{-1}S)$.*

Proof. Since $F^1 > 0$, we get $\det(F^1 - \tilde{p}S) = 0$ if and only if $\det(I - \tilde{p}(F^1)^{-1}S) = 0$. On the other hand, $\tilde{p} \neq 0$ is a root of the last equation if and only if $1/\tilde{p}$ is an eigenvalue of the matrix $(F^1)^{-1}S$. □

Remark 3.5. The result of Corollary 3.4 can be proved by methods of [7, 22].

Remark 3.6. To find a specific trajectory we act as follows. We take an arbitrary initial point $x_0 \in \mathbb{R}^n$. Then we solve equation (2.14) at $x = x_0$ and find its unique positive solution $\Theta(x_0) = \Theta_0$. Then we choose values d_1 and d_2 in accordance with Theorem 3.1 and put $\theta(t) = \Theta(x(t))$. For any perturbation $p(t,x) \in \mathcal{P}_{d_1,d_2}$, the trajectory is a solution of the following Cauchy problem:

$$\begin{cases} \dot{x} = (A_0 + p(t,x)R)x - \frac{1}{2}\, b_0 b_0^* D(\theta) F D(\theta) x, \\ \dot{\theta} = \dfrac{((-F^1 + p(t,x)S)D(\theta)x, D(\theta)x)}{(F^1 D(\theta)x, D(\theta)x)} \\ x(0) = x_0,\ \theta(0) = \Theta_0. \end{cases}$$

Notice that equation (2.14) is solved only once.

4 Examples

4.1 Robust feedback synthesis problem for a two-dimensional system

Let us consider the robust feedback synthesis problem for the system

$$\begin{cases} \dot{x}_1 = (1 + p(t,x_1,x_2))x_2, \\ \dot{x}_2 = u, \end{cases} \tag{4.1}$$

i.e. for system (1.1), where

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

under the constraint on the control of the form $|u| \le 1$. The case when p is a fixed parameter is well known [18]. We consider p as an unknown bounded perturbation: $d_1 \le p(t,x_1,x_2) \le d_2$.

We have

$$F = \begin{pmatrix} 36 & 12 \\ 12 & 6 \end{pmatrix}, D(\Theta) = \begin{pmatrix} \Theta^{-\frac{3}{2}} & 0 \\ 0 & \Theta^{-\frac{1}{2}} \end{pmatrix}, S = \begin{pmatrix} 0 & 36 \\ 36 & 24 \end{pmatrix}, F^1 - \tilde{p}S = \begin{pmatrix} 144 & 36(1-\tilde{p}) \\ 36(1-\tilde{p}) & 12(1-2\tilde{p}) \end{pmatrix}.$$

Hence, $\det(F^1 - \tilde{p}S) = -432\,(3\tilde{p}^2 + 2\tilde{p} - 1) = 0$ at $\tilde{p} = -1$ and $\tilde{p} = 1/3$. Equivalently, the eigenvalues of the matrix $(F^1)^{-1}S = \begin{pmatrix} -3 & -1 \\ 12 & 5 \end{pmatrix}$ are equal to -1 and 3. Then in (3.4) we get

$$d_1^0 = \max\{\gamma_1 - 1,\ (1-\gamma_2)/3\}, \quad d_2^0 = \min\{(1-\gamma_1)/3,\ \gamma_2 - 1\}.$$

Note that whenever γ_1 and γ_2 are close to 1, the values of d_1^0 and d_2^0 are close to zero. E. g., at $\gamma_1 = 0{,}9$ and $\gamma_2 = 1{,}1$ we obtain $d_1^0 \approx -0{,}03$, $d_2^0 \approx 0{,}03$, and the estimate on the time of motion is $10\,\Theta(x_0)/11 \le T(x_0) \le 10\,\Theta(x_0)/9$. Conversely, when γ_1 and γ_2 are far from 1, the values of d_1^0 and d_2^0 have a greater range. E. g., at $\gamma_1 = 0{,}09$; $\gamma_2 = 4$ we have $d_1^0 \approx -0{,}91$; $d_2^0 \approx 0{,}303$, and the estimate on the time of motion is $\Theta(x_0)/4 \le T(x_0) \le 100\,\Theta(x_0)/9$.

