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Decompositions of the Blaschke–Potapov Factors of the Truncated Hausdorff Matrix Moment Problem: The Case of an Odd Number of Moments

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Abstract

In "*Multiplicative Structure of the Resolvent Matrix for the Truncated Matricial Hausdorff Moment Problem*", Operator Theory: Advances and Applications, (2012) by the author, a multiplicative decomposition of resolvent matrix $U^{(2n)}$ for the truncated Hausdorff matrix moment (THMM) problem via Blaschke–Potapov factors $b^{(2j)}$ was obtained. In this work we show that every such Blaschke–Potapov factor can be represented as a product of block tridiagonal matrices containing Stieltjes matrix parameters (SMP) depending on a or b. This SMP are in turn a generalization of the Yu. Dyukarev's Stieltjes parameters introduced in "*Indeterminacy criteria for the Stieltjes matrix moment problem*", Mathematical Notes (2004).

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1 Introduction

Throughout this paper, let q be a positive integer. We will use $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$ and $\mathbb{N}$ to denote the set of all complex numbers, the set of all real numbers, the set of all nonnegative integers, and the set of all positive integers, respectively. The notation $\mathbb{C}^{p\times q}$ stands for the set of all complex $p\times q$ matrices. For the null matrix that belongs to $\mathbb{C}^{p\times q}$ we will write $0_{p\times q}$. We denote by 0_q and I_q the null and identity matrices in $\mathbb{C}^{q\times q}$. In cases where the size of the null and the identity matrix are clear, we will omit the indices.

The main object of the present work are the *Blaschke–Potapov factors* of the truncated Hausdorff matrix moment (THMM) problem in the case of an odd number of moments. Let

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us first recall the statement of the THMM in the case of an odd number of moments:Let a finite sequence of complex $q\times q$ matrices $(s_j)_{j=0}^{2n}$ be given. Find the set $\mathbb{M}_{\geq}^{q}[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{2n}]$ of all nonnegative Hermitian $q\times q$ measures σ which are defined on the Borel σ–algebra $\mathfrak{B}\cap[a,b]$ on $[a,b]$ such that

$$s_j = \int\limits_{[a,b]} t^j d\sigma(t) \tag{1.1}$$

for each integer j with $0 \leq j \leq 2n$.

It was proved in [8, Theorem 1.3] that the THMM problem is solvable if and only if the block matrices $H_{1,n}$ and $H_{2,n-1}$ are positive semidefinite, where

$$H_{1,n} := \widetilde{H}_{0,n}, \quad n \geq 0, \tag{1.2}$$

$$H_{2,n-1} := -ab\widetilde{H}_{0,n-1} + (a+b)\widetilde{H}_{1,n-1} - \widetilde{H}_{2,n-1}, \quad n \geq 1 \tag{1.3}$$

which are defined with the help of the Hankel matrices,

$$\widetilde{H}_{0,j} := \begin{pmatrix} s_0 & s_1 & \dots & s_j \\ s_1 & s_2 & \dots & s_{j+1} \\ \vdots & \vdots & \vdots & \vdots \\ s_j & s_{j+1} & \dots & s_{2j} \end{pmatrix}, \quad \widetilde{H}_{1,j} := \begin{pmatrix} s_1 & s_2 & \dots & s_{j+1} \\ s_2 & s_3 & \dots & s_{j+2} \\ \vdots & \vdots & \vdots & \vdots \\ s_{j+1} & s_{j+2} & \dots & s_{2j+1} \end{pmatrix}$$

and

$$\widetilde{H}_{2,j} := \begin{pmatrix} s_2 & s_3 & \dots & s_{j+2} \\ s_3 & s_4 & \dots & s_{j+3} \\ \vdots & \vdots & \vdots & \vdots \\ s_{j+2} & s_{j+3} & \dots & s_{2j+2} \end{pmatrix}.$$

Definition 1.1. Let the block Hankel matrices $H_{1,j}$ and $H_{2,j-1}$ be defined by (1.2) and (1.3). The sequence $(s_k)_{k=0}^{2j}$ is called Hausdorff positive (resp. nonnegative) on $[a,b]$ if the block Hankel matrices $H_{1,j}$ and $H_{2,j-1}$ are both positive (resp. nonnegative) definite matrices.

Throughout this work, we assume that $(s_j)_{j=0}^{2n}$ is a Hausdorff positive on $[a,b]$ sequence.

Let σ be a $q\times q$ positive measure on $[a,b]$. Then the function $F_\sigma : \mathbb{C}\setminus[a,b] \to \mathbb{C}^{q\times q}$ defined by

$$F_\sigma(z) := \int_{[a,b]} \frac{1}{t-z}\sigma(dt)$$

is called the Stieltjes transform of σ. In view of the Stieltjes–Perron invesion formula the measure σ is uniquely determined by its Stieltjes transform. Following the classical procedure of Stieltjes the above moment problems are handled to describe the set of Stieljtes transforms of all solutions.

The corresponding set of solutions for the THMM problem in the case of an odd number of moments is given with the help of the linear fractional transformation of the form (see [8, Theorem 6.14]):

$$s(z) = (\alpha^{(2n)}(z)\mathbf{p}(z) + \beta^{(2n)}(z)\mathbf{p}(z))(\gamma^{(2n)}(z)\mathbf{p}(z) + \delta^{(2n)}(z)\mathbf{p}(z))^{-1}. \tag{1.4}$$

The pair column$(\mathbf{p},\mathbf{q})$ satisfies certain properties, see [8, Definition 5.2].

Definition 1.2. The $2q\times 2q$ matrix–valued function

$$U^{(2n)} = \begin{pmatrix} \alpha^{(2n)} & \beta^{(2n)} \\ \gamma^{(2n)} & \delta^{(2n)} \end{pmatrix}$$

of the linear fractional transformation (1.4) is called the resolvent matrix of the THMM, where $\alpha^{(2n)}$, $\beta^{(2n)}$, $\gamma^{(2n)}$ and $\delta^{(2n)}$ are $q\times q$ polynomials constructed via the sequence $(s_j)_{j=0}^{2n}$.

In [8, Lemma 6.3] a crucial property of the RM $U^{(2n)}$, namely, that it belongs to the Potapov class:

Definition 1.3. Let

$$J_q := \begin{pmatrix} 0_q & -iI_q \\ iI_q & 0_q \end{pmatrix} \tag{1.5}$$

and $\Pi_+ := \{w \in \mathbb{C} : \operatorname{Im} w \in (0,+\infty)\}$. A matrix–valued entire function $W : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ is said to belong to the Potapov class $\mathfrak{P}_{J_q}$ in Π_+ if

$$J_q - W^*(z) J_q W(z) \geq 0_{2q\times 2q},\ z \in \Pi_+. \tag{1.6}$$

A matrix–valued function W that belongs to $\mathfrak{P}_{J_q}$ is called a J_q–inner function of $\mathfrak{P}_{J_q}$ if

$$J_q - W^*(x) J_q W(x) = 0_{2q\times 2q},\ x \in \mathbb{R}.$$

In [5] the multiplicative representation of the RM of the THMM problem in the case of an odd number of moments was obtained:

$$U^{(2n)}(z) = b^{(0)}(z)\cdot b^{(2)}(z)\cdot\dots\cdot b^{(2n)}(z), \tag{1.7}$$

where $b^{(2j)}$ are $q\times q$ matrices linear with respect to z.

