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Existence of Periodic Solutions to Nonlinear Difference Equations at Full Resonance

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Abstract

The purpose of this paper is to search for periodic solutions to a system of nonlinear difference equations of the form

$$\Delta x(t) = f(\epsilon, t, x(t)).$$

The corresponding linear homogeneous system has an n-dimensional kernel, i.e. the system is at full resonance. We provide sufficient conditions for the existence of periodic solutions based on asymptotic properties of the nonlinearity f when $\epsilon = 0$. To this end, we employ a projection method using the Lyapunov-Schmidt procedure together with Brouwer's fixed point theorem.

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1 Introduction

In this paper, we study the solvability of nonlinear discrete systems of the form

$$\Delta x(t) = f(\epsilon, t, x(t)). \tag{1.1}$$

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In particular, we are interested in finding N-periodic solutions of the above system, where we assume $f(\epsilon,t,x) = f(\epsilon,t+N,x)$. Here, $f = (f_1, f_2, \ldots, f_n)^T$ where $f_i : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ for $i = 1,2,\ldots,n$. Note that the solution space of the corresponding linear homogeneous system is n-dimensional, i.e. the system is at full resonance. Our approach in providing sufficient conditions for the existence of periodic solutions to (1.1) depends significantly on this resonance along with asymptotic properties of the nonlinear function $f(0,t,x)$.

Since the solution space of the associated linear homogeneous equation is non-trivial, we will use a projection scheme (Lyapunov-Schmidt procedure) together with the Brouwer fixed point theorem to analyze the nonlinear problem (1.1). A similar approach has often been used in the study of both continuous and discrete dynamical systems (see, for instance, [1-7,9-13,15-21]). Our results complement previous work in the study of periodic discrete dynamical systems. We allow for higher-dimensional solution spaces of the associated linear problem as well as for more general asymptotic behavior of the nonlinear function f.

2 Preliminaries

For each natural number $N \geq 2$, let X_N be the set of all sequences x from $\{0,1,2,3,...\}$ into $\mathbb{R}^n$ that are N-periodic; that is, $x(l+N) = x(l)$ for every $l \in \{0,1,2,3,...\}$.
For $x \in X_N$, let $||x||_\infty = \sup\{|x(l)| : l = 0,1,2,3,...\}$.
We define $L : X_N \to X_N$ by

$$Lx(t) = \Delta x(t) = x(t+1) - x(t) \text{ for } t = 0,1,2,3,...,$$

and define $F_\epsilon : \mathbb{R}\times X_N \to X_N$ by

$$F_\epsilon(x)(t) = \begin{bmatrix} f_1(\epsilon,t,x(t)) \\ f_2(\epsilon,t,x(t)) \\ \vdots \\ f_n(\epsilon,t,x(t)) \end{bmatrix} \text{ for } t = 0,1,2,3,....$$

We assume each f_i is a continuous map from $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$ into $\mathbb{R}$ for $i = 1,2,\ldots,n$. It follows that F_ϵ is continuous. It is assumed that for some $m \in \mathbb{R}$, $|f_i(0,t,x)| \leq m$ for $i = 1,2,\ldots,n$. Hence, for all $x \in X_N$, $||F_0(x)||_\infty \leq m$.

Our problem, finding periodic solutions to the system

$$\Delta x(t) = f(\epsilon,t,x(t))$$

is equivalent to solving

$$Lx = F_\epsilon(x).$$

Since L is not invertible, we cannot apply the Brouwer Fixed Point Theorem directly. We shall decompose X_N using the methods described in [5]. We find projections, P, of X_N onto $ker(L)$, and E, of X_N onto $Im(L)$, so that we may write $X_N = ker(L) \oplus Im(I-P)$ and $X_N = Im(L) \oplus Im(I-E)$. The projections we use are those devised by Rodríguez [14].