Equation (2.14) for the controllability function takes the form

$$2a_0\Theta^4 = 36x_1^2 + 24\Theta x_1x_2 + 6\Theta^2x_2^2, \tag{4.2}$$

where $0 < a_0 \le 2/f_{22} = 1/3$. Let $a_0 = 1/3$. The control equals

$$u(\Theta, x) = -\frac{6x_1}{\Theta^2} - \frac{3x_2}{\Theta}, \tag{4.3}$$

Put $\gamma_1 = 0{,}09$; $\gamma_2 = 4$, we have $d_1^0 \approx -0{,}91$; $d_2^0 \approx 0{,}303$. Let us choose $d_1 = -0{,}9$; $d_2 = 0{,}3$ then $[-0.9; 0.3] \subset \subset (-0.91, 0.303)$. Let the initial point be equal to $x_0 = (4; -4)$. Then the unique positive solution of (4.2) is $\Theta_0 \approx 9{,}68$. Three trajectories corresponding to $p = -0{,}9$; $p = 0$; $p = 0.3$ are present in Fig. 1. If $p = const$, then the trajectories fill up the area between the trajectories corresponding to $p = -0.9$ and $p = 0.3$.

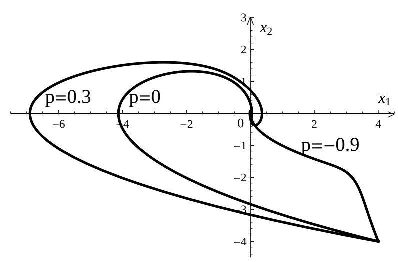

Figure 1. Three trajectories of system (4.1)

As a concrete realization of a perturbation, consider the function

$$p(t, x_1, x_2) = -0.3\sin\left(\frac{(x_1^2 + x_2^2)t}{5}\right). \tag{4.4}$$

The trajectory is given in Fig. 2; the control on the trajectory is given in Fig. 3; the total derivative of the function $\Theta(x)$ with respect to closed-loop system (4.1) is given in Fig. 4. Although the total derivative of the function $\Theta(x)$ satisfies the inequality $-4 \le \dot{\Theta} \le -0.09$, the controllability function is close to linear (given in Fig. 5). We emphasize that in the case then $p(t,x) = 0$ the total derivative of the function $\Theta(x)$ with respect to closed-loop system (4.1) satisfy the equation $\dot{\Theta} = -1$. We may see that the control $u(x)$ satisfies the preassigned constraint $|u(x)| \le 1$. The estimate on the time of motion (3.5) is as follows: $2.42 \le T(x_0) \le 107.57$. The time of motion T is approximately equal to 8.24. Notice that the time of motion is less than Θ_0.

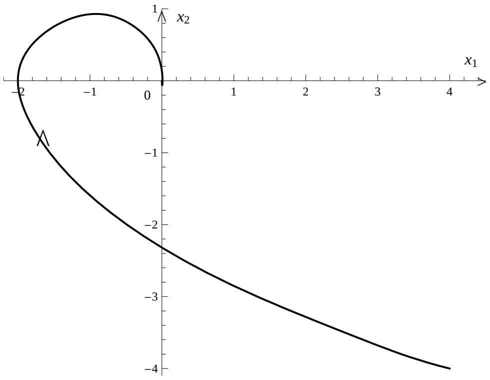

Figure 2. Trajectory for $p(t,x)$ of form (4.4)

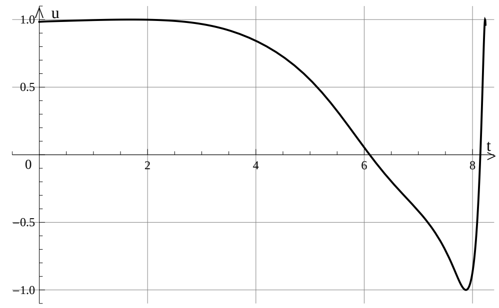

Figure 3. Control on the trajectory for $p(t,x)$ of form (4.4)

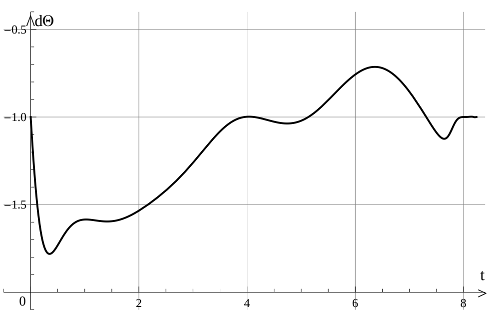

Figure 4. Total derivative of the function $\Theta(x)$ on the trajectory for $p(t,x)$ of form (4.4)