Within the Potapov's framework [20] similar decompositions of RM of matrix interpolation problems were studied in [14], [15], [17], [19], [27] and [28].

1.1 Main results of the present work

The main results of this work are the following:

a) We find two multiplicative decompositions for each Blaschke–Potapov factor $b^{(2j)}$ of the THMM in the case of odd number of moments via two families of Stieltjes parameters depending on the terminal points of the interval $[a,b]$. See Theorem 4.9.

b) As a consequence, in Corollary 4.10 two multiplicative representations of the RM, $U^{(2n)}(z)$ in terms of two families of Stieltjes parameters are given.

c) We prove that the Blaschke–Potapov factors $b^{(2j)}$ belong to the Potapov class $\mathfrak{P}_{J_q}$ in Π_+.

Note that we crucially use the orthogonal polynomials $P_{k,j}$, $\Gamma_{k,j}$ on $[a,b]$ and their second kind polynomials $Q_{k,j}$, $\Theta_{k,j}$, see Definitions A.1 and A.3 as well as [3]. Orthogonal matrix polynomials (OMP) were first studied by M. G. Kreĭn in 1949 [25], [26]. Further research on OMP on the real line was conducted by I. V. Kovalishina [24], A. I. Aptekarev and E. M. Nikishin [1], H. Dym [13], A. Durán [12], H. Dette [11], Damanik/Pushnitski/-Simon [10] and the references therein. See also [18], [22], [23], and [21].

The decomposition of the Blaschke–Potapov of the THMM in the case of an even number of moments is considered in [2].

2 Notation and Preliminaries

Let $R_j : \mathbb{C} \to \mathbb{C}^{(j+1)q\times(j+1)q}$ be given by

$$R_{1,j}(z) := (I_{(j+1)q} - zT_j)^{-1}, \qquad j \geq 0, \tag{2.1}$$

with

$$T_0 := 0_q, \quad T_j := \begin{pmatrix} 0_{q\times jq} & 0_q \\ I_{jq} & 0_{jq\times q} \end{pmatrix}, \qquad j \geq 1. \tag{2.2}$$

Let

$$v_0 := I_q, v_j := \begin{pmatrix} I_q \\ 0_{jq\times q} \end{pmatrix} = \begin{pmatrix} v_{1,j-1} \\ 0_q \end{pmatrix}, \quad \forall j \in \mathbb{N}. \tag{2.3}$$

$$\widehat{s}_j = -abs_j + (a+b)s_{j+1} - s_{j+2}, \quad 0 \leq j \leq 2n-2 \tag{2.4}$$

and

$$y_{[j,k]} := \begin{pmatrix} s_j \\ s_{j+1} \\ \cdots \\ s_k \end{pmatrix}, 0 \leq j \leq k, \ \ \widehat{y}_{[j,k]} := \begin{pmatrix} \widehat{s}_j \\ \widehat{s}_{j+1} \\ \cdots \\ \widehat{s}_k \end{pmatrix} 0 \leq j \leq k. \tag{2.5}$$

Furthermore, let

$$Y_{1,j} := y_{[j,2j-1]}, 1 \leq j \leq n, \quad Y_{2,j} := \widehat{y}_{[j,2j-1]}, 1 \leq j \leq n-1, \tag{2.6}$$

$$u_{1,0} := 0_q, \quad u_{1,j} := \begin{pmatrix} 0_q \\ -y_{[0,j-1]} \end{pmatrix}, 1 \leq j \leq n, \tag{2.7}$$

and

$$u_{2,0} := -(a+b)s_0 + s_1, \quad u_{2,j} := \begin{pmatrix} -(a+b)s_0 + s_1 \\ -\widehat{y}_{[0,j-1]} \end{pmatrix}, 1 \leq j \leq n-1. \tag{2.8}$$

Let $\widehat{H}_{1,j}$ (resp. $\widehat{H}_{2,j-1}$) denote the Schur complement of the block s_{2j} (resp. $-abs_{2j-2} + (a+b)s_{2j-1} - s_{2j}$) of the matrix $H_{1,j}$ (resp. $H_{2,j-1}$):

$$\widehat{H}_{1,0} := H_{1,0}, \quad \widehat{H}_{1,j} := s_{2j} - Y_{1,j}^* H_{1,j-1}^{-1} Y_{1,j}, \ 1 \leq j \leq n, \tag{2.9}$$

$$\widehat{H}_{2,0} := H_{2,0}, \quad \widehat{H}_{2,j} := \widehat{s}_{2j} - Y_{2,j}^* H_{2,j}^{-1} Y_{2,j}. \ 1 \leq j \leq n-1, \tag{2.10}$$

The quantities (2.9) and (2.10) have been defined in [11] for $a = 0$ and $b = 1$.

Definition 2.1. [8, Remark 6.11] Let $(s_k)_{k=0}^{2n}$ be an odd Hausdorff positive on $[a,b]$ sequence. The $2q\times 2q$ matrix polynomial

$$U^{(2j)}(a,b,z) := \begin{pmatrix} \alpha^{(2j)}(a,z) & \beta^{(2j-2)}(a,b,z) \\ \gamma^{(2j)}(a,z) & \delta^{(2j-2)}(a,b,z) \end{pmatrix},\ z\in\mathbb{C},\quad 1\le j\le n, \tag{2.11}$$

is called the RM of the THMM problem in the case of an odd number of moments, where

$$\alpha^{(2j)}(a,z) := I_q - (z-a)u_{1,j}^* R_j^*(\bar{z}) H_{1,j}^{-1} R_j(a) v_j, \tag{2.12}$$
$$\gamma^{(2j)}(a,z) := -(z-a)v_j^* R_j^*(\bar{z}) H_{1,j}^{-1} R_j(a) v_j \tag{2.13}$$

for $0\le j\le n$, and

$$\beta^{(2j)}(a,b,z) := \frac{1}{b-a}(s_0 + (u_{2,j}^* + z s_0 v_j^*) R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a)(u_{2,j} + a v_j s_0)), \tag{2.14}$$
$$\delta^{(2j)}(a,b,z) := \frac{b-z}{b-a}(I_q + (z-a)v_j^* R_j^*(\bar{z}) H_{2,j}^{-1} R_j(a)(u_{2,j} + a v_j s_0)) \tag{2.15}$$

for $0\le j\le n-1$. Below, we will omit the dependence of $U^{(2j)}$ on a and b.