Let

$$(I-E)x(t) = \sum_{l=0}^{N-1} x(l) \text{ for } t = 0,1,2,3,....$$

Let

$$Px(t) = \sum_{l=0}^{N-1} x(l) \text{ for } t = 0,1,2,3,....$$

Remark 2.1. If $\tilde{L}$ is the restriction of L to $Im(I-P)$ then $Im(\tilde{L}) = Im(L)$. $\tilde{L}$, viewed as a map from $Im(I-P)$ into $Im(L)$ is invertible. We denote $(\tilde{L})^{-1}$ by M. One may then verify:

i. M is bounded and linear,

ii. $MLx = (I-P)x$ for all $x \in D(L)$,

iii. $LMh = h$ for all $h \in Im(L)$,

iv. $EL = L$ and $(I-E)L = 0$,

v. $PM = 0$.

Proposition 2.2. *$Lx = F_\epsilon x$ is equivalent to*

$$\begin{cases} x = Px + MEF_\epsilon(x) \\ (I-E)F_\epsilon(Px + MEF_\epsilon(x)) = 0. \end{cases} \tag{2.1}$$

Proof. We have $Lx = F_\epsilon x$ if and only if

$$\begin{cases} E(Lx - F_\epsilon x) = 0 \\ (I-E)(Lx - F_\epsilon x) = 0 \end{cases}$$

if and only if

$$\begin{cases} Lx = EF_\epsilon(x) \\ (I-E)F_\epsilon(x) = 0 \end{cases}$$

if and only if

$$\begin{cases} (I-P)x = MEF_\epsilon(x) \\ (I-E)F_\epsilon(x) = 0 \end{cases}$$

if and only if

$$\begin{cases} x = Px + MEF_\epsilon(x) \\ (I-E)F_\epsilon(Px + MEF_\epsilon(x)) = 0. \end{cases}$$

□

3 Main Results

Since $\ker(L) = \text{span}\{e_1, e_2, \ldots, e_n\}$, where e_i is the ith standard unit basis vector, we may rewrite (2.1) of Proposition 2.2 as the equivalent system of $n+1$ equations

$$\begin{cases} x = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n + MEF_\epsilon(x) \\ 0 = \sum_{l=0}^{N-1} f_i(\epsilon, l, \alpha_1 + [MEF_\epsilon(x)]_1(l), \ldots, \alpha_n + [MEF_\epsilon(x)]_n(l)),\ i = 1, \ldots, n. \end{cases}$$

The proof of Theorem 3.1 relies on techniques that appear in [2,3,4,8,12,13,15,17].

Theorem 3.1. *Suppose that*

(i) $f_i : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ *for* $i = 1,2,\dots,n$ *is continuous, and for some* $m \in \mathbb{R}$, $|f_i(0,t,x)| \le m$ *for* $i = 1,2,\dots,n$.

(ii) For each $i = 1,2,\dots,n$, *there exist constants* $L_i, P_i, N_i > 0$ *such that for all* $x_i > L_i$, $f_i(0,t,x_1,\dots,x_i,\dots,x_n) \ge P_i$ *and* $f_i(0,t,x_1,\dots,-x_i,\dots,x_n) \le -N_i$ *for all* $t = 0,1,2,\dots$ *and all* $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n \in \mathbb{R}$.
Then, there exists an $\epsilon_0 > 0$ *such that for* $\epsilon \in [0,\epsilon_0]$, *there exists at least one periodic solution of*

$$\Delta x(t) = f(\epsilon,t,x(t)).$$

Proof. We define mappings

$$H_1 : \mathbb{R}\times X_N\times\mathbb{R}^n \to X_N,$$

$$H_{i+1} : \mathbb{R}\times X_N\times\mathbb{R}^n \to \mathbb{R},$$

and

$$H : \mathbb{R}\times X_N\times\mathbb{R}^n \to X_N\times\mathbb{R}^n$$

by

$$H_1(\epsilon,x,\alpha_1,\dots,\alpha_n) = \alpha_1 e_1 + \ldots + \alpha_n e_n + MEF_\epsilon(x),$$

and for $i = 1,\dots,n$,

$$H_{i+1}(\epsilon,x,\alpha_1,\dots,\alpha_n) = \alpha_i - \sum_{l=0}^{N-1} f_i(\epsilon,l,\alpha_1+[MEF_\epsilon(x)]_1(l),\dots,\alpha_n+[MEF_\epsilon(x)]_n(l)).$$