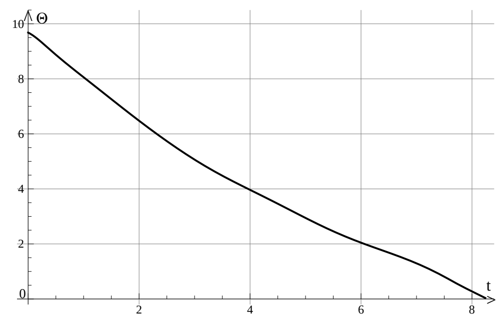

Figure 5. Controllability function $\Theta(x)$ on the trajectory for $p(t,x)$ of form (4.4)

4.2 Robust feedback synthesis problem for a three-dimensional system

Let us consider the robust feedback synthesis problem for the system

$$\begin{cases} \dot{x}_1 = (1 + p(t, x_1, x_2, x_3))x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = u, \end{cases} \tag{4.5}$$

i.e. for the system (1.1), where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $|u| \leq 1$. Here $p(t, x_1, x_2, x_3)$ is an unknown bounded perturbation: $d_1 \leq p(t, x_1, x_2, x_3) \leq d_2$. We have

$$F = \begin{pmatrix} 2400 & 960 & 120 \\ 960 & 420 & 60 \\ 120 & 60 & 12 \end{pmatrix}, D(\Theta) = \begin{pmatrix} \Theta^{-\frac{5}{2}} & 0 & 0 \\ 0 & \Theta^{-\frac{3}{2}} & 0 \\ 0 & 0 & \Theta^{-\frac{1}{2}} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 2400 & 0 \\ 2400 & 1920 & 120 \\ 0 & 120 & 0 \end{pmatrix},$$

$$F^1-\tilde{p}S=\begin{pmatrix} 14400 & 2400(2-\tilde{p}) & 480 \\ 2400(2-\tilde{p}) & 240(7-8\tilde{p}) & 60(3-2\tilde{p}) \\ 480 & 60(3-2\tilde{p}) & 24 \end{pmatrix}.$$

Then we obtain $\det(F^1-\tilde{p}S)=-3456000(20\tilde{p}^2+4\tilde{p}-1)=0$ for $\tilde{p}=(-1\pm\sqrt{6})/10$. Equivalently, the matrix $(F^1)^{-1}S$ has the form

$$(F^1)^{-1}S=\begin{pmatrix} -20 & -17/2 & -1 \\ 80 & 34 & 4 \\ -200 & -80 & -10 \end{pmatrix}$$

and its eigenvalues are equal to $\{2(1\pm\sqrt{6});0\}$. Then in (3.4) we get

$$d_1^0=\max\{(\gamma_1-1)(1+\sqrt{6})/10;\ (1-\gamma_2)(\sqrt{6}-1)/10\},\quad d_2^0=\min\{(1-\gamma_1)(\sqrt{6}-1)/10;\ (\gamma_2-1)(\sqrt{6}+1)/10\}.$$

Equation (2.14) for the controllability function takes the form

$$2a_0\Theta^6=2400x_1^2+1920\Theta x_1x_2+240\Theta^2x_1x_3+420\Theta^2x_2^2+120\Theta^3x_2x_3+12\Theta^4x_3^2, \tag{4.6}$$

where $0<a_0\leq\ 2/f_{33}=1/6$. Let $a_0=1/6$. The control is chosen as

$$u(\Theta,x)=-\frac{60x_1}{\Theta^3}-\frac{30x_2}{\Theta^2}-\frac{6x_3}{\Theta}. \tag{4.7}$$

Put $\gamma_1=0,09$; $\gamma_2=4$, we have $d_1^0\approx-0,313$; $d_2^0\approx 0,131$. Let us choose $d_1=-0,31$; $d_2=0,13$ then $[-0.31,0.13]\subset$ $\subset(-0.313,0.131)$. Let the initial point be equal to $x_0=(4;-4;1)$. Then the unique positive solution of (4.6) is $\Theta_0\approx 18.55$.

Consider a perturbation of the form

$$p(t,x_1,x_2,x_3)=0.13\cos\left(\frac{(x_1^2+x_2^2+x_3^2)t}{70}\right). \tag{4.8}$$

The trajectory components, the control on the trajectory, the total derivative of the function $\Theta(x)$ with respect to system (4.5), the controllability function $\Theta(x)$ are given in Fig. 6-9. We emphasize that in the case then $p(t,x)=0$ the total derivative of the function $\Theta(x)$ with respect to closed-loop system (4.1) satisfy the equation $\dot{\Theta}=-1$. We may see that the control $u(x)$ satisfies the preassigned constraint $|u(x)|\leq 1$. The estimate on the time of motion (3.5) is as follows: $4.63\leq T(x_0)\leq 206.1$. The time of motion T is approximately equal to 18.51. Notice that the time of motion is less then Θ_0.