Note that here we use the representation of the blocks $\beta^{(2j)}$ and $\delta^{(2j)}$ derived in [6, Preposition 3.1] instead of the one introduced in [8, Formula (6.55)] and [8, Formula (6.57)].

The inverse of $H_{1,j}$, for $1\le j\le n$ and the inverse of $H_{2,j}$, for $1\le j\le n-1$ can be written in the form

$$H_{k,j}^{-1} = \begin{pmatrix} H_{k,j-1}^{-1} & 0_{jq\times q} \\ 0_{q\times jq} & 0_q \end{pmatrix} + \begin{pmatrix} -H_{k,j-1}^{-1} Y_{k,j} \\ I_q \end{pmatrix} \widehat{H}_{k,j}^{-1}(-Y_{k,j}^* H_{k,j-1}^{-1}, I_q). \tag{2.16}$$

Let

$$\lambda_0 := I_q,\quad \lambda_j := \begin{pmatrix} 0_{jq\times q} \\ I_q \end{pmatrix},\ j\ge 1. \tag{2.17}$$

The proof of the following Remark follows by direct calculation.

Remark 2.2. Let $(s_k)_{k=0}^{2j}$ be an odd Hausdorff positive on $[a,b]$ sequence. Let $P_{k,j}$ and $Q_{k,j}$ be as in Definition A.1. Let $u_{1,j}$, $u_{2,j}$, $Y_{1,j}$, $Y_{2,j}$, R_j, $H_{1,j}$, $H_{2,j}$, λ_j and $\widehat{s}_j$ be as in (2.7), (2.8), (2.6), (2.1), (1.2), (1.3), (2.17) and (2.4), respectively. Then the following identities are valid:

$$P_{1,j}(z) = z^j I_q - Y_{1,j}^* H_{1,j-1}^{-1} R_{j-1}(z) v_{j-1},\ 1\le j\le n, \tag{2.18}$$
$$P_{2,j}(z) = z^j I_q - Y_{2,j}^* H_{2,j-1}^{-1} R_{j-1}(z) v_{j-1},\ 1\le j\le n-1, \tag{2.19}$$
$$Q_{1,j}(z) = -(-Y_{1,j}^* + z\lambda_j^* H_{1,j-1}) H_{1,j-1}^{-1} R_{j-1}(z) u_{1,j-1} + s_{j-1} 1\le j\le n, \tag{2.20}$$

and

$$Q_{2,j}(z) = -(-Y_{2,j}^* + z\lambda_{j-1}^* H_{2,j-1}) H_{2,j-1}^{-1} R_{j-1}(z)(u_{2,j-1} + z v_{j-1} s_0) + \widehat{s}_{j-1},\ 1\le j\le n-1. \tag{2.21}$$

3 Main Algebraic Identities

In this section we recall and introduce important identities which we use throughout this work. Let

$$u_j := -\operatorname{col}(s_0, s_1, \dots, s_j), \tag{3.1}$$

For each positive integer n, let

$$L_{1,n} := \left(\delta_{j,k+1} I_q\right)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}} \quad \text{and} \quad L_{2,n} := \left(\delta_{j,k} I_q\right)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}}, \tag{3.2}$$

where $\delta_{j,k}$ is the Kronecker symbol: $\delta_{j,k} := 1$ if $j = k$ and $\delta_{j,k} := 0$ if $j \neq k$.

Remark 3.1. Let $u_{1,j}$, v_j, $H_{1,j}$, T_j, λ_j, $L_{1,j}$, $L_{2,j}$, $\widehat{s}_j$, $Y_{1,j}$ and R_j be as in (2.7), (1.2), (2.2), (2.17), (3.2), (2.4), (2.6) and (2.1), respectively. Then the following identities are valid:

$$u_{1,j}^* + v_j^* H_{1,j} T_j^* = 0, \;\; 1 \le j \le n, \tag{3.3}$$

$$\lambda_{j-1} - L_{1,j}^* \lambda_j = 0, \tag{3.4}$$

$$s_0 - v_j^* H_{1,j} v_j = 0, \;\; 1 \le j \le n, \tag{3.5}$$

$$(b-a)T_j^* + (L_{1,j} - bL_{2,j})L_{1,j}^* + v_j v_j^* - (I_{(j+1)q} - aT_j^*) = 0, \;\; 1 \le j \le n, \tag{3.6}$$

$$\widehat{s}_{j-1} + (b-a)s_j + av_j^* H_{1,j}(bT_j^*\lambda_j - 2\lambda_j) + v_j^* Y_{1,j+1} = 0, \tag{3.7}$$

$$\begin{aligned} &a^{j+1}v_j - \Big(-a(L_{1,j} - bL_{2,j})L_{1,j}^*(I - aT_j^*) - a(b-a)T_j^* - a(L_{1,j} - bL_{2,j})L_{1,j}^*\Big)R_j^*(a)\lambda_j \\ &-abT_j^*\lambda_j + 2a\lambda_j = 0. \end{aligned} \tag{3.8}$$

Proof. The identities (3.3), (3.5), (3.6) and (3.7) can be directly calculated using (2.7), (2.3), (1.2), (2.2) and (3.2). □

Proposition 3.2 (Coupling identities)**.** *[8, Proposition 3.4] Let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Then the following identities hold for $1 \le j \le n$:*

$$(u_{2,j-1}^* + as_0 v_{j-1}^*)R_{j-1}^*(a) - v_j^* H_{1,j}(L_{1,j} - bL_{2,j}) = 0, \tag{3.9}$$

$$R_{j-1}(a)(u_{2,j-1} + as_0 v_{j-1})P_{j+1}^*(a) - H_{2,j-1}L_{1,j}^* R_j^*(a)(-H_{1,j}^{-1}Y_{1,j+1} + a\lambda_j) - Y_{2,j} = 0 \tag{3.10}$$

$$R_j(a)v_j(u_{2,j-1}^* + as_0 v_{j-1}^*)R_{j-1}^*(a) - R_j(a)L_{1,j}H_{2,j-1} - H_{1,j}(L_{1,j} - bL_{2,j}) = 0. \tag{3.11}$$

Proposition 3.3 (More coupling identities)**.** *Let $\beta^{(2j)}$, $Q_{2,j}$, $Q_{1,j}$, $P_{1,j}$, $\widehat{H}_{1,j}$ and $\widehat{H}_{2,j}$ be as in (2.14), Definitions A.1, A.3, (2.9) and (2.10), respectively. Then the following identities hold*

$$\beta^{(2j)}(a) - \beta^{(2j-2)}(a) - \frac{1}{b-a}Q_{2,j}^*(a)\widehat{H}_{2,j}^{-1}Q_{2,j}(a) = 0, \;\; 1 \le j \le n-1, \tag{3.12}$$

$$Q_{2,j}^*(a) - (b-a)Q_{1,j+1}^*(a) + (b-a)\beta^{(2j-2)}(a)P_{1,j+1}^*(a) = 0, \;\; 1 \le j \le n-1, \tag{3.13}$$

$$\widehat{H}_{1,1} + \widehat{H}_{2,0} + P_{1,1}(a)Q_{2,0}^*(a) = 0, \tag{3.14}$$

$$\widehat{H}_{1,j+1} + \widehat{H}_{2,j} + P_{1,j+1}(a)Q_{2,j}^*(a) = 0, \; 1 \le j \le n-1. \tag{3.15}$$