H is then given by

$$H(\epsilon,x,\alpha_1,\dots,\alpha_n) = (H_1(\epsilon,x,\alpha_1,\dots,\alpha_n),\dots,H_{n+1}(\epsilon,x,\alpha_1,\dots,\alpha_n)).$$

We shall first analyze the case when $\epsilon = 0$. Note that for $i = 1,2,\dots,n$, $|[MEF_0(x)]_i(l)| \le \|ME\|m$ for every $l \in \{0,1,2,...\}$ and every $x \in X_N$.

Consider $H_{i+1}(0,x,\alpha_1,\dots,\alpha_n)$ for each $i = 1,2,\dots,n$. If α_i is sufficiently large, we may ensure

$$f_i(0,l,\alpha_1+[MEF_0(x)]_1(l),\dots,\alpha_i+[MEF_0(x)]_i(l),\dots,\alpha_n+[MEF_0(x)]_n(l)) \ge P_i > 0$$

and

$$f_i(0,l,\alpha_1+[MEF_0(x)]_1(l),\dots,-\alpha_i+[MEF_0(x)]_i(l),\dots,\alpha_n+[MEF_0(x)]_n(l)) \le -N_i < 0$$

for every $l \in \{0,1,2,...\}$ and every $x \in X_N$. Therefore there is some $\gamma_i > Nm+1 > 0$ such that for all $x \in X_N$ and for all $\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n \in \mathbb{R}$,
$H_{i+1}(0,x,\alpha_1,\dots,\alpha_i,\dots,\alpha_n) < \alpha_i$ and $H_{i+1}(0,x,\alpha_1,\dots,-\alpha_i,\dots,\alpha_n) > -\alpha_i$ for $\alpha_i > \gamma_i$. We let $\delta_i = \gamma_i + Nm + 1$.

Now if $\alpha_i \in [\gamma_i, \delta_i]$, then for all $x \in X_N$ and $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \in \mathbb{R}$ we have $H_{i+1}(0, x, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$

$$
\begin{aligned}
&= \alpha_i - \sum_{l=0}^{N-1} f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \\
&\geq \alpha_i - \sum_{l=0}^{N-1} |f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))| \\
&\geq \alpha_i - Nm \\
&> \alpha_i - \gamma_i \\
&\geq 0
\end{aligned}
$$

and

$H_{i+1}(0, x, \alpha_1, \ldots, -\alpha_i, \ldots, \alpha_n)$

$$
\begin{aligned}
&= -\alpha_i - \sum_{l=0}^{N-1} f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, -\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \\
&\leq -\alpha_i + \sum_{l=0}^{N-1} |f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, -\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))| \\
&\leq -\alpha_i + Nm \\
&< -\alpha_i + \gamma_i \\
&\leq 0.
\end{aligned}
$$

Thus for all $x \in X_N$, $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \in \mathbb{R}$, and $\alpha_i \in [\gamma_i, \delta_i]$,

$$
H_{i+1}(0, x, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n), H_{i+1}(0, x, \alpha_1, \ldots, -\alpha_i, \ldots, \alpha_n) \in [-\alpha_i, \alpha_i] \subseteq [-\delta_i, \delta_i].
$$

Furthermore, if $0 \leq \alpha_i < \gamma_i$, for all $x \in X_N$ and $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \in \mathbb{R}$,

$|H_{i+1}(0, x, \alpha_1, \ldots, \pm\alpha_i, \ldots, \alpha_n)|$

$$
\begin{aligned}
&\leq |\pm\alpha_i| \\
&\quad + \sum_{l=0}^{N-1} |f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \pm\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))| \\
&\leq \gamma_i + Nm \\
&< \delta_i.
\end{aligned}
$$

We have shown that for $\epsilon = 0$, H_{i+1} maps $X_N \times [-\delta_i, \delta_i] \times \mathbb{R}^{n-1}$ into $[-\delta_i, \delta_i]$ for each $i = 1, 2, \ldots, n$.