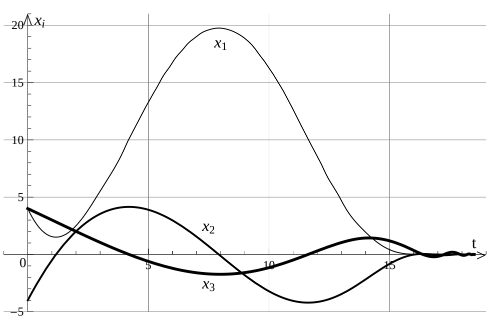

Figure 6. Trajectory components for $p(t,x)$ of form (4.8)

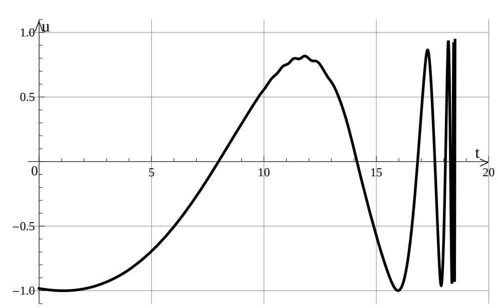

Figure 7. Control on the trajectory for $p(t,x)$ of form (4.8)

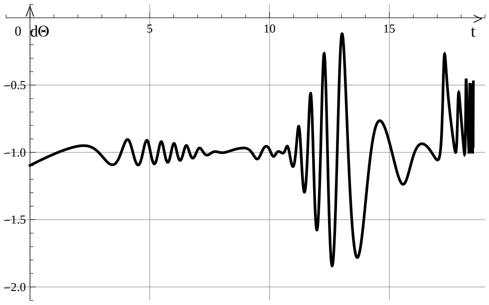

Figure 8. Total derivative of the function $\Theta(x)$ on the trajectory for $p(t,x)$ of form (4.8)

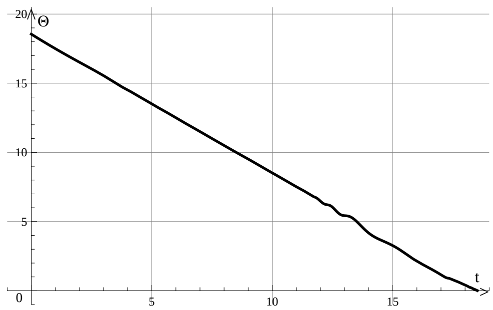

Figure 9. Controllability function $\Theta(x)$ on the trajectory for $p(t,x)$ of form (4.8)

5 Appendix

Lemma 5.1. *Suppose the matrix* $\tilde{\Delta}_n$ *is given by (3.8); then* $\det\tilde{\Delta}_n < 0$.

Proof. Let us fix $n \geq 2$ (the dimension of the system). Let us permute the 1^{st} and the 2^{nd} lines, as well as the 1^{st} and the 2^{nd} columns in $\tilde{\Delta}_n$, then

$$\det\tilde{\Delta}_n = \det\begin{pmatrix} 0 & -f_{11} & -f_{13} & -f_{14} & \dots & -f_{1n} \\ -f_{11} & 2nf_{11} & (2n-2)f_{13} & (2n-3)f_{14} & \dots & (n+1)f_{1n} \\ -f_{13} & (2n-2)f_{13} & (2n-4)f_{33} & (2n-5)f_{34} & \dots & (n-1)f_{3n} \\ -f_{14} & (2n-3)f_{14} & (2n-5)f_{34} & (2n-6)f_{44} & \dots & (n-2)f_{4n} \\ & & & \dots & & \\ -f_{1n} & (n+1)f_{1n} & (n-1)f_{3n} & (n-2)f_{4n} & \dots & 2f_{nn} \end{pmatrix}.$$