Proof. The proof of (3.12) is by direct calculation. Use (2.14), (2.16) (2.21) and the identities

$$R_j(z)=\left(\begin{array}{c|c} R_{j-1}(z) & 0_{(j)q\times q} \\ \hline (z^jI_q,z^{j-1}I_q,\dots,zI_q) & I_q \end{array}\right),\quad \widehat{u}_{2,j}(a)=\begin{pmatrix} \widehat{u}_{2,j-1}(a) \\ -\widehat{s}_{j-2}\end{pmatrix},$$

where $\widehat{u}_{2,j}(a) := u_{2,j-1}+av_{j-1}s_0$. Now we prove (3.13). By using (2.21), (2.20), (2.18) and (2.14) we have

$$\begin{aligned}
&Q^*_{2,j}(a)-(b-a)Q^*_{1,j+1}(a)+(b-a)\beta^{(2j-2)}(a)P^*_{1,j+1}(a)\\
&\quad=\widehat{s}_{j-1}+(u^*_{2,j-1}+as_0v^*_{j-1})R^*_{j-1}(a)H_{2,j-1}Y_{2,j}-a(u^*_{2,j-1}+as_0v^*_{j-1})R^*_{j-1}(a)\lambda_{j-1}\\
&\qquad-(b-a)s_j-(b-a)u^*_{1,j}(a)R^*_j(a)H^{-1}_{1,j}Y_{1,j+1}+a(b-a)u^*_{1,j}(a)R^*_j(a)\lambda_j\\
&\qquad-(s_0+(u^*_{2,j-1}+as_0v_{j-1})R^*_{j-1}(a)H_{2,j-1}R_{j-1}(a)(u_{2,j-1}+av_{j-1}s_0))P^*_{1,j+1}(a)\\
&\quad=\widehat{s}_{j-1}-(b-a)s_j+v^*_jH_{1,j}\Big(-a(L_{1,j}-bL_{2,j})\lambda_{j-1}+(b-a)T_j*R^*_j(a)H^{-1}_{1,j}Y_{1,j+1}\\
&\qquad-a(b-a)T^*_jR^*_j(a)\lambda_j-a^{j+1}v_j+v_jv^*_jR^*_j(a)H^{-1}_{1,j}Y_{1,j+1}-v_jH_{1,j}(L_{1,j}-bL_{2,j})L^*_{1,j}R^*_j(a)\\
&\qquad\cdot H^{-1}_{1,j}Y_{1,j+1}a(L_{1,j}-bL_{2,j})L^*_{1,j}R^*_j(a)\lambda_j\Big)\\
&\quad=\widehat{s}_{j-1}-(b-a)s_j-av^*_jH_{1,j}(bT^*_j\lambda_j-2a\lambda_j)+v^*_jY_{1,j+1}\\
&\quad=0.
\end{aligned}$$

In the second equality we used (3.9), (3.10), (3.11), (3.3) and (3.5), whereas in the third equality we used (3.4), (3.6) and (3.8). In the last equality we used (3.7).

The proof of (3.14) is by direct calculation. Now we prove the identity (3.15). By employing (A.1), (A.2) and (A.5), we have

$$\begin{aligned}
&P_{1,j+1}(a)Q^*_{2,j}(a)+\widehat{H}_{1,j+1}+\widehat{H}_{2,j}\\
&\quad=-(-Y^*_{1,j+1}H^{-1}_{1,j},I_q)R_{j+1}(a)v_{j+1}(u^*_{2,j}+as_0v^*_j)R^*_j(a)\begin{pmatrix} -H^{-1}_{2,j-1}Y_{2,j}\\ I_q\end{pmatrix}\\
&\qquad+\widehat{H}_{1,j+1}+\widehat{H}_{2,j}\\
&\quad=-(-Y^*_{1,j+1}H^{-1}_{1,j},I_q)R_{j+1}(a)L_{1,j+1}H_{2,j}\begin{pmatrix} -H^{-1}_{2,j-1}Y_{2,j}\\ I_q\end{pmatrix}\\
&\qquad-(-Y^*_{1,j+1}H^{-1}_{1,j},I_q)H_{1,j+1}(L_{1,j+1}-bL_{2,j+1})+\widehat{H}_{1,j+1}+\widehat{H}_{2,j}\\
&\quad=-(-Y^*_{1,j+1}H^{-1}_{1,j},I_q)R_{j+1}(a)\begin{pmatrix} 0_{(j+1)q\times q}\\ \widehat{H}_{2,j}\end{pmatrix}\\
&\qquad-(0_{q\times jq},\widehat{H}_{1,j+1})(L_{1,j+1}-bL_{2,j+1})+\widehat{H}_{1,j+1}+\widehat{H}_{2,j}\\
&\quad=0.
\end{aligned}$$

In the second equality we used (3.11), whereas in the third equality we use the following relations:

$$H_{2,j}=\begin{pmatrix} H_{2,j} & Y_{2,j}\\ Y^*_{2,j} & \widehat{s}_{2j-2}\end{pmatrix},\quad H_{1,j}=\begin{pmatrix} H_{1,j} & Y_{1,j}\\ Y^*_{1,j} & s_{2j}\end{pmatrix},$$

(2.9) and (2.10). The last equality is verified by direct calculation. □

4 Two Decompositions of the Blaschke–Potapov Factors in the Case of an Odd Number of Moments

In this section we give a multiplicative representation of the Blaschke–Potapov factors $b^{(2j)}$. of the RM of the THMM.

Definition 4.1. Let $\widehat{H}_{1,j}$, $\widehat{H}_{2,j}$, $P_{1,j}$ and $Q_{2,j}$ be as in (2.9), (2.10), (A.2), (A.1) and (A.5), respectively. Define

$$\begin{aligned} b^{(0)}(z) &:= \begin{pmatrix} I_q & 0_q \\ -(z-a)s_0^{-1} & I_q \end{pmatrix}, \\ b^{(2)}(z) &:= \begin{pmatrix} I_q+(z-a)s_0\widehat{H}_{1,1}^{-1}P_{1,1}(a) & \frac{1}{b-a}\left(s_0+Q_{2,0}^*(\overline{z})\widehat{H}_{2,0}^{-1}Q_{2,0}(a)\right) \\ -(z-a)P_{1,1}^*(a)\widehat{H}_{1,1}^{-1}P_{1,1}(a) & I_q+\frac{z-a}{b-a}P_{1,1}^*(a)\widehat{H}_{2,0}^{-1}Q_{2,0}(a) \end{pmatrix}, \end{aligned} \tag{4.1}$$

and

$$b^{(2j)}(z) := \begin{pmatrix} I_q+\frac{z-a}{b-a}Q_{2,j-1}^*(a)\widehat{H}_{1,j}^{-1}P_{1,j}(a) & \frac{b-z}{(b-a)^2}Q_{2,j-1}^*(a)\widehat{H}_{2,j-1}^{-1}Q_{2,j-1}(a) \\ -(z-a)P_{1,j}^*(a)\widehat{H}_{1,j}^{-1}P_{1,j}(a) & I_q+\frac{z-a}{b-a}P_{1,j}^*(a)\widehat{H}_{2,j-1}^{-1}Q_{2,j-1}(a) \end{pmatrix}, \tag{4.2}$$

for $2 \le j \le n$.