Define $\mathcal{B} = \{(x,\alpha_1,\ldots,\alpha_n) \in X_N \times \mathbb{R}^n : \|x\|_\infty \leq \delta_1 + \ldots + \delta_n + \|ME\|m + 1$ and $|\alpha_i| \leq \delta_i$ for each $i = 1,2,\ldots,n\}$, and note that $\mathcal{B}$ is a non-empty, closed, bounded, convex set.

From the above results, it follows that for $(x,\alpha_1,\ldots,\alpha_n) \in \mathcal{B}$, $H(0,x,\alpha_1,\ldots,\alpha_n) \in \mathcal{B}$. For if $(x,\alpha_1,\ldots,\alpha_n) \in \mathcal{B}$, then $H_{i+1}(0,x,\alpha_1,\ldots,\alpha_n) \in [-\delta_i,\delta_i]$, while

$$\begin{aligned}\|H_1(0,x,\alpha_1,\ldots,\alpha_n)\|_\infty &\leq |\alpha_1|\,\|e_1\|_\infty + \ldots + |\alpha_n|\,\|e_n\|_\infty + \|MEF_0(x)\|_\infty \\ &\leq \delta_1 + \ldots + \delta_n + \|ME\|m \\ &< \delta_1 + \ldots + \delta_n + \|ME\|m + 1.\end{aligned}$$

Since H is a continuous function, the Brouwer Fixed Point Theorem guarantees existence of at least one fixed point of H in $\mathcal{B}$.

Now consider the case when $\epsilon > 0$. We will show that there exists $\epsilon_0 \in \mathbb{R}$ such that for each $\epsilon \leq \epsilon_0$, $H(\epsilon,x,\alpha_1,\ldots,\alpha_n) \in \mathcal{B}$ whenever $(x,\alpha_1,\ldots,\alpha_n) \in \mathcal{B}$.

Choose ϵ small enough so that

$$\Big| f_i(\epsilon,l,\alpha_1 + [MEF_0(x)]_1(l),\ldots,\pm\alpha_i + [MEF_0(x)]_i(l),\ldots,\alpha_n + [MEF_0(x)]_n(l)) -$$
$$f_i(0,l,\alpha_1 + [MEF_0(x)]_1(l),\ldots,\pm\alpha_i + [MEF_0(x)]_i(l),\ldots,\alpha_n + [MEF_0(x)]_n(l)) \Big|$$
$$< \min\{\frac{P_i}{2},\frac{N_i}{2},\frac{1}{N}\}$$

for all $(x,\alpha_1,\ldots,\alpha_n) \in \mathcal{B}$.

Note that we may now assume ϵ lies in some compact interval of $\mathbb{R}$, from which it follows that for all $x \in \mathcal{B}_x = \{\|x\|_\infty \leq \delta_1 + \ldots + \delta_n + \|ME\|m + 1\}$, $\|F_\epsilon(x) - F_0(x)\|_\infty$ can be made arbitrarily small for sufficiently small ϵ. For our purposes, choose ϵ small enough so that for all $x \in \mathcal{B}_x$,

$$\begin{aligned}\|MEF_\epsilon(x)\|_\infty &\leq \|ME\|(\|F_\epsilon(x) - F_0(x)\|_\infty + \|F_0(x)\|_\infty) \\ &\leq \|ME\|(\frac{1}{\|ME\|} + m) \\ &= 1 + \|ME\|m.\end{aligned}$$