For $k = 2,\dots,n$, let us introduce the following matrices

$$\Delta_2 = \begin{pmatrix} 0 & -f_{11} \\ -f_{11} & 2nf_{11} \end{pmatrix}, \quad \Delta_k = \begin{pmatrix} 0 & -f_{11} & -f_{13} & \dots & -f_{1k} \\ -f_{11} & 2nf_{11} & (2n-2)f_{13} & \dots & (k+1)f_{1k} \\ -f_{13} & (2n-2)f_{13} & (2n-4)f_{33} & \dots & (k-1)f_{3k} \\ & & \dots & & \\ -f_{1k} & (k+1)f_{1k} & (k-1)f_{3k} & \dots & (n-k+2)f_{kk} \end{pmatrix}, \ k \geq 3,$$

that is, Δ_k are leading principal submatrices of the matrix Δ_n. Also, consider the matrix F^1, which is defined by (3.2). Let us permute its 1^{st} and the 2^{nd} lines, as well as the 1^{st} and the 2^{nd} columns. As a result, we get the matrix

$$\Phi = \begin{pmatrix} (2n-2)f_{22} & (2n-1)f_{21} & (2n-3)f_{23} & \dots & nf_{2n} \\ (2n-1)f_{12} & 2nf_{11} & (2n-2)f_{13} & \dots & (n+1)f_{1n} \\ (2n-3)f_{23} & (2n-2)f_{13} & (2n-4)f_{33} & \dots & (n-1)f_{3n} \\ & & \dots & & \\ nf_{2n} & (n+1)f_{1n} & (n-1)f_{3n} & \dots & 2f_{nn} \end{pmatrix}.$$

Since $F^1 > 0$, we get $\Phi > 0$. Following [8], for a matrix Z with elements z_{ij} we use the notation

$$Z\begin{pmatrix} i_1,\dots,i_q \\ j_1,\dots,j_q \end{pmatrix} = \det\begin{pmatrix} z_{i_1j_1} & z_{i_2j_2} & \dots & z_{i_1j_q} \\ & & \dots & \\ z_{i_qj_1} & z_{i_qj_2} & \dots & z_{i_qj_q} \end{pmatrix}.$$

Now, let us prove that $\det\Delta_k < 0$ for all $k = 2,\ldots,n$. The proof is by induction on k. For $k = 2$, we get

$$\det\Delta_2 = \det\begin{pmatrix} 0 & -f_{11} \\ -f_{11} & 2nf_{11} \end{pmatrix} = -f_{11}^2.$$

Since the matrix F is positive definite [10, 5, 13], it follows that $f_{11} > 0$, hence, $\det\Delta_2 < 0$.

Suppose $k \geq 3$ and $\det\Delta_i < 0$ for $i = 2,\ldots,k-1$. Introduce the following notation for special minors of the matrix Δ_n and take into account that the elements of Φ coincide with corresponding elements of the matrix Δ_n, except the 1^{st} line and the 1^{st} column:

$$\triangle_{1,k} = \Delta_k\begin{pmatrix} 2,3,\ldots,k \\ 1,2,\ldots,k-1 \end{pmatrix},\ \triangle_{k,1} = \Delta_k\begin{pmatrix} 1,2,\ldots,k-1 \\ 2,3\ldots,k \end{pmatrix},\ \triangle_{k,k} = \Delta_k\begin{pmatrix} 1,2,\ldots,k-1 \\ 1,2,\ldots,k-1 \end{pmatrix} = \det\Delta_{k-1},$$

$$\triangle = \Delta_k\begin{pmatrix} 2,\ldots,k-1 \\ 2,\ldots,k-1 \end{pmatrix} = \Phi\begin{pmatrix} 2,\ldots,k-1 \\ 2,\ldots,k-1 \end{pmatrix},\ \triangle_{1,1} = \Delta_k\begin{pmatrix} 2,\ldots,k \\ 2,\ldots,k \end{pmatrix} = \Phi\begin{pmatrix} 2,\ldots,k \\ 2,\ldots,k \end{pmatrix}.$$

Since $\Phi > 0$, we get $\triangle > 0$ and $\triangle_{1,1} > 0$. Since Δ_n is symmetric, $\triangle_{1,k} = \triangle_{k,1}$.

Now we apply Silvester's determinant identity [8] to the matrix Δ_k, which reads

$$\det\begin{pmatrix} \triangle_{k,k} & \triangle_{1,k} \\ \triangle_{k,1} & \triangle_{1,1} \end{pmatrix} = \det\Delta_k\triangle,$$

i.e.,

$$\triangle_{k,k}\triangle_{1,1} - \triangle_{1,k}^2 = \det\Delta_k\triangle.$$

Since $\triangle > 0$, $\triangle_{1,1} > 0$, and $\triangle_{k,k} < 0$ by the induction hypothesis, it follows that $\det\Delta_k < 0$. The induction arguments complete the proof.

Thus, $\det\Delta_k < 0$ for all $k = 2,\ldots,n$. Since $\det\Delta_n = \det\tilde{\Delta}_n$, we get $\det\tilde{\Delta}_n < 0$. □

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