In [5] it was proved that the RM of the THMM problem can be represented in the form,

$$U^{(2j)}(z) = U^{(2j-2)}(z)b^{(2j)}(z), \quad 1 \le j \le n. \tag{4.3}$$

Here we denote $U^{(0)}(z) := b^{(0)}(z)$.

Definition 4.2. Let $\widehat{H}_{1,j}$, $P_{1,j}$ and $Q_{2,j}$ be as in (2.9), (A.2), (A.1) and (A.5), respectively. Define

$$\begin{aligned} \widetilde{b}^{(0)}(z) &:= b^{(0)}(z), \\ \widetilde{b}^{(2)}(z) &:= \begin{pmatrix} I_q+(z-a)s_0\widehat{H}_{1,1}^{-1}P_{1,1}(a) & (z-a)s_0\widehat{H}_{1,1}^{-1}s_0 \\ -(z-a)P_{1,1}^*(a)\widehat{H}_{1,1}^{-1}P_{1,1}(a) & I_q-(z-a)P_{1,1}^*(a)\widehat{H}_{1,1}^{-1}s_0 \end{pmatrix}, \end{aligned} \tag{4.4}$$

and

$$\widetilde{b}^{(2j)}(z) := \begin{pmatrix} I_q+\frac{z-a}{b-a}Q_{2,j-1}^*(a)\widehat{H}_{1,j}^{-1}P_{1,j}(a) & \frac{z-a}{(b-a)^2}Q_{2,j-1}^*(a)\widehat{H}_{1,j}^{-1}Q_{2,j-1}(a) \\ -(z-a)P_{1,j}^*(a)\widehat{H}_{1,j}^{-1}P_{1,j}(a) & I_q-\frac{z-a}{b-a}P_{1,j}^*(a)\widehat{H}_{1,j}^{-1}Q_{2,j-1}(a) \end{pmatrix}, \tag{4.5}$$

for $2 \le j \le n$.

Lemma 4.3. *Let $b^{(2j)}$ and $\widetilde{b}^{(2j)}$ be as in Definitions 4.1 and 4.2, respectively. Then the following equality holds,*

$$b^{(2j)}(z) = \widetilde{b}^{(2j)}(z)b^{(2j)}(a), \quad j \in \{1,\dots,n\}. \tag{4.6}$$

Proof. By using the notation

$$\begin{pmatrix} b_j^{11} & b_j^{12} \\ b_j^{21} & b_j^{22} \end{pmatrix} := b^{(2j)} \quad \text{and} \quad \begin{pmatrix} \widetilde{b}_j^{11} & \widetilde{b}_j^{12} \\ \widetilde{b}_j^{21} & \widetilde{b}_j^{22} \end{pmatrix} := \widetilde{b}^{(2j)} \tag{4.7}$$

along with the definition of (4.2) and (4.5), it is sufficient to verify the following equalities:

$$b_j^{12}(z) - \widetilde{b}_j^{11}(z) b_j^{12}(a) - \widetilde{b}_j^{12}(z) = 0 \tag{4.8}$$

and

$$b_j^{22}(z) - \widetilde{b}_j^{21}(z) b_j^{12}(a) - \widetilde{b}_j^{22}(z) = 0. \tag{4.9}$$

We prove (4.8). For $j = 1$ we use (3.14) and the obvious identity $(b-a)s_0 + P_{1,1}(a)s_0 - Q_{2,0}(a) = 0$. Consider now $2 \le j \le n$; we then have

$$\begin{aligned}
& b_j^{12}(z) - \widetilde{b}_j^{11}(z) b_j^{12}(a) - \widetilde{b}_j^{12}(z) \\
=& \frac{1}{b-a}\Bigg(\frac{b-z}{b-a} Q_{2,j-1}^*(a)\widehat{H}_{2,j-1}^{-1} Q_{2,j-1}(a) - Q_{2,j-1}^*(a)\widehat{H}_{2,j-1}^{-1} Q_{2,j-1}(a) \\
& - \frac{z-a}{b-a} Q_{2,j-1}^*(a)\widehat{H}_{1,j}^{-1} P_{1,j}(a) Q_{2,j-1}^*(a)\widehat{H}_{2,j-1}^{-1} Q_{2,j-1}(a) \\
& - \frac{z-a}{b-a} Q_{2,j-1}^*(a)\widehat{H}_{2,j-1}^{-1} Q_{2,j-1}(a)\Bigg) \\
=& -\frac{z-a}{(b-a)^2} Q_{2,j-1}^*(a)\left(\widehat{H}_{2,j-1}^{-1} + \widehat{H}_{1,j}^{-1} P_{1,j}(a) Q_{2,j-1}^*(a)\widehat{H}_{2,j-1}^{-1} + \widehat{H}_{2,j}^{-1}\right) Q_{2,j-1}(a) \\
=& -\frac{z-a}{(b-a)^2} Q_{2,j-1}^*(a)\widehat{H}_{1,j}^{-1}\left(\widehat{H}_{1,j} + P_{1,j}(a) Q_{2,j-1}^*(a) + \widehat{H}_{2,j-1}\right)\widehat{H}_{2,j-1}^{-1} Q_{2,j-1}(a) \\
=& 0.
\end{aligned}$$

In the last equality we used (3.15); similarly, one can prove the equality (4.9). Thus the Lemma is proved. □

4.1 First Decomposition of the Blaschke–Potapov Factors in the Case of an Odd Number of Moments

Definition 4.4. Let $(s_k)_{k=0}^{2j}$ be Hausdorff positive on $[a,b]$ sequence. Let v_j, R_j, $H_{1,j}$, $\widetilde{u}_{2,j}$, $Q_{2,j}$ and $P_{1,j}$ be as in (2.3), (2.1), (1.2), (2.8), (A.5) and (A.2), respectively. Denote

$$\begin{aligned}
M_0(a) :=& s_0^{-1}, \\
M_j(a) :=& v_j^* R_j^*(a) H_{1,j}^{-1} R_j(a) v_j - v_{j-1}^* R_{j-1}^*(a) H_{1,j-1}^{-1} R_{j-1}(a) v_{j-1}, \quad 1 \le j \le n, \qquad (4.10) \\
\widetilde{L}_0^{(2n)}(a) :=& \widetilde{u}_{2,0}^* K_{2,0}^{-1} \widetilde{u}_{2,0}, \\
\widetilde{L}_j^{(2n)}(a,b) :=& \frac{1}{b-a} Q_{2,j}^*(a) P_{1,j+1}^{*^{-1}}(a), \quad 1 \le j \le n. \qquad (4.11)
\end{aligned}$$

These matrices are called Stieltjes parameters of the THMM problem in the case of an odd number of moments. Below, we shall omit the dependence of M_j, L_j on a and b.