For each ϵ satisfying the above properties, it now follows that $H(\epsilon,x,\alpha_1,\ldots,\alpha_n) \in \mathcal{B}$ whenever $(x,\alpha_1,\ldots,\alpha_n) \in \mathcal{B}$. First observe that for all $(x,\alpha_1,\ldots,\alpha_n) \in \mathcal{B}$,

$$\begin{aligned}\|H_1(\epsilon,x,\alpha_1,\ldots,\alpha_n)\|_\infty &\leq |\alpha_1|\,\|e_1\|_\infty + \ldots + |\alpha_n|\,\|e_n\|_\infty + \|MEF_\epsilon(x)\|_\infty \\ &\leq \delta_1 + \ldots + \delta_n + \|ME\|m + 1.\end{aligned}$$

Next, for all $\alpha_i \in [\gamma_i, \delta_i]$,

$$
\begin{aligned}
&H_{i+1}(\epsilon, x, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)\\
&\quad = \alpha_i - \sum_{l=0}^{N-1}\Big(f_i(\epsilon, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\Big)\\
&\qquad - \sum_{l=0}^{N-1} f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\\
&\quad \leq \alpha_i - \sum_{l=0}^{N-1}\Big(f_i(\epsilon, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\Big)\\
&\qquad - NP_i\\
&\quad < \alpha_i - \frac{NP_i}{2}\\
&\quad < \alpha_i,
\end{aligned}
$$

while a similar calculation shows that $H_{i+1}(\epsilon, x, \alpha_1, \ldots, -\alpha_i, \ldots, \alpha_n) > -\alpha_i$ for all $\alpha_i \in [\gamma_i, \delta_i]$. Also, for all $\alpha_i \in [\gamma_i, \delta_i]$,

$$
\begin{aligned}
&H_{i+1}(\epsilon, x, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)\\
&\quad = \alpha_i - \sum_{l=0}^{N-1}\Big(f_i(\epsilon, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\Big)\\
&\qquad - \sum_{l=0}^{N-1} f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\\
&\quad \geq \alpha_i - \sum_{l=0}^{N-1}\Big|f_i(\epsilon, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l))\Big|\\
&\qquad - \sum_{l=0}^{N-1}\Big|f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\geq \alpha_i - \sum_{l=0}^{N-1} \Big| f_i(\epsilon, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big| \\
&\qquad - Nm \\
&> \alpha_i - Nm - 1 \\
&> \alpha_i - \gamma_i \\
&\geq 0,
\end{aligned}
$$

while a similar calculation shows that $H_{i+1}(\epsilon, x, \alpha_1, \ldots, -\alpha_i, \ldots, \alpha_n) \leq 0$ for all $\alpha_i \in [\gamma_i, \delta_i]$.

Finally, for all $\alpha_i \in [0, \gamma_i]$,

$$
\begin{aligned}
&|H_{i+1}(\epsilon, x, \alpha_1, \ldots, \pm\alpha_i, \ldots, \alpha_n)| \\
&\leq |\pm\alpha_i| + \sum_{l=0}^{N-1} \Big| f_i(\epsilon, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \pm\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \pm\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big| \\
&\qquad + \sum_{l=0}^{N-1} \Big| f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \pm\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big| \\
&\leq \gamma_i + \sum_{l=0}^{N-1} \Big| f_i(\epsilon, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \pm\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \\
&\qquad - f_i(0, l, \alpha_1 + [MEF_0(x)]_1(l), \ldots, \pm\alpha_i + [MEF_0(x)]_i(l), \ldots, \alpha_n + [MEF_0(x)]_n(l)) \Big| \\
&\qquad + Nm \\
&< \gamma_i + Nm + 1 \\
&= \delta_i.
\end{aligned}
$$

Hence for each ϵ sufficiently small, $H(\epsilon, x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$ whenever $(x, \alpha_1, \ldots, \alpha_n) \in \mathcal{B}$. Since H is a continuous function, the Brouwer Fixed Point Theorem guarantees existence of at least one fixed point of H in $\mathcal{B}$. This fixed point is a periodic solution of

$$\Delta x(t) = f(\epsilon, t, x(t)).$$

□

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