Remark 4.5. The following identity holds:

$$M_j = P_{1,j}^*(a)\widehat{H}_{1,j}^{-1} P_{1,j}(a) = -\Theta_{2,j}^{-1}(a) P_{1,j}(a), \quad 1 \le j \le n. \tag{4.12}$$

Proof. The proof of the first equality of (4.12) follows by direct calculation. Use (2.16). To verify the second equality of (4.12) we use the identity $\widehat{H}_{1,j} = -P_{1,j}(a)\Theta^*_{2,j}(a)$ which was proved in [4, Formula (A.13)]. □

Proposition 4.6. *Let $\widetilde{b}^{(2j)}$, M_j and $\widetilde{L}^{(2n)}_j$ be as in (4.4), (4.5), (4.10) and (4.11), respectively. Then the following equality holds:*

$$\widetilde{b}^{(2j)}(z) = \begin{pmatrix} I_q & -\widetilde{L}^{(2n)}_{j-1} \\ 0_q & I_q \end{pmatrix}\begin{pmatrix} I_q & 0_q \\ -(z-a)M_j & I_q \end{pmatrix}\begin{pmatrix} I_q & \widetilde{L}^{(2n)}_{j-1} \\ 0_q & I_q \end{pmatrix}, \tag{4.13}$$

for $1 \le j \le n$.

Proof. By (4.7) the equality (4.13) is equivalent to the four equalities:

$$\widetilde{b}^{11}_j - I_q - (z-a)\widetilde{L}^{(2n)}_{j-1}M_j = 0, \tag{4.14}$$

$$\widetilde{b}^{12}_j - (z-a)\widetilde{L}^{(2n)}_{j-1}M_j\widetilde{L}^{(2n)}_{j-1} = 0, \tag{4.15}$$

$$\widetilde{b}^{21}_j + (z-a)M_j = 0, \tag{4.16}$$

$$\widetilde{b}^{22}_j - I_q + (z-a)M_j\widetilde{L}^{(2n)}_{j-1} = 0. \tag{4.17}$$

Next we prove (4.14). By using (4.7), (4.11) and (4.12), we obtain

$$\begin{aligned}
&\widetilde{b}^{11}_j - I_q - (z-a)\widetilde{L}^{(2n)}_{j-1}M_j \\
&= \frac{z-a}{b-z}Q^*_{2,j-1}(a)\widehat{H}^{-1}_{1,j}P_{1,j}(a) - \frac{z-a}{b-a}Q^*_{2,j-1}(a)P^{*^{-1}}_{1,j}(a)P^*_{1,j}(a)\widehat{H}^{-1}_{1,j}P_{1,j}(a) \\
&= 0.
\end{aligned}$$

The equalities (4.16), (4.17) are proved in a similar way. Equality (4.16) is verified by definition of the matrices (4.4), (4.5) and (4.12). □

4.2 Second Decomposition of the Blaschke–Potapov Factors in the Case of an Odd Number of Moments

Let $\widetilde{L}^{(2n)}_j$, $\widetilde{u}_{2,j}$, $K_{2,j}$ and R_j be as in (4.11), (2.8), (A.9) and (2.1). Denote

$$\begin{aligned}
L_0(a) &:= \widetilde{L}^{(2n)}_0, \\
L_j(a) &:= \widetilde{u}^*_{2,j}R^*_j(a)K^{-1}_{2,j}R_j(a)\widetilde{u}_{2,j} - \widetilde{u}^*_{2,j-1}R^*_{j-1}(a)K^{-1}_{2,j-1}R_{j-1}(a)\widetilde{u}_{2,j-1},\ 1 \le j \le n-1.
\end{aligned} \tag{4.18}$$

Remark 4.7. The following identities hold:

$$L_j(a) = \Theta^*_{2,j}(a)\widehat{K}^{-1}_{2,j}\Theta_{2,j}(a) = P^{-1}_{1,j+1}(a)\Theta_{2,j}(a), \tag{4.19}$$

$$\widetilde{u}^*_{2,j}R^*_j(a)K^{-1}_{2,j}R_j(a)\widetilde{u}_{2,j} = -Q^*_{1,j+1}(a)P^{*^{-1}}_{1,j+1}(a). \tag{4.20}$$

Proof. The proof the of first equality of (4.19) follows by direct calculation. To verify the second equality of (4.19) we use the identity $\widehat{K}_{2,j} = \Theta_{2,j}(a)P^*_{1,j+1}(a)$ which is proved in [4, Formula (A.16)]. Equality (4.20) appears in [4, Formula (50)]. □

Below, we shall omit the dependence of L_j on a.

Lemma 4.8. *Let* $\widetilde{b}_j^{(2j)}$, $b_j^{(2j)}$ *and* $c^{(2j)}$ *be as in (4.4), (4.5), (4.1), (4.2) and (B.1), respectively. Then the following identity holds for* $2 \le j \le n$*:*

$$\widetilde{b}^{(2j)}(z) = b^{(2j-2)^{-1}}(a)\cdots b^{(2)^{-1}}(a)b^{(0)^{-1}}(a)c^{(2j)}(z)b^{(0)}(a)b^{(2)}(a)\dots b^{(2j-2)}(a). \tag{4.21}$$

Proof. From (2.11), and (1.7) we have,

$$\begin{pmatrix} I_q & \beta^{(2j-2)}(a) \\ 0_q & I_q \end{pmatrix} = b^{(0)}(a)b^{(2)}(a)\dots b^{(2j-2)}(a). \tag{4.22}$$

Furthermore, from fact that

$$\begin{aligned}
\beta^{(2j-2)}(a) - \widetilde{L}_j^{(2n)} - \widetilde{u}^*_{2,j-1}R^*_{j-1}(a)K^{-1}_{2,j-1}R_{j-1}(a)\widetilde{u}_{2,j-1}\\
=&\beta^{(2j-2)}(a) - \frac{1}{b-a}Q^*_{2,j-1}(a)P^{*^{-1}}_{1,j}(a) + Q^*_{1,j}(a)P^{*^{-1}}_{1,j}(a)\\
=&0
\end{aligned}$$

where use used (4.11), (4.20) and (3.13), it readily follows that:

$$\begin{pmatrix} I_q & \widetilde{L}_j^{(2n)} \\ 0_q & I_q \end{pmatrix} = \begin{pmatrix} I_q & \beta^{(2j-2)}(a) \\ 0_q & I_q \end{pmatrix}\begin{pmatrix} I_q & -\widetilde{u}^*_{2,j-1}R^*_{j-1}(a)K^{-1}_{2,j-1}R_{j-1}(a)\widetilde{u}_{2,j-1} \\ 0_q & I_q \end{pmatrix}. \tag{4.23}$$

Finally, by (4.13), (4.23), (B.3), (B.2) and (4.22) it follows (4.21). □

The following is the main result of this work.

Theorem 4.9. *Let* $b^{(2j)}$, M_j, $L_j^{(2n)}$ *and* L_j *be as in (4.1), (4.2), (4.10), (4.11) and (4.18). The following identities hold*
a)

$$b^{(2)}(z) = \begin{pmatrix} I_q & L_0 \\ 0_q & I_q \end{pmatrix}\begin{pmatrix} I_q & 0_q \\ -(z-a)M_1 & I_q \end{pmatrix}\begin{pmatrix} I_q & -L_0 \\ 0_q & I_q \end{pmatrix}b^{(2)}(a), \tag{4.24}$$

$$b^{(2j)}(z) = \overleftarrow{\prod_{k=1}^{j-1}} b_k^{-1}(a)\overleftarrow{\prod_{k=0}^{j-1}}\begin{pmatrix} I_q & L_k \\ 0_q & I_q \end{pmatrix}\begin{pmatrix} I_q & 0_q \\ -(z-a)M_j & I_q \end{pmatrix}\overrightarrow{\prod_{k=0}^{j-1}}\begin{pmatrix} I_q & -L_k \\ 0_q & I_q \end{pmatrix}\overrightarrow{\prod_{k=1}^{j}} b^{(2k)}(a) \tag{4.25}$$

for $2 \le j \le n$.
b)

$$b^{(2j)}(z) = \begin{pmatrix} I_q & -\widetilde{L}_{j-1}^{(2n)} \\ 0_q & I_q \end{pmatrix}\begin{pmatrix} I_q & 0_q \\ -(z-a)M_j & I_q \end{pmatrix}\begin{pmatrix} I_q & \widetilde{L}_{j-1}^{(2n)} \\ 0_q & I_q \end{pmatrix}b^{(2j)}(a). \tag{4.26}$$

for $1 \le j \le n$.

Proof. The proof follows readily from (4.6) and (4.21). □

By employing (1.7), (4.24), (4.25) and (4.26) we obtain the following Corollary.

Corollary 4.10. *The following representation of the resolvent matrix in the case of odd numbers of moments holds:*

$$U^{(2n)}(z) = \overrightarrow{\prod_{k=0}^{n-1}} \left[\begin{pmatrix} I_q & 0_q \\ -(z-a)M_k & I_q \end{pmatrix} \begin{pmatrix} I_q & L_k \\ 0_q & I_q \end{pmatrix} \right] \begin{pmatrix} I_q & 0_q \\ -(z-a)M_n & I_q \end{pmatrix}$$
$$\cdot \prod_{k=0}^{n-1} \begin{pmatrix} I_q & -L_k \\ 0_q & I_q \end{pmatrix} \overrightarrow{\prod_{k=0}^{n}} b^{(2k)}(a). \tag{4.27}$$

$$U^{(2n)}(z) = \begin{pmatrix} I_q & 0_q \\ -(z-a)M_0 & I_q \end{pmatrix} \overrightarrow{\prod_{k=1}^{n}} \left[\begin{pmatrix} I_q & -L_{k-1}^{(2n)} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -(z-a)M_k & I_q \end{pmatrix} \right.$$
$$\left. \cdot \begin{pmatrix} I_q & L_{k-1}^{(2n)} \\ 0_q & I_q \end{pmatrix} b^{(2k)}(a) \right]. \tag{4.28}$$

Remark 4.11. The equality (4.27) is a generalization of the analogous formula for the Stieltjes parameters of the truncated Stieltjes matrix moment problem in the case of an odd number of moments, obtained by Yu. Dyukarev in [16].

Proof. **Step 1.** In both sides of (4.27) set $a = 0$ and b tends to $+\infty$. Observe that

$$\lim_{b\to\infty} \prod_{k=0}^{n-1} \begin{pmatrix} I_q & -L_k(0) \\ 0_q & I_q \end{pmatrix} \overrightarrow{\prod_{k=0}^{n}} b^{(2k)}(a) = I_{2q}.$$

. **Step 2.** Multiply both sides of the equality 4.27 by the matrix $\begin{pmatrix} 0_q & I_q \\ I_q & 0_q \end{pmatrix}$. Thus one obtains the Dyukarev's formula [16, Theorem 7].

□

Observe that the resolvent matrix of the Stieltjes matrix moment problem in the case of an odd number of moments is obtained by applying Step 1 and Step 2.

Proposition 4.12. *Let $b^{(2j)}$ be as in (4.1), (4.2). The Blaschke–Potapov factor $b^{(2j)}$ of the resolvent matrix of the THMM problem in the case of an odd number of moments belongs then to the Potapov class in Π_+.*

Proof. Taking into account that $b^{(2j)^{-1}}(a)$ is a $\widetilde{J}_q$ unitary matrix and the fact that M_j, $L_j^{(2n)}$ are nonnegative Hermitian matrices, we have

$$\widetilde{J}_q - b^{(2j)^*}(z)\widetilde{J}_q b^{(2j)}(z) = -(z-\bar{z}) \begin{pmatrix} I_q \\ L_{j-1}^{(2n)} \end{pmatrix} M_j (I_q, L_{j-1}^{(2n)}) \geq 0,$$

for $\Im z > 0$. Therefore, $b^{(2j)}$ is a Potapov function in Π_+. □

A Orthogonal Matrix Polynomials on $[a,b]$

In this appendix we recall the OMP on $[a,b]$. In [29], (resp. [6]) it was proved that polynomials $\Gamma_{k,j}$ (resp. $P_{k,j}$) for $k=1,2$ are in fact OMP on $[a,b]$. In [3] some properties of second kind polynomials $Q_{k,j}$ and $\Theta_{k,j}$ for $k=1,2$ were discussed. In [9] explicit interrelations between $P_{k,j}$, $\Gamma_{k,j}$ and their second kind polynomials were studied.

A.1 Orthogonal Matrix Polynomials: The Case of Odd Number of Moments

Definition A.1. Let $(s_k)_{k=0}^{2j}$ be an odd positive Hausdorff on $[a,b]$ sequence. Let R_j, v_j, $H_{k,j}$, $Y_{k,j}$, $u_{k,j}$ for $k=1,2$ be as in (2.1), (2.3), (1.2), (1.3), (2.6), (2.7), (2.8). Define

$$P_{1,0}(z):=I_q,\;\; P_{2,0}(z):=I_q,\;\; Q_{1,0}(z):=0_q,\;\; Q_{2,0}(a,b,z):=-(u_{2,0}+z\,s_0), \tag{A.1}$$

$$P_{1,j}(z):=(-Y_{1,j}^{*}H_{1,j-1}^{-1},I_q)R_j(z)v_j,\quad 1\le j\le n, \tag{A.2}$$

$$P_{2,j}(a,b,z):=(-Y_{2,j}^{*}H_{2,j-1}^{-1},I_q)R_j(z)v_j,\quad 1\le j\le n-1, \tag{A.3}$$

$$Q_{1,j}(z):=-(-Y_{1,j}^{*}H_{1,j-1}^{-1},I_q)R_{1,j}(z)u_{1,j},\quad 1\le j\le n \tag{A.4}$$

and

$$Q_{2,j}(a,b,z):=-(-Y_{2,j}^{*}H_{2,j-1}^{-1},I_q)R_j(z)(u_{2,j}+zv_js_0),\quad 1\le j\le n-1. \tag{A.5}$$

A.2 Orthogonal Matrix Polynomials: The Case of Even Number of Moments

Let $n\in\mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q\times q$ matrices. Furthermore, let

$$\widetilde{u}_{1,0}:=s_0,\quad \widetilde{u}_{2,0}:=-s_0 \tag{A.6}$$

and for every $1\le j\le n-1$, let

$$\widetilde{u}_{1,j}:=y_{[0,j]}-b\begin{pmatrix}0_q\\ y_{[0,j-1]}\end{pmatrix},\quad \widetilde{u}_{2,j}:=-y_{[0,j]}+a\begin{pmatrix}0_q\\ y_{[0,j-1]}\end{pmatrix}. \tag{A.7}$$

For $1\le j\le n$ denote

$$\widetilde{Y}_{1,j}:=by_{[j,2j-1]}-y_{[j+1,2j]},\quad \widetilde{Y}_{2,j}:=-a\,y_{[j,2j-1]}+y_{[j+1,2j]}. \tag{A.8}$$

Let

$$K_{1,n}:=b\widetilde{H}_{0,n}-\widetilde{H}_{1,n},\quad K_{2,n}:=-a\widetilde{H}_{0,n}+\widetilde{H}_{1,n},\quad 0\le 2n+1\le m. \tag{A.9}$$

Definition A.2. Let the block Hankel matrices $K_{1,j}$ and $K_{2,j}$ be defined by (A.9). The sequence $(s_k)_{k=0}^{2j+1}$ is called Hausdorff positive (resp. nonnegative) on $[a,b]$ if the block Hankel matrices $K_{1,j}$ and $K_{2,j-1}$ are both positive (resp. nonnegative) definite matrices.

In [7, Theorem 1.3] it was proven that the THMM problem in the case of an even number of moments is solvable if and only if the sequence $(s_k)_{k=0}^{2n+1}$ is Hausdorff nonnegative on $[a,b]$.

We will consider only sequences which are Hausdorff positive on $[a,b]$.

Definition A.3. Let $K_{k,j}$, $\widetilde{u}_{k,j}$, $\widetilde{Y}_{k,j}$, for $k=1,2$, R_j and v_j be as in (A.9), (A.6), (A.7), (A.8), (2.1) and (2.3), respectively. Let $(s_k)_{k=0}^{2j+1}$ be a sequence which is Hausdorff positive on $[a,b]$. Let

$$\Gamma_{1,0}(z) := I_q, \Gamma_{2,0}(z) := I_q, \;\; \Theta_{1,0}(z) := s_0, \Theta_{2,0}(z) := -s_0 \tag{A.10}$$

for all $z\in\mathbb{C}$. For $k\in\{1,2\}$ and $1\le j\le n$ define

$$\Gamma_{1,j}(b,z) := (-\widetilde{Y}_{1,j}^{*}K_{1,j-1}^{-1}, I_q)R_j(z)v_j, \tag{A.11}$$

$$\Gamma_{2,j}(a,z) := (-\widetilde{Y}_{2,j}^{*}K_{2,j-1}^{-1}, I_q)R_j(z)v_j, \tag{A.12}$$

$$\Theta_{1,j}(b,z) := (-\widetilde{Y}_{1,j}^{*}K_{1,j-1}^{-1}, I_q)R_j(z)\widetilde{u}_{1,j}, \tag{A.13}$$

$$\Theta_{2,j}(a,z) := (-\widetilde{Y}_{2,j}^{*}K_{2,j-1}^{-1}, I_q)R_j(z)\widetilde{u}_{2,j} \tag{A.14}$$

for all $z\in\mathbb{C}$.

As in the case of an odd number of moments, we usually omit the dependence of the polynomials $\Gamma_{k,j}$ and $\Theta_{k,j}$ for $k=1,2$ on the parameters a and b.

B Auxiliary Blaschke–Potapov Factors in the Case of Odd Number of Moments

Let $P_{1,j}$, $Q_{1,j}$ and $\widehat{H}_{1,j}$ be as in Definition A.1, Definition A.3 and (2.9), respectively. Denote

$$c^{(2j)}(z) := \begin{pmatrix} I_q+(z-a)Q_{1,j}^{*}(a)\widehat{H}_{1,j}^{-1}P_{1,j}(a) & (z-a)Q_{1,j}^{*}(a)\widehat{H}_{1,j}^{-1}Q_{1,j}(a) \\ -(z-a)P_{1,j}^{*}(a)\widehat{H}_{1,j}^{-1}P_{1,j}(a) & I_q-(z-a)P_{1,j}^{*}(a)\widehat{H}_{1,j}^{-1}Q_{1,j}(a) \end{pmatrix}, \tag{B.1}$$

for $0\le j\le n$.

Proposition B.1. *[4, Theorem 3.2] Let $c^{(2j)}$, M_j and L_j be as in (B.1), (4.10) and (4.18), respectively. Then the following equality holds:*

$$c^{(2j)}(z) = \overleftarrow{\prod_{k=0}^{j-1}}\begin{pmatrix} I_q & L_k \\ 0_q & I_q\end{pmatrix}\begin{pmatrix} I_q & 0_q \\ -(z-a)M_j & I_q\end{pmatrix}\overrightarrow{\prod_{k=0}^{j-1}}\begin{pmatrix} I_q & -L_k \\ 0_q & I_q\end{pmatrix}, \tag{B.2}$$

for $0\le j\le n$.

The proof this Proposition is based in following equalities:

$$\overleftarrow{\prod_{k=0}^{j-1}}\begin{pmatrix} I_q & L_k \\ 0_q & I_q\end{pmatrix} = \begin{pmatrix} I_q & \sum_{k=0}^{j-1}L_k \\ 0_q & I_q\end{pmatrix} = \begin{pmatrix} I_q & \widetilde{u}_{2,j-1}^{*}R_{j-1}^{*}(a)K_{2,j-1}^{-1}R_{j-1}(a)\widetilde{u}_{2,j-1} \\ 0_q & I_q\end{pmatrix}. \tag{B.3}$$

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